THE INVERSE BORN APPROXIMATION: EXACT DETERMINATION OF SHAPE OF CONVEX VOIDS

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ABSTRACT

The Inverse Born Approximation (IBA) to the elastic wave inverse scattering problem is known to give highly accurate results for the shape of complex voids. In this paper we present an argument demonstrating that the IBA is, in fact, exact for determining the size, shape and orientation of a wide class of these scatterers given infinite bandwidth and unlimited aperture information. Essentially, our argument demonstrates how the IBA algorithm picks out the singular contribution to the impulse response function and correctly relates it to the shape of the scatterer. Some specific examples will be used to illustrate the more intuitive aspects of the discussion.

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I. INTRODUCTION

The exact inverse scattering problem, in which the potential varies in an arbitrary way (3D), is in general unsolved. Some recent progress has been made on this problem for the scalar wave Schroedinger equation. However, for the classical wave equation (and in particular for the elastic wave case) the problem remains quite intractable. The inverse scattering problem for the classical wave equation has attracted considerable attention in many areas: geophysics, distance target identification (sonar, radar, etc.) and non-destructive evaluation (NDE). In recent years the 3D inverse classical wave equation has been successfully approached in various limiting cases where a good deal of a priori information is known concerning the nature of the potential. For example, if the potential is known to be weak everywhere, then the potential can be recovered using the Inverse Born Approximation (to a resolution that depends on the strength of the scattering). Further, the physical optics approximation, various practical imaging methods (e.g., the synthetic aperture method) and acoustic backscatter tomography can be shown to determine in certain cases the same computational algorithm as the Inverse Born Approximation (IBA). Hence one may expect this algorithm to have a much wider range of validity than implied by its derivation in the weak scattering limit. In particular, computer experiments have shown that the IBA does a good job of inverting longitudinal to longitudinal scattering data from: (1) a wide class of voids, (2) from flat cracks, and (3) from voids with cracks, as well as (4) from weak scattering inclusions. In this paper we will show in detail that the IBA can be used to exactly determine the size, shape and orientation of convex voids.

The result we will show is stated below. First we will introduce two functions: (1) $A(\omega, \hat{e}_i, \hat{e}_s)$ the longitudinal to longitudinal ($L\rightarrow L$) displacement scattering amplitude; and (2) the time domain, $L\rightarrow L$, impulse response function $R(t, \hat{e}_i, \hat{e}_s)$. Since we are dealing only with the longitudinal component of the displacement, we represent $A(\omega, \hat{e}_i, \hat{e}_s)$ and $R(t, \hat{e}_i, \hat{e}_s)$ as scalars for ease of notation. The direction of the incident wave is denoted by the unit vector $\hat{e}_i$, while the scattering direction is $\hat{e}_s$. The time and angular frequency are denoted by $t$ and $\omega$, respectively. The impulse response function will be defined in the next section. For the moment it is adequate to consider it to be a "normalized displacement" received in the far-field as the result of an incident delta function (time localized) wave striking the flaw. We will consider a convex void (Fig. 1) in an isotropic, otherwise homogeneous elastic material. The material property deviations describing this flaw are proportional to a characteristic function $\gamma(\hat{r})$ which is defined to be 1 inside the flaw and zero outside. Thus, at the boundary there is a surface of discontinuity which outlines the flaw. We will show that if we know the backscattered impulse response function, $R(t, \hat{e}_i, -\hat{e}_i)$, for all times and all directions of incidence (assuming a common origin in
Fig. 1. Cross-sectional view of convex flaw with finite radius of curvature everywhere. The incident wave is shown striking the flaw as the contact gives rise to the initial delta function in $R(t, \hat{e}_1, -\hat{e}_1)$.

time for measurements from different incident directions) then the surface of discontinuity can be exactly determined to within an overall constant using the IBA. This allows the exact determination of the size, shape and orientation of convex voids.

The key to our argument lies in an examination of the direct and inverse scattering problems in the time domain. Figure 2 shows schematically $R(t, \hat{e}_1, -\hat{e}_1)$, the backscattered impulse response function, for a convex void (1) the exact solution and (2) in the Born approximation. As expected from its limited range of validity, the Born result differs rather dramatically from the true solution. Important differences are (1) the appearance of a second, non-physical, delta function in the Born solution, and (2) the exact solution continues for a considerably longer time since multiple reflections and surface waves are, of course, included, unlike the Born. However, there is one striking similarity. Namely in both cases the initial delta function (1) occurs at the same time, and (2) is proportional to $\sqrt{R_1 R_2}$, where $R_1$ and $R_2$ are the local radii of curvature. We will show that the discontinuities in the characteristic function reconstruction arise from the delta function (singular parts) of the impulse response function. Then we will show that the second delta function in the direct Born solution is redundant in determining the shape of the surface of discontinuity.
After having argued that both the exact and Born first delta function are proportional to $\sqrt{R_1 R_2}$, we will show that they must yield the same surface of discontinuity to within an overall constant, completing the proof.

II. TIME DOMAIN SCATTERING

The time domain scattering geometry is shown in Fig. 1. We will consider flaws which are isolated voids with a finite radius of curvature everywhere in an otherwise isotropic, homogeneous elastic material. The incident displacement field impulse (the longitudinal component is considered at all times and it suffices for our purposes to represent it by a scalar) $u^I(\vec{r}, t)$ is given by

$$u^I(\vec{r}, t) = u^0 \delta(t - \vec{r} \cdot \hat{e}_i / c) \tag{2.1}$$

The origin of coordinates is assumed to lie in the interior of the flaw and the zero of time corresponds to the incident impulse crossing the origin of coordinates in the absence of the flaw. The far-field scattered displacement is given as

$$u_s(\vec{r}, t) = \lim_{L \to \infty} \frac{u^0}{L} R(t - L/c, \hat{e}_i, \hat{e}_s) \tag{2.2}$$

Here $L$ is the distance from the origin of coordinates to the receiver. Equation (2.2) serves to define the impulse response function. The scattering amplitude and $R(t, \hat{e}_i, \hat{e}_s)$ are simply related via a Fourier transform.
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\[ R(t, \hat{e}_1, \hat{e}_s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \ A(\omega, \hat{e}_1, -\hat{e}_1) e^{-i\omega t} \]  \hspace{1cm} (2.3)

Fig. 1 shows the initial contact of the plane wave with the surface, for the case of an incident plane wave scattering from a convex flaw with finite local radii of curvature. The resulting initially scattered displacement amplitude is a delta function pulse in the far field whose strength is proportional to \( \sqrt{R_1 R_2} \) evaluated at the point of tangency. This result can be obtained via a numerical series expansion of the incident and scattered fields about the instant of contact.

Kennaugh and Moffat noted for a perfectly conducting body that the initial portion of the impulse response function is given correctly by the Kirchhoff Approximation (Physical Optics) for electromagnetic scattering. We conjecture that such a result also holds for the case of elastic wave scattering from voids. In support of this conjecture we have numerically calculated the frequency domain scattering amplitude for a wide variety of axially symmetric flaws with different \( R_1 \) and \( R_2 \) using the method of optimal truncation (MOOT) recently introduced by Visscher. The impulse response function was then obtained by Fourier transforming the scattering amplitude and the strength of the initial delta function was extracted using parameter estimation theory. In all cases tested the calculated delta function strengths agreed with the Kirchhoff approximation to the accuracy of our calculations (one part in ten thousand). Since the most general flaw considered in this paper can be completely described at the point of initial contact by \( R_1 \) and \( R_2 \), our conjecture is numerically confirmed. Currently we are making an expansion of the time domain integral equation about the time of the initial contact. Initial analytic results agree with the numerical simulation described above. In the Kirchhoff (Physical Optics) Approximation the L+L impulse response originated in the region of contact is

\[ R^K(t, \hat{e}_1, -\hat{e}_1) = \text{const.} \left( \frac{d^2}{dt^2} \right) \int d^3 \hat{r} \ \delta(2\hat{e}_1 \cdot \hat{r} - ct) \ \gamma(\hat{r}) \] \hspace{1cm} (2.4)

Here \( \gamma(\hat{r}) \) is that characteristic function (1 inside the void and zero outside) which we discussed in the introduction. Note that the integral is just the cross-sectional area of the flaw evaluated on the plane defined by the delta function.

Now turning to the direct Born approximation we find that the impulse response is given by
This approximation is the same as that of the Kirchhoff approximation at the time of initial contact. Thus

\[ R^B(t, \hat{e}_1, -\hat{e}_1) = \text{const.} \frac{d^2}{dt^2} \int d^3\hat{r} \delta(2\hat{e}_1 \cdot \hat{r} - ct) \gamma(\hat{r}) \]  

(2.5)

Here we have evaluated the constants in 2.4 and 2.5 and I denotes the scattering at the time of initial contact. Since we have previously argued that the Kirchhoff approximation yields the initial delta function singularity, it follows that to within a factor of two, the Born Approximation determines the initial delta function correctly.

III. EXACT RECONSTRUCTION

In this section we will show that the reconstruction of the characteristic function's surface of discontinuity depends only on the initial delta function of the impulse response function within the IBA. First we demonstrate that the discontinuity arises from the presence of the delta function in the data. Then we consider the IBA using the direct Born approximation for the scattering. We will show that in this case it is only necessary to know the initial delta function. That is, the final delta function will be shown to be redundant. Consequently, since the initial delta function in the direct Born and the exact scattering initial delta functions are proportional, it follows that if the direct scattering amplitude is inverted by the IBA it generates the correct surface of discontinuity.

Now, before considering the inversion of the exact results, we will examine the inversion of the direct Born results by the IBA. The time domain version of the IBA is defined as

\[ \gamma(\hat{r}) = \text{const.} \int d^2\hat{e}_1 R^B(t = 2\hat{e}_1 \cdot \hat{r}/c, \hat{e}_1, -\hat{e}_1) \]  

(3.1)

Within the context of the weak scattering limit this has a very simple interpretation if we remember that \( R^B(t, \hat{e}_1, -\hat{e}_1) \) is proportional to the second time derivative of the flaw's cross-sectional area at time \( t = 2\hat{e}_1 \cdot \hat{r}/c \). Consequently, Eq. (3.1) says that we should add together equally all impulse response functions evaluated at \( t = 2\hat{e}_1 \cdot \hat{r}/c \), i.e., those parts of \( R(t, \hat{e}_1, -\hat{e}_1) \) which are generated by scattering from an infinitesimal volume element centered at \( \hat{r} \). Consider a point immediately inside the flaw shown in Fig. 3 and then one immediately outside. For a point on the inside some of those cross-sectional areas are shown which contribute in Eq. (3.1). Given our geometric constraints, none of the impulse response
functions corresponding to these cross-sectional areas make a delta function contribution, since none of them are tangent to the surface. As we cross the surface we suddenly obtain a delta function contribution since now certain of the contributing cross-sectioned areas are tangent to the surface. Consequently, there arises a discontinuity in the reconstructed characteristic function since the other terms in the integral are smooth and the delta function makes a finite contribution to the integral in Eq. (3.1). To highlight this point we consider a particularly simple example, namely, a spherical flaw. In this case the impulse response function becomes independent of the incident direction, $\hat{e}_i$. Letting $\cos \theta = \hat{e}_i \cdot \hat{r} / |\hat{r}|$ and making the transform $t = 2r \cos \theta / c$ we can integrate over angles in 3.1 and obtain:

$$\gamma(|\hat{r}|) = \frac{\text{const}}{2r/c} \int_{-2r/c}^{2r/c} dt \ R^B(t, \hat{e}_i, -\hat{e}_i)$$  \hspace{1cm} (3.2)$$

Figure 2 shows the approximate Born impulse response function. The inversion algorithm in this case is just an average, starting at $t = 0$ of the impulse response. For $r < R$, the radius, we get a constant average. However, at $r = R$ the average includes the delta functions which brings the reconstructed characteristic function sharply to zero. This illustrates our general point that discontinuities in the reconstruction arise from delta functions in the impulse response function.
Now we will show that the IBA relies only on the information in the initial delta functions to determine the surface of discontinuity. The direct Born approximation for the impulse response function can be written as

\[ R^B(t, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_1) = \alpha(\hat{\mathbf{e}}_1) \delta(2\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{S}}_1(\hat{\mathbf{e}}_1) - ct) \]

+ \beta(\hat{\mathbf{e}}_1) \delta(2\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{S}}_2(\hat{\mathbf{e}}_1) - ct) + f(t, \hat{\mathbf{e}}_1, -\hat{\mathbf{e}}_1) \]  \hspace{1cm} (3.3)

Here \( \hat{\mathbf{S}}_1(\hat{\mathbf{e}}_1) \) is the position vector of the point of tangency of the incident wavefront and the flaw, \( \hat{\mathbf{S}}_2(\hat{\mathbf{e}}_1) \) is the position vector at the point of tangency of the exiting wavefront. Finally \( f(t, \hat{\mathbf{e}}_1, -\hat{\mathbf{e}}_1) \) is the non-singular contribution to the impulse response. Inserting (3.3) into (3.1) we obtain

\[ \gamma(\hat{\mathbf{r}}) = \int d^2\hat{\mathbf{e}}_1 f(t = 2\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{r}}/c, \hat{\mathbf{e}}_1, -\hat{\mathbf{e}}_1) \]

+ \int d^2\hat{\mathbf{e}}_1 \alpha(\hat{\mathbf{e}}_1) \delta(2\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{S}}_1(\hat{\mathbf{e}}_1) - 2\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{r}}) \]  \hspace{1cm} (3.4)

+ \int d^2\hat{\mathbf{e}}_1 \beta(\hat{\mathbf{e}}_1) \delta(2\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{S}}_2(\hat{\mathbf{e}}_1) - 2\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{r}})

We will now show that the final set of delta functions (third term) are redundant, i.e., that

\[ \int d^2\hat{\mathbf{e}}_1 \alpha(\hat{\mathbf{e}}_1) \delta(2\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{S}}_1(\hat{\mathbf{e}}_1) - 2\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{r}}) = \int d^2\hat{\mathbf{e}}_1 \beta(\hat{\mathbf{e}}_1) \delta(2\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{S}}_2(\hat{\mathbf{e}}_1) - 2\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{r}}) \]  \hspace{1cm} (3.5)

From Fig. 3 we see that \( \alpha(\hat{\mathbf{e}}_1) = \beta(-\hat{\mathbf{e}}_1) \). That is, the incident delta function in the \( \hat{\mathbf{e}}_1 \) direction is equal to the exiting delta function in the \( -\hat{\mathbf{e}}_1 \) direction. Changing variables \( \hat{\mathbf{e}}_1 \rightarrow -\hat{\mathbf{e}}_1 \) on the right-hand side and using the equality just noted establishes Eq. (3.5). Thus we have shown: (1) that the surface of discontinuity is determined from the singular structure of the impulse response function; (2) that the initial singularities of the Born \( R(t, \hat{\mathbf{e}}_1, -\hat{\mathbf{e}}_1) \) and the exact result are identical to within an overall constant; and (3) that in reconstructing the characteristic surface the exiting delta function in Eq. (3.4) is redundant. Consequently we have shown that the IBA can be used to exactly determine the size, shape and orientation of all convex voids with a finite radius of curvature everywhere.
IV. DISCUSSION AND CONCLUSIONS

In this paper we have argued that given perfect data the IBA can exactly reconstruct the shape of a convex void. Related results can be obtained for other flaw types. For example, the IBA will generate a surface of discontinuity at the correct boundary of a convex inclusion. However, it may also generate other non-physical surfaces of discontinuity (and other non-analytic functions) due to multiple scattering of the incident beam within the inclusion. The proof is essentially similar to that discussed in Sections II and III. A preliminary survey also indicates that the class of flaws can be extended from convex objects to those with the following features. Namely, at each point of the flaw surface one can draw a normal which extends to infinity and which nowhere crosses the body of the flaw. An example is the flaw shown in Fig. 4, to which the IBA has been successfully applied.

We have also studied crack-like defects with the IBA. The success of the computer experiments and the comparison of the scattering results with the Kirchhoff approximation suggests that this may be a fruitful area to explore for similar results.

The consequences of this result is to greatly expand our understanding of the IBA. As shown schematically in Fig. 5, the IBA is not only valid for weak scatters at all frequencies, but it is also valid for convex voids at high frequencies. Thus it provides a convenient interpolation formula for the inversion algorithm for many flaw types. As a final comment we note that any corrections to the IBA should preserve its validity in both of these regimes.

Fig. 4. Complex Flaw.
Fig. 5. Shows schematically the regions of validity of the IBA for a convex flaw. The cross-section indicates the region of the algorithm, whereas the blank regions indicate partial information in the IBA.

V. REFERENCES

14. G.D. Poe and J.L. Opsal, these proceedings.
DISCUSSION

D.O. Thompson (Ames Laboratory): Would you repeat again your argument as to why the Born works with both the weak and the strong scatter?

J.H. Rose (Ames Laboratory): We have shown, one, that the only delta function that matters in fitting the surface is the initial delta function, and two, that both the Born and the exact result give the delta function exactly at the same time, and they're proportional by a factor of two. So, both theories have exactly the same delta function structure on the initial delta function. I've shown that the final delta function was redundant, so it gives no further information. Therefore, if they both have the same delta function structure and if the inverse transforms the Born delta functions exactly right, then it must transform the exact delta functions exactly right.

R.C. Addison (Rockwell International Science Center): You said that this worked with strong scatterers in the sense that they're voids?

J.H. Rose: Yes.

R.C. Addison: What did you mean by that?

J.H. Rose: I just meant they were voids.

R.C. Addison: Are you considering any other strong scatterers?

J.H. Rose: Any strong scatterer whose signal is dominated by the front surface echo will work approximately. If you have an inclusion and you have other echoes, you might get other possible reconstructed discontinuities. If the front surface delta function dominates everything then, of course, those other delta functions will be very weak, and the other possible rings, halos, would be quite small.

C.M. Fortunko (National Bureau of Standards): Can you get two delta functions?

J.H. Rose: We don't know, because we don't have the numerical calculation yet. You can have a path which comes down, hits the concavity again and comes straight back at you, and that might very well foul things up. On the other hand, the algorithm requires a D second E, an integral for D phi, D cosine theta, and you have to be over a full integral over that, so an isolated delta function doesn't hurt you. You have to get a whole coherent set of delta functions so it is possible that it will work even for some cases of concavity.