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**Abstract**
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**Keywords**
Berry-Esseen theorem, finite population, normal approximation, sampling without replacement, simple random sampling

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# A Sub-Gaussian Berry-Esseen Theorem For the Hypergeometric Distribution

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**ABSTRACT**

In this paper, we derive a necessary and sufficient condition on the parameters of the Hypergeometric distribution for weak convergence to a Normal limit. We establish a Berry-Esseen theorem for the Hypergeometric distribution solely under this necessary and sufficient condition. We further derive a nonuniform Berry-Esseen bound where the tails of the difference between the Hypergeometric and the Normal distribution functions are shown to decay at a sub-Gaussian rate.

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*Keywords* Finite Population, Sampling Without Replacement.

# 1 Introduction

Consider a dichotomous finite population of size $N$ having $M$ individuals of type A and $N - M$ individuals of type B. Suppose a sample of size $n$ is drawn at random, without replacement from this population. Let $X$ denote the number of 'type A'-individuals in the sample. Then, $X$ is said to have the Hypergeometric distribution with parameters $n, M, N$, written as $X \sim Hyp(n; M, N)$. The probability mass function (p.m.f) of $X$ is given by,

$$P(X = x) \equiv P(x; n, M, N) = \begin{cases} \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} & \text{if} \quad x = 0, 1 \ldots, n \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

where, for any two integers $r \geq 1$ and $s$,

$$\binom{r}{s} = \begin{cases} \frac{r!}{s!(r-s)!} & \text{if} \quad 0 \leq s \leq r \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

with $0! = 1$ and $r! = 1 \cdot 2 \cdots r$. Let $f = \frac{n}{N}$ denote the sampling fraction and let $p = \frac{M}{N}$ denote the proportion of the 'type A'-objects in the population. The Hypergeoemetric distribution plays an important role in many areas of Statistics, including sample surveys (e.g., finite population inference), statistical quality control (acceptance sampling plans), etc. Normal approximations to the Hypergeometric probabilities $P(.; n, M, N)$ of (1.1) are classical in the cases where the sampling fraction $f$ and the proportion $p$ are bounded away from 0 and 1; see for example Feller(1971). However, the extreme cases where $f$ or $p$ take values near the boundary values 0 and 1 are very important in sample surveys and quality control applications. In this paper, we investigate the validity and the rate of Normal approximation to the Hypergeometric distribution allowing the parameters $f$ and $p$ to tend to any points in the interval $[0, 1]$, including the boundary points. The main results of the paper give a necessary and sufficient condition on the parameters $f$ and $p$ for a valid Normal approximation. It is shown that a Normal limit for properly centered and scaled version of $X$ holds if and only if

$$Np(1 - p)f(1 - f) \to \infty. \tag{1.3}$$

As a consequence of this, we conclude that for the Normal distribution function to approximate the distribution function of $X$, all four quantities, namely, (i) the number $M$ $(= Np)$ of 'type A'-objects, (ii) the number of 'type B'-objects, $N - M$, (iii) the sample size $n$, as well as (iv) the size of the unselected objects $N - n$ in the population, must tend to infinity.

We also investigate the rate of Normal approximation to the distribution of $X$. Note that $X$ is the sum of a collection of $n$ *dependent* Bernoulli random variables. In Section 2, we establish a Berry-Esseen Theorem on the rate of Normal approximation to the distribution function of $X$ solely under the necessary and sufficient condition (1.3). It is shown that under (1.3) the rate of approximation is $O([Np(1 - p)f(1 - f)]^{-1/2})$. It is also shown in Section 2 that this rate is optimal and can not be improved. Note that the rate $O([Np(1 - p)f(1 - f)]^{-1/2})$ is equivalent to the standard rate $O(n^{-1/2})$ (for sums of $n$ *independent* Bernoulli random variables, say) only when $p$ is bounded away from 0 and 1 and $f$ bounded away from 1. However, for $p$ and $f$ close to these boundary points, the rate of approximation can be substantially slower. In such situations, the dependence of the Bernoulli random variables associated with $X$ has a nontrivial effect on the accuracy of the Normal approximation.

Under somewhat stronger conditions on $f$ and $p$, we also derive a non-uniform version of the Berry-Esseen Theorem. The nonuniform bound shows that in the tails, the error of Normal approximation dies at a sub-Gaussian rate for a wide range of values of $f$ and $p$. As a corollary, we also derive an exponential (sub-Gaussian) probability inequality for the tails of $X$, which may be of independent interest.

The rest of the paper is organized as follows. We conclude Section 1 with a brief literature review. Section 2 introduces the asymptotic framework and contains the results on the validity of the Normal approximation and the Berry-Esseen theorems. Proofs of all the results are given in Section 3.

For results on Normal approximations to Hypergeometric probabilities in the standard cases where the sampling fraction $f$ and the proportion $p$ are bounded away from 0 and 1, see Feller(1971). For general $p$ and $f$, Nicholson (1956) derived some very precise bounds for the point probabilities $P(.;n,M,N)$ using some nonstandard normalizations of the Hypergeometric random variable $X$. General methods for proving the CLT for sample means under sampling without replacement from finite populations are given by Madow (1948), Erdos & Renyi (1959) and Hajek(1960). For results on Berry-Esseen Theorems and Edgeworth expansions for the functions of sample means and U-statistics based on finite population observations, see Babu & Singh (1985), Kokic & Weber (1990), Chen & Sitter (1993), Bloznelis (1999), Bloznelis & Götze (2000), and the references therein.

## 2 Main Results

Let $r$ be a positive integer valued variable and for each $r \in \mathbf{N}$ (where $\mathbf{N} = \{1, 2, \ldots\}$), let $X_r$ be a random variable having the Hypergeometric distribution with parameters $(n_r, M_r, N_r)$. Thus we consider a sequence of dichotomous finite populations indexed by $r$, with the population of objects of type A and the sampling fraction respectively given by,

$$p_r = \frac{M_r}{N_r} \quad \text{and} \quad f_r = \frac{n_r}{N_r} \quad \text{for all} \quad r \in \mathbf{N}. \tag{2.1}$$

To avoid trivialities, all through the paper, we shall assume that

$$1 \le M_r < N_r, \quad 1 \le n_r < N_r \qquad \text{for all} \quad r \in \mathbf{N}, \quad \text{and} \quad N_r^{-1} = o(1) \quad r \to \infty. \tag{2.2}$$

Thus, $p_r$ , $f_r \in (0, 1)$ for all $r \in \mathbf{N}$. Let

$$\sigma_r^2 \equiv N_r p_r q_r f_r (1 - f_r), \tag{2.3}$$

where $q_r = 1 - p_r$. The first result concerns the validity of the Normal approximation to the distribution of $X_r$.

**Theorem 2.1:** Suppose that (2.2) holds and that $X_r \sim Hyp(n_r, M_r, N_r)$, $r \in \mathbf{N}$. Then there exists a Normal random variable $W \sim N(\mu, \sigma^2)$ for some $\mu \in \mathbf{R}$ and $\sigma \in (0, \infty)$ such that

$$\Delta_r \equiv \sup_{x \in \mathbf{R}} \left| P\left(\frac{X_r - n_r p_r}{\sigma_r} \leq x\right) - P\left(W \leq x\right) \right| \longrightarrow 0 \quad \text{as} \quad r \to \infty \tag{2.4}$$

if and only if

$$\sigma_r^2 \to \infty \qquad \text{as} \quad r \to \infty. \tag{2.5}$$

When (2.5) holds, one must have $\mu = 0$ and $\sigma = 1$.

Note that $\sigma_r^2 = n_r p_r q_r (1 - f_r) = \frac{N_r - 1}{N_r} Var(X_r)$. Hence Theorem 2.1 shows that the Normal approximation to the Hypergeometric distribution holds solely under the condition that the variance of the Hypergeometric distribution goes to infinity with $r$. In particular, it is not necessary to impose separate conditions on the asymptotic behavior of the three sequences $\{n_r\}_{\{r \geq 1\}}$, $\{p_r\}_{\{r \geq 1\}}$ and $\{f_r\}_{\{r \geq 1\}}$. A necessary condition for (2.5) is that $n_r \to \infty$ and $(N_r - n_r) \to \infty$ as $r \to \infty$. This follows by noting that $\sigma_r^2 = n_r p_r q_r (1 - f_r) = (N_r - n_r) p_r q_r f_r \leq \min\{n_r, N_r - n_r\}$ for all $r \geq 1$. Thus, for the Normal approximation to hold, both the sample size $n_r$ and the residual sample size $(N_r - n_r)$ must become unbounded as $r \to \infty$. By interchanging the roles of $p_r$ and $q_r$ with $f_r$ and $(1 - f_r)$, it follows that for the validity of the Normal approximation, we must also have

$$M_r \wedge (N_r - M_r) \longrightarrow \infty \quad \text{as} \quad r \to \infty, \tag{2.6}$$

i.e., the number of objects of type A and type B must go to infinity with $r$.

In a seminal paper, Hajek (1968) obtained a necessary and sufficient condition for the CLT for finite population sums, assuming that

$$n_r \wedge N_r - n_r \to \infty \quad \text{as} \quad r \to \infty. \tag{2.7}$$

The observations above imply that this is not a serious restriction; Indeed, in the cases where (2.7) fail, the CLT need not hold.

Condition (2.5) also allows the proportion $p_r$ of 'type A'-objects in the population and the sampling fraction $f_r$ to simultaneously converge to the extreme points 0 and 1 at certain rates. If the sequence $\{f_r\}_{\{r \geq 1\}}$ is bounded away from 0 and 1 and (2.2) holds, then the CLT of Theorem 2.1 holds if and only if (iff)

$$\frac{1}{N_r} = o(q_r \wedge p_r) \quad \text{as} \quad r \to \infty, \tag{2.8}$$

i.e., iff (2.6) holds. Similarly, for $\{p_r\}_{\{r \geq 1\}}$ bounded away from 0 and 1, the CLT holds *iff*

$$\frac{1}{N_r} = o(f_r \wedge (1 - f_r)) \quad \text{as} \quad r \to \infty, \tag{2.9}$$

i.e., iff (2.7) holds. However, when both $\{p_r\}_{\{r \geq 1\}}$ and $\{f_r\}_{\{r \geq 1\}}$ simultaneously converge to some limits in $\{0, 1\}$, neither of (2.8) and (2.9) alone is enough to guarantee the CLT. For example if

$f_r \sim N_r^{-a}$ and $p_r \sim N_r^{-b}$ for some $0 < a, b < 1$, with $a + b > 1$, then (2.8) and (2.9) hold but the Normal approximation of Theorem 2.1 is no longer valid.

Next we obtain a refinement of (2.4) by specifying the rate of convergence of $\Delta_r$ to zero.

**Theorem 2.2:** Suppose that $X_r \sim Hyp(n_r, M_r, N_r)$, $r \in \mathbf{N}$, and that (2.5) holds. Then there exists a constant $C_1 \in (0, \infty)$ such that for all $r \in \mathbf{N}$,

$$\Delta_r \leq \frac{C_1}{\sigma_r}. \tag{2.10}$$

Theorem 2.2 is a uniform Berry-Esseen theorem that shows that under (2.5), the rate of Normal approximation to the Hypergeometric distribution is uniformly $O\left(\sigma_r^{-1}\right)$ as $r \to \infty$. When both the sequences $\{p_r\}_{\{r\geq 1\}}$ and $\{f_r\}_{\{r\geq 1\}}$ are bounded away from 0 and 1, this rate is $O\left(n_r^{-\frac{1}{2}}\right)$, which is the same as the rate of Normal approximation for sums of $n_r$ *independent and identically distributed* (iid) random variables with a finite third moment. Although the Hypergeometric random variable $X_r$ can be written as a sum of $n_r$ *dependent* Bernoulli ($p_r$) variables, the lack of independence of the summands does not affect the rate of Normal approximation as long as the sequence $\{p_r\}_{r\geq 1}$ is bounded away from 0 and 1 and $\{f_r\}_{r\geq 1}$ is bounded away from 1; The rate becomes worse otherwise.

A second important aspect of Theorem 2.2 is that the bound on $\Delta_r$ holds under the same condition (2.5) that is both necessary and sufficient for a Normal limit. Since $\frac{X_r - n_r p_r}{\sigma_r}$ is supported on a lattice with maximal span $\sigma_r^{-1}$, it is not difficult to show that if (2.5) holds, then $\liminf_{r\to\infty} \Delta_r \sigma_r > 0$, i.e., there exists a constant $C_2 \in (0, \infty)$ such that

$$\Delta_r > \frac{C_2}{\sigma_r} \tag{2.11}$$

for all but finitely many $r$'s. Thus, the rate in Theorem 2.2 is *optimal* and can not be improved upon.

The next result gives a non-uniform version of the Berry-Esseen theorem. To state it, let $\phi(\cdot)$ and $\Phi(\cdot)$ respectively denote the density and the distribution function of a standard Normal random variable, i.e., $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})$, $\quad x \in \mathbf{R}$ and $\Phi(x) = \int_{-\infty}^{x} \phi(t)dt$, $\quad x \in \mathbf{R}$. Also let $I(\cdot)$ denote the indicator function. Define

$$\delta_r = \frac{1}{10}(\max(a_{1r}, 2))^{-1}, \qquad r \geq 1, \tag{2.12}$$

where $a_{1r} = \frac{\bar{f}_r + 4}{4(1-\bar{f}_r)}$ and where

$$\bar{f}_r = \begin{cases} f_r & : \text{ if } \ f_r \leq \frac{1}{2} \\ 1 - f_r & : \text{ if } \ f_r > \frac{1}{2}. \end{cases}$$

Then, we have the following result.

**Theorem 2.3:** Suppose that $X_r \sim Hyp(n_r, M_r, N_r), r \in \mathbf{N}$. Assume that $r$ is such that

$$\delta_r \sigma_r > 1. \tag{2.13}$$

Then there exists universal constants $C_3, C_4 \in (0, \infty)$ (not depending on $r, n_r, M_r$ and $N_r$) such that

$$\left| P\left(\frac{X_r - n_r p_r}{\sigma_r} \leq x\right) - \Phi(x) \right| \leq \frac{C_3}{\sigma_r} \frac{1 + |x|^2}{\lambda_r(x)} \exp\left(-C_4 x^2 \lambda_r^2(x)\right) \tag{2.14}$$

for all $x \in \mathbf{R}$, where $\lambda_r(x) = q_r I(x \leq 0) + p_r I(x \geq 0)$.

Theorem 2.3 shows that the error of Normal approximation to the Hypergeometric distribution dies at a sub-Gaussian rate in the tails. The only condition needed for the validity of this bound is (2.13). It is easy to check that

$$\delta_r \in \left(\frac{1}{25}, \frac{1}{20}\right] \tag{2.15}$$

for all $r$ satisfying (2.13). Hence, the bound in (2.14) is available for all $r$ such that $\sigma_r \geq 25$.

An immediate consequence of Theorem 2.3 is the following exponential (sub-Gaussian) probability bound on the tails of $X_r$.

**Corollary 2.4:** Suppose that $X_r \sim Hyp(n_r, M_r, N_r), r \in \mathbf{N}$. Then, there exist universal constants $C_5, C_6 \in (0, \infty)$ (not depending on $r, n_r, M_r, N_r$) such that for all $r$ satisfying (2.13),

$$P\left(\left|\frac{X_r - n_r p_r}{\sigma_r}\right| \geq x\right) \leq \frac{C_5}{(p_r \wedge q_r)^3} \exp\left(-C_6 x^2 [p_r \wedge q_r]^2\right) \quad \text{for all} \quad x \in (0, \infty).$$

# 3 Proofs

We now introduce some notation and notational convention to be used in this section. For real numbers $x, y$, let $x \wedge y = min\{x, y\}$ and $x \vee y = max\{x, y\}$. Let $\lfloor x \rfloor$ denote the largest integer not exceeding $x$, $x \in \mathbf{R}$. For $a \in (0, \infty)$, write $\phi_a(x) = \frac{1}{a}\phi(\frac{x}{a})$ and $\Phi_a(x) = \Phi(\frac{x}{a})$, $x \in \mathbf{R}$, for the density and distribution functions of a $N(0, a^2)$ variable. Write $\phi_a = \phi$ and $\Phi_a = \Phi$ for $a = 1$. Let

$$\Delta_r^*(x) = P\left(\frac{X_r - n_r p_r}{\sigma_r} \leq x\right) - \Phi(x), \quad x \in \mathbf{R}. \tag{3.1}$$

Let $\mathbf{N} = \{1, 2, \ldots\}$, $\mathbf{Z}_+ = \{0, 1, \ldots\}$ and $\mathbf{Z} = \{\ldots, -1, 0, 1, \ldots\}$.

For notational simplicity, we shall drop the suffix $r$ from notation, except when it is important to highlight the dependence on $r$. Thus, we write $n, M, N$ for $n_r, M_r, N_r$ respectively and set $p = \frac{M}{N}$, $q = 1 - p$ and $f = \frac{n}{N}$. We shall use $C$ to denote a generic positive constant that does not depend on $r$. Unless otherwise stated, limits in order symbols are taken by letting $r \to \infty$.

For proving the result, we shall frequently make use of Stirling's approximation (cf. Feller(1971))

$$m! \;=\; \sqrt{2\pi} e^{-m+\epsilon_m} m^{m+\frac{1}{2}} \quad \text{for all} \quad m \in \mathbf{N}, \tag{3.2}$$

where the error term $\epsilon_m$ admits the bound

$$\frac{1}{12m+1} \leq \epsilon_m \leq \frac{1}{12m} \qquad \text{for all} \quad m \in \mathbf{N}.$$

Also note that for $g(y) = \log y$, $y \in (0, \infty)$, the $k$th derivative of $g$ is given by $g^{(k)}(y) = \frac{(-1)^{k-1}(k-1)!}{y^k}$, $y \in (0, \infty)$, $k \in \mathbf{N}$. Hence, for any $k \in \mathbf{N}$ and $\delta \in (0, 1)$,

$$\left| g^{(k)}(1+x) \right| \leq \frac{(k-1)!}{(1-\delta)^k} \qquad \text{for all} \qquad 0 \leq |x| < \delta. \tag{3.3}$$

For Lemma 3.1, let $X \sim Hyp(n; M, N)$ for a *given* set of integers $n, M, N \in \mathbf{N}$ with $1 \leq n \leq (N-1)$, $1 \leq M \leq (N-1)$. Note that this notation is consistent with our convention of dropping the suffix $r$; $X, n, M, N$ in Lemma 3.1 would subsequently represent $X_r, n_r, M_r, N_r$ for a *fixed* $r \in \mathbf{N}$ for which (2.2) holds. Let

$$x_{k,n} = \frac{x - np}{\sqrt{npq}} \quad \text{and} \quad a_{k,n} = \frac{x_{k,n}}{(1-f)\sqrt{npq}}, \quad 0 \leq k \leq n, \tag{3.4}$$

where $f = \frac{n}{N}$, $p = \frac{M}{N}$ and $q = 1 - p$. Lemma 3.1 gives a basic approximation to Hypergeometric probabilities solely under condition (3.5) stated below.

**Lemma 3.1** Suppose that $X \sim Hyp(n; M, N)$ for a *given* set of integers $n, M, N \in \mathbf{N}$ such that

$$0 < f < 1, \quad 0 < p < 1 \quad \text{and} \quad 6(np \wedge nq) \geq 1, \tag{3.5}$$

where $f = \frac{n}{N}$, $p = \frac{M}{N}$ and $q = 1 - p$ are as in (3.4). Then, for any given $\delta \in (0, \frac{1}{2}]$,

$$\log P(k; n, M, N) = -\frac{x_{k,n}^2}{2(1-f)} - \frac{1}{2}\log\left(2\pi npq(1-f)\right) + r_n^*(k) \tag{3.6}$$

for all $k \in \{0, \ldots, n\}$ with $|a_{k,n}| \leq \delta$, where $P(k; n, M, N) = P(X = k)$ (cf. (1.1)) and where the remainder term $r_n^*(k)$ admits the bound

$$\begin{aligned} |r_n^*(k)| \quad \leq \quad & \frac{1}{6npq(1-\delta)(1-f)} + \left[\frac{1}{2}|a_{k,n}| + a_{k,n}^2\left\{\frac{1}{4} + \frac{2\delta}{(1-\delta)^3}\right\}\right] \\ & + |a_{k,n}|^3 npq\left(\frac{f}{4} + 1\right)\left\{\frac{1}{2} + \frac{2(1+\delta)}{(1-\delta)^3}\right\}, \end{aligned} \tag{3.7}$$

provided $|a_{k,n}| \leq \delta$.

**Proof:** For $k \in \{0, 1, \ldots, n\}$,

$$P(k, n, M, N) \quad = \quad \frac{\binom{Np}{k}\binom{Nq}{n-k}}{\binom{N}{n}}$$

$$= \binom{n}{k} p^k q^{n-k} \frac{\prod_{j=1}^{k-1}(1-\frac{j}{Np}) \prod_{j=1}^{n-k-1}(1-\frac{j}{Nq})}{\prod_{j=1}^{n-1}(1-\frac{j}{N})}$$

$$= \binom{n}{k} p^k q^{n-k} \; R(k,n,M,N), \quad \text{say.} \tag{3.8}$$

First consider the denominator of $R(k;n,M,N)$. By (3.2),

$$\begin{aligned}\prod_{j=1}^{n-1}(1-\frac{j}{N}) &= \frac{N!}{(N-n)!N^n} \\ &= \frac{e^{(-N+\epsilon_N)}N^{N+\frac{1}{2}}}{e^{(-(N-n)+\epsilon_{N-n})}(N-n)^{N-n+\frac{1}{2}}} \frac{1}{N^n} \\ &= \frac{e^{(\epsilon_N-\epsilon_{N-n})}e^{-n}}{(1-f)^{N(1-f)+\frac{1}{2}}}.\end{aligned}$$

Similarly, the numerator of $R(k;n,M,N)$ is given by

$$\begin{aligned}\prod_{j=1}^{k-1}(1-\frac{j}{Np}) \prod_{j=1}^{n-k-1}(1-\frac{j}{Nq}) &= \frac{e^{-k}e^{\epsilon_{Np}-\epsilon_{Np-k}}}{(1-\frac{k}{Np})^{Np-k+\frac{1}{2}}} \frac{e^{-(n-k)}e^{\epsilon_{Nq}-\epsilon_{Nq-n+k}}}{(1-\frac{n-k}{Nq})^{Nq-n+k+\frac{1}{2}}} \\ &= \frac{e^{-n}e^{\epsilon_{Np}-\epsilon_{Np-k}+\epsilon_{Nq}-\epsilon_{Nq-n+k}}}{(1-\frac{k}{Np})^{Np-k+\frac{1}{2}}(1-\frac{n-k}{Nq})^{Nq-n+k+\frac{1}{2}}}.\end{aligned}$$

Note that by (3.4),

$$\frac{k}{Np} = f + x_{k,n}\sqrt{\frac{fq}{Np}} \quad \text{and} \quad \frac{n-k}{Nq} = f - x_{k,n}\sqrt{\frac{fp}{Nq}}. \tag{3.9}$$

Hence $R(k;n,M,N)$ can be expressed as

$$\begin{aligned}R(k;n,M,N) = \exp(\epsilon_{Np}-\epsilon_{Np-k}+\epsilon_{Nq}-\epsilon_{Nq-n+k}+\epsilon_{N-n}-\epsilon_N)(1-f)^{N(1-f)+\frac{1}{2}} \\ \times\left\{\left(1-f-x_{k,n}\sqrt{\frac{fq}{Np}}\right)^{Np\left(1-f-x_{k,n}\sqrt{\frac{fq}{Np}}\right)+\frac{1}{2}}\right\} \\ \times\left\{\left(1-f+x_{k,n}\sqrt{\frac{fp}{Nq}}\right)^{Nq\left(1-f+x_{k,n}\sqrt{\frac{fp}{Nq}}\right)+\frac{1}{2}}\right\}.\end{aligned}$$

Next write

$$\begin{aligned}z_{k,n} &= \frac{x_{k,n}\sqrt{\frac{fp}{Nq}}}{1-f}, \quad y_{k,n} = \frac{x_{k,n}\sqrt{\frac{fq}{Np}}}{1-f} \quad \text{and} \\ \epsilon^* &= \epsilon_{Np}-\epsilon_{Np-k}+\epsilon_{Nq}-\epsilon_{Nq-n+k}+\epsilon_{N-n}-\epsilon_N.\end{aligned} \tag{3.10}$$

Then it follows that

$$\begin{aligned}\log R(k;n,M,N) &= \epsilon^* - \frac{\log(1-f)}{2} - \Big(Np(1-f)(1-y_{k,n})+\frac{1}{2}\Big)\log(1-y_{k,n}) \\ &\quad - \Big(Nq(1-f)(1+z_{k,n})+\frac{1}{2}\Big)\log(1+z_{k,n}) \\ &\equiv \epsilon^* - \frac{\log(1-f)}{2} - A_1 - A_2, \qquad \text{say.}\end{aligned} \tag{3.11}$$

Fix $\delta \in (0, 1/2)$. By Taylor's expansion and (3.3),

$$\begin{aligned}
A_1 &= \left(Np(1-f)(1-y_{k,n}) + \frac{1}{2}\right)\log(1-y_{k,n}) \\
&= \left(Np(1-f)(1-y_{k,n}) + \frac{1}{2}\right)\left(-y_{k,n} - \frac{y_{k,n}^2}{2} + r_{1n}(k)\right) \\
&= -y_{k,n}\left(Np(1-f) + \frac{1}{2}\right) - \frac{y_{k,n}^2}{2}\left(\frac{1}{2} - Np(1-f)\right) + r_{2n}(k), \qquad (3.12)
\end{aligned}$$

where $r_{1n}(k)$ and $r_{2n}(k)$ are remainder terms, defined by the equality of the successive expressions. By (3.3), for all $n, k$ satisfying $|y_{k,n}| \le \delta$,

$$\begin{aligned}
|r_{1n}(k)| &\le \frac{2}{(1-\delta)^3}\frac{|y_{k,n}|^3}{3!} \quad \text{and,} \\
|r_{2n}(k)| &\le \frac{Np}{2}(1-f)|y_{k,n}|^3 + \left|Np(1-f)(1-y_{k,n}) + \frac{1}{2}\right| \cdot |r_{1n}(k)|. \qquad (3.13)
\end{aligned}$$

By similar arguments,

$$\begin{aligned}
A_2 &= \left[Nq(1-f)(1+z_{k,n}) + \frac{1}{2}\right]\log(1+z_{k,n}) \\
&= \left(Nq(1-f) + \frac{1}{2}\right)z_{k,n} + \frac{z_{k,n}^2}{2}\left[Nq(1-f) - \frac{1}{2}\right] + r_{3n}(k), \qquad (3.14)
\end{aligned}$$

where for all $n, k$, satisfying $|z_{k,n}| \le \delta$,

$$|r_{3n}(k)| \le Nq(1-f)\frac{|z_{k,n}|^3}{2} + \left|Nq(1-f)(1+z_{k,n}) + \frac{1}{2}\right| \cdot \frac{|z_{k,n}|^3}{3(1-\delta)^3}\,. \qquad (3.15)$$

From, (3.11),(3.12) and (3.14), we have

$$\begin{aligned}
\log R(k; n, M, N) &= \epsilon^* - \frac{1}{2}\log(1-f) - \left[\frac{1}{2}(z_{k,n} - y_{k,n}) + \frac{z_{k,n}^2}{2}\left\{Nq(1-f) - \frac{1}{2}\right\}\right. \\
&\qquad \left. + \frac{y_{k,n}^2}{2}\left\{Np(1-f) - \frac{1}{2}\right\} + r_{2n}(k) + r_{3n}(k)\right] \\
&= \epsilon^* - \frac{1}{2}\log(1-f) - \frac{x_{k,n}^2 f}{2(1-f)} + r_{4n}(k) \qquad (3.16)
\end{aligned}$$

where for all $n, k$ satisfying $(|y_{k,n}| \vee |z_{k,n}|) \le \delta$,

$$|r_{4n}(k)| \le |r_{2n}(k)| + |r_{3n}(k)| + \frac{1}{2}|y_{k,n} - z_{k,n}| + \frac{1}{4}\left(y_{k,n}^2 + z_{k,n}^2\right).$$

Next using Stirling's formula on the binomial term, we have

$$\begin{aligned}
\log\left\{\binom{n}{k}p^k q^{n-k}\right\} &= \log\left\{\frac{e^{(\epsilon_n - \epsilon_k - \epsilon_{n-k})}}{\sqrt{2\pi npq}}\right\} - \left(nq - x_{k,n}\sqrt{npq} + \frac{1}{2}\right)\log\left\{1 - x_{k,n}\sqrt{\frac{p}{nq}}\right\} \\
&\qquad - \left(np + x_{k,n}\sqrt{npq} + \frac{1}{2}\right)\log\left\{1 + x_{k,n}\sqrt{\frac{q}{np}}\right\} \\
&\equiv \epsilon^{**} - \log\sqrt{2\pi npq} - A_3 - A_4, \quad \text{say,} \qquad (3.17)
\end{aligned}$$

where $\epsilon^{**} = \epsilon_n - \epsilon_k - \epsilon_{n-k}$. Next write $\tilde{y}_{k,n} = x_{k,n}\sqrt{\frac{p}{nq}}$ and $\tilde{z}_{k,n} = x_{k,n}\sqrt{\frac{q}{np}}$. Then, by arguments similar to (3.12) and (3.14),

$$\begin{aligned} A_3 &= \left(nq - x_{k,n}\sqrt{npq} + \frac{1}{2}\right)\log\left(1 - x_{k,n}\sqrt{\frac{p}{nq}}\right) \\ &= -\tilde{y}_{k,n}\left(nq + \frac{1}{2}\right) + \frac{\tilde{y}_{k,n}^2}{2}\left(nq - \frac{1}{2}\right) + r_{5n}(k) \end{aligned}$$

and

$$\begin{aligned} A_4 &= \left(np + x_{k,n}\sqrt{npq} + \frac{1}{2}\right)\log\left(1 + x_{k,n}\sqrt{\frac{q}{np}}\right) \\ &= \tilde{z}_{k,n}\left(np + \frac{1}{2}\right) + \frac{\tilde{z}_{k,n}^2}{2}\left(np - \frac{1}{2}\right) + r_{6n}(k) \end{aligned}$$

where for all $k$ and $n$ satisfying $|\tilde{y}_{k,n}| \vee |\tilde{z}_{k,n}| \le \delta$,

$$\begin{aligned} |r_{5n}(k)| + |r_{6n}(k)| &\le \frac{n}{2}\left[q|\tilde{y}_{k,n}|^3 + p|\tilde{z}_{k,n}|^3\right] + \frac{2}{(1-\delta)^3}\left[\left(nq + \frac{1}{2} + nq|\tilde{y}_{k,n}|\right)|\tilde{y}_{k,n}|^3\right. \\ &\quad \left. + \left(np + \frac{1}{2} + np|\tilde{z}_{k,n}|\right)|\tilde{z}_{k,n}|^3\right]. \end{aligned} \tag{3.18}$$

Hence, as in (3.16), it follows that

$$\log\left\{\binom{n}{k}p^k q^{n-k}\right\} = \epsilon^{**} - \log\sqrt{2\pi npq} - \frac{1}{2}x_{k,n}^2 + r_{7n}(k) \tag{3.19}$$

where for all $n, k$ satisfying $|\tilde{y}_{k,n}| \vee |\tilde{z}_{k,n}| \le \delta$,

$$|r_{7n}(k)| \le \left|\frac{1}{2}\left(\tilde{z}_{k,n} + \tilde{y}_{k,n}\right) - \frac{1}{4}\left(\tilde{y}_{k,n}^2 + \tilde{z}_{k,n}^2\right)\right| + |r_{5n}(k)| + |r_{6n}(k)|.$$

Note that

$$\begin{aligned} fq + fp + (1-f)p + (1-f)q &= 1, \\ (fq)^2 + (fp)^2 + ((1-f)p)^2 + ((1-f)q)^2 &= (1-2pq)(1-2(1-f)) < 1, \end{aligned}$$

and by (3.4), $y_{k,n} = fqa_{k,n}$, $z_{k,n} = fpa_{k,n}$, $\tilde{y}_{k,n} = pa_{k,n}$, and $\tilde{z}_{k,n} = qa_{k,n}$. Hence, it follows that

$$\frac{1}{2}\left(|y_{k,n}| + |\tilde{y}_{k,n}| + |z_{k,n}| + |\tilde{z}_{k,n}|\right) + \frac{1}{4}\left(y_{k,n}^2 + \tilde{y}_{k,n}^2 + z_{k,n}^2 + \tilde{z}_{k,n}^2\right) \le \frac{1}{2}|a_{k,n}| + \frac{1}{4}a_{k,n}^2. \tag{3.20}$$

Now, combining (3.8), (3.16) and (3.18) and using (3.20) and the above identities, after some algebra, we get

$$\log P(k; n, M, N) = -\frac{x_{k,n}^2}{2(1-f)} - \frac{1}{2}\log(2\pi npq(1-f)) + r_n^*(k),$$

where for all $k, n$ satisfying $|a_{k,n}| \le \delta$,

$$\begin{aligned} |r_n^*(k) - \epsilon^* - \epsilon^{**}| &\le |r_{4n}(k)| + |r_{7n}(k)| \\ &\le \frac{npq}{2}|a_{k,n}|^3\left[(1-f)(fq)^2 + (1-f)(fp)^2 + p^2 + q^2\right] \end{aligned}$$

$$
\begin{aligned}
&+\frac{2npq}{(1-\delta)^3}|a_{k,n}|^3\Big[(1-f)f^2\left\{(1+\delta fq)q^2+(1+\delta fp)p^2\right\}\\
&+(1+\delta p)p^2+(1+\delta q)q^2\Big]\\
&+\frac{2}{(1-\delta)^3}|a_{k,n}|^3\frac{1}{2}\left[(1+f^3)(p^3+q^3)\right]+\frac{1}{2}|a_{k,n}|+\frac{1}{4}a_{k,n}^2\\
\leq\ &\frac{1}{2}|a_{k,n}|+a_{k,n}^2\left\{\frac{1}{4}+\frac{2\delta}{(1-\delta)^3}\right\}+|a_{k,n}|^3npq\left(\frac{f}{4}+1\right)\left\{\frac{1}{2}+\frac{2(1+\delta)}{(1-\delta)^3}\right\}.
\end{aligned}
\tag{3.21}
$$

Note that for all $k, n$ satisfying $|a_{k,n}| \leq \delta$,

$$
Np-k \geq Np-(np+\delta(1-f)npq) > np\frac{(1-f)}{2} > 0
$$

and

$$
Nq-(n-k) > nq\frac{(1-f)}{2} > 0.
$$

Hence, by the error bound in Stirling's approximation, for all $k, n$ with $|a_{k,n}| \leq \delta$ and $6(np \wedge nq) \geq 1$,

$$
\begin{aligned}
\epsilon^* &\geq \frac{1}{12Np+1}-\frac{1}{12(Np-k)}+\frac{1}{12Nq+1}-\frac{1}{12(Nq-(n-k))}+\frac{1}{12(N-n)+1}-\frac{1}{12N}\\
&\geq -\frac{12k+1}{(12Np+1)(12(Np-k))}-\frac{12(n-k)+1}{(12Nq+1)(12(Nq-n+k))}\\
&\geq -\frac{1}{6Np(1-\delta)(1-f)}-\frac{1}{6Nq(1-\delta)(1-f)}\\
&= -\frac{f}{6npq(1-\delta)(1-f)};
\end{aligned}
$$

$$
\epsilon^* \leq 0+0+\left[\frac{1}{12(N-n)+1}-\frac{1}{12N}\right] \leq \frac{f}{6npq(1-\delta)(1-f)};
$$

$$
\epsilon^{**} \leq \frac{1}{12n}-\frac{1}{12k+1}-\frac{1}{12(n-k)+1} \leq 0; \tag{3.22}
$$

$$
\epsilon^{**} \geq \frac{1}{12n+1}-\frac{1}{12k}-\frac{1}{12(n-k)} \geq -\frac{n}{12k(n-k)} \geq -\frac{1}{6npq(1-\delta)}.
$$

Hence, the lemma follows from (3.21) and the above inequalities.

**Lemma 3.2** Let $g : \mathbf{R} \longrightarrow [0,\infty)$ be such that $g$ is $\uparrow$ on $(-\infty, a)$ and $g$ is $\downarrow$ on $(a,\infty)$ for some $a \in \mathbf{R}$. Then, for any $k \in \mathbf{N}$, $b \in \mathbf{R}$ and $h \in (0,\infty)$,

$$
\sum_{i=o}^{k} g(b+ih) \leq \int_b^{b+hk} g(x)dx + 2hg(x_0), \tag{3.23}
$$

where $g(x_0) = max\{g(b+ih) : i = 0,1,\ldots,k\}$.

**Proof**: For $b \geq a$, by monotonicity,

$$h\sum_{i=0}^{k} g(b+ih) \leq hg(b) + \int_b^{b+hk} g(x)dx\,.$$

For $b < a$, let $k_1 = \sup\{i : b+ih < a\}$ and $b_1 = b+k_1h$. Then,

$$\begin{aligned} h\sum_{i=0}^{k_1} g(b+ih) &\leq \sum_{i=0}^{k_1-1}\int_{b+ih}^{b+(i+1)h} g(x)dx + hg(b+k_1h) \\ &\leq \int_b^{b_1} g(x)dx + hg(b_1). \end{aligned}$$

Hence, for $b < a$ and $k > k_1$,

$$\begin{aligned} h\sum_{i=0}^{k} g(b+ih) &= h\sum_{i=0}^{k_1} g(b+ih) + h\sum_{i=k_1+1}^{k} g(b+ih) \\ &= h\sum_{i=0}^{k_1} g(b+ih) + h\sum_{j=0}^{k-k_1-1} g(b_1+h+jh) \\ &\leq \int_b^{b_1} g(x)dx + hg(b_1) + hg(b_1+h) + \int_{b_1+h}^{b_1+h+(k-k_1-1)h} g(x)dx - hg(b_1) \\ &\leq \int_b^{b+hk} g(x)dx + 2hg(x_0). \end{aligned}$$

For $b < a$ and $k < k_1$, it is easy to check (using the arguments above) that bound (3.23) trivially holds. This completes the proof of the lemma.

**Lemma 3.3** Let $\phi(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})$, $x \in \mathbf{R}$. Then, for any $h \in (0,\infty)$, $b \in [0,\infty)$, $j_0 \in \mathbf{N}$,

$$\begin{aligned} &\left|h\sum_{i=0}^{j_0}\phi(b+ih) - \int_{b-\frac{h}{2}}^{b+(j_0+\frac{1}{2})h}\phi(x)dx\right| \qquad (3.24) \\ \leq\ & \frac{h^2}{12}\left[\int_{b-\frac{h}{2}}^{b+j_0h+\frac{h}{2}} |\phi^{''}(x)|dx + (4+h)\max\left\{|\phi^{''}(x)| : b-\frac{h}{2} < x < b+j_0h+\frac{h}{2}\right\}\right]. \end{aligned}$$

**Proof :** Note that the function $|\phi''(x)| = |x^2-1|\phi(x)$ is even, and on $[0,\infty)$, it is increasing on $[1, 3^{1/2}]$ and decreasing on each of the intervals $[0,1)$ and $(3^{1/2},\infty)$, with the maximum value $\frac{1}{\sqrt{2\pi}}$ at $x=0$ and the minimum value 0 at $x=1$. First suppose that $(b-\frac{h}{2}, b+(j_0+\frac{1}{2})h) \cap \{0,\sqrt{3}\} = \emptyset$. Then, writing $b_i = b+ih$, $i \geq 0$, and using Taylor's expansion, one can show that the leftside of (3.24) is bounded above by

$$\begin{aligned} \sum_{i=0}^{j_0}\left|\int_{b_i-\frac{h}{2}}^{b_i+\frac{h}{2}}\Big(\phi(x)-\phi(b_i)\Big)dx\right| &\leq \frac{1}{2}\sum_{i=0}^{j_0}\int_{b_i-\frac{h}{2}}^{b_i+\frac{h}{2}}(x-b_i)^2\left\{\sup_{y\in(b_i-\frac{h}{2},b_i+\frac{h}{2})}|\phi^{''}(y)|\right\}dx \\ &\leq \frac{1}{2}\sum_{i=0}^{j_0}\left(2\int_0^{\frac{h}{2}} y^2dy\right)\times\left\{\left|\phi^{''}\left(b_i-\frac{h}{2}\right)\right| \vee \left|\phi^{''}\left(b_i+\frac{h}{2}\right)\right|\right\} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{h^3}{24}\sum_{i=0}^{j_0}\left\{\left|\phi''\left(b_i-\frac{h}{2}\right)\right|+\left|\phi''\left(b_i+\frac{h}{2}\right)\right|\right\}\\
&\leq \frac{h^3}{12}\sum_{i=0}^{j_0+1}\left|\phi''\left(b_i-\frac{h}{2}\right)\right| .
\end{aligned}
$$

Hence by two applications of Lemma 3.2, one can show that

$$
h\sum_{i=0}^{j_0+1}\left|\phi''\left(b_i-\frac{h}{2}\right)\right| \;\leq\; \int_{b-\frac{h}{2}}^{b+j_0h+\frac{h}{2}}|\phi''(x)|dx+4\max\{|\phi''(x)|: b-\frac{h}{2}\leq x\leq b+j_0h+\frac{h}{2}\}.
$$

Next consider the case where $0\in[b-\frac{h}{2}, b+\frac{h}{2})$. Then, by Taylor's expansion,

$$
\left|h\phi(b)-\int_{b-\frac{h}{2}}^{b-\frac{h}{2}}\phi(x)dx\right|\leq h^3|\phi''(0)|/24.
$$

Now using similar arguments for the case '$\sqrt{3}\in(b-\frac{h}{2}, b+(j_0+\frac{1}{2})h)\neq\emptyset$' and using the above bounds, one can complete the proof of the lemma.

**Proof of Theorem 2.1:** Suppose that (2.5) holds. Fix $\epsilon\in(0,1)$. By Chebyshev's inequality, for all $r\in\mathbf{N}$,

$$
P\left(\left|\frac{X_r-n_rp_r}{\sigma_r}\right|>\frac{2}{\epsilon}\right)\;\leq\;\frac{\epsilon^2}{4}. \tag{3.25}
$$

By Lemmas 3.1 and 3.3, for any $r\in\mathbf{N}$ with $f_r\leq\frac{1}{2}$,

$$
\begin{aligned}
\Delta_{1r}(\epsilon) &\equiv \sup_{-\frac{2}{\epsilon}\leq a<b\leq\frac{2}{\epsilon}}\left|P\left(a<\frac{X_r-n_rp_r}{\sigma_r}\leq b\right)-[\Phi(b)-\Phi(a)]\right|\\
&\leq \sum_{-\frac{2\sigma_r}{\epsilon}<k-n_rp_r\leq\frac{2\sigma_r}{\epsilon}}\left|P(k;n_r,M_r,N_r)-\frac{1}{\sigma_r}\phi\left(\frac{k-n_rp_r}{\sigma_r}\right)\right|\\
&\quad+\sum_{-\frac{2}{\epsilon}\leq a<b\leq\frac{2}{\epsilon}}\left|\sum_{a\sigma_r<k-n_rp_r\leq b\sigma_r}\frac{1}{\sigma_r}\phi\left(\frac{k-n_rp_r}{\sigma_r}\right)-[\Phi(b)-\Phi(a)]\right|\\
&\leq \frac{C}{\sigma_r^2}\sum_{-\frac{2\sigma_r}{\epsilon}<k-n_rp_r\leq\frac{2\sigma_r}{\epsilon}}\exp\left(\frac{C}{\sigma_r}\right)\exp\left(-\frac{(k-n_rp_r)^2}{\sigma_r^2}\left[\frac{1}{2}-\frac{C}{\sigma_r}\right]\right)\\
&\quad+\frac{C}{\sigma_r^2}\left[\int_{-\infty}^{\infty}|\phi''(x)|dx+1\right]+\frac{2}{\sqrt{2\pi}\sigma_r}\\
&\leq \frac{C}{\sigma_r}\left[\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{4}\right)dx+1\right],
\end{aligned}
$$

provided $\frac{C}{\sigma_r}<\frac{1}{4}$. Hence, there exists an $r_0\in\mathbf{N}$ such that for all $r\geq r_0$ with $f_r\leq\frac{1}{2}$

$$
\Delta_{1r}(\epsilon)<\frac{\epsilon}{4}.
$$

Also by Mill's ratio, $\Phi(-\frac{2}{\epsilon})+1-\Phi(\frac{2}{\epsilon})<\epsilon\phi(\frac{2}{\epsilon})$. Hence, using (3.25) and the above inequalities, it can be shown that for all $r\geq r_0$ with $f_r\leq\frac{1}{2}$,

$$
\Delta_r(\epsilon)<\epsilon. \tag{3.26}
$$

Next suppose that $f_r > \frac{1}{2}$. Consider the collection of $N_r - n_r$ objects that are left after the sample of size $n_r$ has been selected from the population of size $N_r$. Let $Y_r$ =the number of 'type A'-objects in this collection. Then, for all $r \in \mathbf{N}$ and $j \in \mathbf{Z}$,

$$Y_r \sim Hyp(N_r - n_r; M_r, N_r), \quad \text{and} \quad P(X_r = j) = P(Y_r = M_r - j). \tag{3.27}$$

Hence,

$$P(X_r \le k) = \sum_{j=0}^{k} P(X_r = j) = \sum_{j=0}^{k} P(Y_r = M_r - j) = P(Y_r \ge M_r - k).$$

Further, note that $Var(Y_r) = (N_r - n_r)p_r q_r \left(1 - \frac{N_r - n_r}{N_r}\right) = \sigma_r^2$. Hence, for each $x \in \mathbf{R}$,

$$\begin{aligned} P\left(\frac{X_r - n_r p_r}{\sigma_r} \le x\right) &= P\left(X_r \le n_r p_r + x\sigma_r\right) \\ &= P(X_r \le \lfloor n_r p_r + x\sigma_r \rfloor) \\ &= P(Y_r \ge M_r - \lfloor n_r p_r + x\sigma_r \rfloor) \\ &= P\left(\frac{Y_r - (N_r - n_r)p_r}{\sigma_r} \ge \frac{M_r - \lfloor n_r p_r + x\sigma_r \rfloor - (N_r - n_r)p_r}{\sigma_r}\right) \\ &= P(\tilde{Y}_r \ge \check{x}_r) \qquad \text{(say)}, \end{aligned}$$

where $\tilde{Y}_r = \frac{Y_r - (N_r - n_r)p_r}{\sigma_r}$ and $\check{x}_r = \frac{M_r - \lfloor n_r p_r + x\sigma_r \rfloor - (N_r - n_r)p_r}{\sigma_r}$. Note that,

$$\check{x}_r < \frac{1}{\sigma_r}\left[N_r p_r - (n_r p_r + x\sigma_r - 1) - N_r p_r + n_r p_r\right] = -x + \sigma_r^{-1}$$

and similarly, $\check{x}_r \ge -x$. Hence, this implies,

$$P(\tilde{Y}_r < \check{x}_r) \le P(\tilde{Y}_r \le \check{x}_r) \le P(\tilde{Y}_r \le -x + \sigma_r^{-1})$$

and

$$P(\tilde{Y}_r < \check{x}_r) \ge P(\tilde{Y}_r < -x) \ge P(\tilde{Y}_r \le -x - \sigma_r^{-1}).$$

Now using the above identity and inequalities, we have

$$\begin{aligned} &\left| P\left(\frac{X_r - n_r p_r}{\sigma_r} \le x\right) - \Phi(x) \right| = |P(\tilde{Y}_r \ge \check{x}_r) - (1 - \Phi(-x))| = |\Phi(-x) - P(\tilde{Y}_r < \check{x}_r)| \\ &\le \max\{|P(\tilde{Y}_r \le -x - \sigma_r^{-1}) - \Phi(-x - \sigma_r^{-1})|, |P(\tilde{Y}_r \le -x + \sigma_r^{-1}) - \Phi(-x + \sigma_r^{-1})|\} \\ &\quad + \max\{|\Phi(-x) - \Phi(-x - \sigma_r^{-1})|, |\Phi(-x) - \Phi(-x + \sigma_r^{-1})|\}. \end{aligned} \tag{3.28}$$

By repeating the arguments leading to (3.26), it follows that there exists $r_1 \in \mathbf{N}$ such that for all $r \ge r_1$ with $(1 - f_r) \le \frac{1}{2}$,

$$\sup_{x \in \mathbf{R}} |P(\tilde{Y}_r \le x) - \Phi(x)| \le \epsilon. \tag{3.29}$$

Hence, (2.4) now follows from (2.5),(3.26),(3.28) and (3.29), with $W \sim N(0,1)$. In particular, if (2.5) holds, then one must have $\mu = 0$ and $\sigma = 1$.

Conversely, suppose that (2.4) holds for some $\mu \in \mathbf{R}$ and $\sigma \in (0, \infty)$. Then, for any sequences $\{a_r\}_{r\geq 1}$,$\{b_r\}_{r\geq 1} \subset \mathbf{R}$ with $a_r < b_r$ for all $r \geq 1$,

$$\left| P\left(a_r < \frac{X_r - n_r p_r}{\sigma_r} \leq b_r\right) - P(a_r < W \leq b_r)\right| \leq 2\Delta_r \to 0 \quad \text{as} \quad r \to \infty. \tag{3.30}$$

If possible, suppose that $\sigma_r < 1$ infinitely often. Then, we can pick $a_r, b_r \in [-1, 1]$ such that for all such $r$, $a_r - b_r = 1$ and

$$\frac{\lfloor n_r p_r \rfloor - n_r p_r}{\sigma_r} < a_r < b_r < \frac{\lfloor n_r p_r \rfloor + 1 - n_r p_r}{\sigma_r}.$$

Then,

$$P\left(a_r < \frac{X_r - n_r p_r}{\sigma_r} \leq b_r\right) = 0$$

but

$$P\left(a_r < W \leq b_r\right) \geq \inf\{P(a < W \leq b) : a, b \in [-1, 1], b - a = 1\} > 0,$$

infinitely often. This contradicts (3.30). Hence, we may suppose that $\sigma_r \geq 1$ for all but finitely many $r$'s.

Now define $a_r = \frac{\lfloor n_r p_r \rfloor - n_r p_r + \frac{1}{3}}{\sigma_r}$ and $b_r = \frac{\lfloor n_r p_r \rfloor - n_r p_r + \frac{2}{3}}{\sigma_r}$. Since $P(X_r \in \{0, 1, \ldots, n_r\}) = 1$,

$$P\left(a_r < \frac{X_r - n_r p_r}{\sigma_r} \leq b_r\right) = P\left(\lfloor n_r p_r \rfloor + \frac{1}{3} < X_r \leq \lfloor n_r p_r \rfloor + \frac{2}{3}\right) = 0.$$

Next using the definitions of $a_r$, $b_r$, and the fact that '$x - 1 < \lfloor x \rfloor \leq x$ for all $x \in \mathbf{R}$', we get

$$-\frac{2}{3\sigma_r} < a_r < b_r \leq \frac{2}{3\sigma_r}, \quad r \geq 1. \tag{3.31}$$

By (3.30) and (3.31), it follows that

$$\begin{aligned}
\frac{1}{3\sigma_r} \min\{\phi_\sigma(x - \mu) : |x| \leq \frac{2}{3\sigma_r}\} &\leq \int_{a_r}^{b_r} \phi_\sigma(x - \mu) dx \\
&= P(a_r < W \leq b_r) \\
&= \left| P\left(a_r < \frac{X_r - n_r p_r}{\sigma_r} \leq b_r\right) - P(a_r < W \leq b_r)\right| \\
&\longrightarrow 0 \quad \text{as} \quad r \to \infty.
\end{aligned}$$

As a result, $\sigma_r \to \infty$ as $r \to \infty$ and (2.5) holds. This completes the proof of the theorem.

*To ensure economy of space, we shall first give a proof of Theorem 2.3 and then outline the main steps in the proof of Theorem 2.2.*

**Proof of Theorem 2.3:** Let $r \in \mathbf{N}$ be an integer such that (2.13) holds. Since $r$ will be held *fixed* all through the proof, we shall drop $r$ from the notation for simplicity, and write $f_r = f$,

$\sigma_r = \sigma$, $p_r = p$, $q_r = q$, $n_r - n$, etc. First, suppose that $f \le \frac{1}{2}$. Consider the case $x \le 0$. Let $\tilde{x}_k = \frac{x_k}{\sqrt{1-f}} = \frac{k-np}{\sigma}$, $k = 0, 1, \ldots, n$. Define

$$\begin{aligned} K_0 &= \sup\{k \in \mathbf{Z}_+ : \tilde{x}_k \le 0\} \\ K_1 &= \sup\{k \in \mathbf{Z}_+ : \tilde{x}_k \ge -1\} \\ K_2 &= \sup\{k \in \mathbf{Z}_+ : \tilde{x}_k \ge -\delta\sigma\} \quad \text{and} \\ J_x &= \lfloor np + x\sigma \rfloor, x \in \mathbf{R}, \end{aligned}$$

where $\delta \equiv \delta_r \in (0, \frac{1}{2}]$ is as in (2.12). Note that by definition,

$$\begin{aligned} &K_1 - 1 < np - \sigma \le K_1, \qquad K_2 - 1 < np - \delta\sigma^2 \le K_2, \\ &\tilde{x}_j \in [-1, 0] \quad \text{for all} \quad K_1 \le j \le K_0 \qquad \text{and} \quad \tilde{x}_j \in [-\delta\sigma, -1) \quad \text{for all} \quad K_2 \le j < K_1. \end{aligned}$$

Hence, for any $x \in [-\delta\sigma, 0]$,

$$\begin{aligned} &\left| P\left(\frac{X - np}{\sigma} \le x\right) - \Phi(x)\right| = |P(X \le J_x) - \Phi(x)| \\ \le\ & P(X < K_2) + \sum_{j=K_2}^{J_x} \left| P(X = j) - \frac{\phi(\tilde{x}_j)}{\sigma}\right| + \left| \sum_{j=K_2}^{J_x} \frac{\phi(\tilde{x}_j)}{\sigma} - \Phi(x)\right| \\ =\ & I_1 + I_2 + I_3, \qquad \text{say.} \end{aligned} \tag{3.32}$$

Consider $I_2$ for $x \in [-\delta\sigma, -1)$. Note that for $x < -1$, $\frac{J_x - np}{\sigma} \le x < -1$. Hence $J_x < K_1$ and $\tilde{x}_j < -1$ for all $j < J_x$. From Lemma 3.1,

$$\begin{aligned} |r^*(j)| &\le \frac{1}{6\sigma^2(1-\delta)} + \left[\frac{|\tilde{x}_j|^2}{2\sigma} + \frac{|\tilde{x}_j|^2}{\sigma^2}\left\{\frac{1}{4} + \frac{2\delta}{(1-\delta)^3}\right\} + \frac{|\tilde{x}_j|^3}{2\sigma} A\right] \\ &\equiv r^{**}(j), \end{aligned} \tag{3.33}$$

where $A = a_1\left(1 + \frac{4(1+\delta)}{(1-\delta)^3}\right)$ and $a_1 \equiv a_{1r} = \frac{f+4}{4(1-f)}$ (cf. (2.12)). For the given choice of $\delta$, it is easy to verify that $\delta \le \frac{1}{20}$ and $\delta A < .59$. Hence

$$\begin{aligned} |r^*(j)| &\le (0.2)\sigma^{-2} + \frac{\tilde{x}_j^2}{2}\left[\frac{1}{\sigma} + \frac{2}{\sigma^2}(0.3667) + \delta A\right] \\ &\le (0.2)\sigma^{-2} + \frac{\tilde{x}_j^2}{2}\left[\min\{0.86, \frac{6}{5\sigma} + 0.59\}\right]. \end{aligned} \tag{3.34}$$

Now, from (3.33), for all $K_2 \le j < K_1$,

$$|r^*(j)| \le (0.2)\sigma^{-2} + |\tilde{x}_j|^3\left| \left[\frac{1}{2\sigma} + \frac{1}{\sigma^2}(0.3667) + \frac{3a_1}{\sigma}\right] \le 4|\tilde{x}_j|^3 \frac{a_1}{\sigma}. \right. \tag{3.35}$$

Next note that

$$\frac{J_x - np}{\sigma} \le x \in \mathbf{R},$$

$$\int_a^\infty y^3 \exp(-\frac{by^2}{2})dy = \frac{1}{2b^2}(1 + ba^2)e^{-ba^2} \quad \text{for all} \quad a, b \in (0, \infty),$$

and that for any $a \in (0,\infty)$, the function $g(y;a) = y^3\exp(-ay)$, $y \in [0,\infty)$, is increasing on $[0,\sqrt{\frac{3}{2a}}]$, and decreasing on $(\sqrt{\frac{3}{2a}},\infty)$. Hence, by Lemmas 3.1 and 3.2, (3.34) and (3.35), with $c = .07$, we have

$$\begin{aligned}
I_2 &\le \sum_{j=K_2}^{J_x} \left|\frac{\phi(\tilde{x}_j)}{\sigma}\exp(r^*(j)) - \frac{\phi(\tilde{x}_j)}{\sigma}\right| \\
&\le \frac{1}{\sigma}\sum_{j=K_2}^{J_x} \phi(\tilde{x}_j)|r^*(j)|\exp(|r^*(j)|) \\
&\le \frac{4a_1}{\sqrt{2\pi}\sigma^2}\exp(\sigma^{-2})\sum_{j=K_2}^{J_x}|\tilde{x}_j|^3\exp(-c\tilde{x}_j^2) \\
&\le \frac{4a_1\exp(\sigma^{-2})}{\sqrt{2\pi}\sigma}\Big[\int_{\frac{K_2-np}{\sigma}}^{\frac{J_x-np}{\sigma}}|y|^3\exp(-c|y|)dy \\
&\qquad + \frac{2}{\sigma}\max\{|y|^3\exp(-c|y|) : K_2 \le np+\sigma y \le J_x\}\Big] \\
&\le \frac{C}{\sigma(1-f)}\left[(1+x^2)\exp(-cx^2)\right]. \qquad (3.36)
\end{aligned}$$

Also, for $-1 \le x \le 0$, by Lemma 3.1,

$$\begin{aligned}
\Delta_1(x) &\equiv \left|P\left(-1 \le \frac{X-np}{\sigma} \le x\right) - \sum_{j=K_1}^{K_0}\frac{1}{\sigma}\phi(\tilde{x}_j)\right| \\
&\le \sum_{j=K_1}^{K_0}\left|P(X=j) - \frac{1}{\sigma}\phi(\tilde{x}_j)\right| \\
&\le \sum_{j=K_1}^{K_0}\exp\left(-\frac{\tilde{x}_j^2}{2}\right)|r^*(j)|\frac{\exp(|r^*(j)|)}{\sqrt{2\pi}\sigma}.
\end{aligned}$$

For $K_1 \le j \le K_0$, from (3.33) and (3.34),

$$\begin{aligned}
|r^*(j)| &\le \left[\frac{1}{2\sigma}|\tilde{x}_j| + r^{**}(j)\right] \wedge \left[\frac{1}{5\sigma^2} + \frac{1}{2\sigma} + \frac{1}{2\sigma^2}(0.3667) + \frac{A}{2\sigma}\right] \\
&\le \left[\frac{1}{2\sigma} + \frac{1}{5\sigma^2} + (0.43)\tilde{x}_j^2\right] \wedge \left[\frac{1}{2\sigma} + \frac{1}{5\sigma^2} + \frac{0.3667}{\sigma^2} + \frac{3a_1}{\sigma}\right] \\
&\le \frac{1}{\sigma} + \left[(.43)\tilde{x}_j^2\right] \wedge \left[\frac{4a_1}{\sigma}\right].
\end{aligned}$$

Hence, for $-1 \le x \le 0$, noting that $K_0 - K_1 \le \sigma$,

$$\begin{aligned}
|\Delta_1(x)| &\le \sum_{j=K_1}^{K_0}\exp(-\tilde{x}_j^2(0.07))\exp(\sigma^{-1})\frac{5a_1}{\sqrt{2\pi}\sigma^2} \\
&\le (K_0 - K_1)\exp(\sigma^{-1})\frac{5a_1}{\sqrt{2\pi}\sigma^2} \\
&\le \frac{C}{\sigma}. \qquad (3.37)
\end{aligned}$$

Thus, the bound (3.36) on $I_2$ holds for all $x \in [-\delta\sigma, 0]$.

Next consider $I_1$. Note that for $j \in \{0, 1, \ldots, n\}$,

$$\begin{aligned} & P(X = j+1) >=< P(X = j) \\ \Leftrightarrow & \frac{Np - j}{j+1} . \frac{n-j}{Nq - n + j + 1} > \quad = \quad < 1 \\ & \Leftrightarrow j <=> np - \frac{Nq+1}{N+2} . \end{aligned} \tag{3.38}$$

Thus, $P(X = j) < P(X = j+1)$ for all $0 \le j \le np - 1$. Hence, by (3.34) and Lemma 3.1,

$$\begin{aligned} I_1 &= \sum_{j=0}^{K_2 - 1} P(X = j) \\ &< K_2 P(X = K_2) \\ &\le K_2 \frac{1}{\sigma} \phi\left(\tilde{x}_{K_2}\right) \exp(r^*(K_2)) \\ &\le \frac{K_2}{\sqrt{2\pi}\sigma} \exp\left(\frac{1}{5\sigma^2}\right) \exp(-\tilde{x}_{K_2}^2(.07)) \\ &\le \frac{K_2}{\sqrt{2\pi}\sigma} \exp\left(\frac{1}{5\sigma^2}\right) \exp\left(-\left(\delta\sigma - \frac{1}{\sigma}\right)^2 (0.07)\right) \\ &\le \frac{K_2}{\sqrt{2\pi}\sigma} \exp(-\delta^2\sigma^2(0.07) + 2\delta(0.07) + 0.13\sigma^{-2}) \\ &\le \frac{np}{\sqrt{2\pi}\sigma} \exp(-\delta^2\sigma^2(0.07)) \exp(0.014) \\ &\le (q(1-f))^{-1} \sigma \exp(-\delta^2\sigma^2(0.07)). \end{aligned}$$

It is easy to check that,

$$\frac{\sigma \exp(-\delta^2\sigma^2(0.07))}{(1+x^2)\exp(-x^2(0.07))} \le \begin{cases} \frac{2}{(0.07)\delta^2\sigma} & : \text{ if } \; x \in [0, \frac{\delta\sigma}{\sqrt{2}}], \\ \frac{2}{\delta^2\sigma} & : \text{ if } \; x \in [\frac{\delta\sigma}{\sqrt{2}}, \delta\sigma]. \end{cases}$$

Hence, it follows that for all $x \in [-\delta a, 0]$,

$$I_1 \le \frac{C}{\delta^2 q\sigma(1-f)}(1+x^2)\exp(-x^2(0.07)). \tag{3.39}$$

Next note that by definition, $\tilde{x}_{J_x} \le x$ and $\tilde{x}_{K_2} \le -\delta\sigma + \sigma^{-1}$. Hence, for $x \in [-\delta\sigma, 0]$, by Lemma 3.3,

$$\begin{aligned} I_3 \;&\le\; \left|\frac{1}{\sigma}\sum_{j=K_2}^{J_x}\phi(\tilde{x}_j) - \int_{\tilde{x}_{K_2}-(2\sigma)^{-1}}^{\tilde{x}_{J_x}+(2\sigma)^{-1}}\phi(y)dy\right| + \left|\Phi(x) - \Phi\left(\tilde{x}_{J_x} + (2\sigma)^{-1}\right)\right| + \Phi\left(\tilde{x}_{K_2} - (2\sigma)^{-1}\right) \\ &\le\; \frac{1}{12\sigma^2}\left[\int_{-\infty}^{x+\frac{1}{2\sigma}} |\phi''(y)|dy + 5\max\{|\phi''(y)| : -\infty < y < x + \frac{1}{2\sigma}\}\right] \\ &\qquad + \Phi\left(x + \frac{1}{2\sigma}\right) - \Phi\left(x - \frac{1}{2\sigma}\right) + \Phi(-\delta\sigma + \frac{1}{2\sigma}). \end{aligned}$$

Note that for any $a \in (0, \infty)$,

$$\int_a^\infty y^2 e^{\left(-\frac{y^2}{2}\right)} dy \le \frac{1}{a}\int_a^\infty y^3 e^{\left(-\frac{y^2}{2}\right)} dy = \frac{2}{a}\int_{\frac{a^2}{2}}^\infty t e^{-t} dt = \frac{a^2+2}{a} e^{-\frac{a^2}{2}};$$

$$\int_a^\infty y^2 e^{-\frac{y^2}{2}}\,dy \le \int_0^\infty y^2 e^{-\frac{y^2}{2}}\,dy \le \sqrt{\frac{\pi}{2}};$$

$$\max\{|\phi^{''}(y)| : a < y < \infty\} \le \frac{1}{\sqrt{2\pi}} I(0 < a < \sqrt{3}) + |\phi^{''}(a)| I(a \ge \sqrt{3});$$

$$\exp\left(-\frac{\left(a-(2\sigma)^{-1}\right)^2}{2}\right) \le \exp\left(-\frac{a^2}{2}+\frac{a}{2\sigma}\right) \le \exp\left(-\frac{a^2}{2}+\frac{\delta}{2}\right), \quad \text{for all} \quad a \in (0, \delta\sigma).$$

Also note that, for $0 < a \le 1$, $b \in (0, \infty)$,

$$1 - \Phi(b) \le \frac{1}{b}\phi(b),$$

$$1 - \Phi(a) \le \int_a^1 \phi(x)dx + \phi(1) \le \phi(a)(1-a) + \phi(a) = (2-a)\phi(a).$$

Thus, for any $x \in (0, \infty)$,

$$\Phi(x) \le e^{-\frac{x^2}{2}}.$$

Since $(2\sigma)^{-1} < \frac{1}{8}$ and $|y + (2\sigma)^{-1}| \le |y|$ for $y < -\frac{1}{8}$, we have, for all $x \in [-\delta a, 0]$,

$$\begin{aligned}
I_3 \;\le\;& \frac{1}{12\sigma^2}\Big[2I(-2 \le x \le 0) + 5|x|\phi(x + (2\sigma)^{-1})I(-\delta\sigma \le x \le -2) \\
&+ 5\left\{\frac{1}{\sqrt{2\pi}}I(-2 \le x \le 0) + (x^2+1)\phi(x+(2\sigma)^{-1})I(-\delta\sigma \le x \le -2)\right\}\Big] \\
&+ \frac{1}{\sqrt{2\pi}\sigma}I(-2 \le x \le 0) + \frac{1}{\sigma}\phi\left(x + (2\sigma)^{-1}\right)I(-\delta a \le x < -2) + \Phi(-\delta\sigma + (2\sigma)^{-1}) \\
\le\;& \frac{1}{2\sigma}I(-2 \le x \le 0) + 2\left\{\frac{x^2+1}{2\sigma^2} + \frac{1}{\sigma}\right\}\phi\left(x + \frac{1}{2\sigma}\right)I(-\delta\sigma \le x \le -2) + \exp\left(-\frac{\left(\delta\sigma - \frac{1}{2\sigma}\right)^2}{2}\right) \\
\le\;& \frac{C}{\sigma}(1+|x|)\exp\left(-\frac{x^2}{2}\right). \qquad (3.40)
\end{aligned}$$

Next note that

$$P\left(\frac{X-np}{\sigma} \le x\right) = 0 \quad \text{for all} \quad x < -\frac{np}{\sigma}$$

and for $-\frac{np}{\sigma} \le x \le -\delta\sigma$,

$$\begin{aligned}
P\left(\frac{X-np}{\sigma} \le x\right) \le I_1 \;&\le\; (q(1-f))^{-1}\sigma\exp(-\delta^2\sigma^2(0..07)) \\
&=\; (q(1-f))^{-1}\frac{(\delta\sigma)^2}{\delta^2\sigma}\exp\Big(-\delta^2q^2(1-f)^2\Big[\frac{-np}{\sigma}\Big]^2(0.07)\Big) \\
&\le\; (\delta^2 q(1-f)\sigma)^{-1}|x|^2\exp\Big(-\delta^2q^2(1-f)^2x^2(0.07)\Big).
\end{aligned}$$

Hence, for all $x \le -\delta\sigma$,

$$\begin{aligned}
\left|P\left(\frac{X-np}{\sigma} \le x\right) - \Phi(x)\right| \;&\le\; \frac{|x|^2\exp(-\delta^2q^2(1-f)^2x^2(0.07)) + \exp\left(-\frac{x^2}{2}\right)}{\delta q(1-f)\sigma} \\
&\le\; \frac{2}{\delta q(1-f)\sigma}x^2\exp\Big(-\delta^2q^2(1-f)^2x^2(0.07)\Big). \qquad (3.41)
\end{aligned}$$

Now using the fact that $\delta \in \left[\frac{1}{25}, \frac{1}{20}\right]$ for all $f \in (0, \frac{1}{2}]$, from (3.36),(3.37) and (3.39)-(3.41), it follows that there exist numerical constants $C_1$ and $C_2$, not depending on $n, M, N$, such that for all $x \in (-\infty, 0]$,

$$\left|P\left(\frac{X - np}{\sigma} \leq x\right) - \Phi(x)\right| \leq \frac{C_1}{\sigma q}(1 + x^2)\exp(-C_2 q x^2),$$

provided $\delta\sigma > 1$. This proves (2.14) for $x \in (-\infty, 0]$ and $f \leq \frac{1}{2}$.

To prove the theorem for $x \geq 0$ and $f \leq \frac{1}{2}$, define

$$V_r = n_r - X_r, \quad r \in \mathbf{N}.$$

Note that $V_r$ has a Hypergeometric distribution with parameters $n_r, N_r - M_r, N_r$. Further,

$$\frac{X_r - n_r p_r}{\sigma_r} = -\frac{V_r - n_r q_r}{\sigma_r} \quad \text{for all} \quad r \in \mathbf{N}.$$

Hence, the derived bound on the right tails of $\frac{X_r - n_r p_r}{\sigma_r}$, can be obtained by repeating the arguments above with $X_r$ replaced by $V_r$ and $p_r$ replaced by $q_r$ for any $r$ such that $\delta\sigma_r > 1$. This proves (2.14) for $x \in [0, \infty)$ and $f \leq \frac{1}{2}$. The proof of (2.14) for '$f \in [\frac{1}{2}, 1]$ and $x \in \mathbf{R}$' follows by replacing the above arguments with $X_r, f_r$ replaced by $Y_r, 1 - f_r$ respectively and using the bound (3.27) and (3.28). This completes the prrof of the theorem.

**Proof of Theorem 2.2:** As in the proof of Theorem 2.3, first we suppose that $f_r \leq \frac{1}{2}$. By (3.1), (3.32), (3.36), (3.37), and (3.40), it follows that for all $r$ with $\delta_r\sigma_r > 1$,

$$\begin{aligned} \sup_{x \in [-\delta_r\sigma_r, 0]} |\Delta_r^*(x)| &\leq P(X_r < K_2) + \sup_{x \in [-\delta_r\sigma_r, 0]} \{I_2 + I_3\} \\ &\leq P(X_r \leq K_2 - 1) + \frac{C}{\sigma_r}. \end{aligned} \tag{3.42}$$

By Chebyshev's inequality, noting that $K_2 - 1 < n_r p_r - \delta_r \sigma_r^2 \leq K_2$, we have

$$\begin{aligned} P(X_r \leq K_2 - 1) &\leq P\left(\left|\frac{X_r - n_r p_r}{\sigma_r}\right| \geq \left|\frac{K_2 - n_r p_r - 1}{\sigma_r}\right|\right) \\ &\leq \frac{Var(X_r)}{(K_2 - 1 - n_r p_r)^2} \\ &\leq \frac{N_r \sigma_r^2}{N_r - 1}(\delta_r \sigma_r^2)^{-2} \\ &\leq \frac{2}{\delta_r^2 \sigma_r^2}. \end{aligned} \tag{3.43}$$

Also,

$$\begin{aligned} \sup_{-\infty \leq x \leq -\delta_r\sigma_r} |\Delta_r^*(x)| &\leq P(X_r \leq K_2 - 1) + \Phi(-\delta_r\sigma_r) \\ &\leq \frac{C}{\delta_r^2 \sigma_r^2}. \end{aligned} \tag{3.44}$$

Since $\delta_r \geq \frac{1}{22.5}$ for all $r$ with $f_r \leq \frac{1}{2}$, from (3.42)-(3.44), it follows that there exists a universal constant $C_3$ such that for all $r$ with $\delta_r \sigma_r > 1$ and $f_r \leq \frac{1}{2}$,

$$\sup_{x \leq 0} |\Delta_r^*(x)| \leq \frac{C_3}{\sigma_r}.$$

Now retracing the arguments in the proof of Theorem 2.3 for the case "$x \geq 0, f_r \leq \frac{1}{2}$" (with the variable $V_r$) and for the case "$x \in \mathbf{R}, f > \frac{1}{2}$" (with $Y_r$), one can complete the proof of Theorem 2.3.

**Proof of Corollary 2.4**: Use (2.14) and the inequality "$\exp(x) \geq (1 + x)$ for all $x \in (0, \infty)$".

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