Notes on the GridLAB-D Household Equivalent Thermal Parameter Model

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Original Release Date: January 15, 2019  
Revisions: April 23, 2019; May 28, 2019  
Latest Revision: July 15, 2020

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Abstract
GridLAB-D (GLD) is an agent-based platform, developed by researchers at Pacific Northwest National Laboratory, that permits users to accurately simulate the state dynamics of power distribution systems at time scales ranging from sub-seconds to years. The purpose of this study is to present, in careful comprehensive form, a complete analytical state-space control representation for a version of the GLD Household Equivalent Thermal Parameter Model as support documentation for model users. This model is a physically-based implementation of a household with multiple price-responsive and conventional appliances whose thermal dynamics are determined over successive days by resident appliance usage and external weather conditions.

Keywords
Household modeling, electric power distribution system, agent-based platform, GridLAB-D

Disciplines
Economic Theory | Electrical and Computer Engineering | Industrial Organization | Oil, Gas, and Energy | Power and Energy

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Notes on the GridLAB-D Household Equivalent Thermal Parameter Model

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Economics Working Paper #19001, Iowa State University
Last Revised: 15 July 2020

Abstract: GridLAB-D is an agent-based platform, developed by researchers at Pacific Northwest National Laboratory, that permits users to accurately simulate the state dynamics of power distribution systems at time scales ranging from sub-seconds to years. The purpose of this study is to present, in careful comprehensive form, a complete analytical state-space control representation for a version of the GridLAB-D Household Equivalent Thermal Parameter Model as support documentation for model users. This model is a physically-based implementation of a household with multiple price-responsive and conventional appliances whose thermal dynamics are determined over successive days by resident appliance usage and external weather conditions.

1 Overview

As part of ongoing project research at Iowa State University on Transactive Energy System (TES) design, an agent-based computational platform has been developed permitting the evaluation of TES designs for Integrated Transmission and Distribution (ITD) systems. This platform is referred to as the ITD TES Platform V2.0.¹

¹A preliminary version of this platform, V1.0, is formulated and tested in [1].
The ITD TES Platform V2.0 is currently being used to study ITD TES designs for ITD systems with distribution systems populated by households; see, for example, Battula et al. [2]. Figs. 1-3 depict partial agent hierarchies and software components for the platform, as specialized for these household studies.

![Diagram of ITD TES Platform](image)

**Figure 1:** The ITD TES Platform (V2.0) specialized for household studies. Down-pointing arrows indicate “has a” relationships and up-pointing arrows denote “is a” relationships.

As indicated in Fig. 3, key features of the household agents that populate the platform distribution system are currently being implemented using the GridLAB-D (GLD) Household Equivalent Thermal Parameter (ETP) Model.\(^2\) This model is a complex C++ program with many interacting components. While some model documentation is available, it is not comprehensive. For example, it is not easy to distinguish structural elements from data-driven elements. Moreover, physical interpretations and units of measurement are difficult to discern for some key parameters.

The purpose of this study is to present, in careful comprehensive form, a complete analytical representation for the GLD Household ETP Model in standard state-space control form, as support documentation. The first two sections of this study provide preliminary background materials. Section 2 reviews basic terminology regarding classification of variables.

\(^2\)For general introductions to GLD, see [3,4].
Section 3 presents a state-space control model in standard continuous-time form.

Section 4 develops and presents a complete analytic state-space control representation for a version of the GLD Household ETP Model documented in [5–7] and implemented by means
of a C++ program [8]. In this version, each household has an electric Heating, Ventilation, and Air-Conditioning (HVAC) system whose power consumption is managed by an HVAC ON/OFF controller. Each household also has additional appliances modeled by GLD’s ZIP load object [9].

Section 5 discusses the GLD implementation of the continuous-time GLD Household ETP Model. As shown in Section 4, this model is a linear nonhomogeneous differential system with a time-varying coefficient vector. GLD implements a closed-form solution for this linear system in approximate form by assuming forcing terms are constant-valued over successive time-steps of equal length. The pros and cons of applying closed-form solution methods to linearized models as opposed to applying discretization methods to nonlinear models are briefly discussed.

It is then shown how a simple forward finite-difference approximation method could instead be used to implement the GLD Household ETP Model. This method does not require linearity of the underlying differential system. The final part of this section illustrates how this method can be directly applied to the nonlinear continuous-time state-space control model presented in Section 3. However, the accuracy and stability of approximate solution methods for the GLD Household ETP Model remains an important open issue.

Tables listing GLD Household ETP Model user-set parameters, derived parameters, variables, and default values/functions (if any) for user-set parameters are provided in an Appendix at the end of these notes.

2 Preliminary Classification of Variables Terminology

A variable whose value is determined outside of a model M is said to be exogenous relative to M. If an exogenous variable for a model M takes a constant value over time, it is often referred to as a parameter of M. If an exogenous variable for a model M is a function of time, it is often referred to as a forcing term for M.

A variable whose value is determined within a model M is said to be endogenous relative to M. An endogenous variable appearing within the time-\( t \) equations for a model M whose value is determined by these equations is said to be a time-\( t \) endogenous variable for M. An endogenous variable appearing within the time-\( t \) equations for a model M whose value
is determined by means of model-M equations at earlier times \( s < t \) is said to be a time-\( t \) predetermined variable for M. The time-\( t \) predetermined variables for a model constitute the time-\( t \) state variables for this model.

For a model specified over times (or time periods) \( t \geq t^0 \), values for the state variables at the initial time \( t^0 \) need to be exogenously given since there are no modeled relationships prior to this initial time. A control variable for a model M can be either exogenous or endogenous in form. A control variable for a model M is exogenous relative to M if it is set externally, with no dependence on model-M outcomes. A control variable for a model M is endogenous relative to M if it is determined as a function of model-M outcomes.

3 State-Space Control Model: Continuous-Time Form

**Standard Structural Model:** For each \( t \geq t^0 \),

- **Dynamic state equations:** \( \dot{x}(t) = S(u(t), w(t), z(t), x(t) \mid \theta^S) \) (1)
- **Simultaneous equations:** \( 0 = H(u(t), w(t), z(t), x(t) \mid \theta^H) \) (2)
- **Integral equations:** \( x(t) = \int_{t^0}^t \dot{x}(\tau)d\tau + x(t^0) \) (3)

**Variables, Parameters, and Functional Forms:**

\( x(t) = (x_1(t), \ldots, x_N(t)) = \) State vector for time \( t \geq t^0 \)

\( \dot{x}(t) = (\dot{x}_1(t), \ldots, \dot{x}_N(t)) = \) State gradient vector for time \( t \geq t^0 \)

\( u(t) = (u_1(t), \ldots, u_M(t)) = \) Control vector for time \( t \geq t^0 \)

\( w(t) = (w_1(t), \ldots, w_J(t)) = \) Vector of forcing terms for time \( t \geq t^0 \)

\( z(t) = (z_1(t), \ldots, z_L(t)) = \) Vector of endogenous variables for time \( t \geq t^0 \)

\( \theta^S = (\theta^S_1, \ldots, \theta^S_{SV}) = \) Parameter vector

\( \theta^H = (\theta^H_1, \ldots, \theta^H_{HV}) = \) Parameter vector

\( S: \mathbb{R}^{M+J+L+N+SV} \rightarrow \mathbb{R}^N \)
\[ H: \mathbb{R}^{M+J+L+N+HV} \rightarrow \mathbb{R}^L \]

Classification of Variables:

- **Time-t endogenous variables for** \( t \geq t^0 \): \( \dot{x}(t), z(t) \)
- **Time-t predetermined variables for** \( t > t^0 \): \( x(t) \)
- **Exogenous controls and forcing terms for** \( t \geq t^0 \): \( u(t), w(t) \)
- **Exogenous parameters and initial state conditions**: \( \theta^S, \theta^H, \) and \( x(t^0) \)

As indicated in the classification of variables, this illustrative state-space control model has \( N+L \) time-\( t \) endogenous variables at each time \( t \): namely, the \( N \) variables appearing in the vector \( \dot{x}(t) \) and the \( L \) variables appearing in the vector \( z(t) \). In turn, there are \( N+L \) equations provided to solve for these \( N+L \) time-\( t \) endogenous variables: namely, the \( N \) state equations (1) and the \( L \) simultaneous equations (2).

The integral equations (3) ensure that the solved solution-value for \( \dot{x}(t) \) is the derivative of \( x(t) \) for \( t > t^0 \) and the right-derivative of \( x(t^0) \) at \( t = t^0 \). Note that the control variables appearing in \( u(t) \) at each time \( t \) are assumed to be exogenously determined.

4 GLD Household ETP Model: Analytic Formulation

4.1 Overview

In this section we present a complete analytic state-space control formulation for the GLD Household ETP Model based on the GLD documentation [5–7] and the GLD source code [8]. For concreteness, we assume each household has an electric HVAC system running in cooling mode with a linear cooling-capacity curve (the GLD default setting). This HVAC system has a 1-speed fan\(^3\) for the maintenance of air circulation. Each household also has a mix of additional appliances modeled by GLD’s ZIP load object [9]. These additional appliances include: Lights, Plugs, Clothes-Washer, Refrigerator, Dryer, Freezer, Dishwasher, Range, and Microwave. The GLD ZIP load object allows the modeling of voltage dependence for

\(^3\)For later purposes, it is important to note that GLD implements a 1-speed fan to be ON if and only if the HVAC system is ON.
these appliances. The corresponding user energy-consumption profiles for these appliances are constructed from field data, considering weekday and seasonal patterns; these profiles can be accessed at [10].

As will be seen below, many of the parameters appearing in the GLD Household ETP Model are in fact derived as functions of other parameters. The model parameters set directly by the user are classified as user-set parameters.\(^4\) Definitions and units for these user-set parameters are listed in Table 1 in the Appendix.\(^5\) The vector of these user-set model parameters will hereafter be denoted by \(\theta_{\text{user}}\). Definitions and units for model parameters determined as functions of \(\theta_{\text{user}}\), referred to as derived parameters, are listed in Table 2 in the Appendix. The coupled-parameter relationships expressing these derived parameters as functions of user-set parameters are given in Section 4.4.

Finally, some model parameters are internally assigned numerical values by GLD in a manner that cannot be changed or influenced by user-set parameter values. Hereafter these parameters will be referred to as GLD-determined parameters. Some of these GLD-determined parameters represent standard unit conversion factors. However, others appear to be based on structural presumptions or derived as empirical estimates from survey data, and their physical interpretations and units of measurement are not always clearly explained. Ideally, all of the latter parameters should instead be modeled as user-set parameters with GLD-provided default values, giving users a chance to modify/update the values of these parameters in response to changed distribution system conditions. This important issue is not dealt with in the current study.

4.2 Complete Analytic Formulation: Preliminary Developments

As detailed in [6], the GLD Household ETP Model assumes the thermal state of a house at any time \(t\) is given by a state vector \((T_a(t), T_m(t))\), where: \(T_a(t)\) denotes the time-\(t\) inside air temperature; \(T_m(t)\) denotes the time-\(t\) inside mass temperature; and time is measured at the granularity of hours. The thermal dynamics of the house are then represented as a two-
dimensional first-order differential system in \((T_a(t), T_m(t))\) that determines the movement of 
\(T_a(t)\) and \(T_m(t)\) over time.

More precisely, as seen in [6, Eqs. (1)-(2)], the dynamic state equations for the GLD Household ETP Model are expressed in the following linearized form:

\[
\dot{T}_a(t) = \frac{1}{C_a} \left( U_a [ T_o(t) - T_a(t) ] + H_m [ T_m(t) - T_a(t) ] + Q_a(t) \right) ;
\]

\[
\dot{T}_m(t) = \frac{1}{C_m} \left( H_m [ T_a(t) - T_m(t) ] + Q_m(t) \right) ,
\]

where: \(T_o(t)\) denotes outside air temperature at time \(t\); \(Q_a(t)\) denotes the total heat flow rate to inside air mass at time \(t\); and \(Q_m(t)\) denotes the total heat flow rate to inside solid mass at \(t\). Equations (4)-(5) can equivalently be expressed in the following matrix form:

\[
\dot{x}(t) = Mx(t) + Bv(t) ;
\]

where \(M = \begin{bmatrix} -\frac{U_a + H_m}{C_a} & \frac{H_m}{C_a} \\ \frac{H_m}{C_m} & -\frac{H_m}{C_m} \end{bmatrix} \); 
\(x(t) = \begin{bmatrix} T_a(t) \\ T_m(t) \end{bmatrix} ; \)
\(B = \begin{bmatrix} \frac{U_a}{C_a} & \frac{1}{C_a} & 0 \\ 0 & 0 & \frac{1}{C_m} \end{bmatrix} \); 
\(v(t) = \begin{bmatrix} T_o(t) \\ Q_a(t) \\ Q_m(t) \end{bmatrix} .\)

Form (6) expresses the dynamic state equations for the GLD Household ETP Model as a linear nonhomogenous differential system with state vector \(x(t)\), state matrix \(M\), and time-varying coefficient vector \(Bv(t)\).

Nevertheless, it is difficult to glean from the GLD documentation [6] alone the intended structural representations for the time-\(t\) endogenous variables \(Q_a(t)\) and \(Q_m(t)\). By a structural representation is meant a simultaneous-equation system such as (2) that permits these time-\(t\) endogenous variables to be expressed as functions of state variables, control variables, forcing terms, and parameters.

We therefore consulted the GLD documentation [5, 7] and GLD source code [8] to understand better how equations (4) and (5) are augmented in GLD with simultaneous-equation
relationships to obtain a complete structural representation for the thermal dynamics of a household with an HVAC system. This section presents this complete structural representation for the special case in which the household’s HVAC system is an electric system running in cooling mode with a linear cooling-capacity curve.

For this purpose, we will first re-express equations (4) and (5) in the standard continuous-time state-space control model form presented in Section 3. Let the time-$t$ outside temperature (an external weather-related forcing term) be denoted by $w^S(t) = (T_o(t))$. Also, let the time-$t$ endogenous variables appearing in these equations be denoted by $z^S(t) = (Q_a(t), Q_m(t))$. Finally, let the parameters appearing in these equations be denoted by $\theta^S = (U_a, H_m, C_a, C_m)$. Equations (4) and (5) can then be expressed in the following compact form:

$$\dot{x}(t) = S(w^S(t), z^S(t), x(t) | \theta^S) \quad (7)$$

However, the differential system (7) is not yet in complete form due to the appearance of the time-$t$ endogenous variables $z^S(t)$ on the right-hand side. To obtain a complete form, system (7) needs to be augmented with a system of simultaneous equations such as (2) that permit these time-$t$ endogenous variables to be expressed as functions of the time-$t$ state $x(t)$, the time-$t$ control variable $u(t)$, forcing terms, and parameters.

According to the GLD documentation [5, 6], the time-$t$ endogenous variables $Q_a(t)$ and $Q_m(t)$ represent the total heat flow rates to the household’s inside air mass and inside solid mass, respectively. The total heat flow rate $Q_a(t)$ is assumed to be determined by specified fractions of (i) solar radiation ($Q_s(t)$); (ii) the internal heat gain from household occupants and non-HVAC equipment ($Q_i(t)$); and (iii) HVAC system cooling-mode operations ($Q_{hvac}(t)$) as follows:

$$0 = [1 - f_{ac}]Q_{hvac}(t) + [1 - f_s]Q_s(t) + [1 - f_i]Q_i(t) - Q_a(t) , \quad (8)$$

where $f_{ac}$, $f_s$, and $f_i$ are user-set unit-free weight coefficients in $[0,1]$. The heat flow rate $Q_m(t)$ is then assumed to be determined as follows:

$$0 = f_{ac}Q_{hvac}(t) + f_sQ_s(t) + f_iQ_i(t) - Q_m(t) . \quad (9)$$

The weight coefficient $f_{ac}$ is identified as a user-set parameter in the GLD documentation [5, p. 5]. However, $f_{ac}$ is hard-coded to 0 in the GLD source code [8, lines 1808-1809].
As discussed in Pratt [5], the time-\(t\) endogenous variable \(Q_s(t)\) (Btu/hr) appearing in (8) and (9) is determined as a function of the time-\(t\) incident solar radiation \(ISR(t)\) (Btu/hr-ft\(^2\)), an external weather-related forcing term,\(^7\) as follows:

\[
0 = \left[ A_g \cdot SHGC_{\text{nom}} \cdot \text{WET} \right] \cdot ISR(t) - Q_s(t), \quad (10)
\]

where: WET (decimal %) is a user-set parameter; and \(A_g\) (ft\(^2\)) and \(SHGC_{\text{nom}}\) (decimal %) are derived parameters whose derivations as functions of user-set parameters are given below in Section 4.4.

Assuming the HVAC system includes a 1-speed fan for the maintenance of air circulation, \(Q_{\text{hvac}}(t)\) (Btu/hr) in eqs. (8) and (9) is given by:

\[
Q_{\text{hvac}}(t) = \left( -HVACPow(t) + \text{FanPow} \right) \cdot u(t) \quad (11)
\]

where: -\(\lbrack HVACPow(t)\rbrack\) (Btu/hr) denotes heat loss from the ON operation of the HVAC system running in cooling mode; \(\text{FanPow}\) (Btu/hr) denotes heat gain from the ON operation of the 1-speed fan; and \(u(t)\) is a binary 0-1 (OFF/ON) HVAC power-usage control variable. We will next develop with care structural representations for HVACPow\((t)\) and FanPow, i.e., representations expressed solely in terms of user-set parameters, GLD-determined parameters, and forcing terms.

Let \(P^*(t)\) (kW) denote the ON power usage of the HVAC system in cooling mode. Then

\[
\text{HVACPow}(t) = K(t)P^*(t) \quad (12)
\]

where

\[
\begin{align*}
K(t)P^*(t) &= \left( \frac{\text{Voltage\_adj}(t) \cdot \text{DesCoolCap\_adj}(t)}{[1 + LCF(t)]} \right) ; \\
P^*(t) &= \frac{\text{DesCoolCap\_adj}(t) \cdot \text{VF}(t)}{K \cdot \text{COP\_adj}(t)} ; \\
K(t) &= \frac{K \cdot \text{COP\_adj}(t) \cdot \text{Voltage\_adj}(t)}{[1 + LCF(t)] \cdot \text{VF}(t)} ;
\end{align*}
\]

\(^7\)ISR\((t)\) is calculated using the solar\_flux data obtained from the GLD Climate Object. For any time of year, the weather file is processed to estimate the solar radiation incident on a vertical surface oriented in each of eight cardinal directions (based on true south, not magnetic south) from the beam and diffuse components of the global horizontal radiation.
\[
\text{DesCoolCap}_\text{adj}(t) = \text{DesignCoolCap} \cdot [a - b \cdot T_o(t)]; \\
\text{COP}_\text{adj}(t) = \frac{\text{cooling\_COP}}{c + d \cdot T_o(t)}; \\
\text{LCF}(t) = \left( e + \frac{\text{LatCoolFrac}}{[f + \exp(g - m \cdot \text{RH}(t))]} \right); \\
\text{VF}(t) = F_P + F_C \cdot \text{VoltFactorN}(t) + F_Z \cdot [\text{VoltFactorN}(t)]^2; \\
\text{Voltage}_\text{adj}(t) = F_P + F_C \cdot \text{VoltFactorB}(t) + F_Z \cdot [\text{VoltFactorB}(t)]^2; \\
\text{VoltFactorN}(t) = \frac{V_{\text{actual}}(t)}{V_{\text{nominal}}}; \\
\text{VoltFactorB}(t) = \frac{V_{\text{actual}}(t)}{V_{\text{base}}}. 
\]

In eqs. (13)–(17), \text{DesCoolCap}_\text{adj}(t) (\text{Btu/hr}) is determined as a function of the user-set parameter DesignCoolCap (\text{Btu/hr}) and the outside temperature \(T_o(t)\); the term \text{COP}_\text{adj}(t) is a unit-free coefficient of performance factor determined as a function of the user-set parameter Cooling\_COP (unit free) and outside temperature \(T_o(t)\); \(K = 3412\text{Btu/hr-kW}\) is a GLD-determined conversion factor that converts kW to Btu/hr; and \text{LCF}(t) is a unit-free factor determined as a function of the user-set parameter LatCoolFac (unit free) and time-\(t\) relative humidity RH\((t)\).

The parameters \(a, b, c, d, e, f, g, \text{and} \ m\) appearing in eqs. (13)–(17) are GLD-determined parameters whose values are GLD-set as follows: \(a = 1.48924533\) (unit free); \(b = 0.00514995\) (1/\(^\circ\)F); \(c = -0.01363961\) (unit free); \(d = 0.01066989\) (1/\(^\circ\)F); \(e = 0.1\) (unit free), \(f = 1.0\) (unit free), \(g = 4.0\) (unit free), and \(m = 10.0\) (unit free).

The coefficients \(F_P\) (power fraction), \(F_C\) (current fraction), and \(F_Z\) (impedance fraction) appearing in eqs. (18) and (19) are unit-free GLD-determined parameter values given by \(F_P = 0.8, F_C = 0.0, \text{and} \ F_Z = 0.2.\) The term \(V_{\text{actual}}(t)\) (volts) appearing in the numerator of eqs. (20) and (21) is a time-\(t\) forcing term given by the simulated actual voltage at time \(t\) obtained from the GLD meter object in run-time. The term \(V_{\text{nominal}}\) (volts) appearing in the denominator of eq. (20) is a user-set parameter for nominal voltage that the user can set either to 120V or to 240V. In contrast, the term \(V_{\text{base}}\) (volts) appearing in the

\[^8\]These coefficients are GLD-set for an HVAC system but can be set by users for other types of appliances.

\[^9\]\(V_{\text{actual}}(t)\) is jointly determined by the power injections and withdrawals of all resources connected to the GLD-simulated distribution grid. In the current study the effect of any one household’s operations on \(V_{\text{actual}}(t)\) is assumed to be negligible, thus permitting it to be treated as an external forcing term for the household’s thermal dynamics.
denominator of eq. (21) is a GLD-determined parameter that is GLD-set at 240V.

Finally, the parameter FanPow (Btu/hr) is derived from the ON power consumption $P_{fan}$ (kW) of the single-speed fan, as follows:

$$\text{FanPow} = K \cdot P_{fan}$$

(22)

where, as before, $K = 3412 \text{Btu/hr-kW}$ is a GLD-determined conversion factor that converts kW to Btu/hr. In turn,

$$P_{fan} = C \cdot \text{FanDesignPower}$$

(23)

where FanDesignPower (W) is a user-set parameter and $C = 1/1000$ is a GLD-determined conversion factor that converts watts to kW.

The fourteen equations (8)-(21) can be compactly expressed as a 14-dimensional system of time-$t$ simultaneous equations taking the following form:

$$0 = H_1(u(t), w^H_1(t), z(t) | \theta^H_1)$$

(24)

where:

$$u(t) = \text{binary 0-1 (OFF/ON) HVAC power-usage control variable}$$

$$w^H_1(t) = (T_o(t), RH(t), V_{\text{actual}}(t), ISR(t))$$

$$z(t) = (Z_1(t), Z_2(t), Z_3(t))$$

$$Z_1(t) = (Q_a(t), Q_m(t), Q_s(t), Q_i(t), Q_{hvac}(t), \text{HVACPow}(t))$$

$$Z_2(t) = (P^*(t), K(t), \text{DesCoolCap_adj}(t), \text{COP_adj}(t), LCF(t))$$

$$Z_3(t) = (VF(t), \text{Voltage_adj}(t), \text{VoltFactorN}(t), \text{VoltFactorB}(t))$$

$$\theta^H_1 = (\theta^{H11}, \theta^{H12}, \theta^{H13})$$

$$\theta^{H11} = (f_{ac}, f_s, f_i, \text{DesignCoolCap}, \text{Cooling\_COP}, \text{LatCoolFrac}, V_{\text{nominal}})$$

$$\theta^{H12} = (\text{FanPow}, \text{DuctPressureDrop}, \text{DesignCoolAirflow}, \text{DesignHeatAirflow})$$

$$\theta^{H13} = (\text{WET}, A_g, \text{SHGC}_{\text{nom}})$$

The 14-dimensional system of equations (24) determines all time-$t$ endogenous variables in $z(t)$, apart from $Q_i(t)$, as functions of the control variable $u(t)$, the forcing terms $w^H_1(t)$, the parameters in $\theta^H_1$, and $Q_i(t)$. However, $Q_i(t)$ itself is not determined by system (24).
To determine $Q_i(t)$ (Btu/hr), an additional time-$t$ simultaneous equation is needed that expresses $Q_i(t)$ as a function of the time-$t$ control variable $u(t)$, time-$t$ forcing terms, time-$t$ endogenous variables, and parameters. The determination of $Q_i(t)$ is formulated in [5] in general descriptive terms. This general formulation will now be expressed in needed state-space control form, as follows.

Let $p_{eu}(t)$ (kW) denote the current real power for each household non-HVAC\(^\text{10}\) end-use load $eu$, multiplied by the user-set fraction $f_{I_{eu}}$ of this load that is internal to the household. Let $\text{NEU}$ denote the user-set number of household non-HVAC end-use loads. Also, let $\text{NOC}$ denote the user-set number of household occupants, where the sensible heat from each of these occupants is measured by the user-set parameter $\text{SHOC}$ (Btu/hr-occupant).

Finally, let $f_{oc}$ denote the user-set occupancy fraction and $K$ denote the GLD-determined conversion factor given by 3412 Btu/hr-kW. Then:

$$Q_i(t) = K \cdot \left( \sum_{eu=1}^{\text{NEU}} p_{eu}(t) \cdot f_{I_{eu}} \right) + [\text{SHOC} \cdot \text{NOC} \cdot f_{oc}] \quad (25)$$

It is seen from (25) that $Q_i(t)$ depends on NEU forcing terms external to HVAC operations: namely, the NEU elements of the vector $w_{H2}(t) = (p_1(t), \ldots, p_{\text{NEU}}(t))$ giving the time-$t$ real power levels for each of the household’s non-HVAC end-use loads, assumed to be NEU in number. Let $z_{H2}(t) = Q_i(t)$. Let $\mathbf{fI} = (f_{I_{1}}, \ldots, f_{I_{\text{NEU}}})$ denote the NEU-dimensional vector giving the fractions of non-HVAC end-use loads that are internal to the household. Finally, let the vector of user-set parameters for relation (25) be denoted by $\theta^{H2} = (\mathbf{fI}, \text{NEU}, \text{SHOC}, \text{NOC}, f_{oc})$. Given this notation, the time-$t$ simultaneous equation (25) for $Q_i(t)$ can be expressed in the required form as follows:

$$0 = H_2(w_{H2}(t), z_{H2}(t) | \theta^{H2}) \quad (26)$$

Relation (26) completes the basic state-space control model representation for the GLD Household ETP Model.\(^\text{11}\)

\(^{10}\)We have added the “non-HVAC” qualifier here to be consistent with the interpretation of $Q_i(t)$ as internal heat gain arising from household non-HVAC equipment and occupants.

\(^{11}\)Concerns remain about the precise GLD-determination of the time-$t$ forcing terms $w_{H2}(t)$. These time-$t$ forcing terms need to be consistent with: (i) the user’s specification of the household’s appliance mix; (ii) the user’s specification of household occupants at time $t$; and (iii) the user’s specification of occupant methods that affect non-HVAC equipment usage at time $t$. Note that the occupants of a household at any given time $t$ can differ from the resident(s) of a household, i.e., the people who are in residence at the household. Occupants can be temporary visitors. This distinction is important for household welfare calculations.
4.3 Complete Analytic Formulation: Summary Form

Below we provide a complete summary analytic formulation of the GLD Household ETP Model as a state-space control model. This complete analytic description differs from the description of the standard state-space control model presented in Section 3 in one important regard: namely, it incorporates coupled-parameter relationships that show precisely how each derived parameter appearing in the model equations is determined as a function of the user-set parameters listed in Table 1.

Coupled-parameter relationships relating derived to user-set parameters need to be given for the GLD Household ETP Model in order to ensure that all of these parameters are set reasonably for the study at hand. Specifically, the user should set values for the parameters in \( \theta_{user} \) that are sensible compatible settings for the particular household that the user is trying to model. The coupled-parameter relationships should then guarantee that all other parameter settings are sensible and compatible for this household as well.

**GLD Household ETP Model in State-Space Control Form:** For each \( t \geq t^0 \),

**Dynamic state equations:**

\[
\dot{x}(t) = S(w^S(t), z^S(t), x(t) | \theta^S) \quad (27)
\]

**Simultaneous equations:**

\[
0 = H_1(u(t), w^{H1}(t), z^{H1}(t) | \theta^{H1}) \quad (28)
\]

**Simultaneous equation:**

\[
0 = H_2(w^{H2}(t), z^{H2}(t) | \theta^{H2}) \quad (29)
\]

**Integral equations:**

\[
x(t) = \int_{t^0}^{t} \dot{x}(\tau)d\tau + x(t^0) \quad (30)
\]

**Coupled-Parameter Relationships:**

\[
0 = CP_S(\theta^S, \theta_{user}) \quad (31)
\]

\[
0 = CP_{H1}(\theta^{H1}, \theta_{user}) \quad (32)
\]

\[
0 = CP_{H2}(\theta^{H2}, \theta_{user}) \quad (33)
\]

**Variables, Parameters, and Functional Forms** \((t \geq t^0)\):

\[
x(t) = (T_a(t), T_m(t)) = \text{State vector at time } t
\]

\[
\dot{x}(t) = (\dot{T}_a(t), \dot{T}_m(t)) = \text{State gradient vector at time } t
\]

\[
u(t) = \text{Binary 0-1 (OFF/ON) power-usage control variable at time } t
\]

\[
w(t) = (T_o(t), RH(t), V_{\text{actual}}(t), ISR(t), p_1(t), \ldots, p_{\text{NEU}}(t)) = \text{Forcing terms at time } t
\]
$w^S(t) = \left(T_o(t)\right)$ = Forcing term for $S(\cdot)$ in (27) at $t$

$w^{H1}(t) = \left(T_o(t), RH(t), V_{\text{actual}}(t), ISR(t)\right)$ = Forcing terms for $H_1(\cdot)$ in (28) at $t$

$w^{H2}(t) = \left(p_1(t), \ldots, p_{\text{NEU}}(t)\right)$ = Forcing terms for $H_2(\cdot)$ in (29) at $t$

$z^S(t) = \left(Q_a(t), Q_m(t)\right)$ = Time-$t$ endogenous variables for $S(\cdot)$ in (27)

$z^{H1}(t) = \left(z^{H11}(t), z^{H12}(t), z^{H13}(t)\right)$ = Time-$t$ endogenous variables for $H_1(\cdot)$ in (28)

$z^{H11}(t) = \left(Q_a(t), Q_m(t), Q_s(t), Q_i(t), Q_{\text{hvac}}(t), \text{HVAC} \text{Pow}(t)\right)$

$z^{H12}(t) = \left(P^*(t), K(t), \text{DesCoolCap_{adj}}(t), \text{COP_{adj}}(t), \text{LCF}(t)\right)$

$z^{H13}(t) = \left(\text{VF}(t), \text{Voltage_{adj}}(t), \text{VoltFactorN}(t), \text{VoltFactorB}(t)\right)$

$z^{H2}(t) = \left(Q_i(t)\right)$ = Time-$t$ endogenous variable for $H_2(\cdot)$ in (29)

$\theta^S = \left(U_a, H_m, C_a, C_m\right)$ = Parameter vector for $S(\cdot)$ in (27)

$\theta^{H1} = \left(\theta^{H11}, \theta^{H12}, \theta^{H13}\right)$ = Parameter vector for $H_1(\cdot)$ in (28)

$\theta^{H11} = \left(f_{ac}, f_s, f_i, \text{DesignCoolCap}, \text{Cooling\_COP}, \text{LatCoolFrac}, \text{V\_nominal}\right)$

$\theta^{H12} = \left(\text{FanPow}, \text{DuctPressureDrop}, \text{DesignCoolAirflow}, \text{DesignHeatAirflow}\right)$

$\theta^{H13} = \left(\text{WET}, A_g, \text{SHGC_{nom}}\right)$

$\theta^{H2} = \left(\Pi_1, \ldots, \Pi_{\text{NEU}}, \text{NEU}, \text{SHOC}, \text{NOC}, f_{\text{oc}}\right)$ = Parameter vector for $H_2(\cdot)$ in (29)

$\theta_{\text{user}}$ = TV-dimensional vector consisting of all user-set parameters listed in Table 1

$S:R^{S_{J+L+S+N+SV}} \rightarrow R^N$ given by the differential equation system (7)

$H_1:B \times R^{H1_{J+H1}+H1_{N+H1V}} \rightarrow R^{L-1}$ given by the simultaneous-equation system (24)

$H_2:B \times R^{H2_{J+H2}+H2_{N+H2V}} \rightarrow R$ given by the simultaneous equation (26)

$CP_S:R^{SV+TV} \rightarrow R^{SV}$

$CP_{H1}:R^{H1_{IV+TV}} \rightarrow R^{H1_{V}}$
\[ CP_{H2}: R^{H2V+TV} \rightarrow R^{H2V} \]

\[ B = \{0, 1\}, \ J=\text{NEU}+4, \ L=15, \ N=2 \]

\[ SJ=1, \ SL=2, \ SN=2, \ SV=4 \]

\[ H1J=4, \ H1L=15, \ H1N=0, \ H1V=14 \]

\[ H2J=\text{NEU}, \ H2L=1, \ H2N=0, \ H2V=\text{NEU}+4 \]

**Classification of Variables:**

*Time-t endogenous variables for* \( t \geq t^0 \): \( \dot{x}(t), \ z(t) \)

*Time-t predetermined variables for* \( t > t^0 \): \( x(t) \)

*Exogenous controls and forcing terms for* \( t \geq t^0 \): \( u(t), w(t) \)

*Exogenous parameters and initial state conditions:* \( \theta_{\text{user}}, \ \theta^S, \ \theta^H = (\theta^{H1}, \theta^{H2}), \) and \( x(t^0) \)

### 4.4 Coupled-Parameter Relationships for the Analytic Formulation

The coupled-parameter relationships (31) that permit the parameters appearing in the parameter vector \( \theta^S = (U_a, H_m, C_a, C_m) \) for the state differential system (27) to be expressed as functions of the user-set parameters \( \theta_{\text{user}} \) listed in Table 1 are as follows.\(^{12}\)

\[ U_a = \frac{A_c}{R_c} + \frac{A_d}{R_d} + \frac{A_f}{R_f} + \frac{A_g}{R_g} + \frac{A_w}{R_w} + VH_aAhI; \]  
\[ H_m = h_s \left[ (A_{wt} - A_g - A_d) + A_{wt}\text{IWR} + \frac{A_cn_s}{EWR} \right]; \]  
\[ C_a = 3VH_aAh; \]  
\[ C_m = Am_f - 2VH_aAh, \]

\(^{12}\)The expressions (35) and (45) below for \( H_m \) and \( A_w \) are consistent with the GLD documentation [5] and the GLD source code [8]. The \( H_m \) and \( A_w \) expressions appearing in the GLD documentation [6] appear to be incorrect. Also, it is clear from (36) and (38) below that \( C_a \) is a derived parameter. However, the GLD documentation [12] and the GLD source code [8, line 190] incorrectly imply that \( C_a \) is a user-set parameter whose value can be set independently of other user-set parameters.
where:

\[ A = x \times y \times n_s \quad \text{(38)} \]
\[ R = \frac{y}{x} \quad \text{(39)} \]
\[ A_c = \frac{A}{n_s} \times \text{ECR} \quad \text{(40)} \]
\[ A_d = n_d \times A_{1d} \quad \text{(41)} \]
\[ A_f = \frac{A}{n_s} \times \text{EFR} \quad \text{(42)} \]
\[ A_{wt} = 2n_s h \left[1 + R \right] \sqrt{\frac{A}{n_s R}} \quad \text{(43)} \]
\[ A_g = \text{WWR} \times A_{wt} \times \text{EWR} \quad \text{(44)} \]
\[ A_w = (\text{EWR} \times A_{wt}) - (A_d + A_g) \quad \text{(45)} \]
\[ R_g = \text{Value determined from a table in the GLD documentation [11]} \quad \text{(46)} \]

In (34), \( V_{H_a} = 0.018 \) (Btu/ft\(^3\)-oF) is a GLD-determined parameter value denoting volumetric heat capacity of air at standard conditions.\(^{14}\) Also, in (41), \( A_{1d} = 19.5 \) (ft\(^2\)) is a GLD-determined parameter value for the area of one door.

The coupled-parameter relationships (32) that permit the parameters appearing in \( \theta^{H1} = (\theta^{H11}, \theta^{H12}, \theta^{H13}) \) for \( H_1(\cdot) \) in (24) to be expressed as functions of the user-set parameters \( \theta_{\text{user}} \) listed in Table 1 are as follows. First consider \( \theta^{H11} \). The coupled-parameter relationships (32) that functionally relate \( \theta^{H11} \) to \( \theta_{\text{user}} \) are direct one-to-one mappings because all of the parameters appearing in \( \theta^{H11} \) are user-set parameters.

Next, consider \( \theta^{H12} \). The coupled parameter relationships for the derived parameter FanPow are given by (22) and (23). The derived parameter DesignCoolAirflow (cfm) is

\(^{13}\)As indicated below in expressions (38) and (39), \( A \) and \( R \) are derived parameters whose values are commonly dependent on the user-set values for \( x \) and \( y \). The GLD documentation [12] identifies \( A \) and \( R \) as user-set parameters, which incorrectly implies their values can be set independently of each other. Also, as indicated below in expression (46), \( R_g \) is a derived parameter whose value is determined as a function of user-set parameters. However, the GLD documentation [12] and the GLD source code [8, line 408] identify \( R_g \) as a user-set parameter, incorrectly implying that its value can be set independently of the values set for all other other user-set parameters.

\(^{14}\)More precisely, \( V_{H_a} = 0.018 \) (Btu/ft\(^3\)-oF) is calculated as the product of two other GLD-determined parameter values: namely, AirDensity = 0.0735 (lb/ft\(^3\)) and AirHeatCapacityValue = 0.2402 (Btu/lb-oF). See [6, sec. 3.2.1].
given in [5, p. 20] as follows:

\[
\text{DesignCoolAirflow} = \left( \frac{\text{DesignCoolCap}}{[1 + \text{LatCoolFrac}] [F \cdot \text{VH}_a]} \right) \cdot \frac{1}{60}
\] (47)

where

\[
F = [\text{DCT} - \text{CoolSupplyAirTemp}].
\] (48)

The terms LatCoolFrac (unit free), DCT (°F) and CoolSupplyAirTemp (°F) in (47) and (48)
are user-set parameters. The term \(\text{VH}_a\) (Btu/ft\(^3\)-°F) is a GLD-determined parameter with
a GLD-set value given by 0.018; see Footnote 14. Also, DesignHeatAirflow (cfm) is given
in [5, p. 19] as follows:\(^15\)

\[
\text{DesignHeatAirflow} = \left( \frac{\max\{\text{AuxHeatCap}, \text{DesignHeatCap}\}}{G \cdot \text{VH}_a} \right) \cdot \frac{1}{60}
\] (49)

where

\[
G = [\text{HeatSupplyAirTemp} - \text{DesignHeatSetpoint}].
\] (50)

In (49) and (50), AuxHeatCap (Btu/hr), DesignHeatCap (Btu/hr), HeatSupplyAirTemp
(°F), and DesignHeatSetpoint (°F) are all user-set parameters.

Next consider \(\theta_{H13}\). WET is a user-set parameter. The derived parameter \(A_g\) is deter-
mined as a function of user-set parameters by equations (38), (39), (43), and (44). A table in
the GLD documentation [11] indicates that the derived parameter \(\text{SHGC}_{\text{nom}}\) is a function of
combined settings for three user-set parameters: namely, glazing_treatment, glazing_layers,
and window_frame.\(^16\)

Finally, consider \(\theta_{H2}\). The coupled-parameter relationships (33) that functionally relate
\(\theta_{H2}\) to \(\theta_{\text{user}}\) are direct one-to-one mappings because all of the parameters appearing in \(\theta_{H2}\)
are user-set parameters.

4.5 Default Functions for User-Set Parameters

Table 1 provides a complete listing of the user-set parameters for the GLD Household ETP
Model. As seen in Table 4, GLD provides default values for most of these user-set parameters.

\(^{15}\)The expression (49) given below for DesignHeatAirflow is consistent with the GLD document [5, p. 19]
and the GLD source code [8, line 1479].

\(^{16}\)The GLD source code [8, line 190] states that \(\text{SHGC}_{\text{nom}}\) is a user-set parameter, implying incorrectly
that the value of this parameter can be set independently of the values for all other user-set parameters.
However, for the four user-set parameters DesignCoolCap (Btu/hr), DesignHeatCap (Btu/hr), AuxHeatCap (Btu/hr), and FanDesignPower (W), GLD instead provides default functions. More precisely, for these four user-set parameters a user can either directly set their values or use GLD default functions whose arguments are given by GLD-determined parameters, GLD-derived parameters, and/or other user-set parameters.

The GLD default function for DesignCoolCap expresses DesignCoolCap as a function of two derived parameters \( U_a, \) SHGC plus various user-set parameters, as follows:

\[
\text{DesignCoolCap} = \text{Ceil} \left( \frac{\text{DCP}}{6000} \right) \cdot 6000
\]  

where

\[
\text{DCP} = U_a \cdot [1 + \text{LatCoolFrac}] \cdot \text{CDT} - \text{DCT} \cdot [1 + \text{OSF}] + \text{DIG} + [\text{DPS} \cdot \text{SHGC}]
\]  

The derived parameter \( U_a \) is determined as a function of user-set parameters by relationship (34) together with equations (38) through (46). Using the coupled-parameter relations for \( A_g \) and SHGC\(_{\text{nom}}\), the derived parameter SHGC is determined as a function of user-set parameters by substituting out for \( A_g \) and SHGC\(_{\text{nom}}\) in the following relationship:

\[
\text{SHGC} = A_g \cdot \text{SHGC}_{\text{nom}} \cdot \text{WET}
\]  

The GLD default function for DIG (Btu/hr) is given by

\[
\text{DIG} = q \cdot A^r.
\]  

In (54), \( q \) (Btu/hr-ft\(^2\)) and \( r \) (unit free) are GLD-determined parameters with GLD-set values given by \( q = 167.09 \) (Btu/hr-ft\(^2\)) and \( r = 0.442 \) (unit free). Also \( A \) (ft\(^2\)) is a derived parameter determined as the product of the three user-set parameters \( x, y, \) and \( n_a \).

The GLD default functions for DesignHeatCap (Btu/hr) and AuxHeatCap (Btu/hr) and are identical, expressed as follows:

\[
\text{DesignHeatCap} = \text{AuxHeatCap} = \text{Ceil} \left( \frac{\text{HeatCap}}{10000} \right) \cdot 10000
\]  

where

\[
\text{HeatCap} = U_a [1.0 + \text{OSF}] \cdot \text{[DesignHeatSetpoint - HeatDesignTemp]}
\]  

19
In (56), $U_a$ (Btu/hr-°F) is a GLD-derived parameter; see (34). The remaining terms OSF (unit free), DesignHeatSetpoint (°F), and HeatDesignTemp (°F) are user-set parameters.

Finally, the GLD default function for FanDesignPower (W) as a function of user-set parameters, derived parameters and GLD-determined parameters is as follows:\textsuperscript{17}

$$\text{FanDesignPower} = \text{Ceil} \left( n \cdot r \cdot D \cdot E \right) \cdot q$$  \hspace{1cm} (57)

where

$$D = \left[ \text{DuctPressureDrop} \right]$$

$$E = \max \{ \text{DesignCoolAirflow}, \text{DesignHeatAirflow} \}$$

The factor $D$ (DuctPressureDrop) measured in inches of water is a user-set parameter. DesignCoolAirflow (cfm) and DesignHeatAirflow (cfm) in $E$ are derived parameters whose derivations as functions of user-set parameters are given above in Section 4.4. The terms $n$, $q$, and $r$ are GLD-determined parameters whose numerical values are set as follows in the GLD source code:\textsuperscript{18}

$$n = \left[ \frac{8}{(745.7 \times 0.42)} \right];$$

$$q = \frac{745.7}{8 \times 0.88};$$

$$r = 0.117 \text{ Watt}/[\text{cfm-inches of water}].$$

5 GLD Household ETP Model: Implementation

5.1 Overview

The GLD source code [8] indicates that the GLD Household ETP Model is implemented by first determining its closed-form solution and then discretizing the implementation of this closed-form solution by approximating forcing terms as step functions. Specifically, at each time step, the value of each forcing term is held constant at the value it takes on at the beginning of this time step.

\textsuperscript{17}The $\text{Ceil}()$ function in C and C++ returns the smallest possible integer value which is greater than or equal to the given argument.

\textsuperscript{18}The units for $n$ and $q$ are not specified in the GLD source code. However, in order for FanPow in (22) to be measured in Btu/hr, the product $nq$ should be unit free.
This section demonstrates an alternative implementation approach. The GLD Household ETP Model is approximated by means of a simple forward finite-difference method called the Euler Method.\(^{19}\) Similar to the GLD method, the time-step length is assumed to be short enough to permit forcing terms to be held constant at their initial time-step levels during each time-step.

### 5.2 Matrix Representation

The matrix form (6) expresses the GLD Household ETP Model as a linear nonhomogenous differential system with a time-varying coefficient vector \(Bv(t)\). Assuming a known trajectory for \(v(t)\), together with suitable regularity conditions, a closed-form solution for (6) can be analytically determined using various methods. One such method, outlined in the GLD documentation \([13]\), involves first converting this system into a one-dimensional second-order differential system in \(T_a(t)\) with modified boundary conditions, solving for \(T_a(t)\), and then deriving the implied solution value for \(T_m(t)\).

Recall, however, that the linearity of the GLD Household ETP Model is itself a strong initial assumption. Consequently, what one is obtaining is a closed-form solution to a system in approximate linear form. An alternative way to proceed would be to start from an ETP Model represented as a continuous-time nonlinear state-space control model, as expressed in Section 3. Various discretization methods could then be directly applied to this nonlinear system to obtain an approximate discrete-time solution.

Which method — initial linearization or discrete-time approximation — would lead to smaller approximation errors when numerically implemented on a computer depends on a number of critical factors: namely, the extent to which household thermal dynamics are well approximated by a linear differential system such as (6); the determination (approximation) of the vector of time-varying forcing terms; the determination (approximation) of boundary conditions; round-off errors; truncation errors; and error accumulation over time.

---

\(^{19}\)The Euler Method suffices for this purposes of this study. However, reduced approximation error can be obtained by augmenting this first-order “predictor” method with a “corrector” method. For further discussion of approximation methods for ordinary differential systems of equations, see any basic text such as Lambert \([14]\). For online lecture notes, see Süli \([15]\).
5.3 Finite-Difference Approximation Method

Below we illustrate how a relatively simple forward finite-difference method can be used to obtain a discrete-time approximation for the nonlinear continuous-time state space control model expressed in Section 3. As will be seen, this method does not require linearization of the state function \( S(\cdot) \) in (1) or the function \( H(\cdot) \) in (2) that expresses simultaneous-equation relationships. However, it does presume that the time-step length \( \Delta t \) used for the discretization is sufficiently small that the trajectory for the forcing-term vector \( w(t) \) can be well-approximated by a step function over successive steps of equal length \( \Delta t \).

Consider the continuous-time state-space control model in standard form, as presented in Section 3. Let \( t \geq t^0 \) be given, and let \( \Delta t \) (hr) denote a positive time increment, e.g., 1hr/12 equivalent to 300s. Let the gradient \( \dot{x}(t) \) for the state vector \( x(t) \) at each time \( t \) be approximated by the following finite-difference expression:

\[
\dot{x}(t) \approx \frac{x(t + \Delta t) - x(t)}{\Delta t}
\]  

(58)

Substituting (58) in place of \( \dot{x}(t) \) in (1), and manipulating terms, one obtains

\[
x(t + \Delta t) \approx x(t) + S(u(t), w(t), z(t), x(t) \mid \theta^S) \cdot \Delta t
\]  

(59)

For each \( k = 0, 1, \ldots \), let period \( k \) denote the time interval \([t^0 + k\Delta t, t^0 + (k + 1)\Delta t]\). Also, define

\[
F(u_k, w_k, z_k, x_k \mid \theta^S, \Delta t) \equiv x_k + S(u_k, w_k, z_k, x_k \mid \theta^S) \cdot \Delta t
\]  

(60)

where

\[
u_k = u(t^0 + k\Delta t)
\]  

(61)

\[
w_k = w(t^0 + k\Delta t)
\]  

(62)

\[
z_k = z(t^0 + k\Delta t)
\]  

(63)

\[
x_k = x(t^0 + k\Delta t)
\]  

(64)

Then the original continuous-time state space model (1) over times \( t \geq t^0 \) can be expressed in discrete-time approximate form over periods \( k = 0, 1, \ldots \), as follows:
Discrete-time approximation equations for periods $k \geq 0$:

Dynamic state equations: $x_{k+1} = F(u_k, w_k, z_k, x_k | \theta^S, \Delta t)$ (65)

Simultaneous equations: $0 = H(u_k, w_k, z_k, x_k | \theta^H)$ (66)

Variables, parameters, and functional forms:

$u_k = (u_{k1}, \ldots, u_{kM}) \in \mathbb{R}^M$, for periods $k \geq 0$

$w_k = (w_{k1}, \ldots, w_{kJ}) \in \mathbb{R}^J$, for periods $k \geq 0$

$z_k = (z_{k1}, \ldots, z_{kL}) \in \mathbb{R}^L$, for periods $k \geq 0$

$x_k = (x_{k1}, \ldots, x_{kN}) \in \mathbb{R}^N$, for periods $k \geq 0$

$\theta^S = (\theta^S_1, \ldots, \theta^S_{SV}) \in \mathbb{R}^{SV}$

$\theta^H = (\theta^H_1, \ldots, \theta^H_{HV}) \in \mathbb{R}^{HV}$

$F: \mathbb{R}^{M+J+L+N+SV} \rightarrow \mathbb{R}^N$

$H: \mathbb{R}^{M+J+L+N+HV} \rightarrow \mathbb{R}^L$

Classification of variables:

*Period-k endogenous variables for $k \geq 0$: $x_{k+1}, z_k$

*Period-k predetermined variables for $k > 0$: $x_k$

*Exogenous controls and forcing terms for $k \geq 0$: $u_k, w_k$

*Exogenous parameters and initial state conditions: $\theta^S, \theta^H, \Delta t$, and $x_0$

By construction, the above discrete-time approximation converges to the original continuous-time state space model as the period-length $\Delta t$ is decreased towards 0.

**Appendix**

The first three tables, below, provide symbols, descriptions, and units for the GLD Household ETP Model user-set parameters, derived parameters, and time-t variables. The fourth table lists default values/functions (if any) for the user-set parameters.
<table>
<thead>
<tr>
<th>User-Set Parameters</th>
<th>Explanations</th>
</tr>
</thead>
<tbody>
<tr>
<td>AuxHeatCap</td>
<td>Auxiliary heating capacity (Btu/hr)</td>
</tr>
<tr>
<td>base_power</td>
<td>Base real power [kW] of the total load at nominal voltage</td>
</tr>
<tr>
<td>CDT</td>
<td>System cooling design temperature [°F]</td>
</tr>
<tr>
<td>Cooling_COP</td>
<td>Coefficient of performance [unit free] for HVAC system in cooling mode</td>
</tr>
<tr>
<td>CoolSupplyAirTemp</td>
<td>Cooling supply air temperature [°F]</td>
</tr>
<tr>
<td>cooling_system_type</td>
<td>Determines type of HVAC system running in cooling mode (electric, gas)</td>
</tr>
<tr>
<td>current_fraction</td>
<td>Fraction (decimal %) of load that is constant current [p.u.]</td>
</tr>
<tr>
<td>current_pf</td>
<td>Power factor [unit free] for constant current portion of load [p.u.]</td>
</tr>
<tr>
<td>DCT</td>
<td>System design cooling set-point [°F]</td>
</tr>
<tr>
<td>DesignCoolCap</td>
<td>Design cooling capacity [Btu/hr]</td>
</tr>
<tr>
<td>DesignHeatCap</td>
<td>Design heating capacity [Btu/hr]</td>
</tr>
<tr>
<td>DesignHeatSetpoint</td>
<td>Design heating setpoint [°F]</td>
</tr>
<tr>
<td>DIG</td>
<td>System design internal gain [Btu/hr]</td>
</tr>
<tr>
<td>DPS</td>
<td>System design solar load [Btu/hr-ft²]</td>
</tr>
<tr>
<td>DuctPressureDrop</td>
<td>Duct pressure drop [inches of water]</td>
</tr>
<tr>
<td>ECR</td>
<td>Exterior ceiling, fraction (decimal %) of total</td>
</tr>
<tr>
<td>EFR</td>
<td>Exterior floor, fraction (decimal %) of total</td>
</tr>
<tr>
<td>EWR</td>
<td>Exterior wall, fraction (decimal %) of total</td>
</tr>
<tr>
<td>FanDesignPower</td>
<td>Designed maximum power draw [W] of the ventilation fan</td>
</tr>
<tr>
<td>f_s</td>
<td>Fraction (decimal %) of non-HVAC end-use load internal to house</td>
</tr>
<tr>
<td>J_ac, J_s, J_i</td>
<td>Heat gain (decimal %) from ( Q_{ac}(t) ), ( Q_s(t) ), ( Q_i(t) ) to ( Q_m(t) )</td>
</tr>
<tr>
<td>J_pc</td>
<td>Household occupancy fraction (decimal %)</td>
</tr>
<tr>
<td>glass_type</td>
<td>String-coded glass types (GLASS, LOW_E,...)</td>
</tr>
<tr>
<td>glazing_layers</td>
<td>String-coded window glass-layer types (ONE, TWO, ...)</td>
</tr>
<tr>
<td>glazing_treatment</td>
<td>String-coded exterior window reflectivity types</td>
</tr>
<tr>
<td>HeatDesignTemp</td>
<td>Heating design temperature [°F]</td>
</tr>
<tr>
<td>HeatSupplyAirTemp</td>
<td>Heating supply air temperature [°F]</td>
</tr>
<tr>
<td>I</td>
<td>Infiltration volumetric air exchange rate [#times per hr]</td>
</tr>
<tr>
<td>impedance_fraction</td>
<td>Fraction (decimal %) of load that is constant impedance [p.u.]</td>
</tr>
<tr>
<td>impedance_pf</td>
<td>Power factor [unit free] for constant impedance portion of load [p.u.]</td>
</tr>
<tr>
<td>IWR</td>
<td>Interior/exterior wall surface ratio [unit free]</td>
</tr>
<tr>
<td>LatCoolFrac</td>
<td>Fractional cooling-load increase [unit free] due to latent heat</td>
</tr>
<tr>
<td>m_t</td>
<td>Total thermal mass per unit floor area [Btu/°F-ft²]</td>
</tr>
<tr>
<td>m_d</td>
<td>Number (integer) of doors</td>
</tr>
<tr>
<td>n_s</td>
<td>Number (integer) of stories</td>
</tr>
<tr>
<td>NEU</td>
<td>Number (integer) of household non-HVAC end-use loads</td>
</tr>
<tr>
<td>NOC</td>
<td>Number (integer) of household occupants</td>
</tr>
<tr>
<td>OSF</td>
<td>Over-sizing factor [unit free]</td>
</tr>
<tr>
<td>power_fraction</td>
<td>Fraction (decimal percentage) of load that is constant power [p.u.]</td>
</tr>
<tr>
<td>power_pf</td>
<td>Power factor [unit free] for constant power portion of load [p.u.]</td>
</tr>
<tr>
<td>R_c</td>
<td>Thermal resistance [hr-°F-ft²/Btu] of house ceilings</td>
</tr>
<tr>
<td>R_d</td>
<td>Thermal resistance [hr-°F-ft²/Btu] of house doors</td>
</tr>
<tr>
<td>R_f</td>
<td>Thermal resistance [hr-°F-ft²/Btu] of house floors</td>
</tr>
<tr>
<td>R_w</td>
<td>Thermal resistance [hr-°F-ft²/Btu] of house walls</td>
</tr>
<tr>
<td>SHOC</td>
<td>Sensible heat [Btu/hr-occupant] from each occupant</td>
</tr>
<tr>
<td>V_nominal</td>
<td>Nominal rating voltage [volts]</td>
</tr>
<tr>
<td>WET</td>
<td>Window exterior transmission coefficient (decimal %)</td>
</tr>
<tr>
<td>window_frame</td>
<td>String-coded window-frame types [INSULATED, WOOD,...]</td>
</tr>
<tr>
<td>WWR</td>
<td>Window-to-exterior-wall ratio (decimal %)</td>
</tr>
<tr>
<td>x, y, h</td>
<td>Width, length, and height [ft]</td>
</tr>
<tr>
<td>∆T</td>
<td>Time-period length [hr]</td>
</tr>
</tbody>
</table>
### Table 2: GLD Household ETP Model: Derived Parameters

<table>
<thead>
<tr>
<th>Derived Parameters</th>
<th>Explanations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>Floor area $x \times y \times n$ [ft$^3$]</td>
</tr>
<tr>
<td>$A_c$</td>
<td>Net exterior ceiling area [ft$^2$]</td>
</tr>
<tr>
<td>$A_d$</td>
<td>Total door area [ft$^2$]</td>
</tr>
<tr>
<td>$A_f$</td>
<td>Net exterior floor area [ft$^2$]</td>
</tr>
<tr>
<td>$A_g$</td>
<td>Gross window area [ft$^2$]</td>
</tr>
<tr>
<td>$A_w$</td>
<td>Net exterior wall area [ft$^2$]</td>
</tr>
<tr>
<td>$A_{wt}$</td>
<td>Gross exterior wall area [ft$^2$]</td>
</tr>
<tr>
<td>$C_a$</td>
<td>Heat capacity [Btu/°F] of the inside air mass</td>
</tr>
<tr>
<td>$C_m$</td>
<td>Heat capacity [Btu/°F] of the inside solid mass</td>
</tr>
<tr>
<td>DesignCoolAirflow</td>
<td>Design cooling airflow [cfm = ft$^3$/min = cubic feet per minute]</td>
</tr>
<tr>
<td>DesignHeatAirflow</td>
<td>Design heating airflow [cfm = ft$^3$/min = cubic feet per minute]</td>
</tr>
<tr>
<td>FanPow</td>
<td>Heat gain [Btu/hr] from the ON operation of the 1-speed fan</td>
</tr>
<tr>
<td>$H_m$</td>
<td>Thermal conductance [Btu/hr-°F] between inside air &amp; solid masses</td>
</tr>
<tr>
<td>$I_{fan}$</td>
<td>Power draw [kW] of the ventilation fan</td>
</tr>
<tr>
<td>$R$</td>
<td>Floor aspect ratio $y/x$ (unit free)</td>
</tr>
<tr>
<td>$R_g$</td>
<td>Thermal resistance [hr-°F-ft$^2$/Btu] of house windows</td>
</tr>
<tr>
<td>SHGC</td>
<td>Solar heat gain coefficient [ft$^2$]</td>
</tr>
<tr>
<td>SHGC$_{nom}$</td>
<td>Nominal solar heat gain coefficient (decimal %)</td>
</tr>
<tr>
<td>$U_a$</td>
<td>Thermal conductance [Btu/hr-°F] between internal and external air masses</td>
</tr>
</tbody>
</table>

### Table 3: GLD Household ETP Model: Time-$t$ Variables

<table>
<thead>
<tr>
<th>Variables</th>
<th>Explanations</th>
</tr>
</thead>
<tbody>
<tr>
<td>COP$_{adj}(t)$</td>
<td>Coefficient of performance (unit free) adjusted for outside temperature effects</td>
</tr>
<tr>
<td>DesCoolCap$_{adj}(t)$</td>
<td>Design cooling capacity [Btu/hr] adjusted for outdoor temperature effects</td>
</tr>
<tr>
<td>HVACPow$_{(t)}$</td>
<td>Heat gain [Btu/hr] from the ON operation of the HVAC system</td>
</tr>
<tr>
<td>$K(t)$</td>
<td>Coefficient of performance factor [Btu/hr-kW] for the HVAC system</td>
</tr>
<tr>
<td>ISR$_{(t)}$</td>
<td>Incident solar radiation [Btu/hr-ft$^2$]</td>
</tr>
<tr>
<td>LCF$_{(t)}$</td>
<td>Fractional cooling-load increase (unit free) due to latent heat and humidity</td>
</tr>
<tr>
<td>$P^\ast(t)$</td>
<td>Power usage [kW] of the ON HVAC system in cooling mode</td>
</tr>
<tr>
<td>$p_{n(t)}$</td>
<td>Real power [kW] for each non-HVAC end-use load at time</td>
</tr>
<tr>
<td>$Q_a(t)$</td>
<td>Total heat flow rate [Btu/hr] to inside air mass</td>
</tr>
<tr>
<td>$Q_{hvac}(t)$</td>
<td>Heat flow rate [Btu/hr] from HVAC system and fan operations</td>
</tr>
<tr>
<td>$Q_i(t)$</td>
<td>Heat flow rate [Btu/hr] from internal non-HVAC equipment and occupants</td>
</tr>
<tr>
<td>$Q_m(t)$</td>
<td>Total heat flow rate [Btu/hr] to inside solid mass</td>
</tr>
<tr>
<td>$Q_s(t)$</td>
<td>Heat flow rate [Btu/hr] from solar radiation</td>
</tr>
<tr>
<td>RH$_{(t)}$</td>
<td>Relative humidity (decimal %)</td>
</tr>
<tr>
<td>$T_a(t)$</td>
<td>Inside air temperature [°F]</td>
</tr>
<tr>
<td>$T_m(t)$</td>
<td>Inside mass temperature [°F]</td>
</tr>
<tr>
<td>$T_o(t)$</td>
<td>Outside air temperature [°F]</td>
</tr>
<tr>
<td>$u(t)$</td>
<td>Binary 0-1 variable denoting OFF/ON HVAC power usage for cooling</td>
</tr>
<tr>
<td>$V_{actual}(t)$</td>
<td>Simulated-actual time-$t$ voltage [volts] obtained from GLD meter object in runtime</td>
</tr>
<tr>
<td>VF$_{(t)}$</td>
<td>Voltage function (unit free)</td>
</tr>
<tr>
<td>Voltage$_{adj}(t)$</td>
<td>Voltage factor function (unit free)</td>
</tr>
<tr>
<td>VoltFactorB$_{(t)}$</td>
<td>Voltage factor (unit free) calculated using base voltage</td>
</tr>
<tr>
<td>VoltFactorN$_{(t)}$</td>
<td>Voltage factor (unit free) calculated using nominal voltage</td>
</tr>
<tr>
<td>User-Set</td>
<td>Explanations</td>
</tr>
<tr>
<td>-----------------------------</td>
<td>-------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>AuxHeatCap</td>
<td>Auxiliary heating capacity (Btu/hr)</td>
</tr>
<tr>
<td>base_power</td>
<td>Base real power (kW) of the total load at nominal voltage</td>
</tr>
<tr>
<td>CDT</td>
<td>System cooling design temperature [°F]</td>
</tr>
<tr>
<td>Cooling_COP</td>
<td>Coefficient of performance [unit free] for HVAC system in cooling mode</td>
</tr>
<tr>
<td>CoolSupplyAirTemp</td>
<td>Cooling supply air temperature [°F]</td>
</tr>
<tr>
<td>cooling_system_type</td>
<td>HVAC system type running in cooling mode</td>
</tr>
<tr>
<td>current_fraction</td>
<td>Fraction [decimal %] of the load that is constant current (p.u.)</td>
</tr>
<tr>
<td>current_pf</td>
<td>Power factor [unit free] for constant current portion of load (p.u.)</td>
</tr>
<tr>
<td>DCT</td>
<td>System design cooling set-point [°F]</td>
</tr>
<tr>
<td>DesignHeatSetpoint</td>
<td>Design heating setpoint [°F]</td>
</tr>
<tr>
<td>DSG</td>
<td>System design internal gain [Btu/hr]</td>
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<td>DPS</td>
<td>System design solar load [Btu/hr-ft²]</td>
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<td>Duct pressure drop [inches of water]</td>
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<td>Exterior ceiling, fraction [decimal %] of total</td>
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<td>EFR</td>
<td>Exterior floor, fraction [decimal %] of total</td>
</tr>
<tr>
<td>EWR</td>
<td>Exterior wall, fraction [decimal %] of total</td>
</tr>
<tr>
<td>( f_{nu} )</td>
<td>Fraction [decimal %] of non-HVAC end-use load en internal to house</td>
</tr>
<tr>
<td>( f_{oc} )</td>
<td>Household occupancy fraction [decimal %]</td>
</tr>
<tr>
<td>glass_type</td>
<td>String-coded glass types</td>
</tr>
<tr>
<td>glazing_layers</td>
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<tr>
<td>HeatSupplyAirTemp</td>
<td>Heating supply air temperature [°F]</td>
</tr>
<tr>
<td>( h_s )</td>
<td>Interior surface heat transfer coefficient [Btu/hr-°F-ft²]</td>
</tr>
<tr>
<td>( I )</td>
<td>Infiltration volumetric air exchange rate [times per hr]</td>
</tr>
<tr>
<td>impedance_fraction</td>
<td>Fraction [decimal %] of load that is constant impedance (p.u.)</td>
</tr>
<tr>
<td>impedance_pf</td>
<td>Power factor [unit free] for constant impedance portion of load (p.u.)</td>
</tr>
<tr>
<td>IWR</td>
<td>Interior/exterior wall surface ratio [unit free]</td>
</tr>
<tr>
<td>LatCoolFrac</td>
<td>Fractional cooling-load increase [unit free] due to latent heat</td>
</tr>
<tr>
<td>( m_s )</td>
<td>Total thermal mass per unit floor area [Btu/°F-Ft²]</td>
</tr>
<tr>
<td>( n_d )</td>
<td>Number [integer] of doors</td>
</tr>
<tr>
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<td>Number [integer] of stories</td>
</tr>
<tr>
<td>NEU</td>
<td>Number [integer] of household non-HVAC end-use loads</td>
</tr>
<tr>
<td>NOC</td>
<td>Number [integer] of household occupants</td>
</tr>
<tr>
<td>OSF</td>
<td>Over-sizing factor [unit free]</td>
</tr>
<tr>
<td>power_fraction</td>
<td>Fraction [decimal %] of load that is constant power (p.u.)</td>
</tr>
<tr>
<td>power_pf</td>
<td>Power factor [unit free] for constant power portion of load (p.u.)</td>
</tr>
<tr>
<td>( R_c )</td>
<td>Thermal resistance [hr-°F-Ft²/Btu] of house ceilings</td>
</tr>
<tr>
<td>( R_d )</td>
<td>Thermal resistance [hr-°F-Ft²/Btu] of house doors</td>
</tr>
<tr>
<td>( R_f )</td>
<td>Thermal resistance [hr-°F-Ft²/Btu] of house floors</td>
</tr>
<tr>
<td>( R_w )</td>
<td>Thermal resistance [hr-°F-Ft²/Btu] of house walls</td>
</tr>
<tr>
<td>SHOC</td>
<td>Sensible heat [Btu/h-occupant] from each occupant</td>
</tr>
<tr>
<td>( V_{nominal} )</td>
<td>Base voltage [volts]</td>
</tr>
<tr>
<td>WET</td>
<td>Window exterior transmission coefficient [decimal %]</td>
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<tr>
<td>window_frame</td>
<td>String-coded window-frame types</td>
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<td>WWR</td>
<td>Window-to-exterior-wall ratio [decimal %]</td>
</tr>
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<td>Width, depth, and height [ft]</td>
</tr>
<tr>
<td>( \Delta t )</td>
<td>Time-period length [hr]</td>
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References


