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Zero Forcing Propagation Time on Oriented Graphs

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Zero Forcing Propagation Time on Oriented Graphs

Abstract
Zero forcing is an iterative coloring procedure on a graph that starts by initially coloring vertices white and blue and then repeatedly applies the following rule: if any blue vertex has a unique (out-)neighbor that is colored white, then that neighbor is forced to change color from white to blue. An initial set of blue vertices that can force the entire graph to blue is called a zero forcing set. In this paper we consider the minimum number of iterations needed for this color change rule to color all of the vertices blue, also known as the propagation time, for oriented graphs. We produce oriented graphs with both high and low propagation times, consider the possible propagation times for the orientations of a fixed graph, and look at balancing the size of a zero forcing set and the propagation time.

Keywords
zero forcing process, propagation time, oriented graphs, Hessenberg path, throttling

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Algebra | Discrete Mathematics and Combinatorics

Comments

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Zero forcing propagation time on oriented graphs

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\textbf{Abstract}
Zero forcing is an iterative coloring procedure on a graph that starts by initially coloring vertices white and blue and then repeatedly applies the following rule: if any blue vertex has a unique (out-)neighbor that is colored white, then that neighbor is forced to change color from white to blue. An initial set of blue vertices that can force the entire graph to blue is called a zero forcing set. In this paper we consider the minimum number of iterations needed for this color change rule to color all of the vertices blue, also known as the propagation time, for oriented graphs. We produce oriented graphs with both high and low propagation times, consider the possible propagation times for the orientations of a fixed graph, and look at balancing the size of a zero forcing set and the propagation time.

\textbf{Keywords:} zero forcing process, propagation time, oriented graphs, Hessenberg path, throttling

\textbf{2000 MSC:} 05C20, 05C15, 05C57

\section{1. Introduction}
Given a directed graph with no loops (i.e., a simple digraph), there are many possible processes that can be used to simulate information spreading. In the
simplest model, each vertex can have two states, knowing or not knowing (using
the colors blue and white, respectively), and then have a color change rule
for changing a vertex from not knowing to knowing (i.e., changes from white
to blue). For each possible color change rule there are a variety of questions
including finding the minimum number of vertices that if initially colored blue
will eventually change all the vertices blue, or finding the length of time it takes
for a graph to become blue. The goal in this paper is to consider a particular
color change rule, known as zero forcing, and to focus on the amount of time it
takes to turn all the vertices blue, known as propagation time, on digraphs and
specifically oriented graphs.

The zero forcing process on a simple digraph is based on an initial coloring
of each vertex as blue or white and the repeated application of the following
coloring rule: If a blue vertex has exactly one white out-neighbor, then that
out-neighbor will change from white to blue. In terms of rumor spreading, this
can be rephrased in the following way: “If I know a secret and all except one of
my friends knows the same secret, then I will share that secret with my friend
that doesn’t know.”

The process of zero forcing was introduced originally for (undirected) graphs
[2] and extended to digraphs in Barioli et al. [3]. Zero forcing for simple digraphs
was studied in [4, 9]. This process is of interest because there is a relationship
between the minimum number of vertices initially colored blue that can transform
the entire graph to blue (also known as the zero forcing number, and the
geometric multiplicity of the eigenvalue 0 for a matrix associated with a graph.
In general, the zero forcing number can be determined computationally but
is NP-hard [1]; however, it has been determined for several families of graphs
(more information can be found in the recent survey by Fallat and Hogben [8]).

While most of the focus of the literature has been on the determination of
the zero forcing number, another natural question to examine is the amount
of time it takes to turn all of the vertices blue, i.e., the propagation time.
This study was initiated for undirected graphs in Hogben et al. [10] where
extremal configurations were determined (i.e., graphs that propagate as quickly
or as slowly as possible) and in Chilakamarri et al. [7] where propagation time
(there called iteration index) was computed for some families of graphs. This
paper expands the study of propagation time to oriented graphs. In particular,
there are some subtle and important distinctions between undirected graphs
and oriented graphs.

In the remainder of the introduction we introduce the notation and give pre-
cise terminology. In Section 2 we show that the propagation time is not affected
when the direction of each arc in a simple digraph is reversed. In Sections 3 and
4 we consider oriented graphs that have high and low propagation times, respec-
tively. For a given graph $G$ there are many possible orientations and this gives
rise to the following problem: For a given graph $G$, find the propagation time
of $G$ as $G$ ranges over all possible orientations of $G$. In Section 5 we consider
such orientation propagation intervals. Finally, in Section 6 we consider what
happens when we balance the size of the zero forcing set with the propagation
time; in particular we show that, unlike in simple graphs, we cannot always obtain significant savings.

1.1. Terminology and definitions

A simple graph (respectively, simple digraph) is a finite undirected (respectively, directed) graph that does not allow loops or more than one copy of one edge or arc; a simple digraph does allow double arcs, i.e., both the arcs \((u, v)\) and \((v, u)\). We use \(G = (V(G), E(G))\) to denote a simple graph and \(\Gamma = (V(\Gamma), E(\Gamma))\) to denote a simple digraph, where \(V\) and \(E\) are the vertex and edge (or arc) sets, respectively. An oriented graph is a simple digraph in which there are no double arcs, i.e., if \((u, v)\) is an arc in \(\Gamma\) then \((v, u)\) is not an arc in \(\Gamma\). For a simple graph \(G\), we also let \(\overrightarrow{G}\) denote an orientation of \(G\), i.e., \(\overrightarrow{G}\) is an oriented graph where ignoring the orientations of the arcs gives the graph \(G\).

For a digraph \(\Gamma\) having \(u, v \in V(\Gamma)\) and \((u, v) \in E(\Gamma)\), we say that \(v\) is an out-neighbor of \(u\) and that \(u\) is an in-neighbor of \(v\). The set of all in-neighbors of \(v\) is denoted \(N^-(v)\) and the cardinality of \(N^-(v)\) is the in-degree of \(v\), denoted \(\deg^-(v)\). Similarly, the set of all out-neighbors of \(v\) is \(N^+(v)\) and the cardinality of \(N^+(v)\) is the out-degree, denoted \(\deg^+(v)\).

For a simple digraph \(\Gamma\), the zero forcing propagation process can be described as follows. Let \(B \subseteq V(\Gamma)\), let \(B(0) := B\) and iteratively define \(B^{(t+1)}\) as the set of vertices \(w\) where for some \(v \in \bigcup_{i=0}^{t} B^{(i)}\) we have that \(w\) is the unique out-neighbor of \(v\) that is not in \(\bigcup_{i=0}^{t} B^{(i)}\). Here \(B(0)\) represents the initial set of vertices colored blue, and at each stage we color as many vertices blue as possible (i.e., we apply the coloring rule simultaneously to all vertices). We say a set \(B\) is a zero forcing set if \(\bigcup_{i=0}^{t} B^{(i)} = V(\Gamma)\) for some \(t\). Further, the propagation time of \(B\), denoted \(pt(\Gamma, B)\), is the minimum \(t\) so that \(\bigcup_{i=0}^{t} B^{(i)} = V(\Gamma)\) (i.e., the minimum amount of time needed for \(B\) to color the entire graph blue).

One way to achieve fast propagation is to simply let \(B(0) = V(\Gamma)\), and be done at time 0. However, we are primarily interested in the propagation time of minimum sized \(B\). In particular, for a simple digraph \(\Gamma\) we will let \(Z(\Gamma)\) denote the minimum size of a zero forcing set for \(\Gamma\). We then define propagation time as follows:

\[
pt(\Gamma) = \min \{ pt(\Gamma, B) : B \text{ is a minimum zero forcing set} \}.
\]

Example 1.1. Consider the oriented graph \(\overrightarrow{G}\) shown in Figure 1. Since the vertices \(c\), \(d\) and \(f\) are not the out-neighbors of any vertices, they cannot be changed to blue by the coloring rule. Therefore these three vertices must be in every zero forcing set of \(\overrightarrow{G}\). We now show that these three vertices form a zero forcing set (and in particular this is the unique minimum cardinality zero

\textsuperscript{2}For visual simplicity, the arrow is only over the main symbol, e.g., an orientation of \(K_n\) is denoted by \(\overrightarrow{K_n}\) rather than \(\overrightarrow{K^2_n}\).
forcing set), allowing us to conclude $Z(\overrightarrow{G}) = 3$. Suppose that $B^{(0)} = \{c, d, f\}$ and mark these vertices by coloring them blue (see $t = 0$ in Figure 1). Since $e$ is the unique white out-neighbor of $f$ then we can color $e$ blue, but this is the only vertex that can be colored at this time, and so we have $B^{(1)} = \{e\}$. The state of our coloring at time $t = 1$ is shown in Figure 1. Now $b$ is the unique white out-neighbor of $e$ and that $a$ is the unique white out-neighbor of $d$, and so we can color both of them and we have $B^{(2)} = \{a, b\}$. At this stage all the vertices are blue and so the propagation time corresponding to this set is 2. As already noted, $\{c, d, f\}$ is the unique minimum zero forcing set, so $\text{pt}(\overrightarrow{G}) = 2$.

Consider the zero forcing propagation process for a simple digraph $\Gamma$ and zero forcing set $B$. When a white vertex $v$ is the unique white out-neighbor of a blue vertex $u$, then we say that $u$ forces $v$ to change its color, and we write $u \rightarrow v$. Given a set $B$ we can consider the set of arcs that correspond to forces that were used in coloring the graph. This collection of arcs is known as a set of forces, and denoted by $\mathcal{F}$. When there is a white vertex that could be changed to blue by two different in-neighbors we put only one of the corresponding arcs in $\mathcal{F}$. In particular, for a given set $B$ there are possibly many different sets of forces for the propagation process. However, whether or not $B$ is a zero forcing set, and similarly the propagation time, is not dependent on which choices made when including forcing arcs (see [5] and [10] for more information).

The subdigraph $(V, \mathcal{F})$ of $\Gamma = (V, E)$ is a collection of disjoint directed paths, where each vertex in $B$ is the tail of a path. In particular, at each time in the propagation process, at most one vertex is added to each path, and thus $|B^{(t)}| \leq |B|$. The next observation is an immediate consequence.

**Observation 1.2.** For a simple digraph $\Gamma$,

$$\frac{|\Gamma| - Z(\Gamma)}{Z(\Gamma)} \leq \text{pt}(\Gamma) \leq |\Gamma| - Z(\Gamma).$$

Finally, without loss of generality we can assume that our digraphs are connected (meaning the underlying simple graph is connected). This is because once the zero forcing numbers and propagation times on each component are known, the zero forcing number and propagation time on the whole graph are
known. This is summarized in the next observation (the statement about zero forcing number appears in the literature, e.g., [5]).

**Observation 1.3.** For a simple digraph $\Gamma$ with connected components $\Gamma_1, \ldots, \Gamma_h$,

$$Z(\Gamma) = \sum_{i=1}^{h} Z(\Gamma_i)$$

and

$$\text{pt}(\Gamma) = \max_i \text{pt}(\Gamma_i).$$

2. Reversing Arcs

Although our focus is on oriented graphs, the results in this section are true for simple digraphs, so we state them that way. Given a simple digraph $\Gamma$, we let $\Gamma^T$ be the simple digraph where the direction of each arc has been reversed. (Note that the adjacency matrix of $\Gamma^T$ is the transpose of the adjacency matrix of $\Gamma$, which motivates the notation.) Reversing the arcs will generally change what the zero forcing sets are and how they propagate. However, we show in Theorem 2.5 below that $\text{pt}(\Gamma^T) = \text{pt}(\Gamma)$, following the arguments in [10].

Let $\Gamma$ be a simple digraph, $B$ a minimum zero forcing set of $\Gamma$, and $F$ a set of forces of $B$. The terminus of $F$, denoted $\text{Term}(F)$, is the set of vertices that do not perform a force in $F$, i.e., these are the heads of the directed paths formed by $F$ (note that if a vertex in $B$ never forces, then it is both the tail and head on a path with no arcs). Let $\text{Rev}(F)$ correspond to the set of forces found by reversing the direction of each arc of $F$. Note that $F \subseteq E(\Gamma)$ and $\text{Rev}(F) \subseteq E(\Gamma^T)$.

**Proposition 2.1.** [5] Let $\Gamma$ be a simple digraph, $B$ a minimum zero forcing set of $\Gamma$, and $F$ a set of forces of $B$. Then $\text{Term}(F)$ is a zero forcing set for $\Gamma^T$ and $\text{Rev}(F)$ is a set of forces. Hence $Z(\Gamma^T) = Z(\Gamma)$.

We previously have defined propagation in terms of an initial set $B$, but we can also define propagation using the set of forces $F$.

**Definition 2.2.** Let $\Gamma = (V,E)$ be a simple digraph and $B$ a zero forcing set of $\Gamma$. For a set of forces $F$ of $B$ that colors all vertices, define $F^{(0)} = B$. For $t \geq 0$, let $F^{(t+1)}$ be the set of vertices $w$ such that for some $v \in \bigcup_{i=0}^{t} F^{(i)}$, the arc $(v, w)$ appears in $F$, $w \notin \bigcup_{i=0}^{t} F^{(i)}$, and $w$ is the only out-neighbor of $v$ not in $\bigcup_{i=0}^{t} F^{(i)}$. (Note that the set $F$ is a collection of arcs, while the sets $F^{(i)}$ are collections of vertices.) The propagation time of $F$ in $\Gamma$, denoted $\text{pt}(\Gamma, F)$, is the minimum $t$ such that $\bigcup_{i=0}^{t} F^{(i)} = V(\Gamma)$.

We now give a connection between the propagation time given by $F$ in $\Gamma$ and the propagation time given by $\text{Rev}(F)$ in $\Gamma^T$.

**Lemma 2.3.** Let $\Gamma = (V,E)$ be a simple digraph, $B$ a minimum zero forcing set, $F$ a set of forces of $B$, and $1 \leq t \leq \text{pt}(\Gamma)$. If $(v, u) \in F$ with $u \in F^{(\text{pt}(\Gamma)-t+1)}$, then $v \in \bigcup_{i=0}^{t} \text{Rev}(F)^{(i)}$. 

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Proof. By Proposition 2.1 we have \( \text{Term}(\mathcal{F}) \) is a zero forcing set for \( \Gamma^T \) with forcing set \( \text{Rev}(\mathcal{F}) \). We establish the result by induction on \( t \). For \( t = 1 \), let \( u \in \mathcal{F}^{\text{pt}(\Gamma)} \). Then \( u \in \text{Term}(\mathcal{F}) = \text{Rev}(\mathcal{F})^{(0)} \). If \( x \neq v \) is an in-neighbor of \( u \) in \( \mathcal{F} \), then \( x \) cannot force in \( \mathcal{F} \) since \( u \in \mathcal{F}^{\text{pt}(\Gamma)} \). So \( x \in \text{Term}(\mathcal{F}) = \text{Rev}(\mathcal{F})^{(0)} \). Hence, \( v \) is the only white out-neighbor of \( u \) in \( \text{Rev}(\mathcal{F}) \). So \( v \in \text{Rev}(\mathcal{F})^{(1)} \).

Assume that the claim is true for \( 1 \leq s \leq t \). Suppose \( u \in \text{Term}(\Gamma) - (t+1) + 1 \). Then \( v \rightarrow u \) at time \( \text{pt}(\Gamma) - t \), so \( u \) cannot perform a force in \( \mathcal{F} \) until \( \text{pt}(\Gamma) - t + 1 \) or later. Thus \( u \in \bigcup_{i=0}^{t+1} \text{Rev}(\mathcal{F})^{(i)} \) by the induction hypothesis. If \( x \neq v \) is an in-neighbor of \( u \) in \( \mathcal{F} \), then \( x \) cannot perform a force in \( \mathcal{F} \) until \( \text{pt}(\Gamma) - t + 1 \) or later. So \( x \in \bigcup_{i=0}^{t+1} \text{Rev}(\mathcal{F})^{(i)} \). Thus, if \( v \notin \bigcup_{i=0}^{t+1} \text{Rev}(\mathcal{F})^{(i)} \), then \( v \in \text{Rev}(\mathcal{F})^{(t+1)} \), i.e., \( v \in \bigcup_{i=t+1}^{\infty} \text{Rev}(\mathcal{F})^{(i)} \) as desired. \( \square \)

Corollary 2.4. Let \( \Gamma = (V, E) \) be a simple digraph, \( B \) a minimum zero forcing set of \( \Gamma \), and \( \mathcal{F} \) a forcing set of \( B \). Then \( \text{pt}(\Gamma, \text{Rev}(\mathcal{F})) \leq \text{pt}(\Gamma, \mathcal{F}) \).

A minimum zero forcing set \( B \) of \( \Gamma \) is said to be an efficient zero forcing set if \( \text{pt}(\Gamma, B) = \text{pt}(\Gamma) \). A set of forces \( \mathcal{F} \) of an efficient forcing set \( B \) is efficient if \( \text{pt}(\Gamma, \mathcal{F}) = \text{pt}(\Gamma) \).

Theorem 2.5. Let \( \Gamma = (V, E) \) be a simple digraph. Then \( \text{pt}(\Gamma^T, \text{Rev}(\mathcal{F})) \leq \text{pt}(\Gamma) \).

Proof. Choose an efficient zero forcing set \( B \) and efficient set of forces \( \mathcal{F} \), so \( \text{pt}(\Gamma, \mathcal{F}) = \text{pt}(\Gamma) \). Then by Corollary 2.4,

\[ \text{pt}(\Gamma^T) \leq \text{pt}(\Gamma^T, \text{Rev}(\mathcal{F})) \leq \text{pt}(\Gamma, \mathcal{F}) = \text{pt}(\Gamma) \]

By reversing the roles of \( \Gamma \) and \( \Gamma^T \), we also obtain the reverse inequality. \( \square \)

3. High Propagation Times

In this section we focus on oriented graphs that have high propagation times. The two key elements to obtain high propagation time are a small zero forcing set and few simultaneous forces occurring at each time step. Although our primary interest is oriented graphs, much of the literature deals with simple digraphs.

A Hessenberg path is a simple digraph with vertices \( \{v_1, v_2, \ldots, v_n\} \) that contains the arcs \((v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)\), and does not contain any arc of the form \((v_i, v_j)\) with \( j > i + 1 \). Note that no restrictions are placed on arcs of the form \((v_i, v_j)\) with \( i > j + 1 \), i.e., back arcs are allowed. (A single isolated vertex is a Hessenberg path.)

Combining [9, Lemma 2.15] (which shows that \( Z(\Gamma) = 1 \) if and only if \( \Gamma \) is a Hessenberg path) and Observation 1.2 we have the following.

Observation 3.1. For any simple digraph \( \Gamma \), the following are equivalent:

1. \( Z(\Gamma) = 1 \).
2. \( \text{pt}(\Gamma) = |\Gamma| - 1 \).
3. \( \Gamma \) is a Hessenberg path.
One natural question is whether a graph $G$ can be oriented in such a way as to produce a specific propagation time. In studying high propagation time, we ask which graphs $G$ can be oriented to produce $\text{pt}(\overrightarrow{G}) = |G| - 1$. A Hamilton path in a graph $G$ is a subgraph that is a path and includes all vertices of $G$.

**Proposition 3.2.** A graph $G$ has an orientation $\overrightarrow{G}$ with $\text{pt}(\overrightarrow{G}) = |G| - 1$ if and only if $G$ has a Hamilton path.

**Proof.** If $\text{pt}(\overrightarrow{G}) = |G| - 1$, then $\overrightarrow{G}$ is a Hessenberg path on $(v_1, v_2, \ldots, v_n)$, in which case $(v_1, v_2, \ldots, v_n)$ is a Hamilton path of $G$. If $(v_1, v_2, \ldots, v_n)$ is a Hamilton path of $G$, then we can orient $G$ so that $\overrightarrow{G}$ is a Hessenberg path on $(v_1, v_2, \ldots, v_n)$ by choosing the arcs $(v_i, v_{i+1})$ for $i = 1, \ldots, n-1$, and for every edge between $v_i$ and $v_j$ with $j > i + 1$ choosing the back arc $(v_j, v_i)$. Since $\overrightarrow{G}$ is a Hessenberg path, $\text{pt}(\overrightarrow{G}) = |G| - 1$. \hfill \square

A simple digraph $\Gamma$ is a digraph of two parallel Hessenberg paths [5] if $\Gamma$ is not itself a Hessenberg path and the vertices of $\Gamma$ can be partitioned as $V(\Gamma) = V(1) \dot{\cup} V(2)$, with the notation $V(h) = \{ p_1^{(h)}, \ldots, p_s^{(h)} \}$ for $h = 1, 2$, so the following is satisfied:

(i) $P(h) = \Gamma[V(h)]$ is a Hessenberg path, and

(ii) there are no $i, j, k, \ell$ with $i < j, k < \ell$, and $(p_k^{(1)}, p_j^{(2)}), (p_i^{(2)}, p_{i+1}^{(1)}) \in E(\Gamma)$ (in other words, there are no forward crossing arcs between the two Hessenberg paths).

**Theorem 3.3.** [5] For any simple digraph $\Gamma$, $Z(\Gamma) = 2$ if and only if $\Gamma$ is a digraph of two parallel Hessenberg paths.

If $\Gamma$ is a digraph of two parallel Hessenberg paths on the Hessenberg paths $P(h), h = 1, 2$, we let $\Gamma(P^{(1)}, P^{(2)})$ refer to this particular way of decomposing $\Gamma$ as a digraph of two parallel Hessenberg paths. In this case, $B = \{ p_1^{(1)}, p_1^{(2)} \}$ is a minimum zero forcing set for $\Gamma$. Moreover, if $B' = \{ v_1, v_2 \}$ is any minimum zero forcing set for $\Gamma$, $\Gamma$ can be expressed as $\Gamma(P^{(1)}, P^{(2)})$ with $p_1^{(1)} = v_1$ and $p_1^{(2)} = v_2$.

An oriented graph of two parallel Hessenberg paths $\overrightarrow{G}$ is a digraph of two parallel Hessenberg paths with no double arcs. The next statement is now immediate from Observations 1.2 and 3.1, and Theorem 3.3.

**Observation 3.4.** For any oriented graph $\overrightarrow{G}$, if $\text{pt}(\overrightarrow{G}) = |\overrightarrow{G}| - 2$, then $Z(\overrightarrow{G}) = 2$ and $\overrightarrow{G}$ is an oriented graph of two parallel Hessenberg paths.

As is the case for graphs, the converse of Observation 3.4 is false, as shown in the next example.

**Example 3.5.** Let $\overrightarrow{P_4}$ be the oriented graph in Figure 2. Then $Z(\overrightarrow{P_4}) = 2$ because vertices $a$ and $b$ have in-degree zero (and $\overrightarrow{P_4}$ is an oriented graph of two parallel Hessenberg paths), but $\text{pt}(\overrightarrow{P_4}) = 1 \neq |\overrightarrow{P_4}| - 2$. \hfill 7
By Observation 3.4, to achieve $\text{pt}(\vec{G}) = |G| - 2$ it must be the case that $Z(\vec{G}) = 2$, and for every minimum zero forcing set exactly one force occurs at each time step. If $\vec{G}$ is disconnected, then $\text{pt}(\vec{G}) = |G| - 2$ if and only if $\vec{G} = K_1 \cup \vec{H}$ where $\vec{H}$ is a Hessenberg path. Thus, we consider only connected graphs.

We extend the notation and definitions in [10] to oriented graphs, but there are some significant differences caused by the orientation, so the definition of zig-zag path in Definition 3.6 and the conditions in Theorem 3.7 are somewhat different. Suppose $\vec{G}(P^{(1)}, P^{(2)})$ is an oriented graph of two parallel Hessenberg paths. The notation $x \prec y$ means $x$ and $y$ are on the same path $P^{(h)}$ and for some $i < j$, $x = p_i^{(h)}$ and $y = p_j^{(h)}$. For $i > 1$, we say that $p_{i-1}^{(h)} = \text{prev}(p_i^{(h)})$ and next($p_{i-1}^{(h)}$) = $p_i^{(h)}$. Furthermore, alt($z_i$) denotes the out-neighbors of $z_i$ not in the same Hessenberg path as $z_i$.

**Definition 3.6.** An orientation of two parallel Hessenberg paths $\vec{G}(P^{(1)}, P^{(2)})$ is a zig-zag orientation, denoted $\vec{G}(P^{(1)}, P^{(2)}, Q)$, if $\vec{G}(P^{(1)}, P^{(2)})$ contains a directed path $Q = (z_1, z_2, \ldots, z_r)$ satisfying the following conditions:

1. $z_i \in V^{(1)}$ for $i$ odd, $z_i \in V^{(2)}$ for $i$ even.
2. $z_i \prec z_{i+2}$ for $1 \leq i \leq r - 2$.
3. For $1 \leq i \leq r - 1$, if $u \in \text{alt}(z_i)$ then $u \preceq z_{i+1}$.

Subject to the constraint of being an orientation of two parallel Hessenberg paths, extra arcs that are not part of either Hessenberg path or the zig-zag path are permitted.

The following theorem characterizes whether a zero forcing set $B$ of cardinality two achieves $\text{pt}(\vec{G}, B) = |G| - 2$. Note that the zig-zag path described in the theorem, chosen to capture the one force at each step property, may not be the only zig-zag path for this zero forcing set.

**Theorem 3.7.** Let $\vec{G}$ be a connected oriented graph with $Z(\vec{G}) = 2$ and minimum zero forcing set $B$. Then $\text{pt}(\vec{G}, B) = |G| - 2$ if and only if $\vec{G}$ can be written as a zig-zag orientation $\vec{G}(P^{(1)}, P^{(2)}, Q)$ with the following properties:

1. $B = \{p_1^{(1)}, p_1^{(2)}\}$.
2. $z_1 = p_1^{(1)}$.
3. $z_r$ is the last vertex of one of $P^{(1)}$ or $P^{(2)}$, $z_{r-1}$ is not the last vertex of its path, and $z_2$ is not the first vertex of $P^{(2)}$ (by definition, $z_1 = p_1^{(1)}$ is the first vertex of $P^{(1)}$).
4. One of the following must hold:

(a) \( r \geq 2 \) and for all \( u \in \text{alt}(z_r) \), \( u \preceq z_{r-1} \).
(b) If \( z \) is the last vertex in \( \text{alt}(z_r) \) according to the path order, then \( \text{prev}(z) \in \text{alt}(z_r) \cup \{z_{r-1}\} \cup B \).

Proof. Assume \( \overrightarrow{G} \) can be written as a zig-zag orientation \( \overrightarrow{G}(P^{(1)}, P^{(2)}, Q) \) with properties (1)-(3). These properties require that only one force may occur at each time step (and \( P^{(1)}, P^{(2)} \) can be the forcing chains). Thus, \( pt(\overrightarrow{G}, B) = |G| - 2 \).

Conversely, if \( pt(\overrightarrow{G}, B) = |G| - 2 \) and \( Z(\overrightarrow{G}) = 2 \), the set of forces of \( B \) induce two parallel Hessenberg paths \( P^{(1)} \) and \( P^{(2)} \), with \( B = \{p_1^{(1)}, p_1^{(2)}\} \) as the zero forcing set. All forces are of the form \( p_i^{(h)} \rightarrow p_{i+1}^{(h)} \) for some \( i \) and \( h \), and exactly one force occurs at each time step.

We now identify the \( z_i \) vertices for \( Q \) using the propagation process. At any point in the propagation process, only one of the paths is forcing, i.e., is active, and the other cannot force, i.e., is inactive. Without loss of generality we will assume that \( p_1^{(2)} \rightarrow p_2^{(2)} \) is the first force and label \( p_1^{(1)} \) with \( z_1 \). The path \( P^{(2)} \) is now active and will continue to force along that path until some vertex which we label \( z_2 \) is reached and that path \( P^{(2)} \) becomes inactive and \( P^{(1)} \) becomes active. We continue this process of identifying the zig-zag path by choosing as \( z_i \) the vertex that was forced immediately before the path that is active switches. This process of labeling continues until the last vertex of either path is reached and is labeled \( z_r \).

This construction has several nice properties. Since \( z_1 \) could not force until \( z_{i+1} \) turned blue, the directed arc \( (z_i, z_{i+1}) \) must be present; this shows that these arcs form a directed path \( Q \). By definition \( z_1 = p_1^{(1)} \), while the labeling will have \( z_2 = p_k^{(2)} \) for some \( k > 1 \), the \( z_i \) alternate between the two paths, \( z_i < z_{i+2} \), and \( z_r \) is the final vertex on one of the paths while \( z_{r-1} \) is not the final vertex on its path. So to ensure \( \overrightarrow{G}(P^{(1)}, P^{(2)}, Q) \) is a zig-zag orientation, we need to verify condition 4 of Definition 3.6. Suppose \( z_{i+1} < u \) for some \( u \in \text{alt}(z_i) \) with \( 1 \leq i \leq r - 1 \), then at the step right after \( z_{i+1} \) turns blue, \( z_i \) has at least two white out-neighbors, namely \( \text{next}(z_i) \) and \( u \). If this happens, then the forcing process will stop here, since we picked \( z_{i+1} \) as the vertex that cannot conduct the force \( z_{i+1} \rightarrow \text{next}(z_{i+1}) \) right after \( \text{prev}(z_{i+1}) \rightarrow z_{i+1} \). This is a contradiction, so condition 4 of Definition 3.6 holds, and we have our zig-zag orientation.

We now need to verify that this zig-zag orientation satisfies the properties given in the theorem. By construction we have that the first two properties are satisfied, so it remains to show that property 3 also holds. Suppose 3(a) does not hold. Let \( y \) and \( z \) be the last two vertices in \( \text{alt}(z_r) \), according to path order, with \( y \preceq z \). Since 3(a) is not true, \( z \) must exist and \( z_{r-1} \preceq z \) (for convention, define \( z_{r-1} = p_1^{(2)} \) in the case of \( r = 1 \)). If \( y \) does not exist, or \( y \preceq z_{r-1} \), then by the time \( z_r \) is blue, \( \text{prev}(z) \) must be blue already, otherwise two forces, \( z_r \rightarrow z \) and \( z_{r-1} \rightarrow \text{next}(z_{r-1}) \), will occur at the same time. Thus, \( \text{prev}(z) \in B \cup \{z_{r-1}\} \).

If \( z_{r-1} \preceq y \), then \( \text{prev}(z) = y \in \text{alt}(z_r) \), for otherwise by the time \( y \) is blue, two
forces \( z_r \rightarrow z \) and \( y \rightarrow \text{next}(y) \), will happen simultaneously. Therefore, 3(b) is true and, in particular, all properties hold.

**Corollary 3.8.** Let \( \overrightarrow{G} \) be an orientation of a connected graph \( G \) for which \( Z(\overrightarrow{G}) = 2 \). Then \( pt(\overrightarrow{G}) = |G| - 2 \) if and only if for every minimum zero forcing set \( B \), \( \overrightarrow{G} \) can be written as a zig-zag orientation \( \overrightarrow{G}(P^{(1)}, P^{(2)}, Q) \) satisfying the three properties listed in Theorem 3.7.

The oriented graph \( \overrightarrow{G} \) shown in Figure 3 illustrates Corollary 3.8. The set \( B = \{p^{(1)}_1, p^{(2)}_1\} \) is a zero forcing set for \( \overrightarrow{G} \) that satisfies the hypotheses of Theorem 3.7, and \( B \) is the only minimum zero forcing set (since \( p^{(1)}_1 \) and \( p^{(2)}_1 \) both have in-degree zero). Thus \( pt(\overrightarrow{G}) = |G| - 2 \).

In order to guarantee \( pt(\overrightarrow{G}) = |G| - 2 \), it is not enough to be able to write \( \overrightarrow{G} \) as a zig-zag orientation \( \overrightarrow{G}(P^{(1)}, P^{(2)}, Q) \) satisfying the hypotheses of Theorem 3.7 for some \( B = \{p^{(1)}_1, p^{(1)}_2\} \). Although \( pt(\overrightarrow{G}, B) = |G| - 2 \), there may be another minimum zero forcing set \( B' \) for which \( pt(\overrightarrow{G}, B') < |G| - 2 \); Figure 4 presents several such examples.

Figure 3: An oriented graph satisfying Properties 1, 2, and 3(b) in Theorem 3.7

Figure 4: Examples of zig-zag orientations conforming to Theorem 3.7 and showing other minimum zero forcing sets that have propagation time less than \( |G| - 2 \)
Note that these examples have the property that one or both of the initial vertices of the $P^{(i)}$ have positive in-degree. In the case when neither of the initial vertices have any in-arcs, then $B$ is the unique minimum zero forcing set and so it is enough that there is a zig-zag orientation starting with $B$ that satisfies the hypotheses of Theorem 3.7. However, having a unique minimum zero forcing set is not necessary for an oriented graph to have $pt(G, B) = |G| - 2$.

The digraph $\overrightarrow{K}_{1,3}$ shown in Figure 5 has two minimum zero forcing sets and $pt(\overrightarrow{K}_{1,3}) = 2 = 4 - 2$. The problem of giving a complete classification of all oriented graphs with $pt(G) = |G| - 2$ remains open.

![Figure 5: An oriented graph with two minimum zero forcing sets and propagation time $|G| - 2$.](image)

4. Low Propagation Times

The smallest possible propagation time is 0. It is easy to see in a connected oriented graph $\overrightarrow{G}$ of order at least two, $Z(\overrightarrow{G}) \leq |\overrightarrow{G}| - 1$ and $pt(\overrightarrow{G}) \geq 1$. Thus, the only oriented graphs having propagation time equal to 0 are graphs with no edges, i.e., sets of isolated vertices.

We now consider graphs that have an orientation with propagation time one. For such orientation, every vertex is in the zero forcing set or colored in the first time step, so we have the following observation.

**Observation 4.1.** For an oriented graph $\overrightarrow{G}$ with $pt(\overrightarrow{G}) = 1$, $Z(\overrightarrow{G}) \geq \lceil \frac{|\overrightarrow{G}|}{2} \rceil$.

Therefore, the orientation must have a large zero forcing number (though this is not sufficient). Graphs having an orientation with propagation time one are not easy to classify. Many graphs, including trees (see Theorem 4.3) and complete graphs of order at least six (see Theorem 4.6), have such orientations. However, this is not true for all graphs, as the next example shows there is no orientation of $K_4$ with propagation time one.

**Example 4.2.** All possible orientations of $K_4$, up to relabeling, are shown in Figure 6, taken from and labeled as in [11]. It is known that $Z(\overrightarrow{K}_4) = 1$ if $\overrightarrow{K}_4$ is a Hessenberg path (D149, with path $(1, 2, 3, 4)$), in which case $pt(\overrightarrow{K}_4) = 3$; for other orientations $Z(\overrightarrow{K}_4) = 2$ (see [4]). For D115 the only zero forcing set of cardinality two is $B_1 = \{1, 3\}$ and $pt(D115, B_1) = 2$. For D129 there are three possible zero forcing sets of cardinality two, but they are all equivalent by symmetry to $B_2 = \{2, 3\}$, and $pt(D129, B_2) = 2$. Since D122 is the reverse of D129, $pt(D122) = 2$. 

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4.1. Trees

In this section we show that any tree (and hence any forest) can be oriented to have propagation time one, unless it consists entirely of isolated vertices.

**Theorem 4.3.** Let $T$ be a tree on $n \geq 2$ vertices. Then there is an orientation $\overrightarrow{T}$ of $T$ such that $\text{pt}(\overrightarrow{T}) = 1$.

**Proof.** A connected oriented graph $\overrightarrow{G}$ of order at least two has $\text{pt}(\overrightarrow{G}) \geq 1$, so it is sufficient to show that any tree $T$ of order at least two has an orientation $\overrightarrow{T}$ with $\text{pt}(\overrightarrow{T}) \leq 1$. We prove this statement by induction.

For the base case, it can be seen that $\text{pt}(\overrightarrow{T}) = 1$ when $n = 2$. Assume every nontrivial tree with fewer than $n$ vertices can be oriented to have propagation time one and consider a tree $T$ on $n$ vertices. Choose a vertex $y$ such that $\deg(y) \geq 2$ and at most one component of $T - y$ is a smaller tree $T'$ of order two or more; any other components are isolated vertices which we denote by $z_1, z_2, \ldots, z_s$. If there is no component of order two or more, orient the edges of $T$ as $z_i \rightarrow y$, for $i = 1, \ldots, s$, so $Z(\overrightarrow{T}) = n - 1$ and $\text{pt}(\overrightarrow{T}) = 1$. Now assume there is a unique component $T'$ of order at least two. By the induction hypothesis, there is an orientation $\overrightarrow{T'}$ of $T'$ with $\text{pt}(\overrightarrow{T'}) = 1$. Let $B$ be an efficient minimum zero forcing set of $\overrightarrow{T'}$, and let $x$ denote the unique neighbor of $y$ among $V(T')$.

First suppose $x \notin B$. Obtain $\overrightarrow{T}$ from $\overrightarrow{T'}$ by orienting edges of $T$ not in $T'$ so that $\deg^-(y) = 0$, i.e., $x \rightarrow y$ and $z_i \rightarrow y$, $1 \leq i \leq s$. Observe that $B \cup \{z_i\}_{i=1}^s$ is a zero forcing set of $\overrightarrow{T}$ with propagation time one. We show that $|B \cup \{z_i\}_{i=1}^s| = Z(\overrightarrow{T})$. Since $\deg^-(z_i) = 0$ ($1 \leq i \leq s$), any minimum zero forcing set of $\overrightarrow{T}$ must be of the form $\hat{B} \cup \{z_i\}_{i=1}^s$. In particular, $\hat{B}$ must force all vertices of $\overrightarrow{T'}$ without help from $y$ or $\{z_i\}_{i=1}^s$, because $y$ cannot contribute any forces to $V(T')$. Therefore, $|B| \leq |\hat{B}|$ and

$$Z(\overrightarrow{T}) \leq |B \cup \{z_i\}_{i=1}^s| \leq |\hat{B} \cup \{z_i\}_{i=1}^s| = Z(\overrightarrow{T'}).$$

Next suppose $x \in B$. Obtain $\overrightarrow{T}$ from $\overrightarrow{T'}$ by orienting edges of $T$ not in $T'$ so that $\deg^-(y) = 0$, i.e., $y \rightarrow x$ and $z_i \rightarrow z_i$, for $i = 1, \ldots, s$. Observe that $B \cup \{y\} \cup \{z_i\}_{i=1}^{s-1}$ is a zero forcing set of $\overrightarrow{T}$ with propagation time one. We show that $|B \cup \{y\} \cup \{z_i\}_{i=1}^{s-1}| = Z(\overrightarrow{T})$. Since $\deg^-(y) = 0$ and $y$ can force at most one of $\{z_i\}_{i=1}^s$ blue, $y$ and at least $s - 1$ of the $z_i$ must be blue initially.
Thus without loss of generality, a minimum zero forcing set of $\overrightarrow{T}$ has the form $\overrightarrow{B} \cup \{y\} \cup \{z_i\}_{i=1}^{n-1}$. If $z_s \in \overrightarrow{B}$, then $(\overrightarrow{B} \setminus \{z_s\}) \cup \{x\} \cup \{z_i\}_{i=1}^{n-1}$ is a zero forcing set with the same cardinality. Thus we can assume $x \in \overrightarrow{B}$, so $\overrightarrow{B}$ forces all vertices of $\overrightarrow{T}'$ without the help of $\{y\} \cup \{z_i\}_{i=1}^{n-1}$. Therefore, $|\overrightarrow{B}| \leq |\overrightarrow{B}|-|\overrightarrow{y}| \leq |\overrightarrow{B}|-|\overrightarrow{B}| = Z(\overrightarrow{T})$.

This completes the induction, and thus the proof. \hfill \Box

**Corollary 4.4.** If $T$ is a forest that contains an edge, then there is an orientation $\overrightarrow{T}$ of $T$ such that $\text{pt}(\overrightarrow{T}) = 1$.

4.2. **Tournaments**

Trees are the sparsest connected graphs, i.e., those with the smallest possible number of edges. At the opposite extreme are tournaments, which are orientations of complete graphs. However, we will see that for most $n$ there is a tournament on $n$ vertices that has propagation time one. In particular, we see that minimum propagation time is not strongly correlated with density.

**Proposition 4.5.** For $n \neq 2$, there is an orientation $\overrightarrow{K_{2n}}$ with $\text{pt}(\overrightarrow{K_{2n}}) = 1$.

**Proof.** Since the tournament of order 2 has propagation time one, we will assume $n \geq 3$. Let $\mathbb{Z}_n$ be the additive cyclic group of order $n$. We partition the vertices into two parts $U$ and $L$, and index the vertices by $\mathbb{Z}_n$, in other words, $U := \{u_i : i \in \mathbb{Z}_n\}$ and $L := \{\ell_i : i \in \mathbb{Z}_n\}$. Place arcs between these vertices as follows: $A_1 = \{(u_i, \ell_{i-1}) : i \in \mathbb{Z}_n\}$, and $A_2 = \{(\ell_j, u_i) : j \neq i - 1\}$. Also, define $A_3 = \{(u_i, u_j) : j - i \in W\}$, where $W = \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\} \subseteq \mathbb{Z}_n$. If $n$ is odd, then $A_3$ is properly defined. For $n = 2k$, each pair $\{u_i, u_{i+k}\}$ is doubly directed in $A_3$. In this case we modify $A_3$ by randomly choosing one of the arcs $(u_i, u_{i+k})$ or $(u_{i+k}, u_{i})$ for each pair $\{u_i, u_{i+k}\}$. Finally, we define $A_4 = \{(\ell_i, \ell_j) : (u_i, u_j) \in A_3\}$ and

$$E = A_1 \cup A_2 \cup A_3 \cup A_4.$$ 

Thus, $\overrightarrow{K_{2n}} = (V, E)$ is a tournament on $2n$ vertices, where $V = U \cup L$. For convenience, we call $U$ the upper part of $\overrightarrow{K_{2n}}$, and $L$ the lower part of $\overrightarrow{K_{2n}}$, and refer to forcing up or down.

Observe that $U$ forms a zero forcing set of $\overrightarrow{K_{2n}}$, $|U| = n$, and $\text{pt}(\overrightarrow{K_{2n}}, U) = 1$. Thus, it suffices to show that $Z(\overrightarrow{K_{2n}}) = n$. In order to prove this, we let $S \subseteq V$ be a set of cardinality $n - 1$ of blue vertices and show it cannot be a zero forcing set.

**Case 1:** $S \subseteq L$.

By assumption, $n \geq 3$, so every blue vertex has at least two white out-neighbors in $U$; hence $S$ cannot be a zero forcing set.

**Case 2:** $S \subseteq U$.

Since $|S| = n - 1$, by symmetry we assume $u_0$ is the only vertex in $U \setminus S$. Let $X := S \cap N^-(u_0)$ and $Y := S \setminus X$. At the first time step, the vertices in
$Y$ can force downward to color the set $Y_L \subset L$ blue; observe $\ell_{n-1} \notin Y_L$ because the only in-neighbor of $\ell_{n-1}$ in $U$ is $u_0 \notin S$. At this time, no $\ell_i \in Y_L$ can force any $\ell_j$ since $u_0$ is a white out-neighbor of $\ell_i$ (since $i \neq n-1$). Also, we observe that every vertex in $Y$ has no white out-neighbors whereas every vertex in $X$ has two white out-neighbors: One is $u_0$ and one is in $L$. Therefore the only possibility for an additional force is $\ell_i$ forces $u_0$ for some $i$.

Now we consider two cases and show each is impossible. First if $n = 2k + 1$ is odd, then $\ell_i$ has $k$ out-neighbors in $L$. But $Y_L$, the set of blue vertices in $L$, contains only $k$ vertices, including $\ell_i$ itself. So $\ell_i$ must have another white out-neighbor in $L$, making it impossible to force $u_0$. Second, if $n = 2k$ is even, then $|Y|$ can be either $k-1$ or $k$. If it is $k-1$, then we apply the same argument as for $n$ odd. So $u_k$ is an out-neighbor of $u_0$ and thus $Y = \{u_1, u_2, \ldots, u_k\}$. By our construction, $Y_L$ will be $\{\ell_0, \ell_1, \ldots, \ell_{k-1}\}$ and $\ell_k$ is an out-neighbor of $\ell_0$.

Under this assumption, for all $i \neq n-1$, $\ell_i$ has at least one white out-neighbor in $L$, and $u_0 \in N^+(\ell_i)$. Thus for $i \neq n-1$, $\ell_i$ cannot force. Since $\ell_{n-1}$ is white, $\ell_{n-1}$ cannot force. Thus $S$ is not a zero forcing set.

**Case 3:** $S \cap U$ and $S \cap L$ are not empty.

We start by carrying out the following shifting process: If some $u_i \in S \cap U$ has only one white out-neighbor and it is in $U$, called $u_j$, then we replace $S$ by $(S \setminus \{\ell_{i-1}\}) \cup \{u_j\}$. Since in the new set $u_i$ can force $\ell_{i-1}$, it is sufficient to show this new set is not a zero forcing set. Continuing this process, we may assume the set $S$ has the property that if $\ell_{i-1}$ is blue, then either $u_i$ is white or it has no unique white out-neighbor in $U$. After completing the shifting process, assume $S \cap U$ and $S \cap L$ are not empty, or else we may apply Case 1 and Case 2. Let $Y$ denote the set of vertices $u_i$ in $S \cap U$ such that $\ell_{i-1}$ is the only white out-neighbor of $u_i$ (it is possible $Y = \emptyset$).

Having completed the shifting process, we claim that $|S \cap U| = n-2$ and $|S \cap L| = 1$. To see this, first observe that $|S \cap U| \neq n-1$, for otherwise $S \cap L = \emptyset$. Suppose $|S \cap U| \leq n-3$. Then initially there are at least three white vertices in $U$: every vertex in $L$ will have at least two white out-neighbors in $U$, so no vertex in $L$ can perform a force at time $t = 1$. Also, no vertex in $U$ can force a vertex in $U$ since we finished the shifting process. If $Y = \emptyset$, then no forces occur, so assume $Y \neq \emptyset$. The vertices in $Y$ are the only vertices that can perform a force at $t = 1$, and the vertices in $Y$ will force downward to color the set $Y_L \subset L$ blue. Let $u_i \in U$ be a blue vertex. If $u_i \in Y$, then $u_i$ has no white out-neighbor after the forces at time $t = 1$. If $u_i \notin Y$ and $\ell_{i-1}$ is white, then $u_i$ has at least one white out-neighbor in $U \setminus S$. If $u_i \notin Y$ and $\ell_{i-1}$ is blue, then $u_i$ has at least two white out-neighbors in $U \setminus S$, since we finished the shifting process. This means at the second step, a blue vertex in $U$ cannot perform a force. Since each vertex in $L$ still has two or more white out-neighbors in $U$, no vertex in $L$ can perform a force either. This means the whole process stops after time $t = 1$ with some white vertices still remaining, a contradiction.

Without loss of generality, we may now assume $U \setminus S = \{u_0, u_j\}$ for some $j$, $(u_0, u_j)$ is an arc, and $S \cap L = \{\ell_i\}$ for some $i$. If $i = j-1$ and $S$ is a zero forcing set, then $(S \setminus \{\ell_i\}) \cup \{u_j\}$ is also a zero forcing set. This is because $(S \setminus \{\ell_i\}) \cup \{u_j\}$ can immediately carry out the force $u_j \rightarrow \ell_i$. However, $(S \setminus \{\ell_i\}) \cup \{u_j\}$ cannot
be a zero forcing set by Case 2. Next we claim that if \( i \neq j - 1 \), then the process stops after time \( t = 1 \). Since \( n \geq 3 \), \( \ell_i \) has at least one white out-neighbor in each of \( U \) and \( L \), so it cannot perform a force at time \( t = 1 \). The only vertices that can perform forces at \( t = 1 \) are by vertices in \( Y \) (again assume \( Y \neq \emptyset \), since otherwise the process already failed), and these vertices force downward to color \( Y_L \) blue. After the first time step, every vertex in \( Y_L \) has at least two white out-neighbors, \( u_0 \) and \( u_j \). No blue vertex in \( U \) has a unique white out-neighbor (because of the shifting process done originally and the forces done at \( t = 1 \)). If \( u_{i+1} \notin \{u_0, u_j\} \), then \( \ell_i \) has at least two white out-neighbors so cannot force. We have already shown that \( i \neq j - 1 \), leaving the case \( i = n - 1 \). So assume \( i = n - 1 \). We claim \( \ell_i \) has two white out-neighbors \( \ell_0 \) and \( u_j \). This is because the arc \((u_0, u_j)\) implies either \( j = 1 \) or \((u_1, u_j)\) is an arc, but both of these cases mean \( u_1 \) is not in \( Y \) (since either \( u_1 = u_j \) is white, or \( u_1 \neq u_j \) initially has at least two white out-neighbors, \( u_j \) and \( \ell_0 \)) and so \( \ell_0 \) is white after \( t = 1 \). Therefore, the process stops after \( t = 1 \) with white vertices remaining. This completes Case 3.

In every case, a set \( S \) of cardinality \( n - 1 \) cannot be a zero forcing set.

**Theorem 4.6.** For all integers \( n \geq 2 \), \( n \neq 4, 5 \), there is an orientation \( \overrightarrow{K}_n \) for \( K_n \) such that \( \text{pt}(\overrightarrow{K}_n) = 1 \).

**Proof.** We have already seen that this statement is true for even \( n \). For the case \( n = 2m + 1 \), we construct \( \overrightarrow{K}_{2m+1} \) by adding one vertex \( x \) to an orientation of \( \overrightarrow{K}_{2m} \) constructed as in Proposition 4.5, and adding directed arcs from \( x \) to all vertices in \( \overrightarrow{K}_{2m} \).

Since the case \( \overrightarrow{K}_3 \) is trivial, we assume \( m \geq 3 \). In this case, every vertex in \( \overrightarrow{K}_{2m} \) has in-degree (within \( \overrightarrow{K}_{2m} \)) at least one under the construction in Proposition 4.5. Let \( B \) be a minimum zero forcing set for \( \overrightarrow{K}_{2m+1} \). Since \( \deg^- (x) = 0 \), \( x \in B \). Since \( \deg^+(x) = 2m \), \( x \) cannot perform a force until all but one vertex of \( \overrightarrow{K}_{2m} = \overrightarrow{K}_{2m+1} - x \) are blue, at which point another vertex can perform the force, since the in-degree within \( \overrightarrow{K}_{2m} \) is positive for every vertex in \( \overrightarrow{K}_{2m} \).

So \( B \setminus \{x\} \) is a zero forcing set for \( \overrightarrow{K}_{2m} \), implying \( |B \setminus \{x\}| \geq Z(\overrightarrow{K}_{2m}) = m \) and \( |B| \geq m + 1 \). Therefore the propagation time of \( \overrightarrow{K}_{2m+1} \) is one, using the efficient zero forcing set \( B'' := B' \cup \{x\} \), where \( B' \) is an efficient zero forcing set for \( \overrightarrow{K}_{2m} \).

**4.3. Data for small graphs that allow propagation time one**

We say that a simple graph \( G \) allows propagation time one if there is some orientation \( \overrightarrow{G} \) of the graph \( G \) with \( \text{pt}(\overrightarrow{G}) = 1 \). Then natural questions are, “Which graphs allow propagation time one?” and, “How common are such graphs?”

We use \( \min_{\overrightarrow{G}} \text{pt}(\overrightarrow{G}) \) to denote the minimum propagation time of \( \overrightarrow{G} \) where \( \overrightarrow{G} \) runs over all orientations of \( G \). Then ‘\( G \) allows propagation time one’ is equivalent to \( \min_{\overrightarrow{G}} \text{pt}(\overrightarrow{G}) = 1 \). For all connected graphs of order at most nine,
the minimum propagation time over all orientations was determined, and we present the data in Table 1. Note that no graph of order at most nine requires a propagation time of three or greater. At least for small graphs it appears that allowing propagation time one is common.

<table>
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<th>n=4</th>
<th>n=5</th>
<th>n=6</th>
<th>n=7</th>
<th>n=8</th>
<th>n=9</th>
</tr>
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<td>2</td>
<td>5</td>
<td>20</td>
<td>106</td>
<td>820</td>
<td>10746</td>
<td>256568</td>
</tr>
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<td>0</td>
<td>1</td>
<td>1</td>
<td>6</td>
<td>33</td>
<td>371</td>
<td>4512</td>
</tr>
</tbody>
</table>

Table 1: Number of connected graphs on \( n \) vertices with given the propagation time as the minimum over all orientations

We have already established that \( K_4 \) does not allow propagation time one. A similar case analysis shows that \( K_5 \) also does not allow propagation time one. In Figure 7 we give orientations for all the remaining graphs on at most 4 vertices, and have marked corresponding minimum zero forcing sets, to verify that they have \( pt(G) = 1 \). We remark here that the undirected graph underlying the oriented graph in Figure 1 does not allow an orientation with propagation time one.

Figure 7: The connected graphs of order at most four (other than \( K_4 \)) with orientations having propagation time one

In addition to the data given in the table, we have verified by computer that no graph of order 10 requires propagation time of three or greater. This leads to the following open questions.

**Question 4.7.** Does there exist an undirected graph \( G \) with \( \min_{\_\_G} pt(G) \geq 3 \)?

More generally, does there exist an undirected graph \( G \) with \( \min_{\_\_G} pt(G) \geq k \) for \( k \) arbitrarily large?

5. Orientation Propagation Intervals

We have seen in preceding sections that for the path \( P_n \) there are orientations with both high and low propagation times. This leads to the following idea.
**Definition 5.1.** Let $G$ be an undirected graph with $m = \min_G \text{pt}(\overrightarrow{G})$ and $M = \max_G \text{pt}(\overrightarrow{G})$. The interval $[m, M]$ is called the orientation propagation interval, and $G$ has a full orientation propagation interval if for every $k$ such that $m \leq k \leq M$ there is some orientation $\overrightarrow{G}$ such that $\text{pt}(\overrightarrow{G}) = k$.

Determining if a graph has a full orientation propagation interval is non-trivial, even for some simple graphs. The difficulty is that the propagation parameter can be sensitive to small perturbations, as shown in the following example.

**Example 5.2.** Let $n \geq 9$ and $k = \lfloor \frac{n+3}{2} \rfloor$. Consider the two oriented paths on $n$ vertices shown in Figure 8 where the vertices are labeled 1 to $n$ going from left to right. The top path has $Z(\overrightarrow{P}_n) = 2$, $\{1, k\}$ is the unique minimum zero forcing set, and $\overrightarrow{P}_n$ has propagation time $n - 2$. The bottom path has $Z(\overrightarrow{P}_n) = 3$ and $\{1, k, k+1\}$ is a minimum zero forcing set with propagation time of $\lceil \frac{n+3}{2} \rceil$. Thus the reversal of the arc between $k - 2$ and $k - 1$ changed the propagation time by at least $\lfloor \frac{n+1}{2} \rfloor$, which can become arbitrarily large.

![Figure 8: Reversing an arc produces a large change in propagation time](image)

In this section we will show that paths have full orientation propagation time interval, while cycles do not. We comment that the behavior and analysis of orientation propagation time intervals is far from understood.

5.1. Paths have a full orientation propagation time interval

By Theorem 4.3 we know for $P_n$ that there is an orientation with propagation time one, and if we orient the edges of the path as $(i, i+1)$ for $1 \leq i \leq n-1$, then the propagation time is $n - 1$. We will show that for $P_n$ there is an orientation with propagation time $k$ for each $1 \leq k \leq n - 1$. The remaining propagation times are given by the following theorem.

**Theorem 5.3.** Let $P_n$ be the path on $n$ vertices and $1 \leq k \leq n - 1$. Then there is an orientation $\overrightarrow{P}_n$ such that $\text{pt}(\overrightarrow{P}_n) = k$.

**Proof.** We label the path with vertices 1, $\ldots$, $n$ and edges joining $i$ and $i + 1$ for $1 \leq i \leq n - 1$. To achieve a propagation time of $n - 2$, take the orientation $(i, i+1)$ for $1 \leq i \leq n - 3$ together with $(n-1, n-2)$ and $(n-1, n)$. Then this orientation $\overrightarrow{P}_n$ has $Z(\overrightarrow{P}_n) = 2$, and $\{1, n-1\}$ is the unique minimum zero forcing set; furthermore, no simultaneous forces can occur, giving propagation time $n - 2$.

Now assume $2 \leq k \leq n - 3$. We consider the following orientation.
• $(i, i+1)$ for $1 \leq i \leq k+1$ (the initial segment).

• For $k+2 \leq j \leq n-1$ orient the edge between $j$ and $j+1$ by

\[
\begin{cases}
  (j, j+1) & \text{if } j \equiv k \text{ or } k+1 \pmod{4}, \\
  (j+1, j) & \text{if } j \equiv k+2 \text{ or } k+3 \pmod{4}.
\end{cases}
\]

A minimum zero forcing set must contain 1, but no other vertex in the initial segment (as 1 can eventually force the initial segment). In particular, vertex $k+1$ will not turn blue until the $k$th step of the propagation process (the vertex $k+1$ can be turned blue earlier through its other neighbor). Therefore the propagation time of this orientation is at least $k$.

Consider the set $S = \{i : \deg^{-}(i) = 0\}$. The vertices in $S$ must be in a zero forcing set since they cannot be turned blue by a neighbor. If a vertex in $S$ has $\deg^{+}(i) = 2$ then one of the neighbors must also be in the zero forcing set, i.e., only $i$ can change them to blue, but it cannot force both. As a consequence when we look at blocks of consecutive vertices between vertices with $\deg^{-}(i) = 2$ we see that each block will have two elements in the zero forcing set and the block will propagate in time two. Similar analysis shows that the tail will also propagate in time at most two.

We conclude that in two steps all vertices except some of those in the initial segment of the path have been turned blue; however, the initial segment will not finish turning blue until time $k$, so the propagation time of this orientation of $P_n$ is $k$. \qed

5.2. Cycles do not have a full orientation propagation time interval

Not all graphs have a full orientation propagation interval, as the following example shows.

**Example 5.4.** The four orientations on $C_4$ up to isomorphism are shown in Figure 9. No orientation has a propagation time of 2, although there are orientations with propagation times 1 and 3. So $C_4$ does not have a full orientation propagation interval.

![Possible orientations of $C_4$](image)

\[
\begin{align*}
Z(C_4) &= 1, & Z(C_4) &= 2, & Z(C_4) &= 2, & Z(C_4) &= 3, \\
\text{pt}(C_4) &= 3, & \text{pt}(C_4) &= 1, & \text{pt}(C_4) &= 1, & \text{pt}(C_4) &= 1.
\end{align*}
\]

Figure 9: Possible orientations of $C_4$
If we orient the cycle $C_n$ by $(i, i+1)$ for $1 \leq i \leq n$ (where we look at the entries modulo $n$), then any vertex can force the entire graph and has $\text{pt}(\overrightarrow{C_n}) = n - 1$.

Now we reverse the arc between 1 and $n$ to $(1, n)$. Use the zero forcing set $B = \{1, 2\}$. Vertices 3 and $n$ are force in the first step. Then at each subsequent step one vertex is forced, so $B$ has propagation time $n - 3$. Every minimum zero forcing set for this orientation is of the form $\{1, k\}$, and for $k \geq 3$ this set has propagation time $n - 2$. Thus $\text{pt}(\overrightarrow{C_n}) = n - 3$.

However, the intermediate value of $n - 2$ is impossible, as the next result shows. In particular, for $n \geq 4$ the cycle $C_n$ does not have a full orientation propagation time interval.

**Proposition 5.5.** Let $n \geq 4$. Then $\text{pt}(\overrightarrow{C_n}) \neq n - 2$ for any orientation of $C_n$.

**Proof.** Suppose $\overrightarrow{C_n}$ is an orientation of $C_n$ with $\text{pt}(\overrightarrow{C_n}) = n - 2$. By Observation 1.2 we must have that $Z(\overrightarrow{C_n}) = 2$. Moreover, it must be the case that precisely one vertex is forced at each time step for every minimum zero forcing set. Since $Z(\overrightarrow{G}) = 2$, $\overrightarrow{G}$ is a graph of two parallel Hessenberg paths, with two arcs between the two paths so that a cycle is formed. Since the initial vertices vertices $v$ and $w$ of the two Hessenberg paths must be a zero forcing set and cannot both force initially, without loss of generality the arcs must be oriented as shown in Figure 10.

![Figure 10: A cycle oriented as two parallel Hessenberg paths](image)

First suppose an out-neighbor $u$ of $v$ exists in the path containing $v$. Then $\{v, u\}$ is a zero forcing set in which two forces are performed initially, contradicting $\text{pt}(\overrightarrow{G}) = n - 2$. Thus $u$ does not exist and $\{v, w\}$ is a zero forcing set in which two forces are performed initially (since $n \geq 4$, $y$ is not the last vertex in the lower path), contradicting $\text{pt}(\overrightarrow{G}) = n - 2$. \qed

6. **Throttling on Oriented Graphs**

To this point we have focused on the propagation time for zero forcing sets that have minimum cardinality. We can relax the requirement to use minimum zero forcing sets and more generally consider any set that forces the entire graph. In this situation we want to balance the cardinality of the zero forcing set and the speed at which it propagates through the graph. For undirected graphs, throttling was studied in [6].
Definition 6.1. Given an oriented graph $\vec{G}$ and a zero forcing set $B$ of $\vec{G}$, the throttling time of $B$ for $\vec{G}$ is $\text{th}(\vec{G}, B) = \text{pt}(\vec{G}, B) + |B|$. The minimum throttling time of an oriented graph $\vec{G}$ is

$$\text{th}(\vec{G}) = \min\{\text{pt}(\vec{G}, B) + |B| : B \text{ is a zero forcing set of } \vec{G}\}.$$  

In [6] the throttling time of an undirected path $P_n$ was determined to be approximately $2\sqrt{n}$. More generally it was shown that for any fixed value $k$ there is a constant $c_k$ such that if the zero forcing number of a graph on $n$ vertices is at most $k$, then the minimum throttling time is at most $c_k\sqrt{n}$.

Here we look at throttling on complete Hessenberg paths. A complete Hessenberg path is the unique tournament with a zero forcing number of one. More precisely, the complete Hessenberg path of order $n$ has vertex set $\{1, 2, \ldots, n\}$ and the following arcs:

$$\{(i, i + 1) : 1 \leq i \leq n - 1\} \cup \{(i, j) : 3 \leq i \leq n \text{ and } 1 \leq j \leq i - 2\}.$$  

A simple check verifies that for $n \geq 4$ that $\{1\}$ is the unique minimum zero forcing set of this oriented graph. We show an example with $n = 5$ in Figure 11.

We will show that unlike undirected graphs, we cannot guarantee a significant savings in throttling. In particular, for the complete Hessenberg path $\vec{H}$, we have $\text{th}(\vec{H}) = \lfloor 2n/3 \rfloor + 1$. The key step is given in the next lemma, which shows that we cannot engage in a large number of simultaneous forces on the complete Hessenberg path.

Lemma 6.2. Let $\vec{H}$ be a complete Hessenberg path, and $B$ a set of blue vertices. Then $B$ can force at most 2 vertices at any given time step.

Proof. Let $\vec{H}$ be a complete Hessenberg path with vertex set $\{1, 2, \ldots, n\}$ and assume $B$ is a set of blue vertices that forces 3 or more vertices at time step $t$. Assume $a < b < c$ are the largest of the vertices that are forced at time $t$ and thus are white at time $t - 1$. Observe that $c$ can only be forced by vertex $c - 1$ or by some vertex $c + 2$ or greater; but any vertex $c + 2$ or greater has $a$, $b$ and $c$ as white out-neighbors so cannot perform any forces. This means $c - 1$ must force $c$. However, $a < b \leq c - 1$ so $c - 1$ has both $a$ and $c$ as white out-neighbors so cannot perform any forces. This means that no vertex can force vertex $c$, which is a contradiction, and the result follows.  

Corollary 6.3. For any zero forcing set $B$ of the complete Hessenberg path $\vec{H}$, we have $2\text{pt}(\vec{H}, B) + |B| \geq n$.  

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In the remainder of this section we will find it convenient for the proofs to group the vertices of the complete Hessenberg path on \( n \) vertices into sets of three. We will adopt the following notation: \( \ell := \lfloor n/3 \rfloor \) and for \( 1 \leq j \leq \ell \) then \( I_j = \{3j - 2, 3j - 1, 3j\} \) while \( I_{\ell + 1} \) will be the remaining vertices (if any). We also note that any zero forcing set must contain at least one vertex in \( I_1 \), i.e., if not then since every vertex is adjacent to two elements in \( I_1 \) then we could never change \( I_1 \) to blue.

**Proposition 6.4.** If \( \overrightarrow{H} \) is a complete Hessenberg path with \( |\overrightarrow{H}| = n \) then \( \text{th}(\overrightarrow{H}) \leq \lceil 2n/3 \rceil + 1 \).

**Proof.** Define \( B := \{1, 3, 6, 9, \ldots, 3\ell\} \) and note that \( |B| = \lfloor n/3 \rfloor \) + 1. An easy inductive argument show that \( I_j \) will be blue at time \( j \) for \( 1 \leq j \leq \ell \) and \( I_{\ell + 1} \) turns blue at time \( \ell \) or \( \ell + 1 \) if \( |I_{\ell + 1}| = 1 \), or \( |I_{\ell + 1}| = 2 \) respectively. If either \( n \equiv 0 \mod 3 \) or \( n \equiv 1 \mod 3 \) then \( \text{pt}(\overrightarrow{H}, B) = \lfloor n/3 \rfloor \) and

\[
\text{th}(\overrightarrow{H}, B) = 2 \lfloor n/3 \rfloor + 1 = \lceil 2n/3 \rceil + 1.
\]

If \( n \equiv 2 \mod 3 \) then \( \text{pt}(\overrightarrow{H}, B) = \lfloor n/3 \rfloor + 1 \) and

\[
\text{th}(\overrightarrow{H}, B) = 2 \lfloor n/3 \rfloor + 2 < \lceil 2n/3 \rceil + 2.
\]

In all cases \( \text{th}(\overrightarrow{H}, B) \leq \lceil 2n/3 \rceil + 1 \). \( \square \)

**Lemma 6.5.** If \( \overrightarrow{H} \) is a complete Hessenberg path on \( n \) vertices and \( B \) is a zero forcing set such that \( |B| \leq \lfloor n/3 \rfloor \) then there is some time step when exactly one vertex is forced.

**Proof.** Suppose first that for some \( 2 \leq m \leq \ell \) that \( I_m \cap B = \emptyset \). Now we set \( S = I_1 \cup \cdots \cup I_{m-1} \) and \( T = I_m \cup \cdots \cup I_{\ell + 1} \). Since every blue vertex in \( T \) is adjacent to two white vertices in \( I_m \) then nothing in \( T \) can force until \( 3m - 2 \) (the first element of \( T \)) has been forced blue and further when \( 3m - 2 \) has been forced everything in \( S \) has also already been forced (in order for \( 3m - 3 \) to force then \( 1, 2, \ldots, 3m - 5 \) must be blue; if \( 3m - 4 \) is blue before \( 3m - 3 \) forces then the result holds and otherwise \( 3m - 5 \) will force \( 3m - 4 \) at the same time that \( 3m - 3 \) forces \( 3m - 2 \)). At this stage \( 3m - 2 \) can force \( 3m - 1 \) and the only other vertex that can possibly be adjacent to only one white vertex is \( 3m + 1 \) adjacent to \( 3m - 1 \). Therefore, the only vertex that is forced at this stage is \( 3m - 1 \).

If all \( I_m \cap B \neq \emptyset \) then it must be that \( |I_m \cap B| = 1 \) for \( 1 \leq m \leq \ell \). Therefore every vertex numbered 4 or greater has at least two white out-neighbors and so only the single blue vertex in \( I_1 \) can force at the first step. \( \square \)

**Theorem 6.6.** If \( \overrightarrow{H} \) is a complete Hessenberg path then \( \text{th}(\overrightarrow{H}) = \lceil 2n/3 \rceil + 1 \).

**Proof.** An easy verification establishes the result when the complete Hessenberg path \( \overrightarrow{H} \) has 1, 2 or 3 vertices. We will now proceed by induction on \( n \), the number of vertices of \( \overrightarrow{H} \). Assume that \( \text{th}(\overrightarrow{H}) = \lceil 2k/3 \rceil + 1 \) for \( 1 \leq k \leq n - 1 \). Consider
the complete Hessenberg path on \( n \) vertices and for the sake of contradiction we will assume we can find a zero forcing set \( B \) of \( \overrightarrow{H} \) with \( \text{th}(\overrightarrow{H}, B) \leq \lfloor \frac{2n}{3} \rfloor \).

Then by Corollary 6.3 and this assumption we have

\[
 n \leq \text{pt}(\overrightarrow{H}, B) + \text{pt}(\overrightarrow{H}, B) + |B| \leq \text{pt}(\overrightarrow{H}, B) + \lfloor \frac{2n}{3} \rfloor \leq \text{pt}(\overrightarrow{H}, B) + 2n/3.
\]

This tells us that \( n/3 \leq \text{pt}(\overrightarrow{H}, B) \), which in turn implies that \( |B| \leq \lfloor n/3 \rfloor \).

We now consider two cases.

**Case 1:** \( I_m \cap B = \emptyset \) for some \( 2 \leq m \leq \ell \)

Proceeding as in the previous lemma we let \( S = I_1 \cup \ldots \cup I_m-1 \) and \( T = I_m \cup \ldots \cup I_{\ell+1} \) and note that no force happens from the blue vertices in \( T \) until all vertices in \( S \) and \( x := 3m - 2 \) (the first vertex in \( T \)) have been forced.

Once \( x \) has been forced the rest of the forcing can proceed as though \( S \) is not part of the graph. This shows we can split the propagation into two distinct phases and so we have

\[
 \text{pt}(\overrightarrow{H}, B) = \text{pt}(\overrightarrow{H}[S \cup \{x\}], B \cap S) + \text{pt}(\overrightarrow{H}[T], (B \cap T) \cup \{x\}).
\]

Using this we get the following bound:

\[
 \text{th}(\overrightarrow{H}, B) = \text{pt}(\overrightarrow{H}, B) + |B|
 = \text{pt}(\overrightarrow{H}[S \cup \{x\}], B \cap S) + \text{pt}(\overrightarrow{H}[T], (B \cap T) \cup \{x\}) + |B \cap S| + |B \cap T|
 = \text{th}(\overrightarrow{H}[S \cup \{x\}], B \cap S) + \text{th}(\overrightarrow{H}[T], (B \cap T) \cup \{x\}) - 1
 \geq \lfloor \frac{2(|S| + 1)}{3} \rfloor + 1 + \lfloor \frac{2(n - |S|)}{3} \rfloor + 1 - 1
 \geq \lfloor \frac{2n}{3} \rfloor + 1
\]

The \(-1\) term on the third line comes from accounting for \( \{x\} \), and on the fourth line we have used the induction hypothesis. This statement contradicts that \( \text{th}(\overrightarrow{H}, B) \leq \lfloor \frac{2n}{3} \rfloor \), so this case cannot happen.

**Case 2:** \( I_m \cap B \neq \emptyset \) for all \( 1 \leq m \leq \ell \)

In this case we must then have that \( |B| = \lfloor n/3 \rfloor \) and so

\[
 \text{th}(\overrightarrow{H}, B) \geq 1 + \frac{n - \lfloor n/3 \rfloor - 1}{2} + \lfloor n/3 \rfloor.
\]

The first two terms are a lower bound for \( \text{pt}(\overrightarrow{H}, B) \), since by Lemma 6.5 there is some time step in the forcing process when only one force occurs, and since for every other time step at most two forces can occur by Lemma 6.2. Since \( \text{th}(\overrightarrow{H}, B) \) must be a whole number this then implies that \( \text{th}(\overrightarrow{H}, B) \geq \lfloor 2n/3 \rfloor + 1 \), which is again a contradiction to \( \text{th}(\overrightarrow{H}, B) \leq \lfloor 2n/3 \rfloor \).

Therefore we can conclude that \( \text{th}(\overrightarrow{H}, B) \geq \lfloor 2n/3 \rfloor + 1 \) and the construction in Proposition 6.4 is tight. \( \square \)
References


