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# Notes on the GridLAB-D Household Equivalent Thermal Parameter Model

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## **Abstract**

GridLab-D (GLD) is an agent-based platform, developed by researchers at Pacific Northwest National Laboratory, that permits users to accurately simulate the state dynamics of power distribution systems at time scales ranging from sub-seconds to years. The purpose of this study is to present, in careful comprehensive form, a complete analytical state-space control representation for a version of the GLD Household Equivalent Thermal Parameter Model as support documentation for model users. This model is a physically-based implementation of a household with multiple price-responsive and conventional appliances whose thermal dynamics are determined over successive days by resident appliance usage and external weather conditions.

## **Keywords**

Household modeling, electric power distribution system, agent-based platform, GridLAB-D

## **Disciplines**

Behavioral Economics | Econometrics | Statistical Models

# Notes on the GridLAB-D Household Equivalent Thermal Parameter Model

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**Abstract:** GridLab-D (GLD) is an agent-based platform, developed by researchers at Pacific Northwest National Laboratory, that permits users to accurately simulate the state dynamics of power distribution systems at time scales ranging from sub-seconds to years. The purpose of this study is to present, in careful comprehensive form, a complete analytical state-space control representation for a version of the GLD Household Equivalent Thermal Parameter Model as support documentation for model users. This model is a physically-based implementation of a household with multiple price-responsive and conventional appliances whose thermal dynamics are determined over successive days by resident appliance usage and external weather conditions.

## 1 Overview

As part of ongoing project research at Iowa State University on *Transactive Energy System (TES)* design, an agent-based computational platform has been developed permitting the evaluation of TES designs for *Integrated Transmission and Distribution (ITD)* systems. An extended discussion of the first released version of this platform, referred to as ITD TES Platform (V1.0), is provided in ref. [1]. Figs. 1-2 depict partial agent hierarchies for the

platform’s distribution and transmission systems, and Fig. 3 depicts the platform’s principal software components.

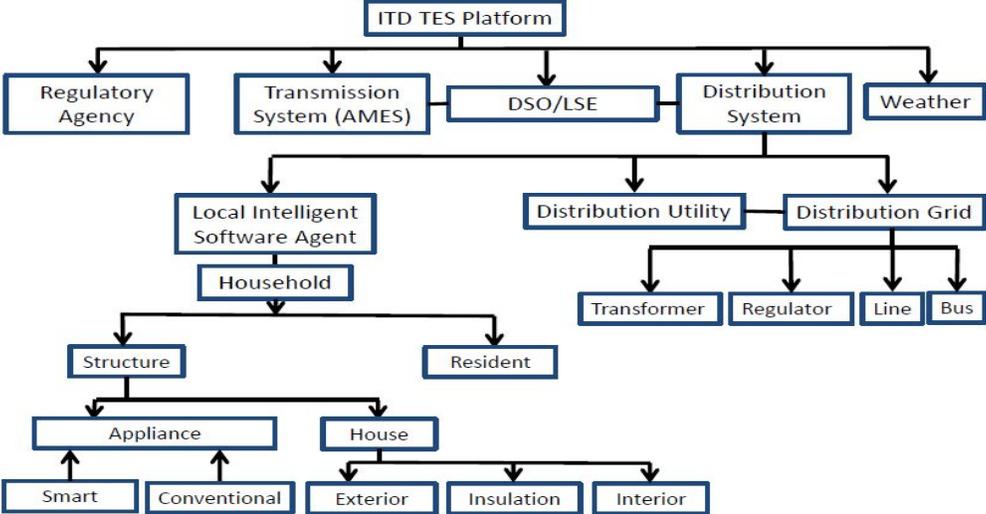


Figure 1: *Partial agent hierarchy for the ITD TES Platform (V1.0)*

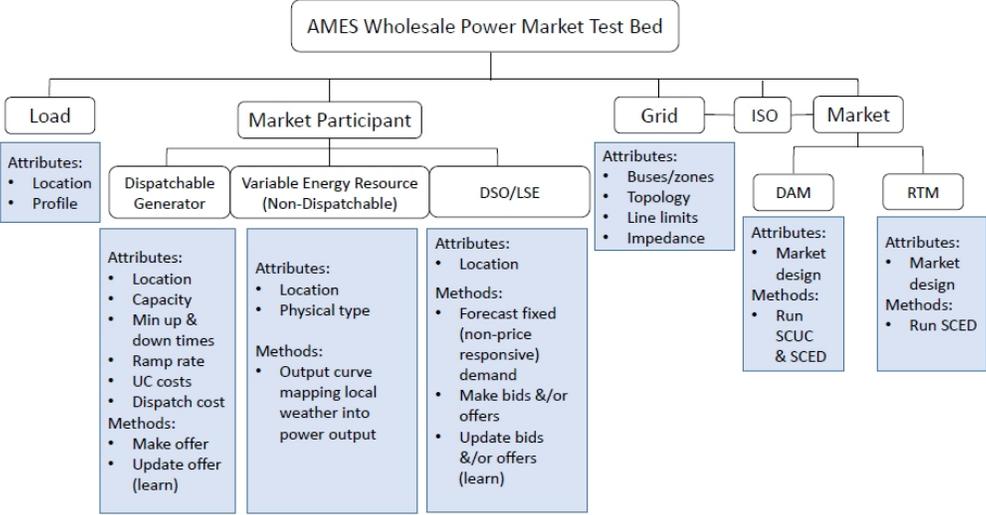


Figure 2: *Partial agent hierarchy for the ITD TES Platform (V1.0) transmission system*

As indicated in Fig. 3, key features of the household agents that populate the platform distribution system are currently being implemented using the *GridLAB-D (GLD) Household Equivalent Thermal Parameter Model*.<sup>1</sup> This model is a complex C++ program with

<sup>1</sup>For general introductions to GLD, see [2,3].

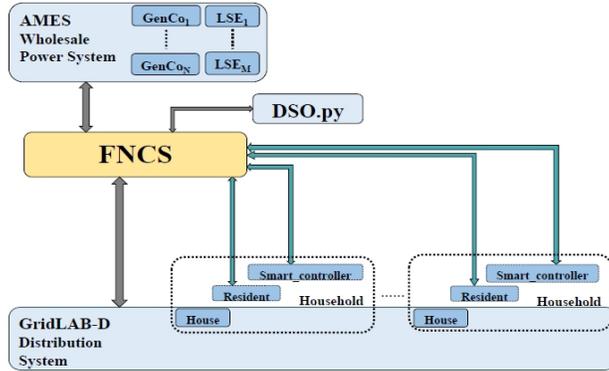


Figure 3: *Principal software components for the ITD TES Platform (V1.0)*

many interacting components. While some model documentation is available, it is not comprehensive. For example, it is not easy to distinguish structural elements from data-driven elements. Moreover, physical interpretations and units of measurement are difficult to discern for some key parameters.

The purpose of this study is to present, in careful comprehensive form, a complete analytical representation for the GLD Household Equivalent Thermal Parameter Model in standard state-space control form, as support documentation for users. The first two sections of this study provide preliminary background materials. Section 2 reviews basic terminology regarding classification of variables. Section 3 presents a state-space control model in standard continuous-time form.

Section 4 develops and presents a complete analytic state-space control representation for a version of the GLD Household ETP Model documented in [4–6] and implemented by means of a C++ program [7]. In this version, each household has an electric Heating, Ventilation, and Air-Conditioning (HVAC) system running in cooling mode with a linear cooling-capacity curve and a 1-speed fan for maintenance of air circulation. Each household also has additional appliances modeled by GLD’s ZIP load object [8]. These additional appliances include: Lights, Plugs, Clothes-Washer, Refrigerator, Dryer, Freezer, Dishwasher, Range, and Microwave. The GLD ZIP load object allows the modeling of voltage dependence for these appliances. The corresponding user energy-consumption profiles for these appliances are constructed from field data, considering weekday and seasonal patterns; these profiles can be accessed at [9].

Section 5 discusses the GLD implementation of the continuous-time GLD Household ETP Model. It is first shown that this model can be represented in matrix form as a continuous-time linear nonhomogeneous differential system with a time-varying coefficient vector. The closed-form solution of such systems is briefly discussed. The pros and cons of applying closed-form solution methods to linearized models as opposed to applying discretization methods to nonlinear models are briefly noted.

The section then notes that GLD implements a closed-form solution for the continuous-time GLD Household ETP Model by assuming forcing terms are constant-valued over successive time steps of equal length. This implementation is mathematically equivalent to approximating the gradient vector of the GLD Household ETP Model by a finite-difference expression over successive time steps of equal length. The final part of this section illustrates how this finite-difference approximation method can be directly applied to the *nonlinear* continuous-time state-space control model presented in Section 3.

Tables listing GLD Household ETP Model user-set parameters, derived parameters, variables, and default values/functions (if any) for user-set parameters are provided in an Appendix at the end of these notes.

## 2 Preliminary Classification of Variables Terminology

A variable whose value is determined *outside* of a model M is said to be *exogenous* relative to M. If an exogenous variable for a model M takes a constant value over time, it is often referred to as a *parameter* of M. If an exogenous variable for a model M is a function of time, it is often referred to as a *forcing term* for M.

A variable whose value is determined *within* a model M is said to be *endogenous* relative to M. An endogenous variable appearing within the time- $t$  equations for a model M whose value is determined by these equations is said to be a *time- $t$  endogenous* variable for M. An endogenous variable appearing within the time- $t$  equations for a model M whose value is determined by means of model-M equations at earlier times  $s < t$  is said to be a *time- $t$  predetermined* variable for M.

The time- $t$  predetermined variables for a model constitute the *time- $t$  state variables* for this model. For a model specified over times (or time periods)  $t \geq t^0$ , values for the state

variables at the initial time  $t^0$  need to be exogenously given since there are no modeled relationships prior to this initial time.

A control variable for a model M can be either exogenous or endogenous in form. A control variable for a model M is exogenous relative to M if it is set externally, with no dependence on model-M outcomes. A control variable for a model M is endogenous relative to M if it is determined as a function of model-M outcomes.

### 3 State-Space Control Model: Continuous-Time Form

**Standard Structural Model:** For each  $t \geq t^0$ ,

$$\text{Dynamic state equations: } \dot{x}(t) = S(u(t), w(t), z(t), x(t) \mid \theta_S) \quad (1)$$

$$\text{Simultaneous equations: } \mathbf{0} = H(u(t), w(t), z(t), x(t) \mid \theta_H) \quad (2)$$

$$\text{Integral equations: } x(t) = \int_{t^0}^t \dot{x}(\tau) d\tau + x(t^0) \quad (3)$$

**Variables, Parameters, and Functional Forms:**

$$x(t) = (x_1(t), \dots, x_N(t)) = \text{State vector for time } t \geq t^0$$

$$\dot{x}(t) = (\dot{x}_1(t), \dots, \dot{x}_N(t)) = \text{State gradient vector for time } t \geq t^0$$

$$u(t) = (u_1(t), \dots, u_M(t)) = \text{Control vector for time } t \geq t^0$$

$$w(t) = (w_1(t), \dots, w_J(t)) = \text{Vector of forcing terms for time } t \geq t^0$$

$$z(t) = (z_1(t), \dots, z_L(t)) = \text{Vector of endogenous variables for time } t \geq t^0$$

$$\theta_S = (\theta_{S1}, \dots, \theta_{SV}) = \text{Parameter vector}$$

$$\theta_H = (\theta_{H1}, \dots, \theta_{HV}) = \text{Parameter vector}$$

$$S: R^{M+J+L+N+SV} \rightarrow R^N$$

$$H: R^{M+J+L+N+HV} \rightarrow R^L$$

**Classification of Variables:**

$$\text{Time-}t \text{ endogenous variables for } t \geq t^0: \dot{x}(t), z(t)$$

*Time- $t$  predetermined variables for  $t > t^0$ :  $x(t)$*

*Exogenous controls and forcing terms for  $t \geq t^0$ :  $u(t), w(t)$*

*Exogenous parameters and initial state conditions:  $\theta_S, \theta_H$ , and  $x(t^0)$*

As indicated in the classification of variables, this illustrative state-space control model has  $N+L$  time- $t$  endogenous variables at each time  $t$ : namely, the  $N$  variables appearing in the vector  $\dot{x}(t)$  and the  $L$  variables appearing in the vector  $z(t)$ . In turn, there are  $N+L$  equations provided to solve for these  $N+L$  time- $t$  endogenous variables: namely, the  $N$  state equations (1) and the  $L$  simultaneous equations (2).

The integral equations (3) ensure that the solved solution-value for  $\dot{x}(t)$  is the derivative of  $x(t)$  for  $t > t^0$  and the right-derivative of  $x(t^0)$  at  $t = t^0$ . Note that the control variables appearing in  $u(t)$  at each time  $t$  are assumed to be exogenously determined.

## 4 GLD Household ETP Model: Analytic Formulation

### 4.1 Overview

In this section we present a complete analytic state-space control formulation for the GLD Household ETP Model based on the GLD documentation [4–6] and the GLD code [7]. For concreteness, we consider the special case of a household with an electric HVAC system running in cooling mode with a linear cooling-capacity curve (the GLD default setting).<sup>2</sup> In addition, we assume that the HVAC system is accompanied by a 1-speed fan<sup>3</sup> for the maintenance of air circulation. Definitions and units for all parameters and variables appearing in the GLD Household ETP Model as formulated in this section are given in Tables 1-4 provided in an Appendix.

More precisely, as will be seen below, many of the parameters appearing in the GLD Household ETP Model equations are in fact derived as functions of other parameters. The model parameters set directly by the user are classified as *user-set parameters*.<sup>4</sup> Definitions

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<sup>2</sup>The GLD settings are `Cool_system_type == electric` and `Cooling_cap_curve == CC_default`.

<sup>3</sup>For later purposes, it is important to note that GLD implements a 1-speed fan to be ON if and only if the HVAC system is ON.

<sup>4</sup>For some parameters a user has a choice either of setting a value for this parameter or using a GLD-provided default value. These parameters are classified here as user-set parameters.

and units for these user-set parameters are listed in Table 1. The vector of these user-set model parameters will hereafter be denoted by  $\theta_{\text{user}}$ . Definitions and units for model parameters determined as functions of  $\theta_{\text{user}}$ , referred to as *derived parameters*, are listed in Table 2. The *coupled-parameter relationships* expressing these derived parameters as functions of user-set parameters are given in Section 4.4.

Finally, some model parameters are internally assigned numerical values by GLD in a manner that cannot be changed or influenced by user-set parameter values. Hereafter these parameters will be referred to as *GLD-determined parameters*. Some of these GLD-determined parameters represent standard unit conversion factors. However, others appear to be based on structural presumptions or derived as empirical estimates from survey data, and their physical interpretations and units of measurement are not always clearly explained. Ideally, all of the latter parameters should instead be modeled as user-set parameters with GLD-provided default values, giving users a chance to modify/update the values of these parameters in response to changed distribution system conditions. This important issue is not dealt with in the current study.

## 4.2 Complete Analytic Formulation: Preliminary Developments

As detailed in [5], the GLD Household ETP Model assumes the thermal state of a house at any time  $t$  is given by a state vector  $(T_a(t), T_m(t))$ , where  $T_a(t)$  denotes the time- $t$  inside air temperature and  $T_m(t)$  denotes the time- $t$  inside mass temperature. The thermal dynamics of the house are then represented as a two-dimensional first-order differential system in  $(T_a(t), T_m(t))$  that determines the movement of  $T_a(t)$  and  $T_m(t)$  over time.

More precisely, as seen in [5, Eqs. (1)-(2)], the dynamic state equations for the GLD Household ETP Model are expressed in the following linearized form:

$$\dot{T}_a(t) = \frac{1}{C_a} \left( U_a [T_o(t) - T_a(t)] + H_m [T_m(t) - T_a(t)] + Q_a(t) \right); \quad (4)$$

$$\dot{T}_m(t) = \frac{1}{C_m} \left( H_m [T_a(t) - T_m(t)] + Q_m(t) \right), \quad (5)$$

where:  $T_o(t)$  denotes outside air temperature at time  $t$ ;  $Q_a(t)$  denotes the total heat flow rate to inside air mass at time  $t$ ; and  $Q_m(t)$  denotes the total heat flow rate to inside solid

mass at  $t$ . Equations (4)-(5) can equivalently be expressed in the following matrix form:

$$\dot{x}(t) = Mx(t) + Bv(t) ; \tag{6}$$

$$\text{where } M = \begin{bmatrix} -\frac{U_a+H_m}{C_a} & \frac{H_m}{C_a} \\ \frac{H_m}{C_m} & -\frac{H_m}{C_m} \end{bmatrix} ;$$

$$x(t) = \begin{bmatrix} T_a(t) \\ T_m(t) \end{bmatrix} ;$$

$$B = \begin{bmatrix} \frac{U_a}{C_a} & \frac{1}{C_a} & 0 \\ 0 & 0 & \frac{1}{C_m} \end{bmatrix} ;$$

$$v(t) = \begin{bmatrix} T_o(t) \\ Q_a(t) \\ Q_m(t) \end{bmatrix} .$$

Form (6) expresses the dynamic state equations for the GLD Household ETP Model as a linear nonhomogenous differential system with state vector  $x(t)$ , state matrix  $M$ , and time-varying coefficient vector  $Bv(t)$ .

Nevertheless, it is difficult to glean from the GLD documentation [5] alone the intended *structural* representations for the time- $t$  endogenous variables  $Q_a(t)$  and  $Q_m(t)$ . By a *structural representation* is meant a simultaneous-equation system such as (2) that permits these time- $t$  endogenous variables to be expressed as functions of state variables, control variables, forcing terms, and parameters.

We therefore consulted the GLD documentation [4,6] and GLD code [7] to understand better how equations (4) and (5) are augmented in GLD with simultaneous-equation relationships to obtain a complete structural representation for the thermal dynamics of a household with an HVAC system. This section presents this complete structural representation for the special case in which the household's HVAC system is an electric system running in cooling mode with a linear cooling-capacity curve.

For this purpose, we will first re-express equations (4) and (5) in the standard continuous-time state-space control model form presented in Section 3. Let the time- $t$  outside temperature (an external weather-related forcing term) be denoted by  $w_S(t) = (T_o(t))$ . Also, let the time- $t$  endogenous variables appearing in these equations be denoted by  $z_S(t) = (Q_a(t), Q_m(t))$ . Finally, let the parameters appearing in these equations be denoted by  $\theta_S = (U_a, H_m, C_a, C_m)$ . Equations (4) and (5) can then be expressed in the following compact

form:

$$\dot{x}(t) = S(w_S(t), z_S(t), x(t) \mid \theta_S) \quad (7)$$

However, the differential system (7) is not yet in complete solvable form due to the appearance of the time- $t$  endogenous variables  $z_S(t)$  on the right-hand side. To obtain a complete solvable form, system (7) needs to be augmented with a system of simultaneous equations such as (2) that permit these time- $t$  endogenous variables to be expressed as functions of the time- $t$  state  $x(t)$ , the time- $t$  control variable  $u(t)$ , forcing terms, and parameters.

According to the GLD documentation [4, 5], the time- $t$  endogenous variables  $Q_a(t)$  and  $Q_m(t)$  represent the total heat flow rates to the household's inside air mass and inside solid mass, respectively. The total heat flow rate  $Q_a(t)$  is assumed to be determined by specified fractions of (i) solar radiation ( $Q_s(t)$ ); (ii) the internal heat gain from household occupants and non-HVAC equipment ( $Q_i(t)$ ); and (iii) HVAC system cooling-mode operations ( $Q_{\text{hvac}}(t)$ ) as follows:

$$0 = [1 - f_{\text{ac}}]Q_{\text{hvac}}(t) + [1 - f_s]Q_s(t) + [1 - f_i]Q_i(t) - Q_a(t) , \quad (8)$$

where  $f_{\text{ac}}$ ,  $f_s$ , and  $f_i$  are user-set unit-free weight coefficients in  $[0,1]$ .<sup>5</sup> The heat flow rate  $Q_m(t)$  is then assumed to be determined as follows:

$$0 = f_{\text{ac}}Q_{\text{hvac}}(t) + f_sQ_s(t) + f_iQ_i(t) - Q_m(t) . \quad (9)$$

Assuming the HVAC system includes a 1-speed fan for the maintenance of air circulation,  $Q_{\text{hvac}}(t)$  (Btu/hr) in eqs. (8) and (9) is given by:

$$Q_{\text{hvac}}(t) = \left( -\text{HVACPow}(t) + \text{FanPow}(t) \right) \cdot u(t) \quad (10)$$

where:  $-\text{[HVACPow}(t)]$  (Btu/hr) denotes heat loss from the ON operation of the HVAC system running in cooling mode;  $\text{FanPow}(t)$  (Btu/hr) denotes heat gain from the ON operation of the 1-speed fan; and  $u(t)$  is a binary 0-1 (OFF/ON) HVAC power-usage control variable. We will next develop with care structural representations for  $\text{HVACPow}(t)$  and  $\text{FanPow}(t)$ , i.e., representations expressed solely in terms of user-set parameters, GLD-determined parameters, and forcing terms.

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<sup>5</sup>The weight coefficient  $f_{\text{ac}}$  is identified as a user-set parameter in the GLD documentation [4, p. 5]. However,  $f_{\text{ac}}$  is hard-coded to 0 in the GLD source code [7, lines 1808-1809].

Let  $P^*(t)$  (kW) denote the ON power usage of the HVAC system in cooling mode. Then

$$\text{HVACPow}(t) = K(t)P^*(t) \quad (11)$$

where

$$K(t)P^*(t) = \left( \frac{\text{Voltage\_adj}(t) \cdot \text{DesCoolCap\_adj}(t)}{[1 + \text{LCF}(t)]} \right);$$

$$P^*(t) = \frac{\text{DesCoolCap\_adj}(t) \cdot \text{VF}(t)}{K \cdot \text{COP\_adj}(t)}; \quad (12)$$

$$K(t) = \frac{K \cdot \text{COP\_adj}(t) \cdot \text{Voltage\_adj}(t)}{[1 + \text{LCF}(t)] \cdot \text{VF}(t)}; \quad (13)$$

$$\text{DesCoolCap\_adj}(t) = \text{DesCoolCap} \cdot [a - b \cdot T_o(t)]; \quad (14)$$

$$\text{COP\_adj}(t) = \frac{\text{cooling\_COP}}{c + d \cdot T_o(t)}; \quad (15)$$

$$\text{LCF}(t) = \left( e + \frac{\text{LatCoolFrac}}{[f + \exp(g - m \cdot \text{RH}(t))]} \right) \quad (16)$$

$$\text{VF}(t) = F_P + F_C \cdot \text{VoltFactorN}(t) + F_Z \cdot [\text{VoltFactorN}(t)]^2; \quad (17)$$

$$\text{Voltage\_adj}(t) = F_P + F_C \cdot \text{VoltFactorB}(t) + F_Z \cdot [\text{VoltFactorB}(t)]^2; \quad (18)$$

$$\text{VoltFactorN}(t) = \left( \frac{V\_actual(t)}{V\_nominal} \right); \quad (19)$$

$$\text{VoltFactorB}(t) = \left( \frac{V\_actual(t)}{V\_base} \right). \quad (20)$$

In eqs. (12)–(16),  $\text{DesCoolCap\_adj}(t)$  (Btu/hr) is determined as a function of the derived parameter  $\text{DesCoolCap}$  (Btu/hr) and the outside temperature  $T_o(t)$ ;  $\text{COP\_adj}(t)$  is a unit-free coefficient of performance factor determined as a function of the user-set parameter  $\text{Cooling\_COP}$  (unit free) and outside temperature  $T_o(t)$ ;  $K = 3412\text{Btu}/[\text{hr}\cdot\text{kW}]$  is a GLD-determined conversion factor that converts kW to Btu/hr; and  $\text{LCF}(t)$  is a unit-free factor determined as a function of the user-set parameter  $\text{LatCoolFac}$  (unit free) and time- $t$  relative humidity  $\text{RH}(t)$ .

The parameters  $a, b, c, d, e, f, g$ , and  $m$  appearing in eqs. (12)–(16) are GLD-determined parameters whose values are GLD-set as follows:  $a = 1.48924533$  (unit free);  $b = 0.00514995$  ( $1/^\circ\text{F}$ );  $c = -0.01363961$  (unit free);  $d = 0.01066989$  ( $1/^\circ\text{F}$ );  $e = 0.1$  (unit free),  $f = 1.0$  (unit free),  $g = 4.0$  (unit free), and  $m = 10.0$  (unit free). The determination of the derived parameter  $\text{DesCoolCap}$  (Btu/hr) in (14) as a function of user-set parameters by means of “coupled-parameter relationships” is given below in Section 4.4.

Finally, the coefficients  $F_P$  (power fraction),  $F_C$  (current fraction), and  $F_Z$  (impedance fraction) appearing in eqs. (17) and (18) are unit-free GLD-determined parameter values given by  $F_P = 0.8$ ,  $F_C = 0.0$ , and  $F_Z = 0.0$ .<sup>6</sup> The numerator  $V\_actual(t)$  (volts) appearing in both equations is a time- $t$  forcing term given by the simulated actual voltage at time  $t$  obtained from the GLD meter object in run-time. The denominator  $V\_nominal$  (volts) in eq. (19) is a user-set parameter for nominal voltage that the user can set either to 120V or 240V. In contrast, the denominator  $V\_base$  (volts) in eq. (20) is a GLD-determined parameter that is GLD-set at 240V.

The structural simultaneous-equation relationships for  $FanPow(t)$  are as follows:<sup>7</sup>

$$FanPow(t) = Ceil(nD) \cdot q \quad (21)$$

where

$$\begin{aligned} D &= [DuctPressureDrop] \cdot E \\ E &= \max\{DesignCoolAirflow, DesignHeatAirflow\} \end{aligned}$$

In (21),  $n$  (1/inches) and  $q$  (unit free) are GLD-determined parameters whose values are GLD-set as follows:

$$\begin{aligned} n &= \left[ \frac{0.117}{[(745.7) \cdot (0.42)]} \right] \cdot 8 \\ q &= \frac{745.7}{[8 \cdot 0.88]} \end{aligned}$$

DuctPressureDrop (inches of water) in (21) is a user-set parameter. The other two factors, DesignCoolAirflow (Btu/hr) and DesignHeatAirflow (Btu/hr), are derived parameters whose derivations as functions of user-set parameters are given below in Section 4.4.

The determination of the time- $t$  endogenous variables  $Q_s(t)$  and  $Q_i(t)$  appearing in equations (8) and (9) is discussed in [4]. These determinations will now be separately explained.

The following relationship is given in [4] for the determination of  $Q_s(t)$  (Btu/hr) as a function of the time- $t$  incident solar radiation  $ISR(t)$  (Btu/hr-ft<sup>2</sup>), an external weather-related forcing term:

$$0 = [A_g \cdot SHGC_{nom} \cdot WET] \cdot ISR(t) - Q_s(t) , \quad (22)$$

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<sup>6</sup>These coefficients are GLD-set for an HVAC system but can be set by users for other types of appliances.

<sup>7</sup>The `ceil()` function in C and C++ returns the smallest possible integer value which is greater than or equal to the given argument.

where: WET (decimal %) is a user-set parameter; and  $A_g$  (ft<sup>2</sup>) and SHGC<sub>nom</sub> (decimal %) are derived parameters whose derivations as functions of user-set parameters are given below in Section 4.4.

The fifteen equations (8)-(22) can be compactly expressed as a 15-dimensional system of time- $t$  simultaneous equations taking the following form:

$$\mathbf{0} = H_1(u(t), w_{H1}(t), z(t) \mid \theta_{H1}) \quad (23)$$

where:

$$\begin{aligned} u(t) &= \text{binary 0-1 (OFF/ON) HVAC power-usage control variable} \\ w_{H1}(t) &= (T_o(t), RH(t), V\_actual(t), ISR(t)) \\ z(t) &= (Z_1(t), Z_2(t), Z_3(t)) \\ Z_1(t) &= (Q_a(t), Q_m(t), Q_s(t), Q_i(t), Q_{hvac}(t), HVACPow(t), FanPow(t)) \\ Z_2(t) &= (P^*(t), K(t), DesCoolCap\_adj(t), COP\_adj(t), LCF(t)) \\ Z_3(t) &= (VF(t), Voltage\_adj(t), VoltFactorN(t), VoltFactorB(t)) \\ \theta_{H1} &= (\theta_{H11}, \theta_{H12}, \theta_{H13}) \\ \theta_{H11} &= (f_{ac}, f_s, f_i, DesCoolCap, Cooling\_COP, LatCoolFrac, V\_nominal) \\ \theta_{H12} &= (DuctPressureDrop, DesignCoolAirflow, DesignHeatAirflow) \\ \theta_{H13} &= (WET, A_g, SHGC_{nom}) \end{aligned}$$

The 15-dimensional system of equations (23) determines all time- $t$  endogenous variables in  $z(t)$ , apart from  $Q_i(t)$ , as functions of the control variable  $u(t)$ , the forcing terms  $w_{H1}(t)$ , the parameters in  $\theta_{H1}$ , and  $Q_i(t)$ . However,  $Q_i(t)$  itself is not determined by system (23).

To determine  $Q_i(t)$  (Btu/hr), an additional time- $t$  simultaneous equation is needed that expresses  $Q_i(t)$  as a function of the time- $t$  control variable  $u(t)$ , time- $t$  forcing terms, time- $t$  endogenous variables, and parameters. The determination of  $Q_i(t)$  is formulated in [4] in general descriptive terms. This general formulation will now be expressed in needed state-space control form, as follows.

Let  $p_{eu}(t)$  (watts) denote the current real power for each household non-HVAC<sup>8</sup> end-use load eu, multiplied by the user-set fraction  $f_{eu}$  of this load that is internal to the household.

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<sup>8</sup>We have added the “non-HVAC” qualifier here to be consistent with the interpretation of  $Q_i$  as internal heat gain arising from household *non-HVAC* equipment and occupants.

Let NEU denote the user-set number of household non-HVAC end-use loads. Also, let NOC denote the user-set number of household occupants, where the sensible heat from each of these occupants is measured by the user-set parameter SHOC (Btu/hr-occupant).

Finally, let  $f_{oc}$  denote the user-set occupancy fraction and C denote the GLD-determined conversion factor given by  $C = K/1000 = 3.412\text{Btu/hr-watt}$ . Then:

$$Q_i(t) = C \cdot \left( \sum_{eu=1}^{NEU} p_{eu}(t) \cdot fl_{eu} \right) + [\text{SHOC} \cdot \text{NOC} \cdot f_{oc}] \quad (24)$$

It is seen from (24) that  $Q_i(t)$  depends on NEU forcing terms external to HVAC operations: namely, the NEU elements of the vector  $w_{H2}(t) = (p_1(t), \dots, p_{NEU}(t))$  giving the time- $t$  real power levels for each of the household's non-HVAC end-use loads, assumed to be NEU in number. Let  $z_{H2}(t) = Q_i(t)$ . Let  $\mathbf{fl} = (fl_1, \dots, fl_{NEU})$  denote the NEU-dimensional vector giving the fractions of non-HVAC end-use loads that are internal to the household. Finally, let the vector of user-set parameters for relation (24) be denoted by  $\theta_{H2} = (\mathbf{fl}, \text{NEU}, \text{SHOC}, \text{NOC}, f_{oc})$ . Given this notation, the time- $t$  simultaneous equation (24) for  $Q_i(t)$  can be expressed in the required form as follows:

$$0 = H_2(w_{H2}(t), z_{H2}(t) \mid \theta_{H2}) \quad (25)$$

Relation (25) completes the basic state-space control model representation for the GLD Household ETP Model.<sup>9</sup>

### 4.3 Complete Analytic Formulation: Summary Form

Below we provide a complete summary analytic formulation of the GLD Household ETP Model as a state-space control model. This complete analytic description differs from the description of the standard state-space control model presented in Section 3 in one important regard: namely, it incorporates *coupled-parameter relationships* that show precisely how each

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<sup>9</sup>Concerns remain about the precise GLD-determination of the time- $t$  forcing terms  $w_{H2}(t)$ . These time- $t$  forcing terms need to be consistent with: (i) the user's specification of the household's appliance mix; (ii) the user's specification of household occupants at time  $t$ ; and (iii) the user's specification of occupant methods that affect non-HVAC equipment usage at time  $t$ . Note that the occupants of a household at any given time  $t$  can differ from the resident(s) of a household, i.e., the people who are in residence at the household. Occupants can be temporary visitors. This distinction is important for household welfare calculations.

derived parameter appearing in the model equations is determined as a function of the user-set parameters listed in Table 1.

Coupled-parameter relationships relating derived to user-set parameters need to be given for the GLD Household ETP Model in order to ensure that all of these parameters are set reasonably for the study at hand. Specifically, the user should set values for the parameters in  $\theta_{\text{user}}$  that are sensible compatible settings for the particular household that the user is trying to model. The coupled-parameter relationships should then guarantee that all other parameter settings are sensible and compatible for this household as well.

**GLD Household ETP Model in State-Space Control Form:** For each  $t \geq t^0$ ,

$$\text{Dynamic state equations:} \quad \dot{x}(t) = S(w_S(t), z_S(t), x(t) \mid \theta_S) \quad (26)$$

$$\text{Simultaneous equations:} \quad \mathbf{0} = H_1(u(t), w_{H1}(t), z_{H1}(t) \mid \theta_{H1}) \quad (27)$$

$$\text{Simultaneous equation:} \quad 0 = H_2(w_{H2}(t), z_{H2}(t) \mid \theta_{H2}) \quad (28)$$

$$\text{Integral equations:} \quad x(t) = \int_{t^0}^t \dot{x}(\tau) d\tau + x(t^0) \quad (29)$$

$$\text{Coupled-Parameter Relationships:} \quad \mathbf{0} = CP_S(\theta_S, \theta_{\text{user}}) \quad (30)$$

$$\mathbf{0} = CP_{H1}(\theta_{H1}, \theta_{\text{user}}) \quad (31)$$

$$\mathbf{0} = CP_{H2}(\theta_{H2}, \theta_{\text{user}}) \quad (32)$$

**Variables, Parameters, and Functional Forms** ( $t \geq t^0$ ):

$x(t) = (T_a(t), T_m(t)) =$  State vector at time  $t$

$\dot{x}(t) = (\dot{T}_a(t), \dot{T}_m(t)) =$  State gradient vector at time  $t$

$u(t) =$  Binary 0-1 (OFF/ON) power-usage control variable at time  $t$

$w(t) = (T_o(t), RH(t), V\_actual(t), ISR(t), p_1(t), \dots, p_{NEU}(t)) =$  Forcing terms at time  $t$

$w_S(t) = (T_o(t)) =$  Forcing term for  $S(\cdot)$  in (26) at  $t$

$w_{H1}(t) = (T_o(t), RH(t), V\_actual(t), ISR(t)) =$  Forcing terms for  $H_1(\cdot)$  in (27) at  $t$

$w_{H2}(t) = (p_1(t), \dots, p_{NEU}(t)) =$  Forcing terms for  $H_2(\cdot)$  in (28) at  $t$

$z_S(t) = (Q_a(t), Q_m(t)) = \text{Time-}t \text{ endogenous variables for } S(\cdot) \text{ in (26)}$

$z_{H_1}(t) = (z_{H_{11}}(t), z_{H_{12}}(t), z_{H_{13}}(t)) = \text{Time-}t \text{ endogenous variables for } H_1(\cdot) \text{ in (27)}$

$z_{H_{11}}(t) = (Q_a(t), Q_m(t), Q_s(t), Q_i(t), Q_{\text{hvac}}(t), \text{HVACPow}(t), \text{FanPow}(t))$

$z_{H_{12}}(t) = (P^*(t), K(t), \text{DesCoolCap\_adj}(t), \text{COP\_adj}(t), \text{LCF}(t))$

$z_{H_{13}}(t) = (\text{VF}(t), \text{Voltage\_adj}(t), \text{VoltFactorN}(t), \text{VoltFactorB}(t))$

$z_{H_2}(t) = (Q_i(t)) = \text{Time-}t \text{ endogenous variable for } H_2(\cdot) \text{ in (28)}$

$\theta_S = (U_a, H_m, C_a, C_m) = \text{Parameter vector for } S(\cdot) \text{ in (26)}$

$\theta_{H_1} = (\theta_{H_{11}}, \theta_{H_{12}}, \theta_{H_{13}}) = \text{Parameter vector for } H_1(\cdot) \text{ in (27)}$

$\theta_{H_{11}} = (f_{\text{ac}}, f_s, f_i, \text{DesCoolCap}, \text{Cooling\_COP}, \text{LatCoolFrac}, V_{\text{nominal}})$

$\theta_{H_{12}} = (\text{DuctPressureDrop}, \text{DesignCoolAirflow}, \text{DesignHeatAirflow})$

$\theta_{H_{13}} = (\text{WET}, A_g, \text{SHGC}_{\text{nom}})$

$\theta_{H_2} = (\text{fl}_1, \dots, \text{fl}_{\text{NEU}}, \text{NEU}, \text{SHOC}, \text{NOC}, f_{\text{oc}}) = \text{Parameter vector for } H_2(\cdot) \text{ in (28)}$

$\theta_{\text{user}} = \text{TV-dimensional vector consisting of all user-set parameters listed in Table 1}$

$S: R^{\text{SJ+SL+SN+SV}} \rightarrow R^{\text{N}}$  given by the differential equation system (7)

$H_1: B \times R^{\text{H1J+H1L+H1N+H1V}} \rightarrow R^{\text{L-1}}$  given by the simultaneous-equation system (23)

$H_2: B \times R^{\text{H2J+H2L+H2N+H2V}} \rightarrow R$  given by the simultaneous equation (25)

$CP_S: R^{\text{SV+TV}} \rightarrow R^{\text{SV}}$

$CP_{H_1}: R^{\text{H1V+TV}} \rightarrow R^{\text{H1V}}$

$CP_{H_2}: R^{\text{H2V+TV}} \rightarrow R^{\text{H2V}}$

$B = \{0, 1\}, J=4+\text{NEU}, L=15, N=2$

$\text{SJ}=1, \text{SL}=2, \text{SN}=2, \text{SV}=4$

H1J=4 , H1L=15 , H1N=0, H1V=13

H2J=NEU , H2L=1 , H2N=0 , H2V=NEU+4

**Classification of Variables:**

*Time- $t$  endogenous variables for  $t \geq t^0$ :  $\dot{x}(t), z(t)$*

*Time- $t$  predetermined variables for  $t > t^0$ :  $x(t)$*

*Exogenous controls and forcing terms for  $t \geq t^0$ :  $u(t), w(t)$*

*Exogenous parameters and initial state conditions:  $\theta_{\text{user}}, \theta_S, \theta_H = (\theta_{H1}, \theta_{H2}),$  and  $x(t^0)$*

**4.4 Coupled-Parameter Relationships for the Analytic Formulation**

The coupled-parameter relationships (30) that permit the parameters appearing in the parameter vector  $\theta_S = (U_a, H_m, C_a, C_m)$  for the state differential system (26) to be expressed as functions of the user-set parameters  $\theta_{\text{user}}$  listed in Table 1 are as follows.<sup>10</sup>

$$U_a = \frac{A_c}{R_c} + \frac{A_d}{R_d} + \frac{A_f}{R_f} + \frac{A_g}{R_g} + \frac{A_w}{R_w} + \text{VH}_a AhI ; \tag{33}$$

$$H_m = h_s \left[ (A_{wt} - A_g - A_d) + A_{wt} \text{IWR} + \frac{A_c n}{\text{ECR}} \right] ; \tag{34}$$

$$C_a = 3\text{VH}_a Ah ; \tag{35}$$

$$C_m = Am_f - 2\text{VH}_a Ah , \tag{36}$$

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<sup>10</sup>The expressions (34) and (44) below for  $H_m$  and  $A_w$  are consistent with the GLD documentation [4] and the GLD source code [7]. The  $H_m$  and  $A_w$  expressions appearing in the GLD documentation [5] appear to be incorrect. Also, it is clear from (35) and (37) below that  $C_a$  is a derived parameter. However, the GLD documentation materials [12,13] and the GLD source code [7, line 190] incorrectly imply that  $C_a$  is a user-set parameter whose value can be set independently of other user-set parameters.

where:<sup>11</sup>

$$A = x \times y \times n ; \quad (37)$$

$$R = y/x ; \quad (38)$$

$$A_c = \frac{A}{n} \times \text{ECR} ; \quad (39)$$

$$A_d = n_d \times A_{1d} ; \quad (40)$$

$$A_f = \frac{A}{n} \times \text{EFR} ; \quad (41)$$

$$A_{wt} = 2nh[1 + R]\sqrt{\frac{A}{nR}} ; \quad (42)$$

$$A_g = \text{WWR} \times A_{wt} \times \text{EWR} ; \quad (43)$$

$$A_w = (\text{EWR} \times A_{wt}) - (A_d + A_g) . \quad (44)$$

$$R_g = \text{Value determined from a table in the GLD documentation [11]} \quad (45)$$

giving setting combinations for `glass_type`, `glazing_layers`, `window_frame`

In (33),  $\text{VH}_a = 0.018$  (Btu/ft<sup>3</sup>-°F) is a GLD-determined parameter value denoting volumetric heat capacity of air at standard conditions.<sup>12</sup> Also, in (40),  $A_{1d} = 12$  (ft<sup>2</sup>) is a GLD-determined parameter value for the area of one door.

The coupled-parameter relationships (31) that permit the parameters appearing in  $\theta_{H1} = (\theta_{H11}, \theta_{H12}, \theta_{H13})$  for  $H_1(\cdot)$  in (23) to be expressed as functions of the user-set parameters  $\theta_{\text{user}}$  listed in Table 1 are as follows.

Starting with  $\theta_{H13}$ , WET is a user-set parameter. The derived parameter  $A_g$  is determined as a function of user-set parameters by equations (37), (38), (42), and (43). A table in the GLD documentation [11] indicates that the derived parameter  $\text{SHGC}_{\text{nom}}$  is a function of combined settings for three user-set parameters: namely, `glazing_treatment`, `glazing_layers`, and `window_frame`.<sup>13</sup>

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<sup>11</sup>As indicated below in expressions (37) and (38),  $A$  and  $R$  are derived parameters whose values are commonly dependent on the user-set values for  $x$  and  $y$ . The GLD documentation materials [12, 15, 16] identify  $A$  and  $R$  as user-set parameters, which incorrectly implies their values can be set independently of each other. Also, as indicated below in expression (45),  $R_g$  is a derived parameter whose value is determined as a function of user-set parameters. However, the GLD documentation materials [12, 14] and the GLD source code [7, line 408] identify  $R_g$  as a user-set parameter, incorrectly implying that its value can be set independently of the values set for all other other user-set parameters.

<sup>12</sup>More precisely,  $\text{VH}_a = 0.018$  (Btu/ft<sup>3</sup>-°F) is calculated as the product of two other GLD-determined parameter values: namely, `AirDensity` = 0.0735 (lb/ft<sup>3</sup>) and `AirHeatCapacityValue` = 0.2402 (Btu/lb-°F). See [5, sec. 3.2.1].

<sup>13</sup>The GLD source code [7, line 190] states that  $\text{SHGC}_{\text{nom}}$  is a user-set parameter, implying incorrectly

Next, consider  $\theta_{H11}$ . The relationship expressing DesCoolCap as a function of two derived parameters ( $U_a$ , SHGC) plus various user-set parameters is as follows:

$$\text{DesCoolCap} = \text{Ceil}\left(\frac{\text{DCP}}{6000}\right) \cdot 6000 \quad (46)$$

where

$$\text{DCP} = U_a \cdot [1 + \text{LatCoolFrac}] [\text{CDT} - \text{DCT}] [1 + \text{OSF}] + \text{DIG} + [\text{DPS} \cdot \text{SHGC}] \quad (47)$$

The derived parameter  $U_a$  is determined as a function of user-set parameters by relationship (33) together with equations (37) through (45). Using the coupled-parameter relations for  $A_g$  and  $\text{SHGC}_{\text{nom}}$ , the derived parameter SHGC is determined as a function of user-set parameters by substituting out for  $A_g$  and  $\text{SHGC}_{\text{nom}}$  in the following relationship:

$$\text{SHGC} = A_g \cdot \text{SHGC}_{\text{nom}} \cdot \text{WET} \quad (48)$$

Now consider  $\theta_{H12}$ . The derived parameter DesignCoolAirflow ( $\text{ft}^3/\text{min}$ ) is given in [4, p. 20] and [10] as follows:

$$\text{DesignCoolAirflow} = \left( \frac{\text{DesCoolCap}}{[1 + \text{LatCoolFrac}] [\text{F} \cdot \text{VH}_a]} \right) \cdot \frac{1}{60} \quad (49)$$

where

$$\text{F} = [\text{DCT} - \text{CoolSupplyAirTemp}] . \quad (50)$$

The terms LatCoolFrac (unit free), DCT ( $^{\circ}\text{F}$ ) and CoolSupplyAirTemp ( $^{\circ}\text{F}$ ) in (49) and (50) are user-set parameters. The term  $\text{VH}_a$  ( $\text{Btu}/\text{ft}^3\text{-}^{\circ}\text{F}$ ) is a GLD-determined parameter with a GLD-set value given by 0.018; see Footnote 12. Also, DesignHeatAirflow ( $\text{ft}^3/\text{min}$ ) is given in [4, p. 19] as follows:<sup>14</sup>

$$\text{DesignHeatAirflow} = \left( \frac{\max\{\text{DesignHeatCapacity}, \text{AuxHeatCapacity}\}}{\text{G} \cdot \text{VH}_a} \right) \cdot \frac{1}{60} \quad (51)$$

where

$$\text{G} = [\text{HeatSupplyAirTemp} - \text{DesignHeatSetpoint}] . \quad (52)$$

---

that the value of this parameter can be set independently of the values for all other user-set parameters.

<sup>14</sup>The seems to be error in the expression for DesignHeatAirflow given in the GLD documentation [10]. The expression (51) given below for DesignHeatAirflow is consistent with the GLD document [4, p. 19] and the GLD source code [7, line 1479].

In (51), DesignHeatCapacity (Btu/hr), AuxHeatCapacity (Btu/hr), HeatSupplyAirTemp ( $^{\circ}\text{F}$ ), and DesignHeatSetpoint ( $^{\circ}\text{F}$ ) are all user-set parameters.

Finally, consider  $\theta_{\text{H2}}$ . The coupled-parameter relationships (32) that functionally relate  $\theta_{\text{H2}}$  to  $\theta_{\text{user}}$  are direct one-to-one mappings because all of the parameters appearing in  $\theta_{\text{H2}}$  are user-set parameters.

## 4.5 Default Functions for User-Set Parameters

Table 1 provides a complete listing of the user-set parameters for the GLD Household ETP Model. As seen in Table 4, GLD provides default values for most of these user-set parameters.

However, for the three user-set parameters DIG (Btu/hr), DesignHeatCapacity (Btu/hr), and AuxHeatCapacity (Btu/hr), GLD instead provides default *functions*. More precisely, for these three user-set parameters a user can *either* directly set their values *or* use GLD default functions whose arguments are given by GLD-determined parameters, GLD-derived parameters, and/or other user-set parameters.

Specifically, the GLD default function for DIG (Btu/hr) is given by

$$\text{DIG} = q \cdot A^r . \quad (53)$$

In (53),  $q$  (Btu/hr-ft<sup>2</sup>) and  $r$  (unit free) are GLD-determined parameters with GLD-set values given by  $q = 167.09$  (Btu/hr-ft<sup>2</sup>) and  $r = 0.442$  (unit free). Also  $A$  (ft<sup>2</sup>) is a derived parameter determined as the product of the three user-set parameters  $x$ ,  $y$ , and  $n$ .

Finally, the GLD default functions for DesignHeatCapacity (Btu/hr) and AuxHeatCapacity (Btu/hr) are identical, expressed as follows:

$$\begin{aligned} \text{DesignHeatCapacity} &= \text{AuxHeatCapacity} & (54) \\ &= U_a[1.0 + \text{OSF}][\text{DesignHeatSetpoint} - \text{HeatDesignTemp}] \end{aligned}$$

In (54),  $U_a$  (Btu/hr- $^{\circ}\text{F}$ ) is a GLD-derived parameter; see (33). The remaining terms OSF (unit free), DesignHeatSetpoint ( $^{\circ}\text{F}$ ), and HeatDesignTemp ( $^{\circ}\text{F}$ ) are user-set parameters.

## 5 GLD Household ETP Model: Implementation

### 5.1 Overview

The GLD code [7] indicates that the GLD Household ETP Model is implemented by first determining its closed-form solution and then discretizing the implementation of this closed-form solution by approximating forcing terms as step functions. Specifically, at each time step, the value of each forcing term is held constant at the value it takes on at the beginning of this time step. As will be seen in Section 5.3 below, this approach is equivalent to approximating the state vector gradient  $\dot{x}(t)$  by a finite-difference expression.

### 5.2 Matrix Representation

The matrix form (6) expresses the GLD Household ETP Model as a linear nonhomogeneous differential system with a time-varying coefficient vector  $Bv(t)$ . Assuming a known trajectory for  $v(t)$ , a closed-form solution for (6) can be analytically determined using various methods. One such method, outlined in the GLD documentation [20], involves first converting this system into a one-dimensional second-order differential system in  $T_a(t)$  with modified boundary conditions, solving for  $T_a(t)$ , and then deriving the implied solution value for  $T_m(t)$ .

Recall, however, that the *linearity* of the GLD Household ETP Model is itself a strong initial assumption. Consequently, what one is obtaining is a closed-form solution to a system in approximate linear form. An alternative way to proceed would be to start from an ETP Model represented as a continuous-time *nonlinear* state-space control model, as expressed in Section 3. Various discretization methods could then be directly applied to this nonlinear system to obtain an approximate discrete-time solution.

Which method – initial linearization or discrete-time approximation – would lead to smaller approximation errors when numerically implemented on a computer depends on a number of critical factors: namely, the extent to which household thermal dynamics are well approximated by a *linear* differential system such as (6); the determination (approximation) of the vector of time-varying forcing terms; the determination (approximation) of boundary conditions; round-off errors; truncation errors; and error accumulation over time.

### 5.3 Finite-Difference Approximation Method

Below we illustrate how a relatively simple finite-difference method can be used to obtain a discrete-time approximation for the *nonlinear* continuous-time state space control model expressed in Section 3. As will be seen, this method does not require linearization of the state function  $S(\cdot)$  in (1) or the function  $H(\cdot)$  in (2) that expresses simultaneous-equation relationships. However, it does presume that the time-step length  $\Delta t$  used for the discretization is sufficiently small that the trajectory for the forcing-term vector  $w(t)$  can be well-approximated by a step function over successive steps of equal length  $\Delta t$ .

Consider the continuous-time state-space control model in standard form, as presented in Section 3. Let  $t \geq t^0$  be given, and let  $\Delta t$  denote a positive time increment whose length is measured in some given time unit (e.g., hours). Let the gradient  $\dot{x}(t)$  for the state vector  $x(t)$  at each time  $t$  be approximated by the following finite-difference expression:

$$\dot{x}(t) \approx \frac{x(t + \Delta t) - x(t)}{\Delta t} \quad (55)$$

Substituting (55) in place of  $\dot{x}(t)$  in (1), and manipulating terms, one obtains

$$x(t + \Delta t) \approx x_k + S(u(t), w(t), z(t), x(t) \mid \theta_S) \cdot \Delta t \quad (56)$$

For each  $k = 0, 1, \dots$ , let *period*  $k$  denote the time interval  $[t^0 + k\Delta t, t^0 + (k + 1)\Delta t)$ . Also, define

$$F(u_k, w_k, z_k, x_k \mid \theta_S, \Delta t) \equiv x_k + S(u_k, w_k, z_k, x_k \mid \theta_S) \cdot \Delta t \quad (57)$$

where

$$u_k = u(t^0 + k\Delta t) \quad (58)$$

$$w_k = w(t^0 + k\Delta t) \quad (59)$$

$$z_k = z(t^0 + k\Delta t) \quad (60)$$

$$x_k = x(t^0 + k\Delta t) \quad (61)$$

Then the original continuous-time state space model (1) over times  $t \geq t^0$  can be expressed in discrete-time approximate form over periods  $k = 0, 1, \dots$ , as follows:

**Discrete-time approximation equations for periods  $k \geq 0$  :**

$$\text{Dynamic state equations: } x_{k+1} = F(u_k, w_k, z_k, x_k \mid \theta_S, \Delta t) \quad (62)$$

$$\text{Simultaneous equations: } \mathbf{0} = H(u_k, w_k, z_k, x_k \mid \theta_H) \quad (63)$$

**Variables, parameters, and functional forms:**

$$u_k = (u_{k1}, \dots, u_{kM}) \in R^M, \text{ for periods } k \geq 0$$

$$w_k = (w_{k1}, \dots, w_{kJ}) \in R^J, \text{ for periods } k \geq 0$$

$$z_k = (z_{k1}, \dots, z_{kL}) \in R^L, \text{ for periods } k \geq 0$$

$$x_k = (x_{k1}, \dots, x_{kN}) \in R^N, \text{ for periods } k \geq 0$$

$$\theta_S = (\theta_{S1}, \dots, \theta_{SV}) \in R^I$$

$$\theta_H = (\theta_{H1}, \dots, \theta_{HV}) \in R^V$$

$$F: R^{M+J+L+N+SV} \rightarrow R^N$$

$$H: R^{M+J+L+N+HV} \rightarrow R^I$$

**Classification of variables:**

*Period- $k$  endogenous variables for  $k \geq 0$ :  $x_{k+1}, z_k$*

*Period- $k$  predetermined variables for  $k > 0$ :  $x_k$*

*Exogenous controls and forcing terms for  $k \geq 0$ :  $u_k, w_k$*

*Exogenous parameters and initial state conditions:  $\theta_S, \theta_H, \Delta t$ , and  $x_0$*

By construction, the above discrete-time approximation converges to the original continuous-time state space model as the period-length  $\Delta t$  is decreased towards 0.

# Appendix

The first three tables, below, provide symbols, descriptions, and units for the GLD Household ETP Model user-set parameters, derived parameters, and time- $t$  variables. The fourth table lists default values/functions (if any) for the user-set parameters.

Table 1: GLD Household ETP Model: User-Set Parameters

User-Set Parameters	Explanations
AuxHeatCapacity	Auxiliary heating capacity (Btu/hr)
CDT	System cooling design temperature ( $^{\circ}$ F)
cooling_COP	Coefficient of performance (unit free) for HVAC system in cooling mode
CoolSupplyAirTemp	Cooling supply air temperature ( $^{\circ}$ F)
DCT	System design cooling set-point ( $^{\circ}$ F)
DesignHeatCapacity	Design heating capacity (Btu/hr)
DesignHeatSetpoint	Design heating setpoint ( $^{\circ}$ F)
DIG	System design internal gain (Btu/hr)
DPS	System design solar load (Btu/hr-ft <sup>2</sup> )
DuctPressureDrop	Duct pressure drop (inches of water)
ECR	Exterior ceiling, fraction of total (decimal %)
EFR	Exterior floor, fraction of total (decimal %)
EWR	Exterior wall, fraction of total (decimal %)
f <sub>eu</sub>	Fraction of non-HVAC end-use load eu internal to household (decimal %)
f <sub>ac</sub> , f <sub>s</sub> , f <sub>i</sub> ,	Heat gain (decimal %) from ( $Q_{hvac}(t)$ , $Q_s(t)$ , $Q_i(t)$ ) to $Q_m(t)$
f <sub>oc</sub>	Household occupancy fraction (decimal %)
glass_type	String-coded glass types (GLASS, LOW_E,...)
glazing_layers	String-coded window glass-layer types (ONE, TWO, ...)
glazing_treatment	String-coded exterior window reflectivity types
HeatDesignTemp	Heating design temperature ( $^{\circ}$ F)
HeatSupplyAirTemp	Heating supply air temperature ( $^{\circ}$ F)
h <sub>s</sub>	Interior surface heat transfer coefficient (Btu/hr- $^{\circ}$ F-ft <sup>2</sup> )
I	Infiltration volumetric air exchange rate (#times per hr)
IWR	Interior/exterior wall surface ratio (unit free)
LatCoolFrac	Fractional cooling-load increase (unit free) due to latent heat
m <sub>f</sub>	Total thermal mass, per unit floor area (Btu/ $^{\circ}$ F-ft <sup>2</sup> )
n	Number of stories (integer)
n <sub>d</sub>	Number of doors (integer)
NEU	Number of household non-HVAC end-use loads (integer)
NOC	Number of household occupants (integer)
OSF	Over-sizing factor (unit free)
R <sub>c</sub>	R-value for ceilings (hr- $^{\circ}$ F-ft <sup>2</sup> /Btu)
R <sub>d</sub>	R-value for doors (hr- $^{\circ}$ F-ft <sup>2</sup> /Btu)
R <sub>f</sub>	R-value for floors (hr- $^{\circ}$ F-ft <sup>2</sup> /Btu)
R <sub>w</sub>	R-value for walls (hr- $^{\circ}$ F-ft <sup>2</sup> /Btu)
SHOC	Sensible heat (Btu/hr-occupant) from each occupant
V_nominal	Nominal rating voltage (volts)
WET	Window exterior transmission coefficient (decimal %)
window_frame	String-coded window-frame types (INSULATED, WOOD, ...)
WWR	Window-to-exterior-wall ratio (decimal %)
x, y, h	Width, length, and height (ft)
$\Delta t$	Time-period length (seconds)

Table 2: GLD Household ETP Model: Derived Parameters

Derived Parameters	Explanations
$A$	Floor area $x \times y \times n$ (ft <sup>2</sup> )
$A_c$	Net exterior ceiling area (ft <sup>2</sup> )
$A_d$	Total door area (ft <sup>2</sup> )
$A_f$	Net exterior floor area (ft <sup>2</sup> )
$A_g$	Gross window area (ft <sup>2</sup> )
$A_w$	Net exterior wall area (ft <sup>2</sup> )
$A_{wt}$	Gross exterior wall area (ft <sup>2</sup> )
$C_a$	Heat capacity (BTU/°F) of the inside air mass
$C_m$	Heat capacity (BTU/°F) of the inside solid mass
DesCoolCap	Design cooling capacity (Btu/hr)
DesignCoolAirflow	Design cooling airflow (cfm = ft <sup>3</sup> /m)
DesignHeat Airflow	Design heating airflow (cfm = ft <sup>3</sup> /m)
$H_m$	Thermal conductance (Btu/hr-°F) between inside air & solid masses
$R$	Floor aspect ratio $y/x$ (unit free)
$R_g$	R-value for windows (hr-°F-ft <sup>2</sup> /Btu)
SHGC	Solar heat gain coefficient (ft <sup>2</sup> )
SHGC <sub>nom</sub>	Nominal solar heat gain coefficient (decimal %)
$U_a$	Thermal conductance (Btu/hr-°F) between internal and external air masses

Table 3: GLD Household ETP Model: Time- $t$  Variables

Variables	Explanations
COP_adj( $t$ )	Coefficient of performance (unit free) adjusted for outside temperature effects
DesCoolCap_adj( $t$ )	Design cooling capacity (Btu/hr) adjusted for outdoor temperature effects
FanPow( $t$ )	Heat gain (Btu/hr) from the ON operation of the 1-speed fan
HVACPow( $t$ )	Heat gain (Btu/hr) from the ON operation of the HVAC system
$K(t)$	Coefficient of performance factor (Btu/hr-kW) for the HVAC system
ISR( $t$ )	Incident solar radiation (Btu/hr-ft <sup>2</sup> )
LCF( $t$ )	Fractional cooling-load increase (unit free) due to latent heat and humidity
$P^*(t)$	Power usage (kW) of the ON HVAC system in cooling mode
$p_{eu}(t)$	Real power (W) for each non-HVAC end-use load eu at time
$Q_a(t)$	Total heat flow rate (Btu/hr) to inside air mass
$Q_{hvac}(t)$	Heat flow rate (Btu/hr) from HVAC system and fan operations
$Q_i(t)$	Heat flow rate (Btu/hr) from internal non-HVAC equipment and occupants
$Q_m(t)$	Total heat flow rate (Btu/hr) to inside solid mass
$Q_s(t)$	Heat flow rate (Btu/hr) from solar radiation
RH( $t$ )	Relative humidity (unit free)
$T_a(t)$	Inside air temperature (°F)
$T_m(t)$	Inside mass temperature (°F)
$T_o(t)$	Outside air temperature (°F)
$u(t)$	Binary 0-1 variable denoting OFF/ON HVAC power usage for cooling
V_actual( $t$ )	Simulated-actual time- $t$ voltage (volts) obtained from GLD meter object in run-time
VF( $t$ )	Voltage function (unit free)
Voltage_adj( $t$ )	Voltage factor function (unit free)
VoltFactorB( $t$ )	Voltage factor (unit free) calculated using base voltage
VoltFactorN( $t$ )	Voltage factor (unit free) calculated using nominal voltage

Table 4: GLD Default Values or Functions (if Any) for User-Set Parameters

User-Set	Explanations	GLD Default
AuxHeatCapacity	Auxiliary heating capacity (Btu/hr)	Default Fct.
CDT	System cooling design temperature ( $^{\circ}\text{F}$ )	95
cooling_COP	Coefficient of performance (unit free) for HVAC system in cooling mode	3.5
CoolSupplyAirTemp	Cooling supply air temperature ( $^{\circ}\text{F}$ )	50
DCT	System design cooling set-point ( $^{\circ}\text{F}$ )	75
DesignHeatCapacity	Design heating capacity (Btu/hr)	Default Fct.
DesignHeatSetpoint	Design heating setpoint ( $^{\circ}\text{F}$ )	70
DIG	System design internal gain (Btu/hr)	Default Fct.
DPS	System design solar load (Btu/hr-ft <sup>2</sup> )	195.0
DuctPressureDrop	Duct pressure drop (inches of water)	0.50
ECR	Exterior ceiling, fraction of total (decimal %)	1.0
EFR	Exterior floor, fraction of total (decimal %)	1.0
EWR	Exterior wall, fraction of total (decimal %)	1.0
f <sub>eu</sub>	Fraction of non-HVAC end-use load eu internal to household (decimal %)	0.9 (ZIP load only)
f <sub>ac</sub> , f <sub>s</sub> , f <sub>i</sub> ,	Heat gain (decimal %) from ( $Q_{\text{hvac}}(t)$ , $Q_s(t)$ , $Q_i(t)$ ) to $Q_m(t)$	0.0, 0.5, 0.5
f <sub>oc</sub>	Household occupancy fraction (decimal %)	0.0
glass_type	String-coded glass types	LOW_E_GLASS
glazing_layers	String-coded window glass-layer types	TWO
glazing_treatment	String-coded exterior window reflectivity types	CLEAR
HeatDesignTemp	Heating design temperature ( $^{\circ}\text{F}$ )	Climate Object ('record.high')
HeatSupplyAirTemp	Heating supply air temperature ( $^{\circ}\text{F}$ )	150
h <sub>s</sub>	Interior surface heat transfer coefficient (Btu/hr- $^{\circ}\text{F}$ -ft <sup>2</sup> )	1.46
I	Infiltration volumetric air exchange rate (#times per hr)	0.5
IWR	Interior/exterior wall surface ratio (unit free)	1.5
LatCoolFrac	Fractional cooling-load increase (unit free) due to latent heat	0.3
m <sub>f</sub>	Total thermal mass, per unit floor area (Btu/ $^{\circ}\text{F}$ -ft <sup>2</sup> )	2.0
n	Number of stories (integer)	1.0
n <sub>d</sub>	Number of doors (integer)	4.0
NEU	Number of household non-HVAC end-use loads (integer)	
NOC	Number of household occupants (integer)	4
OSF	Over-sizing factor (unit free)	0.0
R <sub>c</sub>	R-value for ceilings (hr- $^{\circ}\text{F}$ -ft <sup>2</sup> /Btu)	30.0
R <sub>d</sub>	R-value for doors (hr- $^{\circ}\text{F}$ -ft <sup>2</sup> /Btu)	5.0
R <sub>f</sub>	R-value for floors (hr- $^{\circ}\text{F}$ -ft <sup>2</sup> /Btu)	22.0
R <sub>w</sub>	R-value for walls (hr- $^{\circ}\text{F}$ -ft <sup>2</sup> /Btu)	19.0
SHOC	Sensible heat (Btu/hr-occupant) from each occupant	400.0
V_nominal	Base voltage (volts)	120 or 240
WET	Window exterior transmission coefficient (decimal %)	0.6
window_frame	String-coded window-frame types	THERMAL_BREAK
WWR	Window-to-exterior-wall ratio (decimal %)	0.15
x, y, h	Width, depth, and height (ft)	-, -, 8.0
$\Delta t$	Time-period length (seconds)	1

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