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Using variability related to families of spectral estimators for mixed random processes

Abstract

Traditionally, characterization of spectral information for wide sense stationary processes has been addressed by identifying a single best spectral estimator from a given family. If one were to observe significant variability in neighboring spectral estimators then the level of confidence in the chosen estimator would naturally be lessened. Such variability naturally occurs in the case of a mixed random process, since the influence of the point spectrum in a spectral density characterization arises in the form of approximations of Dirac delta functions. In this work we investigate the nature of the variability of the point spectrum related to three families of spectral estimators: Fourier transform of the truncated unbiased correlation estimator, the truncated periodogram, and the autoregressive estimator. We show that tones are a significant source of bias and variability. This is done in the context of Dirichlet and Fejer kernels, and with respect to order rates. We offer some expressions for estimating statistical and arithmetic variability. Finally, we include an example concerning helicopter vibration. These results are especially pertinent to mechanical systems settings wherein harmonic content is prevalent.

Disciplines

Aerospace Engineering

Using Variability Related to Families of Spectral Estimators for Mixed Random Processes

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Traditionally, characterization of spectral information for wide sense stationary processes has been addressed by identifying a single best spectral estimator from a given family. If one were to observe significant variability in neighboring spectral estimators then the level of confidence in the chosen estimator would naturally be lessened. Such variability naturally occurs in the case of a mixed random process, since the influence of the point spectrum in a spectral density characterization arises in the form of approximations of Dirac delta functions. In this work we investigate the nature of the variability of the point spectrum related to three families of spectral estimators: Fourier transform of the truncated unbiased correlation estimator, the truncated periodogram, and the autoregressive estimator. We show that tones are a significant source of bias and variability. This is done in the context of Dirichlet and Fejer kernels, and with respect to order rates. We offer some expressions for estimating statistical and arithmetic variability. Finally, we include an example concerning helicopter vibration. These results are especially pertinent to mechanical systems settings wherein harmonic content is prevalent.
[DOI: 10.1115/1.1409257]

Keywords: Mixed Spectrum, Autoregression, Periodogram, Statistics, Harmonics

1 Introduction

Spectral estimation has played a major role in a wide variety of theoretical and application areas of science and engineering since the advent of modern computing and the FFT in the mid-1960s. Traditionally, the idea has been to find the "best" spectral estimator. Often the desire was to balance resolution and variability. This is the idea behind both averaging of periodograms and autoregressive (AR) order selection methods. Perhaps because of the limitations and expense of computing resources in the early years it was natural to rely on such selection methods. But it is also natural to question this entire approach if the variability within the family of spectra under consideration is significant. It may well be that neighboring spectra exhibit measurable variability at certain frequencies, while not at others. In fact, this is exactly the case at and near frequencies corresponding to the point spectrum, when the random process includes a deterministic as well as regular component. Sinusoids are the most common source of point spectrum. Given an infinite number of correlation lags, they would appear as Dirac delta functions. But if the spectral family is indexed by the number of correlation lags used, as the case in periodogram, AR and other methods, then the influence of the point spectrum will be seen as peaks whose values are, in and of themselves meaningless, and as spectral leakage.

With the advances in computing resources it is now far easier to both compute and analyze a large family of spectral estimators than it was even 15 years ago. Even so, this family-based approach to statistically reliable spectral estimation has received very little attention; in spite of the fact that it has been suggested for over 15–20 years now. For example, in [1] and [2] the use of periodograms with successively larger windows is proposed. The idea is that if the spectral information remains insensitive to the window size changes then one can have greater confidence in it. This work is intended to contribute a better understanding of the variability of spectral estimator families, with particular attention

to the cases that caused by the presence of point spectrum. Specifically, we address three families: the truncated Fourier transform, the averaged periodogram, and the AR spectra. This variability will be addressed in two stages. In Section 3 we will investigate the variability of these families when the autocorrelation information is exact. This will reflect the order-dependent theoretical spectral variability. It is also valuable in its own right, since there are many applications where the amount of available data far exceeds the range of reliable correlation lags that one might consider. In Section 4 we address the statistical variability associated with lagged-product estimates of the correlation information. The value of the sample mean and corresponding variance is one way of using a family of spectra, as suggested in [1] and [2]. The value of this information is the subject of Section 5. In Section 6 we apply the results of the previous sections to vibration data from a helicopter drive train. Our summary and conclusions are given in Section 7. We now proceed to motivate our investigation and describe the types of processes we are concerned with.

2 The Structure of PSD Estimators for Mixed Spectrum Processes

We consider wide sense stationary (*wss*) random processes of the form

$$x(t) = s(t) + \varepsilon(t) \tag{1}$$

where the signal, $s(t)$, is composed of sinusoids with deterministic amplitudes and frequencies $\{A_k, \omega_k\}$, and with independent phases each distributed uniformly over $[0, 2\pi]$. The noise, $\varepsilon(t)$, is regular, and is assumed to have a continuous power spectral density (PSD), $S_\varepsilon(\omega)$. The PSD of (1) is given by

$$S_x(\omega) = \sum_k \frac{A_k^2}{4} \delta(\omega \pm \omega_k) + S_\varepsilon(\omega) \tag{2}$$

where $\delta(\omega)$ is the well known Dirac-δ function. Consequently, (2) is only defined in the sense of a generalized function, in that only its integral, the cumulative PSD, is well defined with jumps at the signal frequencies. The model (1) is fundamental to mechanical

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systems, such as rotating machinery. Typically, processes associated with such systems include harmonics as well as highly colored spectral components.

Let $\{R_x(\tau)\}_{-(n-1)}^{n-1}$ be the theoretical correlation information through the nth lag. Then the theoretical Fourier transform (FT) spectral estimator is given by

$$S_{FPT(n)}(\omega) = \sum_{\tau=-(n-1)}^{n-1} R_x(\tau) e^{-i\omega\tau} \tag{3}$$

We remark that in (3) and throughout the remainder of this work it is assumed that the sampling interval is 1 second, so that all frequencies are in the interval $[0, \pi)$. It is commonly assumed that (3) will converge to (2) when the number of lags, n, approaches infinity. In the absence of tones this will generally be true. But when tones are present it is *not* true, as will be shown. One solution to this problem turns out to be to use an average of FT(n) spectra for a range of values of n. In addition to solving this problem, the use of such a family offers information that the use of any single PSD cannot offer, namely spectral variability with respect to the number of lags. This is in addition to the statistical variability associated with the use of estimated correlation lags. The theoretical truncated periodogram, PER(n), is one such spectrum. Another common PSD estimator which uses the same n correlation lags is the AR(n) spectrum. The specific form of the theoretical AR(n) spectrum is well known (e.g. [3]), and so it will not be repeated here. In contrast to the FT(n) spectrum, (3), the theoretical PER(n) and AR(n) spectra converge at almost every frequency (except at the point spectrum frequencies) to the continuous spectrum as $n \to \infty$. The FT(n), and PER(n), and AR(n) theoretical spectra all exhibit order-dependent variability due to the presence of tones, and become unbounded at the tone frequencies as $n \to \infty$. The use of estimated correlation information introduces statistical variability, in addition to the arithmetic variability that we will investigate in the next section. Before doing so, however, we offer the following example to provide more motivation for our investigation of the utility of a family of spectral estimators.

Example 1. In this example we consider a process (1) consisting of a single sinusoid, plus a regular component. The theoretical PSD is given in Fig. 1. It includes the δ-function associated with the tone. The peak in the continuous spectrum was selected to simulate a strong system resonance, while the dip corresponds to an anti-resonance. This structure is commonplace in mechanical systems settings.

Assuming that a sufficiently large number of measurements is available (as the case with rotating machinery operating at constant speed) allows us to justify the use of theoretical correlation information. Figure 1 includes the $\pm 2\sigma$ arithmetic variability (dashed lines) of the family of theoretical FT(n) spectra for $n = 32, 33, \ldots, 1024$. While not immediately obvious from Fig. 1, this variability reflects the fact that all of these spectra accurately capture the spectral resonance region, while none of them capture either the antiresonance or the tone. It is well known that the spectral leakage associated with the tone is the source of local variability. But here it is also responsible for non-local bias and variability in regions where the PSD magnitude is not significant. In the region near the tone both bias and variability are meaningless, since in the PSD domain a tone is a Dirac-δ function. The practical implications of this are that the estimated amplitude will converge to infinity as $n \to \infty$, and, consequently, so will the variability of the family of estimators.

A major point of the following sections is to examine the above behavior in detail. Hopefully, this example has hinted at the potential value of using a family of spectral estimators, as opposed to a single "best" estimator, as is traditionally done. By observing the behavior of variability over increasingly larger ranges of n (termed *window closing* in [1] and [2]), it is possible to gain greater confidence of the spectral structure. For example, in the resonance region there is very little variability, so that one can
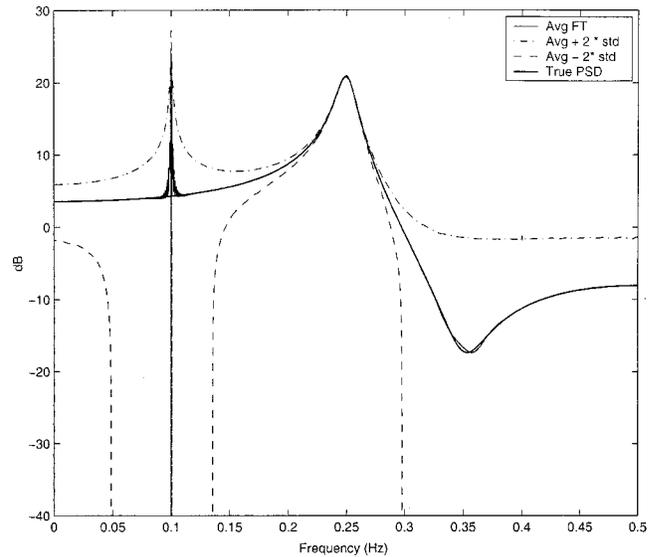
Fig. 1 Arithmetic mean and 2-σ curves corresponding to use of a family of theoretical FT(n) spectra. [Note: Where not shown, the lower 2-σ curve is $-\infty$.]

presume that this region is well characterized without concern for any order selection rules. In the region of the tone the variability range increases, suggesting that this region is not appropriate for characterization by a FT(n) spectrum for any value of n. This suggestion requires clarification. In many situations, such as developing noise and vibration specifications for mechanical systems, the window size, n, is required to be a specific value. In such situations where everyone uses the same window size, type, number of averages, etc. the FT(n), PER(n), and AR(n) PSD estimators can provide proper spectral distribution information over frequency intervals. But just as often, if not more so, the value for n is not fixed. As n grows so does the peak of any tone associated with a PSD. This behavior does not appear to have bothered many people, since it has been demonstrated routinely in most of the high resolution spectral research conducted over the past 35 years, in the context of the two-sinusoid plus white noise setting (e.g., [1]). But in the context of using a family of spectra, as suggested in [1] and [2], one would conclude that any region involving tones should be viewed as unreliable. In the realm of mechanical systems, and in particular, rotating machinery, this would adversely affect spectral analysis, as a whole. This could lead one to apply spectral decomposition tools such as [4] to eliminate this problem. But we will not address this approach in this work, since we are concerned here with the common procedure of analysis of the mixed spectrum, as it is. The FT(n) spectrum is perhaps not as popular as the PER(n) spectrum. Traditional reasons for this range from the fact that it can lead to negative PSD estimates, to the fact that the side-lobe behavior associated with the rectangular windowing operation results in excessive local spectral smearing. The above example suggests that the FT(n) family is not well-suited for accurate characterization of anti-resonance structure when tones are present *anywhere* in the spectrum.

To further motivate the following sections we offer the performance of a family of AR(n) spectra. Using a minimum number of orders $n = 5, 6, \ldots, 10$ in Fig. 2(a) produces less variability in the anti-resonance region than the FT(n) family does. Furthermore, by using orders $n = 20, \ldots, 100$ (far fewer and lower than the FT(n)) Fig. 2(b) indicates not only lack of bias, but minimal variability everywhere except at the tone region. Also, the size of that region is lessening with the use of higher lags (in contrast to the FT(n) spectra). Thus, one can conclude that for accurate anti-
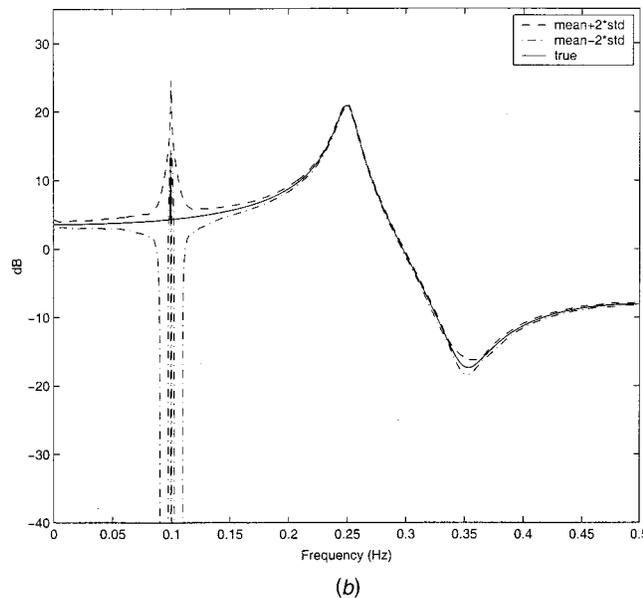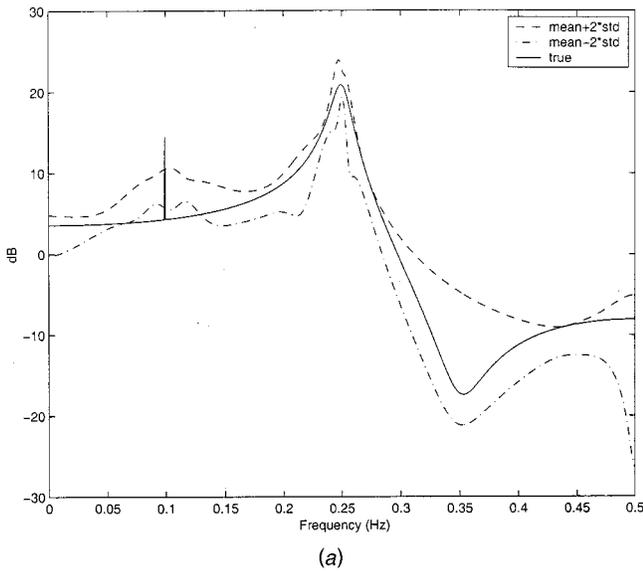
Fig. 2 (*a*) **2-σ curves for an average of 6 AR(n) spectra, n =5:1:10;** (*b*) **2-σ curves for an average of 81 AR(n) spectra, n =20:1:100.**

resonance characterization the $AR(n)$ family is more appropriate than the $FT(n)$ family. In the region near the tone the variability is much more localized than that of the $FT(n)$ family. Even so, since the $AR(n)$ family is a family of PSD estimators, at the tone frequency the amplitude will converge to infinity as $n \rightarrow \infty$, as was the case with the $FT(n)$ family. Our analysis will reveal, however, that the rate of convergence is markedly faster than that of the $FT(n)$ family. This suggests yet another use of spectral families; namely to use their convergence properties to identify tone components. We will discuss this point in greater detail.

The above example utilized theoretical correlation information. Hence, the bias and variability may be said to be arithmetic in nature, as opposed to statistical variability related to using estimates of the correlation lags. We now proceed to a more detailed discussion of the arithmetic variability associated with the theoretical $FT(n)$, $PER(n)$, and $AR(n)$ spectra, which utilize perfect correlation information. The justification for this is in the fact that often in mechanical system analysis one has access to extremely

large amounts of data, in relation to the number, n, of estimated correlation lags used for analysis. In such cases it may be reasonable to presume that the correlation information is highly reliable. We will, at times during this discussion, restrict our attention to the situation of a single tone plus white noise. The reason for this is twofold. First, the presence of a tone can have a significant effect on spectral variability. Second, by use of band pass filtering it is sometimes possible to restrict the region of interest such that within that region the noise spectrum is relatively flat.

3 Arithmetic Variability of Theoretical FT(n), PER(n), and AR(n) Spectral Families

In this section we investigate the arithmetic variability associated with the use of theoretical correlation information for three spectral families. Two of these, namely the $FT(n)$ and $AR(n)$, have been discussed above. The third is the family of truncated periodograms, which we denote as the $PER(n)$ family. This is by far the most popular family of spectral estimators in use in practically all areas of science and engineering. We will obtain quantitative expressions for both the bias and variance. These will entail order-dependent terms, which will provide growth rate information in relation to tones.

We restrict our attention here to the case of the model (1) with a single sinusoid:

$$x(t) = A \sin(\omega_o t + \theta) + \varepsilon(t). \tag{4}$$

To be sure, the two-tone problem is an important and common one. But such a setting would significantly complicate the analysis, possibly to the point of distraction from our main goal, which is to gain a better understanding of the variability of a spectral family in relation to a mixed spectrum setting. So little attention has been paid to this problem that we believe it is appropriate here to restrict our investigation to the more simple setting (4), in order to achieve our goal.

3.1 Variability of the Theoretical FT(n) Spectra. For the model (4), Eq. (3) takes the well known form (e.g. [2]):

$$S_{FT(n)}(\omega) = \frac{A^2}{4} D_n(\omega - \omega_0) + \frac{A^2}{4} D_n(\omega + \omega_0) + S_\varepsilon \otimes D_n(\omega) \tag{5}$$

where $D_n(\omega) = \sin((2n-1)\omega/2)/\sin(\omega/2)$ is the Dirichlet kernel [3] associated with the $2n-1$ point rectangular window. Now,

$$\lim_{n \rightarrow \infty} S_{FT(n)}(\omega) \neq \frac{A^2}{4} \delta(\omega \pm \omega_o) + S_\varepsilon(\omega) \tag{6}$$

Notice that the middle relation in (6) is not an equality, but rather a *nonequality*. This reflects the fact that the family of Dirichlet functions does not converge (*anywhere*) as $n \rightarrow \infty$. This is exemplified in Fig. 3. There, we see that because the Dirichlet function is proportional to n, even though the major side lobes move closer to the tone frequency for increasing n, they are also increasing in size. Consequently, we see that at frequencies sufficiently far from this frequency the Dirichlet peak values do not decrease with increasing n. This fact, while certainly not new, seems to have been ignored in the vast majority of books and papers on the subject where the spectral density is defined via the limiting Fourier transform of the correlation function, using a finite number of lags, as in (3). Even though, relatively speaking, the energy at the origin will overwhelm that in other regions, so that it may *appear* that the sequence is converging to a Dirac-δ function, it is not. In fact, at any fixed frequency other than that of the tone, the sequence of functions will neither converge nor diverge as $n \rightarrow \infty$. Rather it will oscillate with bounded variation. This is of sufficient importance when assessing variability, and is so often ignored in systems and signals publications, that we now present a formalized statement of this for the family of rectangular windows.
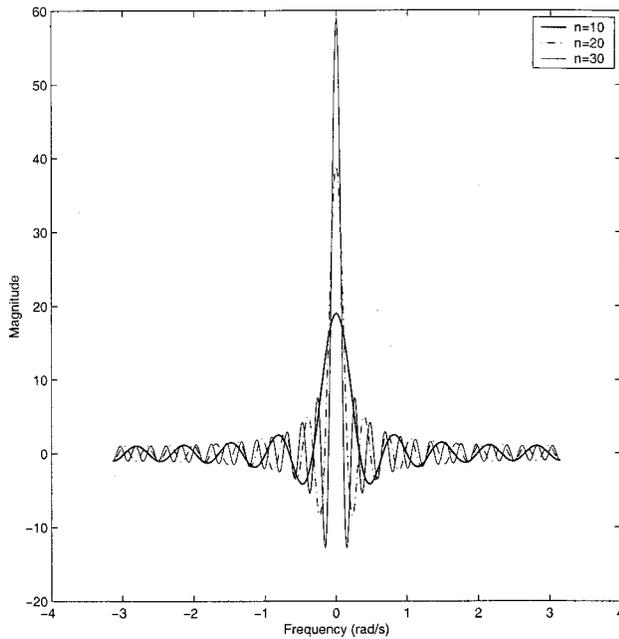
Fig. 3   Example of Dirichlet kernel for $n = 10$, 20, and 30

Fig. 4   Average of FT(n) theoretical spectra for the range 1 $\rightarrow n_{max}$ for n_{max}=100, 200 and 500

Result 1. The Fourier transform pair: $f(t) = 1 \Leftrightarrow F(\omega) = \delta(\omega)$ is not necessarily true. It depends on the selection of functions of which $f(t)$ is the pointwise limit. In particular, let $w_{(n)}(t) = 1$ for $-(n-1) \leq t \leq n-1$, and let it equal zero otherwise. Let $W_{(n)}(\omega) = M s_{\tau = -(n-1)}^{n-1} w(t) e^{i\omega t}$ be the Fourier Transform of $w_{(n)}(t)$. Then $\lim_{n \to \infty} W_n(\omega) = \lim_{n \to \infty} D_n(\omega)$ exist nowhere. Specifically, $W_{(n)}(0)$ is O(n), while for $|\omega| \geq 0$ $W_{(n)}(\omega)$ oscillates (as a function of n) between $\pm 1/\sin(\omega/2)$.

For the more mathematically inclined reader, we remark that a rigorous definition of the Dirac-δ function as the limit of a family of functions requires the use of a family which is suitably well-behaved ([5,6]). The discontinuities at the ends of the rectangular function family are well known to be not well-behaved, to the extent that they yield what is commonly known as the Gibb's phenomenon. The fact is that one can never collect an infinite amount of information, be it data or correlation information. And so the form of truncation becomes important if one desires to correctly infer the outcome, were all the information available.

Recall, that this work is concerned with families of spectra. Because of the undesirable properties associated with the collection of FT(n) spectra which utilize the expected value of the unbiased correlation estimator, it may be of interest to investigate whether averaging them can offer any advantage. For a collection of theoretical spectra $\{S_{FT(n)}(\omega)\}_{n=n_o}^{n_1}$, we now formally define the arithmetic average and variability of the collection of FT(n) spectra over the indices n_o to n_1, respectively, as

$$S_{FT\overline{(n_o,n_1)}}(\omega) = \frac{1}{n_1 - n_o + 1} \sum_{n=n_o}^{n_1} S_{FT(n)}(\omega) \qquad (7a)$$

$$\gamma_{FT(n_o,n_1)}(\omega) = \frac{1}{n_1 - n_o + 1} \sum_{n=n_o}^{n_1} [S_{FT(n)}(\omega) - S_{\overline{FT(n_o,n_1)}}(\omega)]^2 \qquad (7b)$$

It was just noted that the FT(n) spectrum has the undesirable property that the leakage influence associated with a tone will persist independent of n. This is illustrated in Fig. 4, which is related to *Example 1*. We see that not only is this family of averaged FT(n) spectra converging at the anti-resonance location, it also yields progressively localized leakage in the vicinity of the
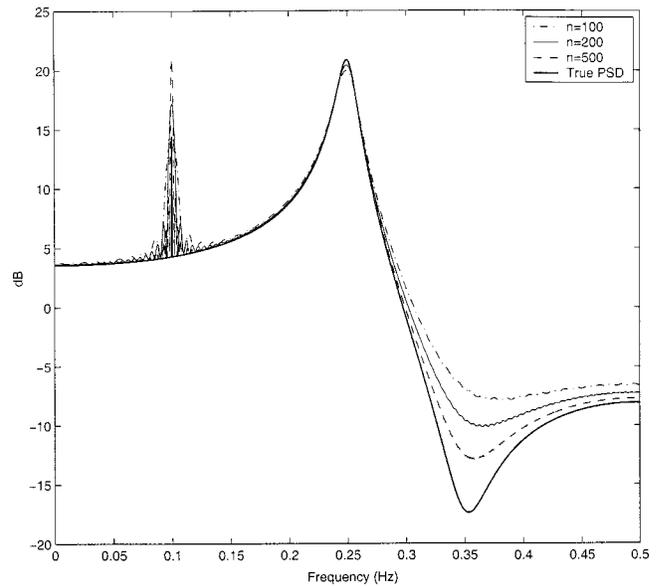
tone. Thus, as suggested previously by [1,2,4] and others, the use of a family can provide an advantage over the use of any single spectra. One could proceed to use this averaging procedure, as opposed to the use of a single FT(n) spectrum, even though it would be more computational. The following result shows that this averaging procedure may be implemented without the need to perform the averaging computation.

Result 2. ([7] p. 16). For $n_o = 1$ and $n_1 = n$ Eq. (7a) may be expressed as

$$S_{\overline{FT(n)}}(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_x(\nu) K_n(\omega - \nu) d\nu = S_{PER(n)}(\omega) \qquad (8)$$

Here, $K_n(\omega) = 1/n (\sin(n\omega/2)/\sin(\omega/2))^2$ is known as Fejer's kernel [3]. Also, $S_{PER(n)}(\omega)$ is the expected value of the popular periodogram spectral estimator, and uses the biased lagged-product correlation estimates, as opposed to the unbiased ones used in (3). To be precise, (8) is not the expected value of the periodogram unless the order, n, is identical to the data record size, N. More generally, for n less than N it is referred to as the *truncated* periodogram. For convenience we will not make such a distinction unless it is necessary. While perhaps not evident from (7a), this kernel is exactly the *average* of the collection of Dirichlet kernels from 1 to n. The fact that the rightmost equality in (8) corresponds to the Fourier transform of the theoretical correlation function that has been windowed using a *triangular*, or Bartlett window is well known. What the leftmost equality shows is that in (7a), when summation indices range from 1 to n, we obtain the expected value of the periodogram estimator. This observation is a long known result, but one that is seldom noted in most books on signals and systems. Traditionally, the Bartlett window is used to reduce the intensity of the spectral side lobes associated with leakage. Our observation suggests that it should also be used to ensure proper behavior of the family of spectra in the case of a mixed process such as (4); namely, convergence as $n \to \infty$.

Both the Fejer and Dirichlet kernels grow at a rate O(n) at a tone frequency. But in contrast to the Dirichlet family, the Fejer family converges to a Dirac-δ function; that is, the leakage away from the tone frequency goes to zero as $n \to \infty$. This is illustrated in Fig. 5(a). In the absence of any tones, if the noise PSD is continuous, then (7a), which is a Cesaro mean, will converge uniformly (in ω) to the true PSD as $n_1 \to \infty$ ([7] p. 16). Since the
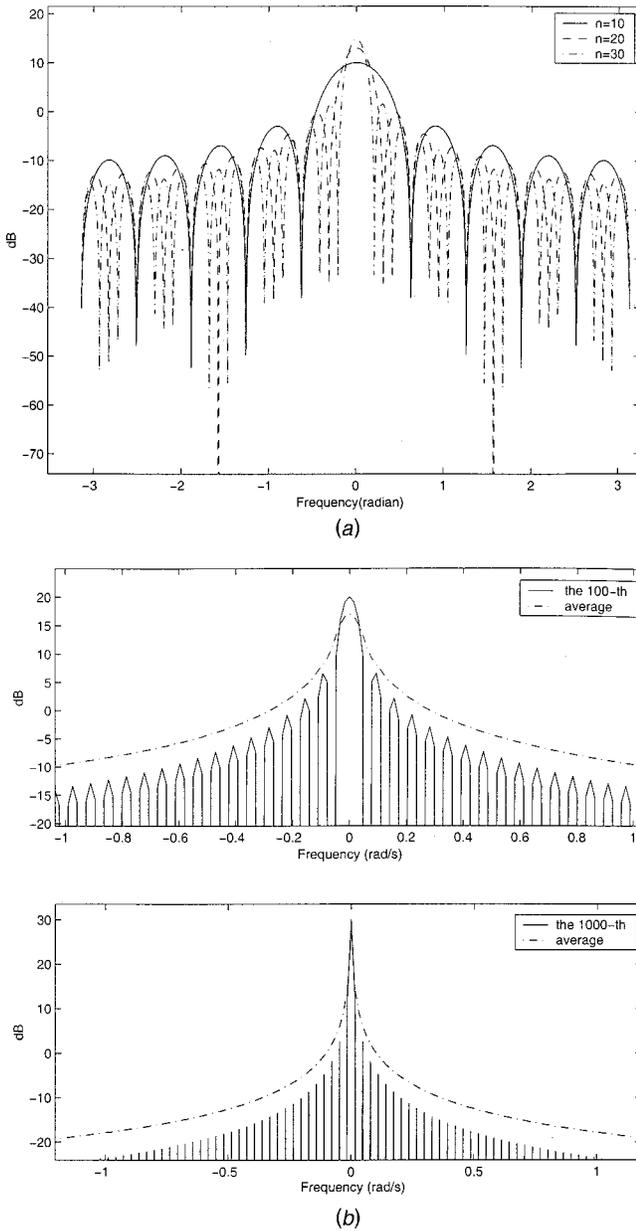
Fig. 5 (*a*) **Example of Fejer kernel for** $n=10$, **20 and 30 in dB;** (*b*) **comparison of the** n^{th} **and the average of the first n Fejer's kernels for** $n=100$ **and 1000.**

(generalized) PSD limit in (8) is absolutely integrable, it follows [7] that even though convergence will fail at the tone frequency, it will take place elsewhere.

When n is sufficiently large, the expression for the arithmetic variance, 7(*b*), of the $\{FT(n)\}$ family of theoretical spectra can be approximated as the following:

$$\gamma_{FT}(\omega) \cong \frac{A^4}{16} \cdot \frac{1}{\sin\left(\frac{\omega - \omega_o}{2}\right)^2} \quad \text{for} \quad |\omega - \omega_0| \gg 0 \quad (9a)$$

$$\gamma_{FT}(\omega_o) \simeq \frac{A^4}{16} \cdot O(n^2) \quad \text{for} \quad \omega = \omega_0 \quad (9b)$$

As mentioned above, at a tone frequency the variability of any PSD family will grow with increasing order, n, as is the case in (9*b*). The fact that the variability of the FT(n) family persists at

frequencies removed from the tone, as given in (9*a*), make that family undesirable for use in the case of a mixed process. We now proceed to a discussion of the variability of the more desirable and commonly used PER(n) family.

3.2 Variability of the Theoretical PER(n) Spectra. The theoretical PER(n) spectrum, which is exactly the expected value of the truncated periodogram, is given by

$$S_{PER(n)}(\omega) = \sum_{\tau=-(n-1)}^{n-1} R_x(\tau) B_n(\tau) e^{-i\omega\tau} \quad (10)$$

where $B_n(\tau) = (n - |\tau|)/n$ is the triangular, or Barlett window, whose Fourier transform is the nth Fejer kernel, $K_n(\omega)$. In the manner of (5), it may be expressed as

$$S_{PER(n)}(\omega) = \frac{A^2}{4} K_n(\omega - \omega_o) + \frac{A^2}{4} K_n(\omega + \omega_o) + S_\epsilon \otimes K_n(\omega) \quad (11)$$

Notice that if $\omega_0 = 2\pi k_0/n$ for any integer k_0 then the first two terms in (11) vanish at all discrete computation frequencies other than ω_0; that is, the tone spectral leakage will not distort the continuous spectrum information at those frequencies. Unfortunately, in practice one seldom has control over such precise placement of the tones in relation to the sampling frequency.

An explicit expression for the arithmetic variability of the PER(n) family can be obtained in the same manner as was done for the FT(n) family. We begin by noting that since the Fejer kernels are O(n) at the tone frequency, then so is the arithmetic average of the theoretical PER(n) spectra. And since, unlike the Dirichlet family, the Fejer family is O($1/n$) at frequencies removed from the tone frequency, then the arithmetic variability of the PER(n) family is O($1/n$). What follows is a more quantitative description of this behavior.

For a collection of theoretical spectra $\{S_{PER(k)}(\omega)\}_{k=1}^n$, we define the arithmetic average and variability of the collection of PER(n) spectra over the indices 1 to n in the usual way. The average over the theretical periodogram family is simply the average over the Fejer kernels, which for large n is approximately

$$\psi_{(n)}(\omega) = \frac{1}{n} \sum_{k=1}^n K_{(k)}(\omega) \cong \frac{1}{1 - \cos(\omega)} \left[\frac{\ln n}{n} + \frac{1}{2n^2} \right.$$

$$\left. - \frac{1}{2n} \ln\left(\frac{1}{2(1 - \cos(\omega))} \right) \right] \quad \text{for} \quad \omega \neq 0 \quad (12a)$$

$$\simeq (n+1)/2 \quad \text{for} \quad \omega = 0 \quad (12b)$$

for $\omega \in [0, 2\pi)$. Therefore the averaged PER(n) spectra is

$$S_{\overline{PER(n)}}(\omega) = \frac{A^2}{4} \psi_{(n)}(\omega - \omega_o) + \frac{A^2}{4} \psi_n(\omega + \omega_o) + S_\varepsilon \otimes \psi_{(n)}(\omega) \quad (13)$$

From (12), as expected, the $\overline{PER(n)}$ spectrum is growing at a rate of O(n) at the $\omega = \pm\omega_0$. At frequencies away from the sinusoid, the $\overline{PER(n)}$ spectrum will converge to true spectrum at a rate of O($\ln(n)/n$). It follows that the kernel $\psi_{(n)}(\omega)$ will converge to the Dirac-δ function.

For the case of white noise, since the last term in (13) is simply σ_ε^2, (13) is controlled entirely by (12). The 3-dB bandwidth of the average of the first n Fejer kernels is two to three times greater than that of a single PER(n) spectrum. Figure 5(*b*) shows the comparison of the nth and the average of the first n Fejer's kernels for values of $n = 100$ and 1000. In contrast to the nth Fejer's kernel, the average of Fejer's kernels does not have side lobes. At frequencies far away from $\omega = 0$ the average kernel converges to zero as $n \to \infty$.

For a process consisting of white noise plus a sinusoid, it is also straightforward to show that for large n the arithmetic variance of the collection $\{S_{PER(k)}(\omega)\}_{k=1}^n$ is approximately:

$$Var_{\overline{PER(n)}}(\omega) \cong \frac{A^4}{16}\left[\gamma_{K(n)}(\omega-\omega_0)+\gamma_{K(n)}(\omega+\omega_0)\right]$$
$$(14a)$$

where

$$\gamma_{K(n)}(\omega) = \frac{1}{n}\sum_{k=1}^{n}(K_k(\omega)-\psi_{(n)}(\omega))^2$$

$$\cong \frac{1}{n(1-\cos\omega)^2}$$

$$\times\left[\frac{\pi^2}{4}-\frac{1}{2}\left(\frac{\pi^2}{6}-\frac{\pi\omega}{2}+\frac{\omega^2}{4}\right)-\left(\frac{\pi^2}{8}-\frac{\pi\omega}{4}\right)\right.$$

$$\left.-\frac{1}{n}\left(\ln n+\frac{1}{2n}-\frac{1}{2}\ln\left(\frac{1}{2(1-\cos\omega)}\right)\right)^2\right]$$

$$\text{for}\quad \omega\neq 0$$

$$\cong (n^2-1)/12 \quad \text{for}\quad \omega=0 \qquad (14b)$$

In particular, at the tone frequency the square root of (14a) becomes

$$\sigma_{PER(n)}(\omega_o)\cong \frac{A^2}{4}*\frac{n}{3.464} \qquad (14c)$$

Thus, at the tone frequency the standard deviation of Fejer kernel family increases by 3 dB as n doubles (at rate O(n)), while at frequencies removed from the tone it decreases at a rate of 1.5dB per doubling (at rate O($1/n$)).

3.3 Variability of the Theoretical AR(n) Spectra. We now turn to the arithmetic mean and variability of the AR(n) family of spectra for a mixed process. These two quantities are defined exactly as they were in (7) for the FT(n) family. Because the AR(n) spectrum is based on prediction of the correlation lags of orders greater than n, we do not have the situation where a kernel function (which is independent of the noise spectral structure) may be analyzed. For this reason we will restrict our attention in this section to processes of the form (4), where the noise is white. In this case, it can be shown that the theoretical AR(n) spectrum for single (complex) sinusoid plus white noise is given by

$$S_{AR(n)}(\omega) = \sigma_\varepsilon^2 \left/ \left|1-\frac{\rho}{1+\rho n}D_n(\omega-\omega_o)\right|^2\right. \qquad (15)$$

Here $\rho\triangleq(A^2/4)/\sigma_\varepsilon^2$ From (15) it follows trivially that for $\rho n\gg 1$:

$$S_{AR(n)}(\omega_o)\cong \sigma_\varepsilon^2|1+\rho n|^2\cong \frac{(A^2/4)^2 n^2}{\sigma_\varepsilon^2} \qquad (16a)$$

and

$$S_{AR(n)}(\omega)\cong \sigma_\varepsilon^2/|1-(1/n)D_n(\omega-\omega_o)|^2, \quad |\omega-\omega_o|\gg 0 \qquad (16b)$$

Equation (16a) states that at the tone frequency the theoretical AR(n) spectrum is proportional to n^2, and to the *local* SNR, defined as $(A^2/4)/(\sigma_\varepsilon^2/n)$ for $\rho n\gg 1$. At the tone frequency the AR(n) spectrum is O(n^2). Equation (16b) states that for $\rho n\gg 1$, at frequencies sufficiently far from the tone frequency the AR(n) spectrum will be close to the noise spectrum, and will, in fact, converge to it as $n\to\infty$. This is due to the fact that the Dirichlet kernel is scaled by a factor of $1/n$, so that it converges to one at the tone frequency and to zero elsewhere. It is possible to gain more insight into the rate of convergence of (16b) by expressing the difference between it and the limit σ_ε^2:

$$\sigma_\varepsilon^2-S_{AR(n)}(\omega)=\sigma_\varepsilon^2\left(\frac{(1/n)K_n(\omega-\omega_o)+(2/n)D_n(\omega-\omega_o)}{1+(1/n)K_n(\omega-\omega_o)+(2/n)D_n(\omega-\omega_o)}\right)$$
$$(17)$$

We know that at frequencies removed from the tone frequency the Fejer kernel converges to zero at rate O($1/n$). Thus, the error (17) is dominated by the Dirichlet term in the numerator. Since the error is O($1/n$) it follows that both the arithmetic average and standard deviation of the collection $\{\text{AR}(k)\}_{k=1}^n$ are O($1/n$).

To evaluate the utility of these results for the case of colored noise, we consider Fig. 6 corresponding to *Example 1*. They include the arithmetic mean $\overline{S}_{AR}^{(n)}(\omega)$ and standard deviation $\sigma_{AR}^{(n)}(\omega)$ of AR spectral family for order ranges 2 to $n=20, 40, 80$, and 160. Figure 6(a) shows that the average converges to the continuous spectrum at all frequencies removed from the tone frequency, while at the tone frequency the rate of growth is O(n^2). Figure 6(b) shows that at frequencies removed from the tone frequency the variance decreases at a rate O($1/n$), while at the tone frequency it increases at a rate of O(n^4).

To summarize this section, we have provided order-related rates of behavior for the FT(n), PER(n), and AR(n) theoretical spectra, as well as rates related to their arithmetic means and standard deviations. This was in the context of mixed spectrum processes of the form (4). At the tone frequency it was noted that the FT(n) and PER(n) spectra grow at a rate O(n), while the AR(n) grows at a rate O(n^2). At frequencies removed from the tone, the variability of the FT(n) family is O(1); that is, it never converges to the true spectrum. The variability of the PER(n) spectra are O($1/n$). We now proceed to address the statistical variability of the families.
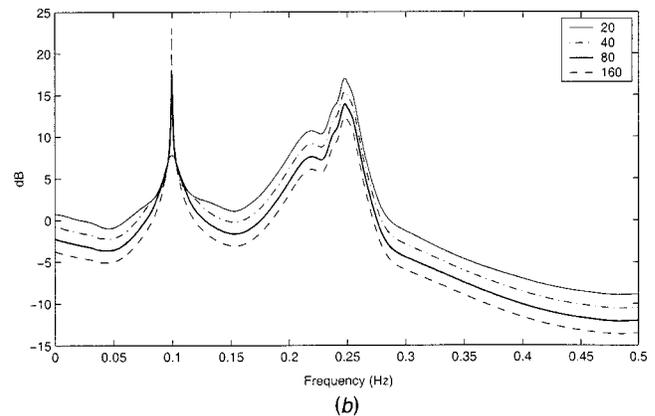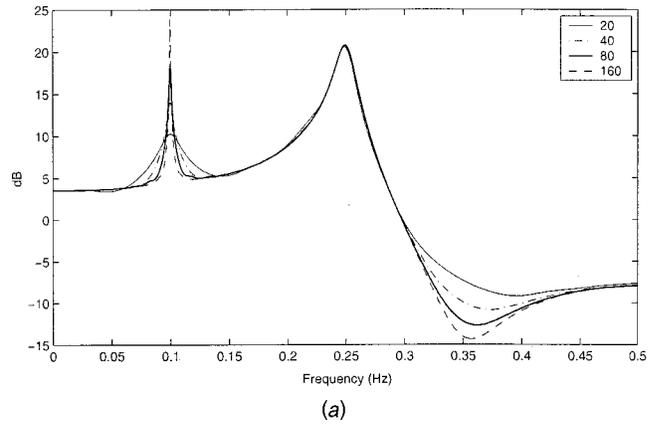

(a)

(b)

Fig. 6 Arithmetic (a) mean and (b) standard deviation of collection of {AR(k): k=2,m} spectra for m=20, 40, 80, and 160

4 Variability of the Estimated FT(n), PER(n), and AR(n) Spectra

There is a wealth of literature on the statistical properties of the marginal FT(n) and AR(n) spectral estimators in the case of regular random processes (see e.g. [8] for references). One can argue that in view of the orthogonalizing property of the frequency decomposition those results should hold in all but the local regions associated with sinusoids. However, we saw in the last section that those local regions can extend over a significant area. In this section we summarize results, having to do with the statistical variability of FT(n), PER(n) and AR(n) spectral estimators in the case of random processes with mixed spectrum. Both of these rely on the statistics of the lagged-product correlation estimator. The following result [9] provides this. Let

$$\hat{R}_x(\tau) = \frac{1}{N} \sum_{n=1}^{N-\tau} x(n)x(n+\tau) \tag{18}$$

denote the *biased* lagged-product estimator $R_x(\tau)$. Let $R_x = [R_x(0), \ldots, R_x(n-1)]^{tr}$ and let $\hat{R}_x$ denote the estimator of it. Then

Result 3. [9] $\sqrt{N}(\hat{R}_x - R_x) - N(0, \Sigma)$ as $N \to \infty$.

The form of the covariance, Σ, given in [9] is lengthy and does not offer much insight. The following expression of [10] affords insight in the frequency domain:

$$\Sigma = \frac{2}{2\pi} \int_{-\pi}^{\pi} \{4\pi(A^2/4)\delta(\omega \pm \omega_o) + S_\varepsilon(\omega)\} S_\varepsilon(\omega) \gamma(\omega) \gamma(\omega)^{tr} d\omega \tag{19}$$

where $\gamma(\omega)^{tr} = [1, \cos(\omega), \ldots, \cos(n-1)\omega]$. This expression will be useful in characterizing the statistical variability of both the FT(n) and AR(n) families. It reveals the direct contribution of the tone at the tone frequency, as well as how it contributes to variability at other frequencies.

4.1 Statistical Variability of the FT(n) Family of Spectral Estimators.
In keeping with (3), we define the FT(n) spectral estimator as

$$\hat{S}_{FT(n)}(\omega) = \sum_{\tau=-(n-1)}^{n-1} \hat{R}(\tau)e^{-j\omega\tau} \tag{20}$$

Recall that (18) is biased. The bias factor equals $1 - \tau/N$, so that for $\tau \ll N$ the bias will be small. Thus, in the case where $n \ll N$ the mean of (20) is approximately equal to the theoretical FT(n) spectrum (3). In many applications involving mechanical systems one has access to a very large amount of data relative to the number of correlation lags selected for spectral analysis. Proceeding under this assumption, and then to compute the variance of (20), notice that it may be expressed as

$$\hat{S}_{FT(n)}(\omega) = 2\gamma(\omega)^{tr}\hat{R}_x - \hat{R}_x(0) \tag{21}$$

Since (21) holds, as well, when the estimated correlations are replaced by the theoretical ones, and since $\hat{R}_x(0)$ is unbiased for $R_x(0)$, it follows that

$$Var\{\hat{S}_{FT(n)}(\omega)\} = (4/N)E\{[\gamma(\omega)^{tr}(\hat{R}_x - R_x)]^2\}$$
$$= (4/N)\gamma(\omega)^{tr}\Sigma\gamma(\omega). \tag{22}$$

Substituting (19) and (22) gives

$$Var\{\hat{S}_{FT(n)}(\omega)\} = \frac{4}{2\pi N} \int_{-\pi}^{\pi} \{4\pi(A^2/4)\delta(\nu \pm \omega_o)$$
$$+ S_\varepsilon(\nu)\} S_\varepsilon(\nu)|\gamma(\nu)^{tr}\gamma(\omega)|^2 d\nu \tag{23}$$

To simplify (23) notice that

$$|\gamma(\nu)^{tr}\gamma(\omega)|^2 \cong (n/4)\Delta_n^2(\omega,\nu)K_n(\omega-\nu)$$
$$- \Delta_n(\omega,\nu)D_n(\omega-\nu)+1 \tag{24}$$

where $\Delta_n(\omega,\nu) = \cos[(n-1)(\omega-\nu)/2]$. The approximate equality in (24) relies on the fact that n is sufficiently large so that for a given ω the above kernels contribute a negligible amount at negative frequencies. Substituting (24) into (21) yields the following new result.

Result 4. For a process of the form (4) where the noise is white with variance σ_ε^2, the variance of the FT(n) PSD estimator (20) is given approximately as
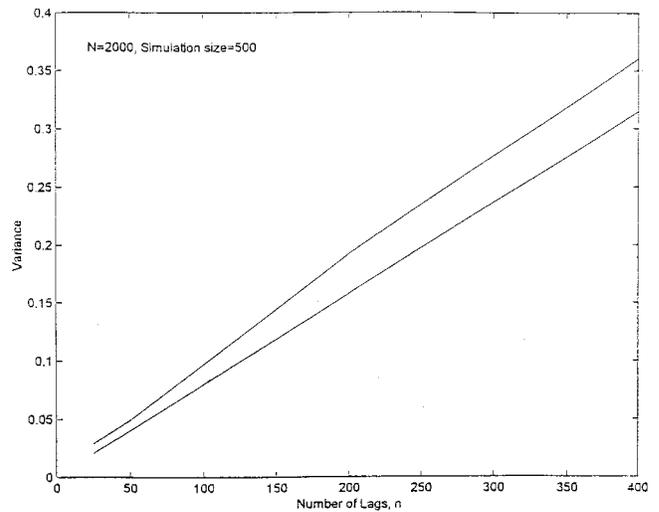
$$Var\{\hat{S}_{FT(n)}(\omega_o)\} \cong [(A^2/2)\pi n^2 + 2\sigma_\varepsilon^4]/N \tag{25a}$$

$$Var\{\hat{S}_{FT(n)}(\omega)\} \cong (2 + n\pi/2)\sigma_\varepsilon^4/N, \quad \text{for } \omega \neq \omega_o \tag{25b}$$

If n and N are selected such that $n/N \to 0$ as $n, N \to \infty$, then (25b) yields the well-known variance value of $2\sigma_\varepsilon^4/N$ at frequencies removed from the tone location. At the tone location we see from (25a) that the statistical variability for large n is dominated by the tone arithmetic variability, and is $O(n^2)$. Figure 7 com-
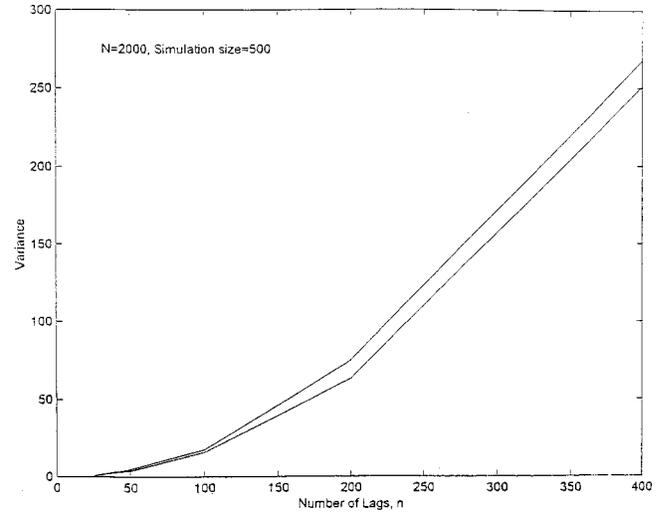
(a)

(b)

Fig. 7 (*a*) Comparison of simulation results against predictions (25) at tone's frequency f=0.25 Hz, (*b*) Comparison of simulation results against predictions (25) at noise frequency f=0.1 Hz.

pares (25) with variances obtained from running 500 simulations. The record size was $N = 2000$, and for each record correlation lag estimates up to order $n = 400$ were computed using (18). The agreement is reasonable, but both of the predicted quantities in (25) are slightly less than the observed values. We believe that this is, in part, due to neglecting the influence of the Dirichlet kernels, in favor of the Fejer's kernels in obtaining (24). It is not unexpected that at the tone frequency the variance becomes dominated by the arithmetic variance of the tone as $n \to \infty$. This will also be seen to be the case with the PER(n) and AR(n) estimators.

4.2 Statistical Variability of the PER(n) Family of Spectral Estimators.

The expected value of the PER(n) spectral estimator associated with (4) is given by (11). For regular random processes (i.e., without tones) the statistical variance of this estimator is well known. At this point we must recall that we are considering the truncated periodogram, where the number of utilized correlation lags, n, is less than the record size, N. Rather than using (18) in (10) it is more common to compute the truncated periodogram as an average of N/n periodograms associated with the contiguous data. Since the single periodogram variance approximately equals to $S^2(\omega)$ for a colored noise process, the variance of this average of N/n periodogram estimators will converge to zero as $N/n \to \infty$, and becomes approximately

$$Var\{\hat{S}_{PER(n)}(\omega)\} \cong nS_\varepsilon^2(\omega)/N, \quad \text{for} \ |\omega - \omega_o| \gg 0 \quad (26)$$

Notice that (26) is constant in the case of the non-truncated periodogram ($n = N$). This is the well known inconsistency property of the periodogram. When a tone is present then one can show that, because of the nature of the Fejer kernel, for sufficiently large order, n, the tone influence will be localized about ω_o. In that region the statistical variance will be dominated by the arithmetic variance, which is O(n^2) regardless of N. It should be expected that when n is sufficiently large then (26) will hold approximately at frequencies removed from the tone frequency. But there is one more very important point to mention. Commonly, it has been speculated that (26) will hold for reasonably narrowband processes [1,2,8]. When conducting PSD analysis it is often presumed that one is dealing with a purely regular process. In this situation the estimate of (26) is obtained by replacing the theoretical, and unknown spectrum, with the estimated one. While not shown here, it turns out that (26) holds very well even in the case of a tone, which is the limit of a narrowband process. If one were to replace the noise spectrum in (26) by (11), then one obtains the variance expression

$$Var\{\hat{S}^{assumed}_{PER(n)}/(\omega)\} = n\left[\frac{A^2}{4}K_n(\omega - \omega_o) + \frac{A^2}{4}K_n(\omega + \omega_o)\right.$$
$$\left. + S_\varepsilon \otimes K_n(\omega)\right]^2 \Big/ N \quad (27a)$$

Clearly, away from the tone frequency (27a) yields a value close to the correct variance (26) for large n. At the tone frequency, if we ignore the negative frequency contribution (27a) becomes

$$Var\{\hat{S}^{assumed}_{PER(n)}(\omega_o)\} = n\left[\frac{A_o^2}{4}n + \sigma_\varepsilon^2\right]^2 \Big/ N \quad (27b)$$

However, in actuality, for a mixed process of sinusoid plus normal white noise with form (2), where $S_\varepsilon(\omega) = \sigma_\varepsilon^2$, if the tone's frequency ω_o is exactly at a bin frequency, it is straightforward to show that the variance of the PER(n) estimators at the tone's frequency ω_o is:

$$Var\{\hat{S}_{PER(n)}(\omega_o)\} = \frac{nS_\varepsilon^2(\omega_o)}{N}\left(1 + \frac{nA_0^2}{2S_\varepsilon(\omega_o)}\right) \quad (28)$$

where, $A_0^2/(2S_\varepsilon(\omega_o)/n)$ is the local SNR at the tone frequency. Comparing (27b) with (28), we see that the presumed variance (obtained for example, using *Matlab*) will behave similar to the

true variance as a function of N, which is O($1/N$). However, there are notable differences. For example, as a function of order, n, the presumed variance will behave as O(n^3), while the true variance will behave as O(n^2). For a large number of averages, N/n, the chi-squared distribution used in the *Matlab* estimation of a specified $(1 - \alpha)$% confidence interval (CI) can be approximated using a normal distribution via the central limit theorem. In this case the presumed 2-σ CI will use (27b), while the actual one will use (28). In either case, it will still be a factor of $2N/n$ above and below the estimated PSD, as would be the case for noise alone. However, at tone frequency it is ill-defined in the sense that if N and n are increased in a way such that N/n is held constant, then the CI will change accordingly at the tone frequency, while remaining the same at other frequencies. Simply, this is because the PSD estimate of the tone itself is ill-defined, as discussed above.

4.3 Statistical Variability of the Ar(n) Family of Spectral Estimators.

Result 3 above was significant in that it was the first characterization of the statistics of the lagged-product autocorrelation estimator for processes with mixed spectrum. It was the essential ingredient for obtaining the following result for the family of AR(n) spectral estimators. Let $\hat{S}_{AR(n)}(\omega)$ denote the AR(n) PSD estimator based on the estimated autocorrelation lags given by (15). The following result is from [10]:

Result 5 [10].

$$\sqrt{N}[\hat{S}_{AR(n)}(\omega) - S_{AR(n)}(\omega)] \overset{d}{\to} N(0, \Omega_n) \quad (29)$$

The form of Ω_n is quite involved, and so is not repeated here for the sake of brevity. The interested reader may refer to [10]. In order to illustrate (29) we offer Fig. 8(*a*), which illustrates how the standard deviation depends on the AR order, n, both at the tone frequency and at a frequency removed from it. We see that at frequencies removed from the tone frequency, it increases sharply for values of the AR order, n, typically used in spectral analysis, while at higher orders the rate becomes more constant, and is approximately 1.5 dB per order doubling. We also notice that it is very insensitive to the SNR. As noted above, the complex nature of Ω_p in (24) precludes its use to predict this rate. However, in view of the orthogonalizing role that kernels such as the Dirichlet and Fejer type play, it is possible to obtain a simple approximation for it. This expression is from [11], for the case of a process with no tones. It is given by

$$\Omega_n \cong 2nS_\varepsilon^2(\omega)/N, \quad |\omega - \omega_o| \gg 0 \quad (30)$$

The variance expression in (30) is extremely simple relative to that in [10] for the variance in (29). Furthermore, it does predict the noted rate. A comparison of (26) and (30) shows that for a given order, n, the AR estimator variance is three times greater than the periodogram. The fact is, however, that due to the poor resolution of the latter in favor of the former, the value of n used in periodogram analysis is usually orders of magnitude larger than that used in AR analysis. In the context of *Example 1*, Fig. 8(*b*) illustrates the variability of an AR(40) PSD estimate based on a record of size $N = 5000$.

5 Statistical Properties of Averages of Families of PSD Estimators

The last two sections addressed the general behavior of families of FT(n), PER(n), and AR(n) spectra, in terms of arithmetic and statistical variability. We will investigate the behavior of averages of a given family, and in particular, what advantages might be gained. In this section we investigate the possible advantages of using an average of PSD estimators of a given type, as opposed to a single one. As noted in *Example 1* of Sec. 2, it can provide some level of increased confidence in the order selection process. In section 3 we discovered that by averaging FT(n) spectra, one arrives at an estimator which has more desirable properties. The main difficulty with conducting an analytical study of statistical

(a)

(b)
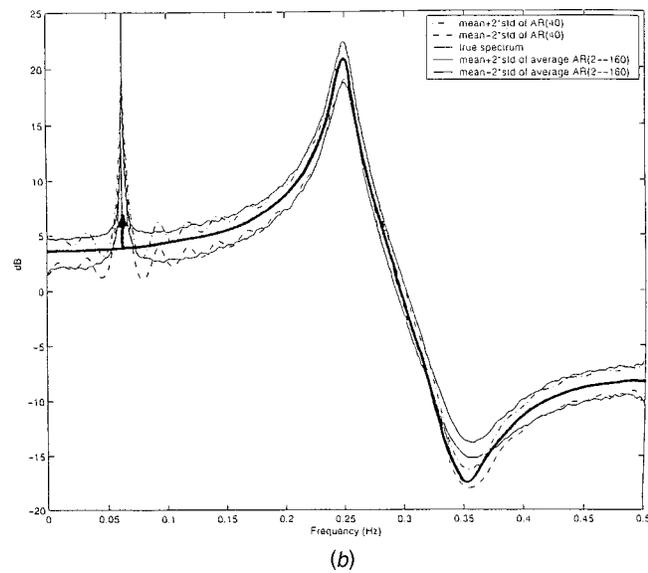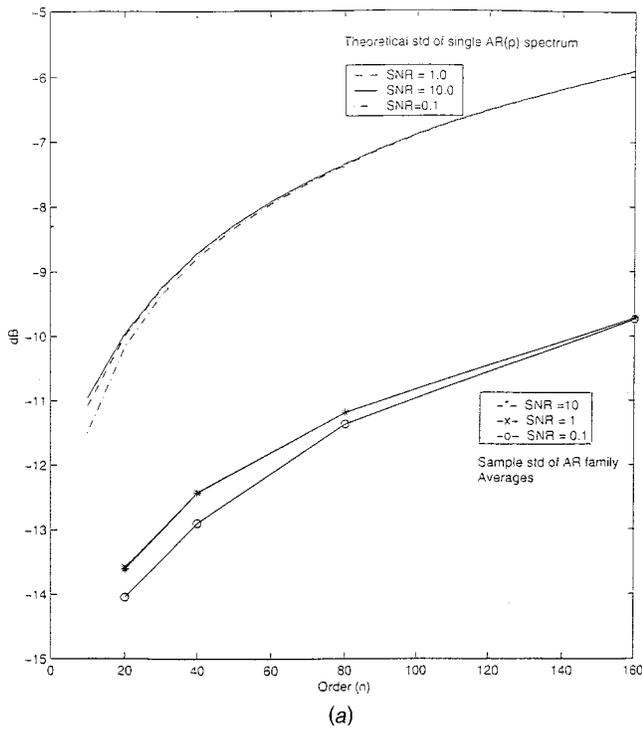
Fig. 8 (*a*) **Evaluation of the variance expression in (28) as a function of model order at a noise frequency. Also shown is the estimated variability of the average of AR spectra (See Section 5 for related discussion).** (*b*) **Comparison of predicted (using (28)) 2-σ regions associated with an AR(40) model, and an average of AR(p) models for p=2:160. Also shown is the estimated variability of the average of AR spectra (see Section 5 for related discussion).**

properties of an average of estimators is that of obtaining all of the joint statistics. For this reason, we will here resort to the use of simulations. Specifically, to estimate the mean and variance information associated with a family, we will use 200 realizations. Each realization includes 10,000 samples of the process (4). We investigate three data sets of different SNR = 0.1, 1.0, and 10. We keep the noise power constant while changing the power of the sinusoid to change SNR. These simulations will be used for the FT(n), PER(n), and AR(n) investigations.

5.1 The FT(n) Spectra and Their Averages. From (8), we have that the average of the $\{FT(k)\}_{k=1}^{n}$ spectra is simply the PER(n) spectrum. Hence, here we will denote this average by the latter.

Comparison of the Means of the FT(n) and PER(n) Spectra. In this subsection there is no need to resort to simulations, since we have expressions for the means of both spectra. The means of these two spectra are given more generally by (5) and (11), respectively. Because our current investigation focuses on white noise, only the last term in each of these equations is altered. Specifically, the terms are simply replaced by the noise variance, σ_{ε}^2. Thus, the advantage offered by averaging is simply that the Dirichlet kernel is replaced by the more desirable Fejer's kernel. They are equal at the tone frequency, and away from it the latter will be closer to the true noise spectrum than the former.

Comparison of the Standard Deviations of the FT(n) and PER(n) Spectra. For n sufficiently large the arithmetic standard deviations of these collections of theoretical spectra can be obtained from (9) and (14). The statistical variances of each of these single spectra for a given n can be obtained from (25) and (26), (28). But since the PER(n) spectrum is exactly the average of the $\{FT(k)\}_{k=1}^{n}$ spectra, clearly, it includes the arithmetic variance of this collection as a portion of its statistical variance. So the advantage of averaging, in terms of variance reduction at frequencies removed from the tone (recall, at the tone both the mean and variance, being functions of n, are both converging to infinity with

(a)

(b)
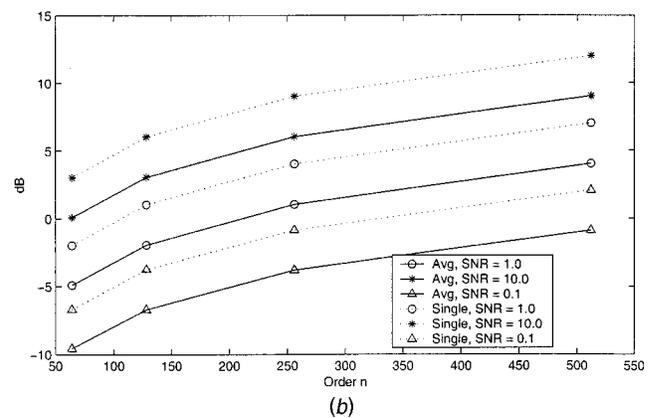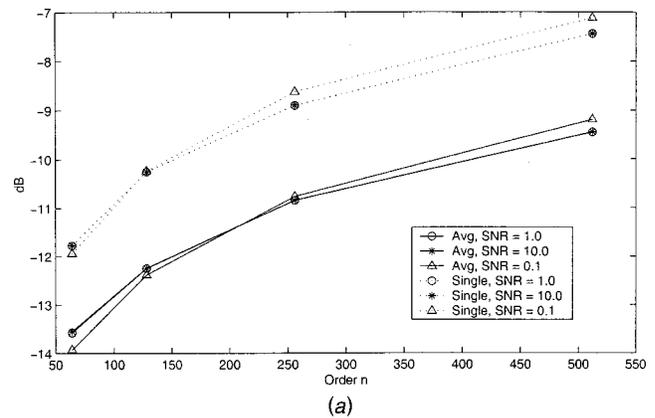
Fig. 9 (*a*) **Comparison of statistical standard deviation of single order PER(n) and the averaged PER(n) spectral estimates for selected order n at nontone frequency f=0.1 Hz.** (*b*) **Comparison of statistical standard deviation of single order PER(n) and the averaged PER(n) spectral estimates for selected order n at tone frequency f=0.3 Hz.**

(a)

(b)
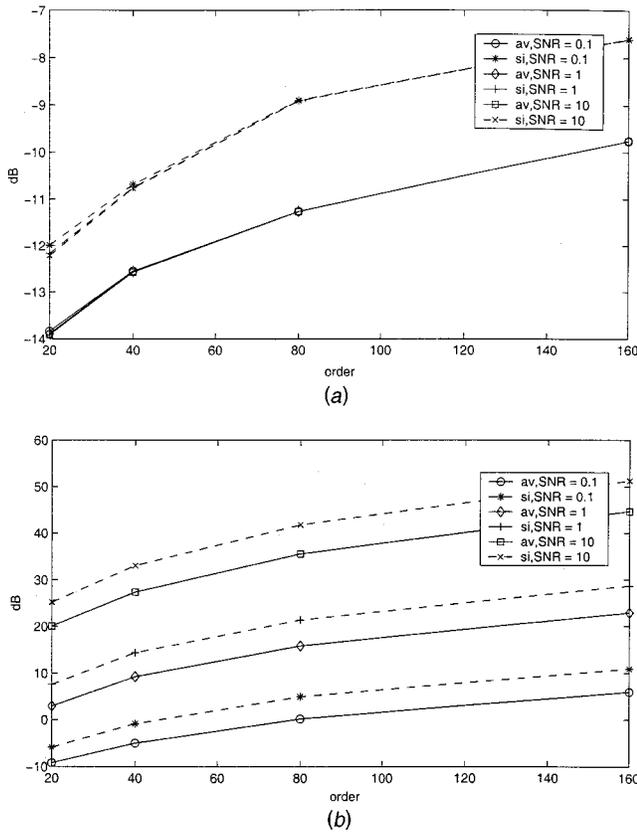
Fig. 10 (*a*) **Comparison of statistical standard deviation of single order AR(n) and the averaged AR(n) spectral estimates for selected order n at non-tone frequency f = 0.1 Hz.** (*b*) **Comparison of statistical standard deviation of single order AR(n) and the averaged AR(n) spectral estimates for selected order n at tone frequency f = 0.3 Hz.**

increasing n) is obtained by simply comparing (25b) to (26). The averaging procedure offers only 1 dB of reduction in the standard deviation.

5.2 The PER(n) Spectra and Their Averages. The average of a collection $\{PER(k)\}_{k=1}^{n}$ spectra does not, to our knowledge, have a well known closed form, as was the case in the last subsection. Hence, here we are forced to conduct simulations in order to estimate the statistical variability of the $\overline{PER(n)}$ spectral estimator.

Comparison of the Means of the PER(n) and $\overline{PER(n)}$ Spectra. It follows from (12a) that at nontone frequencies, the mean of $\overline{PER(n)}$ estimate converges to the true spectrum at rate $O(\ln(n)/n)$, which is slower than the convergence rate of the mean of single order PER(n) estimate, which is O(1/n). At the tone frequency, it follows from (12b) and (11) that the mean of $\overline{PER(n)}$ estimate is half that of $\overline{PER(n)}$, or 3dB smaller. Because the joint statistics have no bearing on the mean of the $\overline{PER(n)}$ estimator, we were able to compute the mean, (13), without the need for simulations.

Comparison of the Standard Deviations of the PER(n) and $\overline{PER(n)}$ Spectra. To evaluate the potential advantage in terms of variance reduction, we offer Fig. 9(a) shows that the statistical standard deviations of both $\overline{PER(n)}$ and PER(n) spectral estimates are nearly independent of SNR and both increase by 1.5 dB per order doubling at non-tone frequencies. But we gain about 2 dB or 37% decrease in standard deviation by averaging for each selected order. At the tone frequency, according to (28), the statistical variance of PER(n) estimate is approximately proportional

to n^2 and to SNR when the noise power are constant. At the tone frequency, Fig. 9(b) illustrates (28), and in particular, that the statistical standard deviation of single order PER(n) estimate increase by 3 dB as n doubles and by 5 dB as SNR increases by 10 times. From this figure we observe that the statistical standard deviation of $\overline{PER(n)}$ estimate has similar behavior to that of the single order PER(n) estimate, while offering a reduction of 3 dB or 50% in standard deviation by averaging at tone frequency.

5.3 The AR(n) Spectra and Their Averages. The simulations were run for $n = 20, 40, 80, 160$. While (15) or (16) may be used to arrive at a form for the mean of the $\overline{AR(n)}$ spectral estimator, we will forego this exercise, and simply note that at all frequencies except that of the tone the average will converge in the mean to the true spectrum, while at the tone frequency it will approach to infinity at a rate of O(n^2). This is the same behavior as that of the mean of the single order AR(n) estimator for large n.

Comparison of the Standard Deviations of the AR(n) and $\overline{AR(n)}$ Spectra. At $\omega \neq \omega_0$, (30) predicts that the statistical variance of $\overline{AR(n)}$ spectral estimate is approximately proportional to the noise power square and to the order n. Thus, we see about 1.5 dB increase as n doubles in Fig. 10(a) and the standard deviation is the same for the three datasets with different SNRs while the noise power keeps constant. The statistical standard deviation of the $\overline{AR(n)}$ estimate is about 2 dB smaller than that of the corresponding single order $\overline{AR(n)}$ spectral estimate. At $\omega = \omega_0$, the standard deviation of $\overline{AR(n)}$ spectral estimate is, as mentioned below (29), quite complex, and not amenable to analysis. The simulation results in Fig. 10(b) show it increases by about 7.5 dB as n doubles and for large n, it increases by 20 dB as SNR increases by 10 times which could also be approximately predicted from (30), if we replace the noise power spectrum with the sinusoid's power spectrum at the tone's frequency in (30). Thus the statistical variance at $\omega = \omega_0$ is approximately, proportional to the square of SNR and to n^5. We gain more than 2 dB decrease in statistical standard deviation by averaging and the gain will be higher by increasing SNR and the averaging order n.

6 Application to The Westland Helicopter Vibration Data

The last section suggested that there might exist little gain, in terms of variance reduction, by averaging either $\{PER(k)\}_{k=1}^{n}$ or $\{AR(k)\}_{k=1}^{n}$ collections, with the exception being that averaging the former can eliminate spectral oscillations in the side lobe leakage. So, in this section our focus will be limited to the variability of the collections, as was the case in the example in Section 2. Furthermore, we will not include the FT(n) family here, as its properties are, in our opinion, not sufficiently attractive for use in mixed spectral analysis using spectral families.

In order to illuminate the value of the variability of the collections, we make a comparison of 95% (or 2-σ) confidence intervals of the PER(n) estimators for $n = 256$ and 1024 and of the AR(n) estimators for $n = 20$ and 40 spectral in relation to real vibration data from Westland Helicopter data set (file w3003001.bin). The file size is 412,464. The data were sampled at 103,116.8 Hz, which we have normalized to 1 Hz. The data time duration is thus, T = 4 seconds, which corresponds to a maximum (1/T) frequency resolution of 0.25 Hz. Since we utilize a normalized the sampling frequency, 1 Hz is actually 103,116.8 Hz. The spectral structure is complicated and contains many sinusoids plus highly colored noise. For this reason we heterodyned the data (including decimation by factor of 10) to restrict the range of interest to the normalized frequency range of [0.16-0.21]Hz.
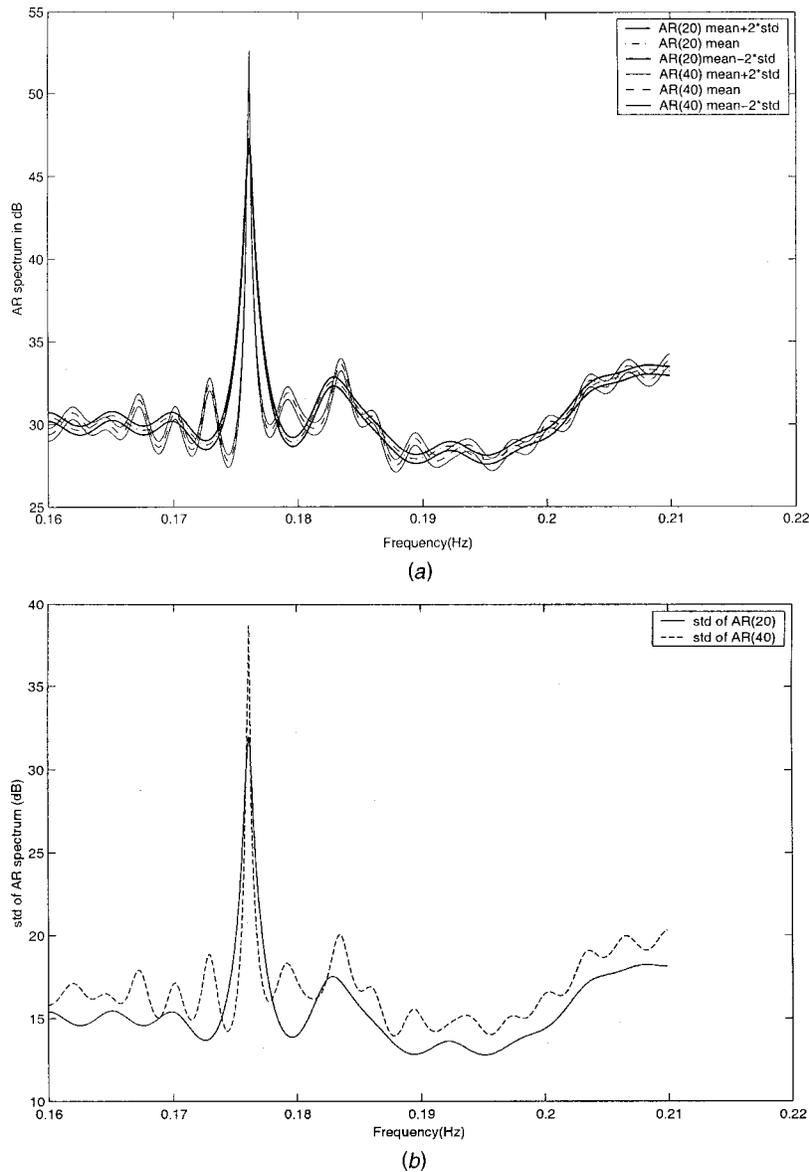
Comparison of 95% (or 2-σ) Confidence Intervals Corresponding to the PER(n) Estimates. Figure 11 shows that the 95% CI (Fig. 11(a)) and standard deviation (Fig. 11(b)) of PER(256) and PER(1024) PSD estimators calculated using *Matlab*, which ap-
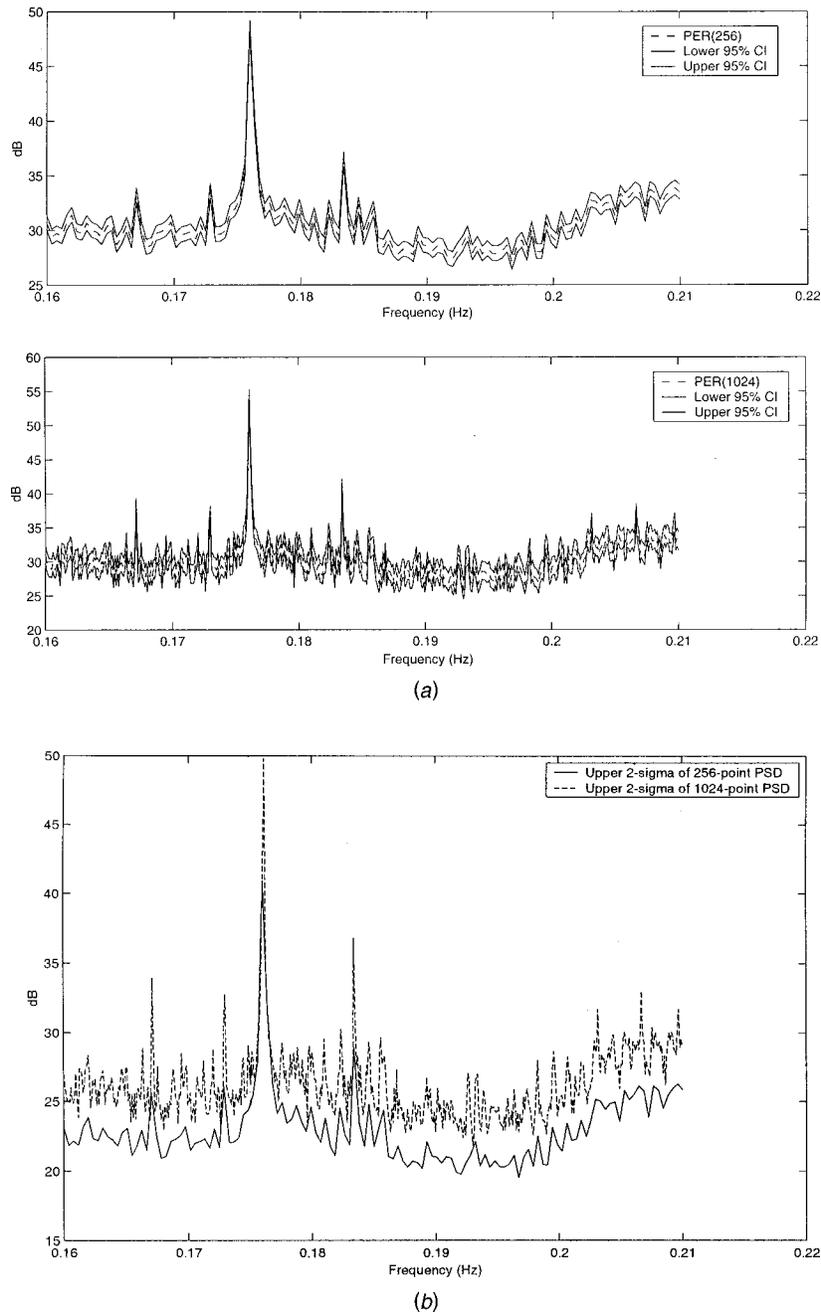
Fig. 11 (*a*) **Comparison of 95% (or 2-σ) confidence interval of PER(n) for n =256 and 1024 for Westland data.** (*b*) **Comparison of 2-sigma of PER(n) for n =256 and 1024 for Westland data.**

plies Kay's formula([2]). The orders 256 and 1024 is chosen because people traditionally use these two orders in periodogram-based spectral analysis. As N is large enough, by the central limit theorem, the variance of the PER(n) from estimate can be predicted by (26) at nontone frequencies, and by and (27a) at tone frequencies. Figure 11(a) and Fig. 11(b) show that at 0.1671 Hz, 0.1729 Hz, 0.1761 Hz, and 0.1834 Hz, the standard deviations of the PER(1024) estimate are approximately 8 dB, 7 dB, 9 dB, and 8 dB, respectively, larger than those of the PER(256) estimate. At other frequencies, we only see about 3 dB increase from PER(256) to Per(1024). Now, (27a) indicates that for large fixed N, if the local SNR is large, the variance of PER(n) at tone's frequency would be O(n^5), corresponding to a 9 dB increase of standard deviation from PER(256) to PER(1024). Thus, from Fig.

11 we can be reasonably confident that there is a sinusoid at 0.1761 Hz. At the other three frequencies, this increase of standard deviation is close to 9 dB. We need to note that first, at these three frequencies, the local SNR is much smaller than that at 0.1761 Hz. Second, if the tone frequency is not exactly at bin frequency, the extent of the increase would be a little less than 9 dB. So, the spectra suggest that each of the three frequencies may well also correspond to tones. It is shown in (26) that at nontone's frequency, the variance of PER(n) is just O(n), which is 3 dB increase from the standard deviation of PER(256) to that of PER(1024). Therefore, at other frequencies, it is just regular process. If we compare the behavior of the means of the PER(256) and PER(1024) estimates with (11), the same conclusion follows. At the four tone frequencies the mean increases by approximately

Fig. 12 (*a*) **Comparison of 95% (or 2-σ) confidence interval of AR(n) for n = 20 and 40 for Westland data,** (*b*) **Comparison of standard deviation of AR(n) spectral estimates for n=20 and 40 for Westland data.**

6 dB from the PER(256) to the PER(1024) estimate, while remaining essentially unchanged at other frequencies.

Comparison of 95% (or 2-σ) Confidence Intervals Corresponding to the AR(n) Estimates. The AR(20) and AR(40) spectral estimates were selected since their orders are in the range that would be obtained using most of the popular order selection rules (e.g. [2]). The 95% C.I. of AR(20) and AR(40) estimates are plotted in Fig. 12(*a*). They were obtained by using (30) to estimate the standard deviation, along with a normality assumption. According to (30), for large *N*, at a tone frequency, the standard deviation of AR(n) spectral estimator should be O(n^5). While at a frequency away from the tone, it should be O(n). Figure 12(*b*) shows that at 0.1761 Hz, the standard deviation of AR(40) spectral estimate is 7.5 dB larger than that of the AR(20) spectral estimate. However, at frequencies removed from this frequency [outside of the interval (0.17, 0.18)] the standard deviation increase is only about 1.5 dB. While at several frequencies between 0.167 Hz and 0.183 Hz, there is more than 3 dB difference between the standard deviations of the two spectral estimators, their uniform spacing suggests that they are a consequence of the distribution of extraneous model poles. In Fig 12(*a*), we see a 6 dB increase in the mean at the frequency, 0.176 Hz By (16), this amount of increase is consistent with the presence of a tone. At other frequencies the two confidence intervals are nearly identical. Hence, in those regions one might assume the spectrum is not only devoid of tones, but is well characterized independent of the model order.

7 Summary and Conclusions

This work was concerned with the properties of families of spectral estimators. The families included the FT(n), PER(n), and AR(n) PSD estimators, where n denotes the number of correlation lags used. The interest was to identify what factors control both arithmetic and statistical variability within a family. The processes considered were those composed of tones and colored noise. For very low orders the arithmetic variability will be closely related to the noise color. For this reason, we elected to focus predominantly on the case of white noise. In this setting the

arithmetic variability associated with spectral bias is caused by the presence of the tone. We reported behavior of both bias and variability in terms of order dependence rates. Some of the results reported have been known, but perhaps not well known. The undesirable properties of the $FT(n)$ spectrum, stemming from the Dirichlet kernel, were such that it was decided to see if averaging could improve matters. This average is, in fact, the tapered (i.e. averaged) periodogram which is so popular. The AR family was noted to have $O(n^2)$ behavior in both mean and standard deviation at tone's frequency. This is in contrast to the tapered periodogram, whose behavior is $O(n)$. We presented expressions which allow one to estimate the mean and variance information in certain cases. We then used this information to guide brief investigation of the utility of using a family (and, in particular, an average) as opposed to a single spectrum. It was observed that averaging can offer potential for reduced statistical variability in certain situations. We also noted that the use of a family, as opposed to a single spectrum, can reduce sensitivity of results to order selection factors. Finally, while many of our results emphasized rates of convergence we did, in fact, provide a number of equations which would allow one to obtain more quantitative statistical behavior of both spectral estimators and their averages. An important and immediate application of these results is the problem of detecting tones. For example, of the three spectral estimators it was noted that only the $AR(n)$ estimator converged to infinity at a rate $O(n^2)$ at a tone frequency. Both the $FT(n)$ and the $PER(n)$ estimators had a rate of only $O(n)$. Thus, even though the vast majority of tone detection algorithms are based on the latter, it is quite possible that the former could offer significant improvement. This was, in fact, the basis for [10].

Acknowledgment

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