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Efficiency with Endogenous Population and Fixed Resources*

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Abstract

This article studies socially optimal allocations, in the first-best sense, in environments characterized by fixed resources and endogenous fertility. Individuals in our environment are fully rational and altruistic toward their descendants, the social planner is benevolent, and there is full information. Our model allows for rich heterogeneity of abilities, preferences for children, and costs of raising children. We find that efficient allocations in the endogenous fertility case differ significantly from its exogenous fertility counterpart. In particular, optimal steady-state population is proportional to the amount of fixed resources and to the level of technology while steady state individual consumption is independent of these variables, a sort of "Malthusian stagnation" result. Furthermore, efficient allocations exhibit inequality, differential fertility, random consumption, and a higher population density of poorer individuals. We prove a version of the second welfare theorem: efficient allocations can be decentralized through competitive markets and an initial distribution of property rights over the fixed resources.

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JEL Classification: D04, D10, D63, D64, D80, D91, E10, E60, I30, J13, N00, 011, 040, Q01.

1 Introduction

There is growing interest in understanding equilibrium and efficiency properties of economies characterized by endogenous fertility (e.g. Golosov et al. 2007, Conde-Ruiz et. al. 2010, Hosseini et. al. 2013, Schoonbroodt and Tertilt 2014, Pérez-Nievas et. al. 2018). As this literature makes clear, usual notions of efficiency may not apply when population is endogenous. This article contributes to this literature by studying in detail a case of major historical importance: the Malthusian case. In particular, we investigate the properties of socially optimal allocations, in the first-best sense, in environments characterized by fixed resources and endogenous fertility.

Malthusian models have recently gained renewing interest as part of a larger literature seeking to provide a unified theory of economic growth, from prehistoric to modern times (e.g., Becker, Murphy, and Tamura 1990, Jones 1999, Galor and Weil 2000, Lucas 2002, Hansen and Prescott

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2002, Doepke 2004). The focus of this literature has been mostly positive rather than normative: to describe mechanisms for the stagnation of living standards even in the presence of technological progress. But the fundamental issue of efficiency in Malthusian economies, which is key to formulating policy recommendations and better understand the extent to which the "Malthusian trap" could have been avoided, has received scarce attention. This paper seeks to fill this gap.

Our model economy is populated by a large number of finitely-lived fully rational individuals who are altruistic toward their descendants. Individuals are of different types and a type determines characteristics such as labor skills, the rate of time preference, and the ability to raise children. Types are stochastic and determined at birth. We formulate the problem faced by a benevolent social planner who cares about the welfare of all potential individuals, present and future. The planner directly allocates consumption and number of children to individuals in all generations and states subject to aggregate resource constraints, promise-keeping constraints, population dynamics, and a fixed amount of natural resources. For short, we call "land" the fixed resource. The economy is closed, there is no capital accumulation nor migration. Furthermore, there is neither underlying frictions, such as private information or moral hazard, so that the focus is on first-best allocations. The rich structure of the model allows us to study questions of aggregate efficiency as well as distributional issues such as optimal inequality, social mobility and social classes. Distributional considerations can be particularly challenging. Lucas (2002) has shown that inequality could be hard to sustain as an equilibrium result in Malthusian economies. We show that inequality arises naturally in our environment.

The following are the main findings. First, we show that stagnation of the Malthusian type is efficient. Specifically, steady-state consumption is independent of the amount of land, and under general conditions, the level of technology. As a result, land discoveries, such as the ones discussed by Malthus, lead to more steady-state population but no additional consumption. A similar prediction holds true for technological advancements as long as the production function is Cobb-Douglas or technological progress is of land-augmenting, the type of progress that is more needed because land is the limiting factor.

The source of the stagnation is a well-known prediction of endogenous fertility models, according to which optimal consumption is proportional to the net costs of raising a child. For example, Becker and Barro find that "*when people are more costly to produce, it is optimal to endow each person produced with a higher level of consumption. In effect, it pays to raise the 'utilization rate' (in the sense of a higher c) when costs of production of descendants are greater* (Becker and Barro, 1988, pg. 10)." We show that this link between optimal consumption and the net cost of raising children also holds for a benevolent planner and under more general conditions. The crux of the proof of stagnation is to show that neither land discoveries, nor technological progress, alter the steady-state net cost of raising a child, and in particular the marginal product of labor, which is needed to value both the parental time costs of children and children's marginal output.

Second, we show that efficient allocations exhibit social classes. Only types with the highest rate of time preference have positive population shares and consumption shares in steady state. Furthermore, unlike the exogenous fertility case, it is generally not efficient to equalize consumption among types, even if their Pareto weights are identical, nor to eliminate consumption risk. Efficient consumption is stochastic even in the absence of aggregate risk. These results are further implications of consumption being a function of the net cost of raising children. In an efficient allocation, poor individuals are the ones with the lowest net costs of raising children.[1]

Third, there is an inverse relationship between consumption and population size: the lower the

[1] Córdoba and Liu (2014, 2016) and Cordoba, Liu and Ripoll (2016) provide further characterizations of models with endogenous fertility and idiosyncractic risk.

consumption of a type the larger its population share. It is efficient to let individuals with lower net costs of having children to reproduce more; but it is also optimal to endow their children with lower consumption, one that is proportional to those net costs. As a result, there are more poor individuals than rich individuals in an efficient allocation. Furthermore, population differences among types are larger than their corresponding consumption differences. The factor controlling the differences depends positively on the elasticity of parental altruism to the number of children, and negatively on the intergenerational elasticity of substitution.[2]

Fourth, fertility differs among types. Optimal fertility depends on parental types but also on grandparent types. Given grandparent types, parents with low consumption have more children than parents with high consumption. Given parent types, consumption rich grandparents have more grandchildren than consumption poor grandparents.

Fifth, steady-state allocations, and in particular the land-labor ratio, generally depends on initial conditions. The efficient steady state depends on the initial distribution of population and on Pareto weights. This is unlike the neoclassical growth model in which the efficient capital-labor ratio, or modified golden rule level of capital, is independent of initial conditions and Pareto weights. Malthusian economies thus do not exhibit a clear separation between efficiency and distribution.

At the light of these results, efficient allocations could rationalize three key aspects of Malthusian economies: (i) stagnation of individual consumption in the presence of technological progress and/or improvements in the availability of land; (2) social classes, inequality, and widespread poverty; (3) differential fertilities. These results could also help explain why the so-called Malthusian trap was so pervasive in pre-industrial societies. Even in the best case scenario of an economy populated by loving rational parents, and governed by an all-powerful benevolent rational planner, stagnation could still naturally arise, as well as social classes and differential fertility. Our results also suggest that is not irrational animal spirits, as suggested by Malthus, which ultimately explains the stagnation. Stagnation can be the result of a social optimal choice between the quality and quantity of life in the presence of limited natural resources.

Our paper is related to Golosov, Jones and Tertilt (2007) who have shown that population is efficient in dynastic altruistic models of endogenous fertility and fixed land of the Barro-Becker type. They do not derive results about stagnation, the distribution of consumption and population, nor differential fertility. Moreover, they elaborate on the Pareto concept of efficiency while we study efficiency from the point of view of utilitarian social planners. We discuss the connection between the two concepts using our framework.

Lucas (2002) studies equilibrium in Malthusian economies populated by altruistic fully rational parents. His focus is on simple representative economies where fertility is equal across groups, in steady state. He shows that stagnation arises under certain conditions. Lucas discusses the difficulties in generating social classes. He is able to generate classes by assuming heterogeneity in the degree of time preference and binding saving constraints. As a result, the equilibrium with social classes is not efficient in his model. We are able to generate efficient social classes and differential fertility by allowing individuals to differ in their labor skills and costs of raising children.

Our paper also relates to Dasgupta (2005) who studies the optimal population in an endowment economy with fixed resources. He does not consider the cost of raising children and focuses on the special case of generation-relative utilitarianism. Our model is richer in production, altruism, and the technology of raising children. Nerlove, Razin and Sadka (1986) show that the population in the competitive equilibrium is efficient under two possible externalities. First, a larger population helps

[2] The intergenerational elasticity of substitution is analogous to the intertemporal elasticity of substitution but applied to different generations rather than different periods. See Cordoba and Ripoll (2018).

to provide more public goods such as national defense. Second, larger population reduces wage rate if there is a fixed amount of land. Eckstein, Stern, and Wolpin (1988) show that population can stabilize and non-subsistence consumption arises in the equilibrium when fertility choices are endogenously introduced into a model with a fixed amount of land. Parents exhibit warm glow altruism while our paper builds on pure altruism. De la Croix (2012) studies sustainable population by proposing non-cooperative bargaining between clans living on an island with limited resources. Children in his model act like an investment good for parents' old-age support.

The rest of the paper is organized as follows. Section 2 sets up the general stochastic model. Section 3 studies the deterministic representative agent version of the general model and derives the main stagnation results. Section 4 considers deterministic heterogeneity and derives key results regarding the distribution of population and consumption across types, as well as the importance of initial conditions for the steady state. Section 5 studies the full stochastic model and derives the key result for differential fertility, consumption, and population. Section 6 concludes. Proofs are provided in the Appendix.

2 Motivating case: Barro-Becker with Fixed Resources

This section derives a set of baseline results that arise when fertility is endogenous and some essential factors of production are in fixed supply. In particular, we consider equilibrium allocations in an agricultural society populated by Barro-Becker families who have access to a Cobb-Duglass technology in land and labor.[3] Land should be understood more generally as to include all resources in fixed supply. Golosov et. al. (2007) have shown that the equilibrium of such an economy is $A-$efficient and $P-$efficient.[4] They did not explore other properties of the equilibrium allocations.

We highlight three qualitative differences between the endogenous and the exogenous fertility versions of the model. First, steady-state individual consumption is unaffected by technological progress or discovery of new resources when fertility is endogenous while it fully responds when fertility is exogenous. Second, steady-state population fully responds to technological progress or discoveries of new resources when fertility is endogenous while, by assumption, population is unaffected in the exogenous fertility case. Third, any initial inequality in land holdings vanishes when population is endogenous while it perpetuates when population is exogenous. The first two predictions of the model are consistent with stylized facts of the Malthusian era, as described for example by Ashraf and Galor (2011), while the third is not. The next section significantly generalizes the benchmark model of this section by considering social optima, rather competitive equilibria, under more general preferences, technologies, heterogeneity, stochastic components, and social mobility.

Households. The economy under consideration has a mass 1 of agents at time 0. Let $i \in [0,1]$ denotes an individual in the beginning of time, who is also the head of dynasty i. Individuals are initially endowed with k_{i0} (≥ 0) units of land. The aggregate amount of land is fixed and given by $K = \int_0^1 k_{i0} di$. Time is discrete: $t = 0, 1, 2, \ldots$.

Individuals live for two periods, one as children and one as adults. Children do not consume. A time$-t$ adult from dynasty i consumes c_{it} and has n_{it} children. Let r_t be the rental rate of land and q_t be its price. There are three costs of raising a child: a goods cost η units per-child, a time cost λ units of labor per child, and the cost of providing k_{it+1} units of land per child, $q_t k_{it+1}$. Adults are subject to a budget constraint of the form:

$$c_{it} + (\eta + q_t k_{it+1})\, n_{it} \leq w_t\,(1 - \lambda n_{it}) + (r_t + q_t)\, k_{it} \quad \text{for } t \geq 0,$$

[3] Dynastic preferences are modelled as in Becker and Barro (1988) and Barro and Becker (1989).

[4] We discuss these concepts in Section 3.

where w_t is the wage rate and $1-\lambda n_{it}$ is the labor supply.

Parents are assumed to be altruistic toward their children. In particular, the lifetime utility of a time$-t$ adult, U_{it}, takes the following recursive time-additive form due to Barro and Becker (1989) and Becker and Barro (1988):

$$U_{it}=\frac{c_{it}^{\xi}}{\xi}+\beta n_{it}^{\psi}U_{it+1},\ \xi\in(0,1)\ \text{ and }\psi\in(\xi,1)\,. \tag{1}$$

Let N_{it} be the size of dynasty i at time t and $N_t=\int_0^1 N_{it}di$ be the total population at time t. N_{it} is defined as

$$N_{it}=\prod_{s=0}^{t-1}n_{is}\text{ for }i\in[0,1]\ \text{ and }t=1,2,...\ . \tag{2}$$

The dynastic problem can then be described as

$$\max_{\{c_{it},k_{it+1},N_{it+1}\}_{t=0}^{\infty},}\sum_{t=0}^{\infty}\beta^{t}N_{it}^{\psi}u\left(c_{it}\right) \tag{3}$$

subject to

$$c_{i,t}+\left(\eta+w_t\lambda+q_tk_{it+1}\right)\frac{N_{it+1}}{N_{it}}\leq w_t+\left(r_t+q_t\right)k_{it}\text{ for }t\geq0, \tag{4}$$

given $k_{i0}>0$. The objective function in (3) is obtained from (1) by recursively eliminating U_{it+s} for $s>0$ and imposing standard boundedness conditions. The budget constraint uses the result that $n_{it}=\frac{N_{it+1}}{N_{it}}$.

Firms. Firms produce using the Cobb-Douglas technology $F\left(K,L;A\right)=AK^{\alpha}L^{1-\alpha}$, which is constant returns in land, K, and labor, L. Parameter A refers to the state of technology. Firms hire labor and rent capital in competitive labor markets.

Resource constraints. The following are the resource constraints of the economy, for $t\geq0$:

$$\begin{aligned}\int_0^1 N_{it}k_{it}di &\leq K;\\ L_t &\leq \int_0^1 N_{it}\left(1-\lambda n_{it}\right)di;\ \text{and}\\ \int_0^1 N_{it}c_{it}di+\eta\int_0^1 N_{it+1}di &\leq F\left(K_t,L_t,A\right).\end{aligned} \tag{5}$$

The first equation states that total individual holdings of land are no larger than the total amount of land available. Assuming that each individual has one unit of labor endowment, the second equation states that labor supply equals total population minus the time cost of raising children. The last equation states that total consumption is no larger than total production.

The following definition of competitive equilibrium is standard.

Definition of equilibrium: Given an initial distribution of land and population, $\{k_{i0},N_{i0}\}_{i\in[0,1]}$, a competitive equilibrium are sequences of prices, $\{q_t,r_t,w_t\}_{t=0}^{\infty}$ and allocations $\left\{c_{it}^{*},n_{it}^{*},k_{it+1}^{*},N_{it+1}^{*}\right\}_{t=0,i\in[0,1]}^{\infty}$ such that (i) given prices, the allocations solve the dynastic problem; (ii) land, labor, and good markets clear.

Equilibrium: Let $R_{t+1}\equiv\frac{r_{t+1}+q_{t+1}}{q_t}$ be the gross return. The following four equations characterize the determination of land returns, consumption, fertility and wages for $t>0$:[5]

$$\left(\frac{c_{it}^{*}}{c_{it+1}^{*}}\right)^{\xi-1}=\beta\left(n_{it}^{*}\right)^{\psi-1}R_{t+1}=\beta\left(n_{t}^{*}\right)^{\psi-1}R_{t+1}\text{ for }t\geq0\text{ and }i\in[0,1]\,, \tag{6}$$

$$c_{it+1}^{*}=c_{t+1}^{*}=\frac{\xi}{\psi-\xi}\left[R_{t+1}\left(\eta+w_t\lambda\right)-w_{t+1}\right]\text{ for }t>0\text{ and }i\in[0,1]\,, \tag{7}$$

[5] Section 7 provides the solution to a generalized version of this model.

$$c_{it}^{*}=A\left(\frac{K}{L_t^{*}}\right)^{\alpha}(1-\lambda n_{it}^{*})-\eta n_{it}^{*}\quad \text{for } t\geq 0, \tag{8}$$

$$w_t^{*}=(1-\alpha)\,A\left(\frac{K}{L_t^{*}}\right)^{\alpha}\quad \text{for } t\geq 0. \tag{9}$$

The first equation is an intergenerational version of the Euler Equation with the special feature that the discount factor, $\beta n_{it}^{\psi-1}$, depends on the number of children. The second equation is derived by combining first-order conditions for fertility and land holdings plus the budget constraint. It states that consumption is equal for all individuals after period 0 and proportional to the net future value costs of raising a child: $R_{t+1}\left(\eta+w_t\lambda\right)-w_{t+1}$. Becker and Barro (1988) first derived this result for a representative agent economy while Bosi et al (2011) extended it to an heterogeneous agent economy. Becker and Barro describe this fundamental finding as follows: "*When people are more costly to produce, it is optimal to endow each person produced with a higher level of consumption. In effect, it pays to raise the 'utilization rate' (in the sense of a higher c_i) when costs of production of descendants are greater." (Barro and Becker, 1989, pg. 484).*

Equation (8) is the resource constraint in per-capita terms while Equation (9) defines equilibrium wages. Plugging (6) into (7), it follows that fertility is the same for all dynasties, that is $n_{it}=n_t$ for all $t>0$. Thus, initial differences in land holding among individuals do not persist through differences in consumption or fertility of their descendants. Instead, they persist through differences in population sizes, N_{it}^{*}. This is in contrast with the exogenous population version of the model in which any initial inequality in land holdings would translate into persistent consumption differences.[6]

Steady state. For concreteness we now focus on steady state solutions of the type $c_t^{*}=c^{*}$, $n_t^{*}=n^{*}$, $N_{it}^{*}=N_i^{*}$, $N^{*}=N^{*}$, $L^{*}=(1-\lambda)\,N^{*}$, $w_t=w^{*}=F_L\left(K,L^{*};A\right)$ and $R_t=R^{*}$. A steady state population requires fertility, n^{*}, to be 1. In that case, (6) reduces to the standard gross return determination, $\beta R^{*}=1$, while equations (7)-(9) can be written as the following system of two equations in two unknowns, c^{*} and w^{*}:

$$c^{*}=\frac{\xi}{\psi-\xi}\left[\eta/\beta+(\lambda/\beta-1)\,w^{*}\right],\ \text{and} \tag{10}$$

$$w^{*}=\frac{1-\alpha}{1-\lambda}\left(c^{*}+\eta\right). \tag{11}$$

Surprisingly, neither A nor K appear in this system. Consequently, steady-state consumption and wages are independent of the technological level, A, or the fixed amount of resources, K, when fertility is endogenous. In contrast, both consumption and wages increase with A and K when fertility is exogenous (see Footnote 6).

Substituting (11) into (10) and solving for c^{*} results in:

$$c^{*}=c\left(\underset{+}{\eta},\underset{+}{\lambda},\underset{-}{\beta},\underset{-}{\psi},\underset{+}{\xi}\right)=\frac{1/\beta-(1-\lambda/\beta)\,\frac{1-\alpha}{1-\lambda}}{\frac{\psi-\xi}{\xi}+(1-\lambda/\beta)\,\frac{1-\alpha}{1-\lambda}}\eta. \tag{12}$$

According to this equation, consumption is a positive function of the goods and time costs of raising children. Use equations (9) into (11) to find the following solution for population:

$$N^{*}=N\left(\underset{+}{A},\underset{+}{K},\underset{-}{\eta},\underset{-}{\lambda},\underset{+}{\beta},\underset{+}{\psi},\underset{-}{\xi}\right)=\left[\frac{(1-\lambda)^{1-\alpha}A}{c\left(\eta,\lambda,\beta,\psi,\xi\right)+\eta}\right]^{1/\alpha}K. \tag{13}$$

Steady-state population responds negatively to consumption, and its determinants, but in addition, it increases with technology and resource availability.

[6] If $n=1$ is imposed then the equilibrium would satisfy $k_{it}=k_{i0}$, $c_{it}=c_i=w\,(1-\lambda)+rk_{i0}-\eta$, $w=F_L\left(K,(1-\lambda)\,N;A\right)$ and $r=F_K\left(K,(1-\lambda)\,N;A\right)$ for all $t\geq 0$.

Next, we investigate the extent to which the predictions of the canonical model of this section are robust to various generalizations. A natural reaction to the above results is that they are just a special case that arises due to the combination of power utility, power altruism, and a Cobb-Douglas technology. Moreover, if the stagnation results could be overturned then the prediction of an equalitarian society may not follow.

3 Socially Efficient Allocations

This section considers social optima for more general specifications of preferences, technologies, and rich heterogeneity in earning abilities, ability to raise children and altruism toward children, as well as stochastic intergenerational transmission of abilities and altruism. We focus on planner's solutions rather than market equilibria for at least two reasons. First, characterizing first-best allocations is of interest on its own. One may suspect, for example, that some of the baseline results, such as consumption not responding to technological progress, may be due to some type of market failure. Second, planner's solutions are often simpler to obtain than market solutions particularly when fertility is endogenous and parents are fully altruistic. In this case, Barro-Becker preferences provide a tractable benchmark but little is known for more general altruistic preferences. A contribution of the paper is to provide a methodology to study Pareto allocations when fertility is endogenous. Section 4 addresses issues of descentralization. Issues of stability are discussed in the Appendix.

We find that the first two features of the equilibrium solution are generally also properties of the planner's solution: consumption remains unresponsive to discoveries of new natural resources or to technological progress in the Cobb-Douglas case, or to land-augmenting technological progress for a more general formulation of the production function. Long-run equality, however, is not a robust feature. We are able to generate long-run inequality and social mobility in the stochastic version of the model. In this regard, our approach is very different from Lucas (2002). He is able to generate two social classes, with no possibility of social mobility, by assuming that the rate of time discount is a non-monotonic function of consumption. This non-standard feature is violated, for example, by the Barro-Becker model. We instead rely on stochastic types which result, even in the full information case, in inequality and social mobility.

3.1 Resource constraints

The production technology is described by the function $F(K, L; A)$ where K is a fixed amount of land, L is labor, and A is a technological parameter. Suppose F is constant returns to scale in K and L. Let $\alpha\left(\frac{K}{L}, A\right) \equiv \frac{F_K(K,L;A)K}{F(K,L;A)}$ denote the land share of output. As in section 2, the economy is populated by large numbers of dynastic altruistic individuals who live for two periods, one as a child and one as an adult. Children do not consume. Individuals are heterogeneous in terms of labor skills, rates of time preference, and ability to raise children. In particular, individuals draw a random signal, or type, $\omega \in \Omega \equiv \{\omega_1, \omega_2, ..., \omega_K\}$, upon birth which defines his or her type. Given parental signal ω_t, child's signals ω_{t+1} are drawn from the Markov chain $\pi(\omega', \omega) = \Pr(\omega_{t+1} = \omega' | \omega_t = \omega)$. Assume π is irreducible. Let $\omega^t = [\omega_0, \omega_1, ..., \omega_t] \in \Omega^{t+1}$ represents a particular family history of signals up to time t while $c_t(\omega^t)$ and $n_t(\omega^t)$ denote consumption and fertility of an individual with that family history. Effective labor supply, $l(\omega)$, degree of altruism, $\Phi(n, \omega)$, and the goods and time costs of raising a child, $\eta(\omega)$ and $\lambda(\omega)$, are then functions of an individual's type.

Let $N_t(\omega^t)$ be the population with history ω^t and $N_t \equiv \sum_{\omega^t} N_t(\omega^t)$ be total population at time t. Initial levels of population of each type, $N_0(\omega_i)$, $\omega_i \in \Omega$, are given. Assuming a law of large numbers, population with history $\omega^{t+1} \in \Omega^{t+2}$ obeys the following law of motion:

$$N_{t+1}\left(\omega^{t+1}\right) = N_t\left(\omega^t\right) n_t\left(\omega^t\right) \pi\left(\omega_{t+1}, \omega_t\right) \text{ for } t \geq 0. \tag{14}$$

Fertility rates are assumed to be subject to a biological maximum of $\overline{n}$. The potential population at history ω^{t+1} is therefore $\overline{N}_{t+1}\left(\omega^{t+1}\right) = \overline{N}_t\left(\omega^t\right)\overline{n}\pi\left(\omega_{t+1}, \omega_t\right)$ with $\overline{N}_0\left(\omega^0\right) = N_0\left(\omega_0\right)$. Aggregate labor supply satisfies

$$L_t = \sum_{\omega^t} N_t\left(\omega^t\right) l\left(\omega_t\right)\left[1 - \lambda_t\left(\omega_t\right) n_t\left(\omega^t\right)\right] \text{ for } t \geq 0, \tag{15}$$

where $l\left(\omega_t\right)\left[1 - \lambda_t\left(\omega_t\right) n_t\left(\omega^t\right)\right]$ is effective individual labor supply of a particular type once time costs of raising children and individual's ability are taken into account. Finally, aggregate resource constraints are given by

$$F(K, L_t; A) = \sum_{\omega^t} N_t\left(\omega^t\right)\left[c_t\left(\omega^t\right) + \eta\left(\omega_t\right) n_t\left(\omega^t\right)\right] \text{ for } t \geq 0. \tag{16}$$

3.2 Individual welfare

Parents are assumed to be altruistic toward their children. The lifetime utility of an individual born at time $t \geq 0$, history ω^t, $U_t\left(\omega^t\right)$, is of the expected-utility form:

$$\begin{aligned} U_t\left(\omega^t\right) &= u\left(c_t\left(\omega^t\right)\right) + \Phi\left(n_t\left(\omega^t\right), \omega\right) E\left[U_{t+1}\left(\omega^{t+1}\right)|\omega^t\right] \\ &\quad + \left(\Phi\left(\overline{n}, \omega\right) - \Phi\left(n_t\left(\omega^t\right), \omega\right)\right)\underline{U}, \end{aligned} \tag{17}$$

where $u(\cdot)$ is the utility flow from consumption, $\Phi(\cdot, \omega)$[7] is the weight that a parent of type ω attaches to the welfare of her n born children, $\Phi\left(\overline{n}, \omega\right) - \Phi\left(n, \omega\right)$ is the weight attached to the unborn children, $E\left[U_{t+1}\left(\omega^{t+1}\right)|\omega^t\right]$ is the expected utility of a born child conditional on parental history and $\underline{U}$ is the utility of an unborn child as perceived by the parent. Function u satisfies $u' > 0$ and $u'' < 0$. The population ethics literature refers to $\underline{U}$ as the "neutral" utility level, a level above which a life is worth living (Blackorby *et al.* 2005, pg. 25).

Equation (17) describes parents as social planners at the family level. This is particularly clear in the special case $\Phi(n, \omega) = n$. The more general function $\Phi(\cdot, \omega)$ allows for flexible weights and time discounting. While $\Phi(n, \omega)$ is the total weight of the n born children, $\Phi_n(n, \omega)$ is the marginal weight assigned to the $n-$child where $n \in [0, \overline{n}]$. We assume $\Phi_n(n, , \omega) > 0$ and $\Phi_{nn}(n, \omega) \leq 0$ so that parents are altruistic toward each child and marginal altruism is non-increasing. These preferences are discussed in Cordoba and Ripoll (2011) who show that (17) satisfies a fundamental axiom of altruism. Specifically, parental utility increases with the number of born children if and only if children are better off born than unborn in expected value, that is, $E\left[U_{t+1}\left(\omega^{t+1}\right)|\omega^t\right] > \underline{U}$.

Let $\beta(\omega) \equiv \Phi(1, \omega)$ be the discount factor, $\xi(c) \equiv \frac{u'(c)c}{u(c)}$ be the elasticity of the utility flow and $\psi(n, \omega) \equiv \frac{\Phi'(n,\omega)n}{\Phi(n,\omega)}$ be the elasticity of the altruistic function. Barro-Becker preferences are an special case obtained when $u(c) = \frac{c^\xi}{\xi}$, $\Phi(n, \omega) = \beta n^\psi$, $\underline{U} = 0$, $\xi \in (0, 1)$ and $\psi \in (\xi, 1)$.

3.3 Social Welfare

The planner is envisioned as the ultimate parent, someone who cares about the welfare of all potential individuals in the society. Consistent with (17), it is natural to consider a social welfare function that takes the following generalized total utilitarian form:

$$\sum_{t=0}^{\infty} \delta^t \left[\sum_{\omega^t} \Psi\left(N_t\left(\omega^t\right)\right) U_t\left(\omega^t\right) + \left(\sum_{\omega^t} \Psi\left(\overline{N}_t\left(\omega^t\right)\right) - \sum_{\omega^t} \Psi\left(N_t\left(\omega^t\right)\right)\right)\underline{U}\right]. \tag{18}$$

[7] In the rest of this paper we sometimes omit the second argument for succinct writing.

The parameter $0 < \delta < 1$ reflects time discounting while the function $\Psi\left(N_t\left(\omega^t\right)\right)$, satisfying $\Psi'(\cdot) \geq 0$ and $\Psi''(\cdot) \leq 0$, is the weight that the social planner puts on group $N_t\left(\omega^t\right)$. The special case $\Psi\left(N_t\left(\omega^t\right)\right) = N_t\left(\omega^t\right)$ describes a classical total utilitarian planner while $\Psi\left(N_t\left(\omega^t\right)\right) = 1$ describes an Millian average utilitarian. The function $\Psi(N) = N^{\psi_p}$ is the natural counterpart of Barro-Becker's altruism but applied to the planner. Denote by $\psi_P(N) = \frac{\Psi'(N)N}{\Psi(N)}$ the elasticity of function Ψ.

The case $\delta = 0$ is defined as

$$\sum_{\omega} \Psi\left(N_0(\omega)\right) U_0(\omega).$$

It refers to a planner who cares only about the initial generation but also future generations to the extent that the initial generation does. In this case social discounting equals private discounting and the problem becomes one of dynastic maximization. $\delta > 0$ refers to a planner who is more patient than individuals, as in Farhi and Werning (2007).

Our social welfare function does not allow for arbitrary Pareto weights, and thus we do not trace the full Pareto frontier. However, the function is tractable and includes the key relevant cases usually considered in the literature: classical utilitarianism and Mills utilitarianism. By construction, the maximizing allocation will be Pareto efficient, or $P-$efficient following Golosov et al. (2007) terminology. In particular, the welfare of all potential individuals, born and unborn, is explicitly considered by the planner. Golosov et al. (2007) emphasize a second efficiency concept, $A-$efficiency, which takes into account only the welfare of individuals born in all feasible allocations. As shown recently by Pérez-Nievas et al. (forthcoming), $A-$efficiency is equivalent to dynastic maximization, which in our our model corresponds to the case $\delta = 0$, if potential people are identified by the date they may be born rather than by a birth order rule, as assumed by Golosov et. al. (2007). Since our model allows for $\delta \geq 0$, then both $A-$efficiency and $P-$efficient allocations are characterized.

The welfare function (18) is a version of NG's (1986) *number-dampened total utility* generalized to include multiple periods and time discounting. The standard reasoning for considering number dampening, or alternative social criteria such as the "critical-level utilitarianism" of Blackorby and Donalson (1984), is to avoid the Repugnant Conclusion. An allocation is "repugnant" when it entails maximum population and minimum utility, or immiseration. The Repugnant Conclusion does not apply in our environment, as shown below, because parental rights are explicitly considered and children are costly to raise (Hammond 1988).

Although our main results hold for a standard total utilitarian, it is useful to consider the number-dampening case for two reasons: (i) it is natural given that parents in our model exhibit such behavior, of diminishing returns to family size; and (ii) it turns out to be important for time consistency and uniqueness of the steady state.

The following assumption bounds the extent to which the planner cares about future generations.

Assumption 1. $\delta < \beta(\omega)$ for all ω.

The role of Assumption 1 is tractability. The assumption is not particularly restrictive because it still allows for the planner to care about future generations more than parents do. We leave the more complicated case $\delta \geq \beta(\omega)$ for the Appendix. We can now define the planner's problem.

Definition Given an initial distribution of population $\{N_0(\omega)\}_{\omega\in\Omega}$, the planner chooses sequences $\left\{U_t\left(\omega^t\right), c_t\left(\omega^t\right), n_t\left(\omega^t\right), N_{t+1}\left(\omega^{t+1}\right), L_t\right\}_{\omega^t\in\Omega^{t+1}, t\geq 0}$ to maximize social welfare (18) subject to sequences of resource constraints (16), labor supply (15), laws of motions for population (14) and individual welfare (17).

We assume throughout that the planner's problem is well defined, that the solution is unique, and refer to its solution as the optimal or efficient allocation. We also follow the standard practice in population ethics of normalizing the utility level $\underline{U}$ to zero (e.g., Blackorby *et al.* 2005, pg. 25). This means that, in the mind of parents and the planner, a life is worth living if and only if $U_t(\omega^t) \geq 0$. Since U_t can be written as a discounted sum of utility flows, we assume that $u(c) \geq 0$.

For clarity, it is convenient to write the Lagrangian corresponding to the planner's problem:

$$\begin{aligned}
\pounds \;=\; & \sum_{t=0} \delta^t \sum_{\omega^t} \Psi\left(N_t\left(\omega^t\right)\right) U_t\left(\omega^t\right) \\
& + \sum_{t=0}^{\infty} \sum_{\omega^t} \theta_t\left(\omega^t\right) N_t\left(\omega^t\right) \left[u(c_t\left(\omega^t\right)) + \Phi\left(n_t\left(\omega^t\right), \omega\right) E_t U_{t+1}\left(\omega^{t+1}\right) - U_t\left(\omega^t\right)\right] \\
& + \sum_{t=0}^{\infty} \sum_{\omega^{t+1}} \gamma_{t+1}\left(\omega^{t+1}\right) \left[N_{t+1}\left(\omega^{t+1}\right) - n_t\left(\omega^t\right) \pi\left(\omega_{t+1}, \omega_t\right) N_t\left(\omega^t\right)\right] \\
& + \sum_{t=0}^{\infty} \mu_t \left[F(K, L_t; A) - \sum_{\omega^t} N_t\left(\omega^t\right) \left[c_t\left(\omega^t\right) + \eta\left(\omega_t\right) n_t\left(\omega^t\right)\right]\right] \\
& + \sum_{t=0}^{\infty} \kappa_t \left[\sum_{\omega^t} N_t\left(\omega^t\right) l\left(\omega_t\right) \left[1 - \lambda_t\left(\omega_t\right) n_t\left(\omega^t\right)\right] - L_t\right],
\end{aligned}$$

where $\left\{\theta_t\left(\omega^t\right), \gamma_{t+1}\left(\omega^{t+1}\right), \mu_t, \kappa_t\right\}_{\omega^t \in \Omega^{t+1}, t \geq 0}$ are non-negative multipliers. We assume parameter values are such that solutions are interior.[8] The first restriction of the problem resembles a promise keeping constraint while the remaining restrictions are resource constraints. The first order conditions with respect to $\{U_t\left(\omega^t\right), N_{t+1}\left(\omega^{t+1}\right), n_t\left(\omega^t\right), c_t\left(\omega^t\right), L_t\}_{\omega^t \in \Omega^{t+1}, t \geq 0}$ are:[9]

$$\theta_0\left(\omega_0\right) N_0\left(\omega_0\right) = \Psi\left(N_0\left(\omega_0\right)\right), \tag{19}$$

$$\begin{aligned}
\theta_{t+1}\left(\omega^{t+1}\right) N_{t+1}\left(\omega^{t+1}\right) \;=\; & \delta^{t+1} \Psi\left(N_{t+1}\left(\omega^{t+1}\right)\right) \\
& + \theta_t\left(\omega^t\right) N_t\left(\omega^t\right) \Phi\left(n_t\left(\omega^t\right), \omega\right) \pi\left(\omega_{t+1}, \omega_t\right),
\end{aligned} \tag{20}$$

$$\begin{aligned}
& \delta^{t+1} \Psi'\left(N_{t+1}\left(\omega^{t+1}\right)\right) U_{t+1}\left(\omega^{t+1}\right) \\
& + \kappa_{t+1} l\left(\omega_{t+1}\right) \left[1 - \lambda_{t+1}\left(\omega_{t+1}\right) n_{t+1}\left(\omega^{t+1}\right)\right] + \gamma_{t+1}\left(\omega^{t+1}\right) \\
= \; & \mu_{t+1} \left[c_{t+1}\left(\omega^{t+1}\right) + \eta\left(\omega_{t+1}\right) n_{t+1}\left(\omega^{t+1}\right)\right] \\
& + n_{t+1}\left(\omega^{t+1}\right) \sum_{\omega^{t+2} | \omega^{t+1}} \gamma_{t+2}\left(\omega^{t+2}\right) \pi\left(\omega^{t+2}, \omega^{t+1}\right),
\end{aligned} \tag{21}$$

$$\begin{aligned}
& \theta_t\left(\omega^t\right) \Phi_n\left(n_t\left(\omega^t\right), \omega\right) E_t U_{t+1}\left(\omega^{t+1}\right) \\
= \; & \mu_t \eta\left(\omega_t\right) + \kappa_t l\left(\omega_t\right) \lambda_t\left(\omega_t\right) + \sum_{\omega^{t+1} | \omega^t} \gamma_{t+1}\left(\omega^{t+1}\right) \pi\left(\omega^{t+1}, \omega^t\right),
\end{aligned} \tag{22}$$

$$\theta_t\left(\omega^t\right) u'(c_t\left(\omega^t\right)) = \mu_t, \tag{23}$$

$$\mu_t F_{Lt} = \kappa_t. \tag{24}$$

where

$$F_{L,t} = \left(1 - \alpha\left(\frac{K}{L_t}, A\right)\right) \frac{F\left(K, L_t; A\right)}{L_t} \tag{25}$$

This system of equations together with (14), (15), (16), (17) and proper transversality conditions fully describe interior efficient allocations. Equation (19) states that the initial social value of

[8] For example, the Barro-Becker model possesses an interior solution under certain parameter restrictions.

[9] To avoid cumbersome notation, we do not introduce new notation to identify optimal allocations. Allocations from now on should be regarded as optimal.

providing utility to a particular group, $\theta_0(\omega_0) N_0(\omega_0)$, depends on the exogenous Pareto weight of group $N_0(\omega_0)$. Equation (20) then allows us to trace the evolution of this value. The right hand side of the equation is the marginal benefit of promising utility $U_{t+1}(\omega^{t+1})$ while the left hand side is its marginal cost. Notice that if the Markov chain π is irreducible, the planner eventually assigns social value, and therefore provides utility, to individuals of all types just because all dynasties eventually have descendants of every type. This would not be the case if π is reducible.

Equation (21) equates marginal benefits to marginal costs of having people. To better understand this expression, assume for a moment that population is not constrained by (14), for example, because the planner have access to an infinite pool of immigrants. In that case $\gamma_{t+1}(\omega^{t+1}) = 0$ for all t and ω^{t+1}. In other words, γ is the value of an immigrant. The marginal benefit of an additional individual of type ω^{t+1} includes her direct effect in social welfare, $\delta^{t+1}\Psi'(N_{t+1}(\omega^{t+1})) U_{t+1}$ plus her effect in the labor supply, $\kappa_{t+1} l(\omega_{t+1}) [1 - \lambda_{t+1}(\omega_{t+1}) n_{t+1}(\omega^{t+1})]$, while the marginal cost includes the costs of providing consumption and children to the individual, $\mu_{t+1}[c_{t+1}(\omega^{t+1}) + \eta(\omega_{t+1}) n_{t+1}(\omega^{t+1})]$. Adding restriction (14) makes the individual more valuable in the amount $\gamma_{t+1}(\omega^{t+1})$ because it relaxes the population constraint at $t+1$, but also increases marginal cost because the planner needs to endow the individual with children at $t+2$.

The condition for optimal fertility is (22). The marginal benefit of a child for an altruistic parent with history ω^t is the expected utility of the child, $E_t U_{t+1}$, times the weight that the parent attaches to the child, $\Phi_n(n_t(\omega^t), \omega)$. The marginal benefit for the planner is this amount times $\theta_t(\omega^t)$. The corresponding marginal cost of the child for the planner includes good costs, $\mu_t \eta(\omega_t)$, time costs, $\kappa_t l(\omega_t) \lambda(\omega_t)$, and the expected shadow costs of a descendant, $\sum_{\omega^{t+1}|\omega^t} \gamma_{t+1}(\omega^{t+1}) \pi(\omega^{t+1}|\omega^t)$.

To characterize the solution of this system we focus primarily on the steady state and proceed in three steps. First, we characterize the deterministic case with only one type (Section 4), then the case with multiple but deterministic types (Section 5) and finally the stationary solution with stochastic types. We show that Malthusian stagnation generally arises when technological progress is of the land-augmenting type meaning that steady-state optimal consumption and fertility choices are independent of K and A. We also characterize the optimal composition of population, the potential dependence of the steady-state land-labor ratio on initial conditions, and fertility differentials among types.

4 Deterministic case with one type

This section considers the representative agent case with only one type. Let $n(\omega) = n$, $\lambda(\omega) = \lambda$, $\eta(\omega) = \eta$, and $l(\omega) = 1$ for simplicity. In this case the resource constraint (16) reduces to:

$$F\left(\frac{K}{N_t}, 1 - \lambda n_t; A\right) = c_t + \eta n_t. \tag{26}$$

Moreover, using (23), (24), and (26), equations (19) to (22) simplify to:

$$\Psi(N_0) = \theta_0 N_0,$$

$$\delta^{t+1}\Psi(N_{t+1}) + \theta_t N_t \Phi(n_t) = \theta_{t+1} N_{t+1}, \tag{27}$$

$$\delta^{t+1}\Psi'(N_{t+1}) U_{t+1} + \gamma_{t+1} = \mu_{t+1} F_{K,t+1} \frac{K}{N_{t+1}} + n_{t+1}\gamma_{t+2}, \text{ and} \tag{28}$$

$$\Phi'(n_t) \frac{U_{t+1}}{u'(c_t)} = \eta + F_{L,t}\lambda + \frac{\gamma_{t+1}}{\gamma_t}\frac{\gamma_t}{\mu_t}. \tag{29}$$

Equation (28) is obtained from (10) after using (24), (26) and the constant returns to scale assumption. Equation (29) is obtained from (22), (23), and (24).[10]. The following proposition

[10] In deterministic model with one type, we write $\Phi(n)$ as $\Phi(n, \omega)$ because everyone has the same ability.

provides a sharp characterization of consumption for all periods, except period 0, for the special case $\Phi(n) = \beta n^{\psi}$.

Proposition 1 *Assume $\Phi(n) = \beta n^{\psi}$, $0 < \psi < 1$, and let $a_m \equiv \left(\frac{\delta}{\beta}\right)^m \frac{\Psi(N_m)}{\Phi(N_m)}$. Then efficient consumption satisfies:*

$$c_{t+1} = \frac{\xi(c_{t+1})}{\psi - \xi(c_{t+1})}\left[\frac{\mu_t}{\mu_{t+1}}(\eta + F_{L,t}\lambda) - F_{L,t+1} - \frac{a_{t+1}}{\sum_{m=0}^{t+1} a_m}\frac{U_{t+1}}{u'(c_{t+1})}(\psi_P(N_{t+1}) - \psi)\right] \tag{30}$$

This expression is similar, and in fact, generalizes equation (7). The term $\frac{\mu_t}{\mu_{t+1}}(\eta + F_{L,t}\lambda) - F_{L,t+1}$ is the net future cost of raising a child from the planner's perspective. The main difference is the last term in the brackets. This term equals zero in three cases: when $\psi_P = \psi$, $\delta = 0$, or $t \to \infty$. In essence, when planner's preferences differ from those of individuals, say because $\delta > 0$ or $\psi_P(N_{t+1}) > \psi$, then consumption is adjusted downward to free resources in order to expand population. But those adjustments are temporary given that $\delta < \beta$ is assumed. In the limit, consumption becomes a sole function of the net future cost of raising a child.

4.1 Steady state

Consider the steady state situation in which N, c, U and L are constant and $n = 1$. The following result holds for general functions F, u and Φ.

Lemma 2 *Steady state consumption satisfies:*

$$c = \frac{\xi(c)}{\psi(1) - \xi(c)}\left[\eta/\beta + (\lambda/\beta - 1)F_L\right]. \tag{31}$$

This expression generalizes (7). It shows that consumption is a function of the net costs of raising a child: $(\eta + \lambda F_L)/\beta - F_L$. The parametric restriction $\psi(1) > \xi(c)$ is needed for consumption to be positive. An implication of this result is that immiseration and the Repugnant Conclusion, of $c = 0$ and $N = \infty$, is not optimal unless the net cost of children is zero.

The resource constraint, equation (26), can be written as:

$$F_L = \frac{1 - \alpha(k; A)}{1 - \lambda}(c + \eta). \tag{32}$$

where $k = \frac{K}{L}$. Equations (31) and (32) form a system of two equations in three unknowns: c, F_L and α. In the case of a Cobb-Douglas production function the land share, α, is a parameter and these two equations can be used to solve for consumption, c, and the marginal product of labor, F_L, independently of K and A, as in Section 2.

In the more general case, equation ((25) is needed to close the system. It can be written as:

$$F_L = (1 - \alpha(k; A))\, F(k, 1; A). \tag{33}$$

Equations (31), (32) and (33) form a system of three equations in three unknowns: c, F_L and k. Since K does not appear in this system then the solutions for c, F_L and k are independent of the amount of land for any constant returns to scale production function. Given k, steady state population can be solved as $N = \frac{K}{(1-\lambda)k}$.

Finally, if A is a land-augmenting parameter then the land share is a function of $\widehat{k} = \frac{AK}{L}$. In that case equations (32) and (33) can be written as $F_L = \frac{1-\alpha(\widehat{k})}{1-\lambda}(c + \eta)$ and $F_L = \left(1 - \alpha\left(\widehat{k}\right)\right) F\left(\widehat{k}, 1\right)$, and the solutions for c, F_L and $\widehat{k}$ will be independent of A. Given $\widehat{k}$, steady state population is given by $N = \frac{AK}{(1-\lambda)\widehat{k}}$.

The following proposition summarizes these findings.

Proposition 3 *In steady state efficient consumption is independent of the amount of land while efficient population increases proportionally with the amount of land. Furthermore, if technological progress is land augmenting then efficient consumption is independent of the level of technology while efficient population increases proportionally with the level of technology.*

5 Deterministic case with multiple types

Consider now the case of multiple deterministic types. Specifically, suppose $\omega^t = [\omega, \omega, \omega ...]$ or just $\omega^t = \omega$ for short. We assume in this section $\delta = 0$. This restriction is without much loss of generality since similar steady state results would be obtained as long as $\delta < \beta(\omega)$, as shown in the previous section. For tractability we also restrict altruism to be of the Barro-Becker form, $\Phi(n, \omega) = \beta(\omega) n^{\psi}$, but allow general formulations for u and F.

We show the following results in this section. First, if $\beta(\omega)$ is different for different types then their population sizes grow at different rates and in steady state only the most patient groups, the ones with highest $\beta(\omega)$, survive. This result implies that efficient social classes cannot be sustained by persistent differences in rates of time preference. Lucas (2002) is able to generate social classes using such a mechanism in a competitive equilibrium with savings constraints which suggests that social classes are not efficient, in the first best sense, in his model.

As an alternative to Lucas (2002), we are able to generate multiple social classes using a more standard mechanism based on heterogeneity in labor skills, $l(\omega)$, and the cost of raising children, $\eta(\omega)$ and $\lambda(\omega)$. This is the second main result of the section. Efficiency requires providing more consumption to individuals with higher costs of raising children. Consumption also increases with labor ability, $l(\omega)$, but only if $\lambda(\omega) > \beta(\omega)$, that is, only if the time costs of raising children are sufficiently high. Otherwise, the efficient allocation involves the high skilled having lower consumption.[11]

Third, we show that the relative population size of a type is inversely related to its relative consumption. Therefore, the population of the poor is larger than the population of the middle class and so on. The planner thus faces a quantity-quality trade-off: she can deliver certain level of welfare by allocating number of children and/or consumption. If children are particularly costly to raise for a certain group, then the planner optimally delivers welfare more through consumption than through children and vice versa.

Fourth, in the deterministic steady state of this section all types have one child and therefore steady-state welfare differences among types only arise from differences in consumption. As a result, types with lower consumption are worse-off than types with higher consumption. All benefits from a larger population accrue only to early members of the dynasty at the expense of later members.

5.1 Dynamics

The following lemma characterizes the evolution of efficient population sizes of different types over time.

Lemma 4 *Let $(\omega, \omega') \in \Omega$. Efficient population sizes satisfy:*

$$\frac{N_t(\omega)}{N_t(\omega')} = \left(\frac{N_0(\omega)}{N_0(\omega')}\right)^{-\frac{\psi}{1-\psi}} \left[\frac{\Psi(N_0(\omega))}{\Psi(N_0(\omega'))} \frac{\beta(\omega)^t u'(c_t(\omega))}{\beta(\omega')^t u'(c_t(\omega'))}\right]^{\frac{1}{1-\psi}}. \tag{34}$$

We characterize the steady state next.

[11] This result could rationalize, for example, why high skilled women may end up having more children and low consumption compared with an equally skilled man. Extending the model to introduce gender differences is a promising agenda for future research.

5.2 Steady state

5.2.1 Distribution of population

Consider now a steady state in which consumption, population shares and population are constant. This requires $n(\omega) = 1$ for all types. In that case, equation (34) simplifies to:

$$\frac{N(\omega)}{N(\omega')} = \left(\frac{\beta(\omega)}{\beta(\omega')}\right)^{\frac{t}{1-\psi}} \left(\frac{N_0(\omega)}{N_0(\omega')}\right)^{-\frac{\psi}{1-\psi}} \left[\frac{\Psi(N_0(\omega))}{\Psi(N_0(\omega'))}\frac{u'(c(\omega))}{u'(c(\omega'))}\right]^{\frac{1}{1-\psi}}. \tag{35}$$

We can now state our next main result which is apparent from this equation.

Proposition 5 *In an interior steady state: (i) Only the most patient types, the ones with the highest $\beta(\omega)$, have positive mass; (ii) The distribution of population depends on the initial distribution unless $\Psi(N_0) = N_0^{\psi}$; In particular, it depends on the initial distribution in the classical utilitarian case; (iii) The relative population size of a particular type is inversely related to its per-capita consumption.*

The proof of this proposition is trivial and hence omitted. The first part of the proposition states that impatient types eventually disappear from the economy. Children are like an investment for altruistic parents as they deliver a stream of future utility flows. Impatient individuals discount future streams more heavily and therefore value children less than patient individuals do. As a result, it is efficient for the planner to provide more consumption to current individuals in exchange for fewer future family members.[12]

The second part of Proposition 5 states that the steady-state distribution of population depends on initial conditions, a result that is analogous to the dependence of the steady-state wealth distribution on initial conditions in the neoclassical growth model (Chatterjee 1994). However, as we see below, this dependence has more profound implications in Malthusian economics because the steady state aggregate land-labor ratio and steady state population depends on initial conditions, and Pareto weights, as well. This is in contrast to the neoclassical growth model where the golden rule level of capital is independent of initial conditions and Pareto weights. Efficiency and distribution are interdependent in Malthusian economies unless Pareto weights are of the form $\Psi(N) = N^{\psi}$, that is, Pareto weights resemble parental weights.

The third part of Proposition 5 shows a fundamental prediction of endogenous population models: an inverse relationship between population size and per-capita consumption. The lower the consumption of a type the larger its share of the total population. The reason is that the planner needs to deliver welfare by providing consumption and children to parents. Whenever the planner chooses to use one channel then it downplays the other.

We still need to solve for consumption to fully derive the consequences of this inverse relationship. For the rest of this section it is convenient to assume a specific functional form for $\Psi(\cdot)$, $\Psi(N) = N^{\psi_p}$, and restrict attention to the set of most patient types, $\Omega_p \subseteq \Omega$. That is, $\beta(\omega) = \beta$ for all $\omega \in \Omega_p$ and $\beta \geq \beta(\omega)$ for all $\omega \in \Omega$. Equation (35) thus simplifies to:

$$\frac{N(\omega)}{N(\omega')} = \left(\frac{N_0(\omega)}{N_0(\omega')}\right)^{\frac{\psi_p-\psi}{1-\psi}} \left(\frac{u'(c(\omega))}{u'(c(\omega'))}\right)^{\frac{1}{1-\psi}}, \ \omega \in \Omega_p. \tag{36}$$

Thus, the long term composition of the population depends on the initial distribution unless $\psi_p = \psi$. Moreover, the initial distribution of population tend to persist if $\psi_p > \psi$. Classical

[12] This result also helps qualify a commonly held view according to which the poor are inherently more impatient, less willing to save, and that their large families somehow reflects their impatience. According to our model, if the poor were really impatient, they would have fewer children and their type would eventually disappear from the population.

utilitarianism is represented by $\psi_p = 1$. In this case, the steady state distribution never resembles the initial distribution unless consumptions are equal across types which is not the case in general, as we show below.[13]

The following lemma characterizes the steady state distribution of population in terms of consumptions.

Lemma 6 *Let* $g(\omega) \equiv \frac{N(\omega)}{N}$. *Then*

$$g(\omega) = \frac{N_0(\omega)^{\frac{\psi_p - \psi}{1-\psi}} u'(c(\omega))^{1/(1-\psi)}}{\sum_{\omega' \in \Omega_p} N_0(\omega')^{\frac{\psi_p - \psi}{1-\psi}} u'(c(\omega'))^{1/(1-\psi)}} \quad \text{for all } \omega \in \Omega_p. \tag{37}$$

The lemma is important because it provides a simple description of the steady-state distribution of population in terms of the given initial distribution and steady-state consumptions.

5.2.2 Consumption

One can show, similarly to the first part of Proposition 5, that only the most patient types have positive consumption in a steady state. According to (23), for consumption to be constant $\frac{\theta_{t+1}(\omega)}{\theta_t(\omega)} = \frac{\mu_{t+1}}{\mu_t}$ is required. Otherwise, $\frac{\theta_{t+1}(\omega)}{\theta_t(\omega)} < \frac{\mu_{t+1}}{\mu_t}$ refers to a type for which consumption falls, and vice versa. Therefore, only the types with the highest ratio $\frac{\theta_{t+1}(\omega)}{\theta_t(\omega)}$ have positive steady state consumption. Moreover, according to (20), $\frac{\theta_{t+1}(\omega)}{\theta_t(\omega)} = \Phi(1,\omega) = \beta(\omega)$ at steady state. Therefore, $\frac{\theta_{t+1}(\omega)}{\theta_t(\omega)}$ is the highest for all $\omega \in \Omega_p$.

The following lemma provides the solution for consumptions in terms of the marginal product of labor.

Lemma 7 *Efficient consumption satisfies:*

$$c(\omega) = \frac{\xi(c(\omega))/\beta}{\psi(1,\omega) - \xi(c(\omega))} \left[\eta(\omega) + (\lambda(\omega) - \beta) F_L l(\omega)\right] \text{ for } \omega \in \Omega_p. \tag{38}$$

Equation (38), analogous to (31), shows that consumption is proportional to the net financial cost of a child. In particular, consumption is larger for types with a higher cost of raising children, either a higher goods cost $\eta(\omega)$ and/or a higher time cost $\lambda(\omega)$. The relationship between skills, $l(\omega)$, and consumption is slightly more complicated. If $\lambda(\omega) > \beta$ then efficient consumption is higher for highly skilled individuals. But if $\lambda(\omega) < \beta$, then efficient consumption is actually lower for the high skilled.

We can now state our next main result which follows from (35) and (38).

Proposition 8 *Steady-state efficient allocations exhibit inequality of consumptions and populations. Types with low consumption have a larger population.*

Proposition 8 is important for at least three reasons. First, as is discussed by Lucas (2002), obtaining an efficient allocation with heterogeneous social classes in Malthusian economies is not trivial yet important. Lucas's solution, which relies on differences in time discounting, generates inefficient social classes in presence of binding constraints. Different discount factors would still lead to only one social group surviving at steady state in an efficient allocation. Second, the efficient allocation can rationalize a distribution of social classes in which the poor are a larger fraction of the population. Third, the proposition also states that in a world where the planner can choose which types to reproduce or not it is not optimal to end a lineage just because it is of

[13] In the utilitarian case, efficient allocations are not time consistent because re-optimizing starting with an initial steady state distribution of population results in a different steady state distribution.

lower skill or poorer. This is in contrast to the literature that is in favor of limiting the fertility of the poor (e.g., Chu and Koo, 1990). Only impatient types disappear from an efficient allocation.

It is possible to obtain a final solution for consumptions and relative population sizes without knowing the marginal product of labor in the following special Barro-Becker case.

Example 9 *Suppose* $u(c) = c^{\xi}/\xi$ *with* $\xi \in (0,1)$, $\Phi(n) = \beta n^{\psi}$, $\psi \in (\xi, 1)$, $\Psi(N) = N^{\psi}$ *and* $\lambda(\omega) = \beta$. *Then*

$$c(\omega) = \frac{\xi}{\psi - \xi} \frac{\eta(\omega)}{\beta} \text{ and } \frac{N(\omega)}{N(\omega')} = \left(\frac{\eta(\omega')}{\eta(\omega)} \right)^{\frac{1-\xi}{1-\psi}}.$$

In this example, consumption is proportional to the goods cost of raising a child, $\eta(\omega)$, while the exponent $\frac{1-\xi}{1-\psi} \in (1, \infty)$ controls the extent to which consumption inequality translates into population inequality. Since the restriction $\psi > \xi$ is needed for an interior solution, the exponent is larger than 1. Therefore, population inequality is larger than consumption inequality. For example, if consumption of the rich is 5 times that of the poor, $\frac{\eta(\omega')}{\eta(\omega)} = 5$, and $\frac{1-\xi}{1-\psi} = 2$ then the population of the poor is 25 times that of the rich. The planner in this example is more willing to accept a large share of poor individuals when intergenerational substitution of consumption is particularly low (ξ is low) and/or parental altruism does not decrease sharply with family size (ψ is high).

5.2.3 Average output

A full solution requires to find the marginal product of labor which itself requires a solution for the land-labor ratio. For this purpose, rewrite the steady-state resource constraint as

$$LF(K/L, 1; A) = N \sum_{\omega} g(\omega) [c(\omega) + \eta(\omega)],$$

Furthermore, total labor supply relative to population is expressed, at steady state, by

$$\frac{L}{N} = \sum_{\omega} g(\omega) l(\omega) [1 - \lambda(\omega)]. \tag{39}$$

Dividing these two equations yields

$$\frac{F(K, L; A)}{L} = F\left(\frac{K}{L}, 1; A\right) = \frac{\sum_{\omega} g(\omega) [c(\omega) + \eta(\omega)]}{\sum_{\omega} g(\omega) l(\omega) [1 - \lambda(\omega)]}. \tag{40}$$

The system of three set of equations, (37), (38) and (40), together with $F_L = (1-\alpha) \frac{F(K,L;A)}{L}$, can then be used to solve for the following unknowns: $g(\omega), c(\omega)$ and L.

5.2.4 Stagnation

Combining (38) and (40), and then using the definition of $\alpha\left(\frac{K}{L}, A\right)$, one obtains:

$$c(\omega) = \frac{\xi(c(\omega))/\beta}{\psi(1,\omega) - \xi(c(\omega))} \left[\eta(\omega) + (\lambda(\omega) - \beta) \left(1 - \alpha\left(\frac{K}{L}, A\right)\right) \frac{\sum_{\omega} g(\omega) [c(\omega) + \eta(\omega)]}{\sum_{\omega} g(\omega) l(\omega) [1 - \lambda(\omega)]} l(\omega) \right] \tag{41}$$

Equations (37) and (41) can be used to solve for $c(\omega)$ and $g(\omega)$. Notice that $\alpha\left(\frac{K}{L}, A\right)$ depends on the ratio of $\frac{K}{L}$. L increases to respond to an increase in K. When the technological progress is land augmenting, the increase in A does not affect $\alpha\left(\frac{K}{L}, A\right)$. Hence $c(\omega)$ and $g(\omega)$ are independent of K, and they are also independent of A if the technological progress is land augmenting. Once these two variables are solved for then (40) can be used to solve for L and (39) for N. The following proposition summarizes these results. The proof is similar to that of Proposition 3 and hence omitted.

Proposition 10 *Suppose $\delta = 0$, $\Phi(n,\omega) = \beta n^{\psi}$ and the steady state is interior. Then: (i) in steady state optimal consumption is independent of the amount of land and optimal population is proportional to the amount of land; (ii) if technological progress is land augmenting then optimal consumption is independent of the level of technology and population increases proportionally with the level of technology; and (iii) optimal allocations depend on the initial distribution of population unless $\Psi(N_0) = N_0^{\psi}$.*

6 Stochastic case

The deterministic version of the model considered so far counterfactually predicts equal fertility among different social groups. Malthus, however, observed that fertility rates were higher among the poor. We now show that a version of the model with stochastic types can generate differential fertility. For tractability we once again assume $\delta = 0$ and use the Barro-Becker functional forms: $\Phi(n,\omega) = \beta n^{\psi}$ and $u(c) = c^{\xi}/\xi$. Equation (20) can be simplified, using equation (23) and the law of motion for population, equation (14), as:

$$\frac{\Phi\left(n_t\left(\omega^t\right)\right)}{n_t\left(\omega^t\right)} = \frac{\mu_{t+1}}{\mu_t}\frac{u'(c_t\left(\omega^t\right))}{u'(c_{t+1}\left(\omega^{t+1}\right))}. \tag{42}$$

An implication of this equation is that all children within a family have the same consumption:

$$c_{t+1}\left(\left[\omega^t, \omega_{t+1}\right]\right) = c_{t+1}\left(\omega^t\right) \text{ for all } \omega_{t+1} \in \Omega. \tag{43}$$

The following lemma shows that optimal consumption allocations are history independent and satisfy a formulation similar to that of (31) or (38). In particular, the consumption of a child is proportional to the expected net costs of raising that child.

Proposition 11 *Given $N_0\left(\omega^0\right)$, optimal allocations $\left\{c_0\left(\omega^0\right)\right\}_{\omega^0\in\Omega}$, $\left\{c_{t+1}\left(\omega^{t+1}\right)\right\}_{t=0,\omega^{t+1}\in\Omega^{t+2}}^{\infty}$, $\left\{n_t\left(\omega^t\right)\right\}_{t=1,\omega^t\in\Omega^{t+1}}^{\infty}$, $\left\{N_{t+1}\left(\omega^{t+1}\right)\right\}_{t=0,\omega^{t+1}\in\Omega^{t+2}}^{\infty}$, $\left\{L_t\right\}_{t=0}^{\infty}$, and $\left\{\frac{\mu_t}{\mu_{t+1}}\right\}_{t=0}^{\infty}$ are solved by the following system:*

$$\begin{aligned}
\frac{u'(c_0\left(\tilde{\omega}^0\right))}{u'(c_0\left(\omega^0\right))} &= \frac{\Psi\left(N_0\left(\omega^0\right)\right)}{N_0\left(\omega^0\right)}\frac{N_0\left(\tilde{\omega}^0\right)}{\Psi\left(N_0\left(\tilde{\omega}^0\right)\right)}, \quad \textit{for all } \omega^0, \tilde{\omega}^0 \in \Omega, \\
c_{t+1}\left(\omega^{t+1}\right) &= \frac{\xi}{\psi - \xi}\left\{\begin{array}{c} \frac{\mu_t}{\mu_{t+1}}\left[\eta\left(\omega_t\right) + F_L(K, L_t; A) l\left(\omega_t\right)\lambda_t\left(\omega_t\right)\right] \\ -F_L(K, L_{t+1}; A) E_t\left(l\left(\omega_{t+1}\right)|\omega_t\right) \end{array}\right\} \quad \textit{for } t \geq 0, \\
\Phi'\left(n_t\left(\omega^t\right)\right) &= \psi\frac{\mu_{t+1}}{\mu_t}\frac{u'(c_t\left(\omega^t\right))}{u'(c_{t+1}\left(\omega^{t+1}\right))}, \quad \textit{where } n_0\left(\omega_{-1}, \omega_0\right) = n_0\left(\omega_0\right) \textit{ for } t \geq 0,
\end{aligned} \tag{44}$$

(14), (15), and (16). The transversality condition for population is

$$\begin{aligned}
&\lim_{T\to\infty} \beta^T N_T\left(\omega^T\right)^{\psi-1} c_T\left(\omega^T\right)^{\xi-1} \pi\left(\omega^T, \omega^0\right)^{1-\psi}\frac{N_{T+1}\left(\omega^{T+1}\right)}{\pi\left(\omega_{T+1}, \omega_T\right)}\left(\begin{array}{c} \eta\left(\omega_T\right) + \\ F_2(K, L_T; A)\lambda_T\left(\omega_T\right) l\left(\omega_T\right) \end{array}\right) \\
&= 0
\end{aligned}$$

Notice that according to the proposition $c_{t+1}\left(\omega^{t+1}\right) = c_{t+1}\left(\omega_t\right)$ and $n_t\left(\omega^t\right) = n_t\left(\omega_{t-1}, \omega_t\right)$ so that efficient consumption is not history dependent and efficient fertility depends only on parents' own ability and grandparents' ability. Similarly, substituting (44) into (42), it follows that $n_t\left(\omega^t\right) = n_t\left(\omega_{t-1}, \omega_t\right)$ so that the number of children only depends on parental and grand-parental types.

6.1 Steady state

Consider now stationary steady state allocations in which $n_t\left(\omega_{t-1}, \omega_t\right) = n\left(\omega_{t-1}, \omega_t\right)$, $c_t\left(\omega_{t-1}\right) = c\left(\omega_{t-1}\right)$, $N_t\left(\omega^t\right) = N\left(\omega_{t-1}, \omega_t\right)$ and $N_t = N$. Let $Q_t \equiv \frac{\mu_t}{\mu_{t+1}}$ be the planner's shadow gross return and with a little bit abuse of notation let $g(\omega_{t-1},\omega_t) \equiv \frac{N(\omega_{t-1},\omega_t)}{N_t}$ be the population share with recent history (ω_{t-1},ω_t). The following lemma summarizes the system of equations and unknowns describing stationary steady state.

Lemma 12 *Steady state allocations,* $c(\omega)$, $n(\omega_{-1},\omega)$, $g(\omega_{-1},\omega)$, Q, L *and* N *are solved from the following systems of equations:*

$$c(\omega) = \frac{\xi Q}{\psi - \xi}\left[\eta(\omega) + F_L l(\omega)\lambda(\omega) - F_L E\left[l(\omega_{+1})|\omega\right]/Q\right], \tag{45}$$

$$n(\omega_{-1},\omega) = \left[\beta Q \frac{u'(c(\omega))}{u'(c(\omega_{-1}))}\right]^{\frac{1}{1-\psi}}, \tag{46}$$

$$g(\omega,\omega_{+1}) = \sum_{\omega_{-1}} n(\omega_{-1},\omega)\pi(\omega_{+1},\omega)g(\omega_{-1},\omega), \tag{47}$$

$$\sum_{\omega_{-1}}\sum_{\omega} g(\omega_{-1},\omega)n(\omega_{-1},\omega_t) = 1, \tag{48}$$

$$F\left(\frac{K}{L},1;A\right) = \frac{N}{L}\sum_{\omega_{-1}}\sum_{\omega} g(\omega_{-1},\omega)\left[c(\omega_{-1}) + \eta(\omega)n(\omega_{-1},\omega)\right], \tag{49}$$

$$\frac{L}{N} = \sum_{\omega_{-1}}\sum_{\omega} g(\omega_{-1},\omega)l(\omega)(1-\lambda(\omega)n(\omega_{-1},\omega)). \tag{50}$$

where $F_L = \left(1-\alpha\left(\frac{K}{L},A\right)\right)F\left(\frac{K}{L},1;A\right)$.

Equation (45) shows the consumption of an individual whose parent is of type ω. Consumption is positively associated with parental costs of raising children and parental skills, and it is negatively associated with the expected skills of the child. Equation (46) shows fertility differentials among different types. Optimal fertility depends on parental and grandparent's types. Given grandparent's types, parents with low consumption have more children than parents with high consumption. Also, given parental types, consumption rich grandparents have more grandchildren than consumption poor grandparents. Equation (48), which in principle serves to solve Q, restricts fertility to be one on average. Equations (49) and (50) are resource constraints of goods and labor.

The next proposition shows that the stagnation property still holds in the stochastic case.

Proposition 13 *Suppose the steady-state is interior. Then, steady state optimal consumption is independent of the amount of land and optimal population increases proportionally with the amount of land. Furthermore, if technological progress is land augmenting then optimal consumption is independent of the level of technology and population increases proportionally with the level of technology.*

To summarize, in addition to stagnation, the key properties of the stochastic steady state are differential fertility and heterogeneous social groups. Moreover, all types, or social groups, are represented in a steady state even if their initial population is zero as long as π is non-reducible.

7 Decentralization

This section extends Section 2 to consider stochastic signals. We show that when $\delta = 0$, the social planner's problem can be decentralized by a competitive market economy with a fixed amount of land. The basic environment is the same with the social planner's problem. Parents are altruistic toward children in the form of Barro-Becker. We follow previous notations except for adding a superscript c to allocations of consumption, fertility, population labor and land to represent competitive equilibrium allocations. Let $k_t^c(\omega^t)$ denotes the amount of land each adult living in period t is endowed with when his family history is $\omega^t \in \Omega^{t+1}$. It can be regarded as the bequest from parents and can be traded at the price p_t. Land can also be rented at the rental rate r_t. Parents are allowed to sign a contingent contract based on children's type ω_{t+1}, that is buying or selling land for the next generation depending on each one's realization of ability. Let $q_t(\omega_{t+1}, \omega_t)$ be the time t price of one unit of land contingent on the time $t+1$'s realization of child's ability to be ω_{t+1} and the time t's realization to be ω_t. People work in a competitive labor market. Let $w_t(\omega_t)$ be the wage of type ω_t at time t.

Initial parents maximize their own dynasty's welfare:

$$\max_{\{c_t(\omega^t), n_t(\omega^t), k_{t+1}(\omega^{t+1})\}_{t=0,\omega^t\in\Omega^{t+1}}^{\infty}} E\left[\sum_{t=0}^{\infty}\sum_{\omega^t\in\Omega^{t+1}}\prod_{j=0}^{t-1}\Phi\left(n_j\left(\omega^j\right),\omega^j\right)u\left(c_t\left(\omega^t\right)\right)\right]$$

Households are subject to the following constraints:

$$\begin{aligned} & c_t\left(\omega^t\right)+\eta\left(\omega_t\right)n_t\left(\omega^t\right)+n_t\left(\omega^t\right)\sum_{\omega_{t+1}}q_t\left(\omega_{t+1},\omega_t\right)k_{t+1}\left(\omega^{t+1}\right) \\ \leq \quad & w_t\left(\omega_t\right)\left(1-\lambda\left(\omega_t\right)n_t\left(\omega^t\right)\right)+\left(r_t+p_t\right)k_t\left(\omega^t\right) \quad \text{for } \omega^t\in\Omega^{t+1}, t\geq 0, \end{aligned} \tag{51}$$

where initial population and land holding $\left\{N_0\left(\omega^0\right), k_0\left(\omega^0\right)\right\}_{\omega^0\in\Omega}$ is given. Assume $\Phi(n,\omega)=\beta n^{\psi}$ and $u(c)=c^{\xi}/\xi$.

Firms hire labor on a competitive labor market, and people rent land to firms on a competitive land market. The competitive equilibrium of this problem is characterized by the following proposition.

Proposition 14 *Given prices* $\left\{r_t, \{w_t(\omega^t)\}_{\omega^t\in\Omega^{t+1}}\right\}_{t=0}^{\infty}$, *and initial level of capital and population* $\left\{k_0\left(\omega^0\right), N_0\left(\omega^0\right)\right\}_{\omega^0\in\Omega}$, *equilibrium allocations* $\left\{c_0^c\left(\omega^0\right)\right\}_{\omega^0\in\Omega}$, $\left\{c_{t+1}^c\left(\omega^{t+1}\right)\right\}_{t=0,\omega^{t+1}\in\Omega^{t+2}}^{\infty}$, $\{n_t^c(\omega^t)\}_{t=0,\omega^t\in\Omega^{t+1}}^{\infty}$, $\left\{k_{t+1}^c\left(\omega^{t+1}\right),\ N_{t+1}^c\left(\omega^{t+1}\right)\right\}_{t=0,\omega^{t+1}\in\Omega^{t+2}}^{\infty}$, *and* $\{L_t^c\}_{t=0}^{\infty}$ *are solved by the following system of equations*

$$c_{t+1}^c\left(\omega^{t+1}\right)=\frac{\xi}{\psi-\xi}\left[\frac{r_{t+1}+p_{t+1}}{p_t}\left[\eta\left(\omega_t\right)+w_t\left(\omega_t\right)\lambda\left(\omega_t\right)\right]-E_t\left[w_{t+1}\left(\omega_{t+1}\right)\right]\right] \tag{52}$$

for $t\geq 0$,

$$\frac{\Phi\left(n_t^c\left(\omega^t\right)\right)}{n_t^c\left(\omega^t\right)}=\frac{p_t}{r_{t+1}+p_{t+1}}\frac{u'(c_t^c\left(\omega^t\right))}{u'(c_{t+1}^c\left(\omega^{t+1}\right))} \text{ for } t\geq 0, \tag{53}$$

(51), (14), (15), and the non-Ponzi game condition

$$\lim_{T\to\infty}\beta^T\frac{N_T^c\left(\omega^T\right)^{\psi-1}c_T^c\left(\omega^T\right)^{\xi-1}\left(\eta\left(\omega_T\right)+\lambda\left(\omega_T\right)w_T\left(\omega_T\right)\right)}{\pi\left(\omega^T,\omega^0\right)^{\psi-1}\pi\left(\omega^{T+1},\omega^T\right)}N_{T+1}^c\left(\omega^{T+1}\right)=0$$

where equilibrium prices $\left\{r_t, \{w_t(\omega_t)\}_{\omega_t\in\Omega}\right\}_{t=0}^{\infty}$ *are given by* $r_t=F_K\left(K,L_t^c;A\right)$ *and* $w_t\left(\omega_t\right)=F_L\left(K,L_t^c;A\right)l\left(\omega_t\right)$. $\{p_t\}_{t=0}^{\infty}$ *is determined by land market equilibrium,* $K=\sum_{\omega^t\in\Omega^{t+1}}N_t^c\left(\omega^t\right)k_t^c\left(\omega^t\right)$ *for all* $t\geq 0$. $\{q_t(\omega_{t+1},\omega_t)\}_{t=0}^{\infty}$ *satisfies* $q_t\left(\omega_{t+1},\omega_t\right)=p_t\pi\left(\omega_{t+1},\omega_t\right)$ *for all* $t\geq 0$.

Rewrite (53) and iterate, we obtain

$$p_t=\frac{\Phi\left(n_t^c\left(\omega^t\right)\right)}{n_t^c\left(\omega^t\right)}\frac{u'(c_{t+1}^c\left(\omega^{t+1}\right))}{u'(c_t^c\left(\omega^t\right))}\left(r_{t+1}+p_{t+1}\right) \tag{54}$$

Iterate (54), we obtain

$$\begin{aligned} p_t \quad = \quad & E_t\left[\sum_{j=0}^{\infty}\left(\prod_{k=0}^{j}\frac{\Phi\left(n_{t+k}^c\left(\omega^{t+k}\right)\right)}{n_{t+k}^c\left(\omega^{t+k}\right)}\right)\frac{u'(c_{t+j+1}^c\left(\omega^{t+j+1}\right))}{u'(c_t^c\left(\omega^t\right))}\right]r_{t+j+1} \\ & +\lim_{T\to\infty}E_t\left[\left(\prod_{k=0}^{\infty}\frac{\Phi\left(n_{t+k}^c\left(\omega^{t+k}\right)\right)}{n_{t+k}^c\left(\omega^{t+k}\right)}\right)\frac{u'(c_{t+T+1}^c\left(\omega^{t+T+1}\right))}{u'(c_t^c\left(\omega^t\right))}\right]p_{t+T+1}. \end{aligned}$$

To guarantee boundedness of the dynasty's welfare, the second term is 0, so that

$$p_t = E_t\left[\sum_{j=0}^{\infty}\left(\prod_{k=0}^{j}\frac{\Phi\left(n_{t+k}^c\left(\omega^{t+k}\right)\right)}{n_{t+k}^c\left(\omega^{t+k}\right)}\right)\frac{u'(c_{t+j+1}^c\left(\omega^{t+j+1}\right))}{u'(c_t^c\left(\omega^t\right))}\right]r_{t+j+1}.$$

Price is the present value of rents discounted by a rate that depends on the standard discount factor in Φ as well as the fertility rates.

Corollary 15 *The competitive equilibrium consumption of every individual depends on parental type but not on one's own type, e.g.* $c_{t+1}^c\left(\omega^{t+1}\right) = c_{t+1}^c\left(\omega_t\right)$. *As a result, fertility depends both on parents' and grandparents' types, e.g.* $n_t^c\left(\omega^t\right) = n_t^c\left(\omega_{t-1}, \omega_t\right)$[14].

The proof of this corollary is straightforward according to equations of (52) and (53), and it is omitted. Next, we show that the social planner's problem can be decentralized by a competitive equilibrium with an initial land distribution.

Proposition 16 *Given initial population* $\left\{N_0\left(\omega^0\right)\right\}_{\omega^0\in\Omega}$ *and the social planner's altruism on people living in the first period,* $\left\{\Psi\left(N\left(\omega^0\right)\right)\right\}_{\omega^0\in\Omega}$, *there exits a competitive equilibrium with an initial land distribution* $\left\{k_0\left(\omega^0\right)\right\}_{\omega^0\in\Omega}$ *and equilibrium prices satisfying* $\frac{r_{t+1}+p_{t+1}}{p_t} = \frac{\mu_t}{\mu_{t+1}}$ *that decentralizes the social planner's problem.*

8 Concluding comments

This article fills a gap in our understanding of efficiency in economies characterized by endogenous fertility and fixed resources. We propose and implement a novel methodology that allow us to provide a sharp and more general characterization of socially optimal allocations in these environments. We find that efficient allocations under endogenous fertility differ sharply from those derived under exogenous fertility. This finding underscores the importance of better understanding social optima when population is endogenous and responds to economic incentives.

The pre-industrial world was to a large extent Malthusian. As documented by Ashraf and Galor (2011), periods characterized by improvements in technology or in the availability of land eventually lead to a larger but not richer population. This is remarkable given the diversity of political, social, religious, geographical, cultural, and economic environments they considered, some arguably more advanced than others. We find that stagnation, inequality, high population of the poor and differential fertility can naturally arise as an optimal social choice. Our findings could shed light on why the Malthusian "trap" was so pervasive in pre-industrial societies. We also show that is not the irrational animal spirit of human beings, as suggested by Malthus, what ultimately explains the stagnation. Stagnation can be the result of an optimal choice between the quality and quantity of life in the presence of limited natural resources.

Finally, we expect that our methodology will further facilitate the integration of demographics and macroeconomics.

[14] $n_0\left(\omega_{-1}, \omega_0\right) = n_0\left(\omega_0\right)$

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Appendix

A.1. Proofs of Propositions and Lemmas

Proof of Proposition 1. In the deterministic case, the first order conditions with respect to n_t, N_{t+1}, U_t, and c_t are

$$\theta_t \Phi_n(n_t) U_{t+1} = \mu_t \eta + \kappa_t l \lambda_t + \gamma_{t+1}, \tag{55}$$

$$\begin{aligned} & \delta^{t+1} \Psi'(N_{t+1}) U_{t+1} + \kappa_{t+1} l [1 - \lambda_{t+1} n_{t+1}] + \gamma_{t+1} \\ = \; & \mu_{t+1} [c_{t+1} + \eta n_{t+1}] + n_{t+1} \gamma_{t+2} \end{aligned} \tag{56}$$

$$\theta_{t+1} N_{t+1} = \delta^{t+1} \Psi(N_{t+1}) + \theta_t N_t \Phi(n_t) \tag{57}$$

$$\theta_t u'(c_t) = \mu_t$$

Substitute out γ_{t+1} and γ_{t+2} of (56) using (55), and then use (24) and (23) to substitute out κ_{t+1} and μ_t, respectively, we obtain

$$\begin{aligned} & \delta^{t+1} \Psi'(N_{t+1}) U_{t+1} + \theta_t \Phi_n(n_t) U_{t+1} \\ = \; & \theta_{t+1} u'(c_{t+1}) \left(c_{t+1} - F_{L,t+1} l + \frac{\mu_t}{\mu_{t+1}} (\eta + F_{L,t} l \lambda_t) \right) \\ & + \theta_{t+1} \frac{n_{t+1} \Phi_n(n_{t+1})}{\Phi(n_{t+1})} (U_{t+1} - u(c_{t+1})) \end{aligned} \tag{58}$$

Iterate (57), we can get

$$\begin{aligned} \theta_{t+1} &= \delta^{t+1} \frac{\Psi(N_{t+1})}{N_{t+1}} + \frac{\Phi(n_t)}{n_t} \left[\delta^t \frac{\Psi(N_t)}{N_t} + \frac{\theta_{t-1} \Phi(n_{t-1})}{n_{t-1}} \right] \\ &= \sum_{m=1}^{t+1} \delta^m \frac{\Psi(N_m)}{N_m} \prod_{j=m}^{t} \frac{\Phi(n_j)}{n_j} + \prod_{j=0}^{t} \frac{\Phi(n_j)}{n_j} \theta_0 \\ &= \sum_{m=0}^{t+1} \delta^m \frac{\Psi(N_m)}{N_m} \prod_{j=m}^{t} \frac{\Phi(n_j)}{n_j} \end{aligned}$$

Given that the altruism function takes Barro-Becker's form $\Phi(n) = \beta n^{\psi}$,

$$\theta_{t+1} = \beta^{t+1} N_{t+1}^{\psi-1} \sum_{m=0}^{t+1} (\delta/\beta)^m \frac{\Psi(N_m)}{N_m^{\psi}}$$

Plug this result into (58),

$$\begin{aligned} & \delta^{t+1} \Psi'(N_{t+1}) U_{t+1} + \beta^{t+1} N_{t+1}^{\psi-1} \frac{n_t \Phi_n(n_t)}{\Phi(n_t)} U_{t+1} \sum_{m=0}^{t} (\delta/\beta)^m \frac{\Psi(N_m)}{N_m^{\psi}} \\ = \; & \beta^{t+1} N_{t+1}^{\psi-1} \sum_{m=0}^{t+1} (\delta/\beta)^m \frac{\Psi(N_m)}{N_m^{\psi}} u'(c_{t+1}) \left(c_{t+1} - F_{L,t+1} l + \frac{\mu_t}{\mu_{t+1}} (\eta + F_{L,t} l \lambda_t) \right) \\ & + \beta^{t+1} N_{t+1}^{\psi-1} \left(\sum_{m=0}^{t+1} (\delta/\beta)^m \frac{\Psi(N_m)}{N_m^{\psi}} \right) \frac{n_{t+1} \Phi_n(n_{t+1})}{\Phi(n_{t+1})} U_{t+1} \\ & - \beta^{t+1} N_{t+1}^{\psi-1} \left(\sum_{m=0}^{t+1} (\delta/\beta)^m \frac{\Psi(N_m)}{N_m^{\psi}} \right) \frac{n_{t+1} \Phi_n(n_{t+1})}{\Phi(n_{t+1})} u(c_{t+1}) \end{aligned}$$

Under the assumed form of $\Phi(\cdot)$, $\psi = \frac{n_t \Phi_n(n_t)}{\Phi(n_t)} = \frac{n_{t+1} \Phi_n(n_{t+1})}{\Phi(n_{t+1})}$, we can simplify terms and solve consumption as (1). ■

Proof of Lemma 2. Consider a steady state situation in which N and c are constant. In that case $n = 1$ and (27) can be written as $\theta_{t+1}/\theta_t = \beta + \frac{\delta^{t+1} \Psi(N)}{N \theta_t} \geq \beta$. Under Assumption 1, the ratio $\frac{\delta^{t+1}\Psi(N)}{N\theta_t}$ goes to zero in the limit. It is easy to show that the Lagrange multipliers grow at constant rate at steady state. To see this, first look at (23) and (24), which implies that θ_t, μ_t, and κ_t grow at the same rate at steady state. Using the steady state version of (22) we can see that

$$\frac{\theta_{t+1}}{\theta_t} = \frac{\theta_{t+1}\Phi_n(1)U}{\theta_t\Phi_n(1)U} = \frac{\frac{\mu_{t+1}}{\mu_t}\mu_t\eta + \frac{\kappa_{t+1}}{\kappa_t}\kappa_t l\lambda + \frac{\gamma_{t+2}}{\gamma_t+1}\gamma_{t+1}}{\mu_t\eta + \kappa_t l\lambda + \gamma_{t+1}}$$

so

$$\frac{\theta_{t+1}}{\theta_t}\mu_t\eta + \frac{\theta_{t+1}}{\theta_t}\kappa_t l\lambda + \frac{\theta_{t+1}}{\theta_t}\gamma_{t+1} = \frac{\mu_{t+1}}{\mu_t}\mu_t\eta + \frac{\kappa_{t+1}}{\kappa_t}\kappa_t l\lambda + \frac{\gamma_{t+2}}{\gamma_{t+1}}\gamma_{t+1}$$

Since θ_t, μ_t, and κ_t grow at the same rate at steady state, this equation can be reduced to

$$\frac{\theta_{t+1}}{\theta_t} = \frac{\gamma_{t+2}}{\gamma_{t+1}}$$

Then γ_t grows at the same rate with θ_t, μ_t, and κ_t at the steady state, which implies that $\frac{\gamma_t}{\mu_t}$ is a constant at steady state. Hence $\frac{\gamma_{t+1}}{\gamma_t}$ is a constant by (29), and so are other Lagrange parameters. Therefore $\theta_{t+1}/\theta_t = \mu_{t+1}/\mu_t = \gamma_{t+1}/\gamma_t = \kappa_{t+1}/\kappa_t = \beta$. We can solve $\frac{\gamma_t}{\mu_t}$ as

$$\frac{\gamma_t}{\mu_t} = \frac{F_K \frac{K}{N}}{1-\beta} = \frac{F\left(\frac{K}{N}, 1-\lambda; A\right) - F_L(1-\lambda)}{1-\beta}. \tag{59}$$

On the other hand, equation (17) simplifies in steady state to $U = \frac{u(c)}{1-\beta}$. Therefore, (29) can be written, using the results obtained for U, $\frac{\gamma_t}{\mu_t}$, $\frac{\gamma_{t+1}}{\gamma_t}$ and the definitions of ψ and ξ[15] as:

$$\beta\psi(1)c/\xi(c) = (1-\beta)\eta + (\lambda-\beta)F_L + \beta F\left(\frac{K}{N}, 1-\lambda; A\right). \tag{60}$$

This equation together with the resource constraint (26) can be used to solve consumption as

$$c = \frac{\xi(c)}{\psi(1) - \xi(c)}\left[\frac{\mu_t}{\mu_{t+1}}(\eta + F_L\lambda) - F_L\right].$$

(23) and (27) imply $\frac{\mu_{t+1}}{\mu_t} = \beta$ at steady state, then we obtain (31). ■

Proof of Proposition 3. For the first part of this proposition A is given. Then c, MPL, and $\frac{K}{L}$ are fully determined by the three equations (31), (32) and (33). Hence consumption is independent of the amount of land while efficient population increases proportionally with the amount of land. For the second part when $F\left(\frac{K}{N}, 1-\lambda, A\right) = \hat{F}\left(\frac{AK}{N}, 1-\lambda\right)$, $F(K,L,A) = \hat{F}(AK,L)$,then

$$\alpha\left(\frac{K}{L}, A\right) = \frac{A\hat{F}_1(AK,L)K}{\hat{F}(AK,L)} = \frac{\hat{F}_1(AK/L,1)}{\hat{F}(AK/L,1)}\frac{AK}{L} = \hat{\alpha}\left(\frac{AK}{L}\right)$$

$$c = \frac{\xi(c)/\beta}{\psi - \xi(c)}[\eta + (\lambda-\beta) * MPL] \tag{61}$$

$$MPL = \frac{1-\hat{\alpha}\left(\frac{AK}{L}\right)}{1-\lambda}(c+\eta) \tag{62}$$

where

$$MPL = \left(1 - \hat{\alpha}\left(\frac{AK}{L}\right)\right)\hat{F}\left(\frac{AK}{L}, 1\right). \tag{63}$$

Then c, MPL and $\frac{AK}{L}$ are solved by (61), (62) and (63), which are all independent of A. ■

Proof of Lemma 4. Let $s_t(\omega) \equiv \theta_t(\omega)N_t(\omega)$. Equation (20), given that $\delta = 0$ is assumed, can then be written as $s_{t+1}(\omega) = s_t(\omega)\Phi(n_t(\omega))$, in particular,

$$s_1(\omega) = s_0(\omega)\Phi(n_0(\omega)),\ s_2(\omega) = s_0(\omega)\prod_{i=0}^{1}\Phi(n_i(\omega)).$$

[15] In the deterministic case with one type, we write $\psi(1,\omega)$ and $\xi(c(\omega))$ as $\psi(1)$ and $\xi(c)$, respectively.

More generally, $s_t(\omega) = s_0(\omega) \prod_{i=0}^{t-1} \Phi(n_i(\omega))$. Recall that $\Phi(n) = \beta(\omega) n^{\psi}$ is assumed in part IV, it follows that:

$$s_t(\omega) = s_0(\omega) \beta(\omega)^t \left(\prod_{i=0}^{t-1} n_i(\omega) \right)^{\psi} = \theta_0(\omega) (N_0(\omega))^{1-\psi} \beta(\omega)^t (N_t(\omega))^{\psi}. \tag{64}$$

Now (23) can be written as $\mu_t N_t(\omega) = s_t(\omega) u'(c_t(\omega))$. Therefore

$$\frac{N_t(\omega)}{N_t(\omega')} = \frac{s_t(\omega) u'(c_t(\omega))}{s_t(\omega') u'(c_t(\omega'))}.$$

Substituting (64) into this equation gives

$$\frac{N_t(\omega)}{N_t(\omega')} = \frac{\theta_0(\omega) (N_0(\omega))^{1-\psi} \beta(\omega)^t (N_t(\omega))^{\psi} u'(c_t(\omega))}{\theta_0(\omega') (N_0(\omega'))^{1-\psi} \beta(\omega')^t (N_t(\omega'))^{\psi} u'(c_t(\omega'))}.$$

Finally, use (19) to substitute $\theta_0(\omega)$ and solve for $\frac{N_t(\omega)}{N_t(\omega')}$ to obtain (34). ■

Proof of Lemma 6. According to equation (36), and let $\omega' = \omega_0$,

$$N(\omega) = N(\omega_0) \left(\frac{N_0(\omega)}{N_0(\omega_0)} \right)^{\frac{\psi_p - \psi}{1-\psi}} \frac{u'(c(\omega))^{1/(1-\psi)}}{u'(c(\omega_0))^{1/(1-\psi)}}.$$

Adding $N(\omega)$ over ω,

$$N = \sum_{\omega} N(\omega) = \frac{N(\omega_0)}{N_0(\omega_0)^{\frac{\psi_p - \psi}{1-\psi}} u'(c(\omega_0))^{1/(1-\psi)}} \sum_{\omega} N_0(\omega)^{\frac{\psi_p - \psi}{1-\psi}} u'(c(\omega))^{1/(1-\psi)}$$

and therefore

$$g(\omega_0) = \frac{N(\omega_0)}{N} = \frac{N_0(\omega_0)^{\frac{\psi_p - \psi}{1-\psi}} u'(c(\omega_0))^{1/(1-\psi)}}{\sum_{\omega} N_0(\omega)^{\frac{\psi_p - \psi}{1-\psi}} u'(c(\omega))^{1/(1-\psi)}} \text{ for all } \omega_0 \in \Omega_p.$$

■

Proof of Lemma 7. Rewrite (21) using (24) and evaluate it at the steady state:

$$1 = \frac{\mu_{t+1}}{\gamma_{t+1}(\omega)} [c(\omega) + \eta(\omega) - F_L l(\omega)(1 - \lambda(\omega))] + \frac{\gamma_{t+2}(\omega)}{\gamma_{t+1}(\omega)}.$$

Since $\frac{\gamma_{t+2}(\omega)}{\gamma_{t+1}(\omega)}$ is constant in steady state then $\frac{\mu_{t+1}}{\gamma_{t+1}(\omega)}$ needs to be constant for this equation to hold, which means that $\frac{\gamma_{t+1}(\omega)}{\gamma_t(\omega)} = \frac{\mu_{t+1}}{\mu_t} = \beta$. The last equality holds by (23) and (27). Therefore, the previous equation can be written as:

$$\frac{\gamma_t(\omega)}{\mu_t} = \frac{1}{1-\beta} [c(\omega) + \eta(\omega) - F_L l(\omega)(1 - \lambda(\omega))]. \tag{65}$$

This expression states that the value of an immigrant in terms of goods, $\frac{\gamma_t(\omega)}{\mu_t}$, is the net present value of the net cost. In steady state $U(c(\omega)) = \frac{u(c(\omega))}{1-\beta}$. Use this result, (23) and (24) to rewrite (22) as:

$$\frac{\Phi'(1)}{1-\beta} \frac{u(c(\omega))}{u'(c(\omega))} = \eta(\omega) + F_L l(\omega) \lambda(\omega) + \beta \frac{\gamma_t(\omega)}{\mu_t}. \tag{66}$$

One can combine (65) and (66) to solve for consumption as

$$c(\omega) = \frac{\xi(c(\omega))}{\Phi'(1) - \beta \xi(c(\omega))} [\eta(\omega) + (\lambda(\omega) - \beta) F_L l(\omega)].$$

Using $\Phi(n) = \beta n^{\psi}$ provides the result. ■

Proof of Proposition 11. When $\delta = 0$, the first order conditions with respect to

$$\left\{U_0\left(\omega^0\right), U_{t+1}\left(\omega^{t+1}\right), N_{t+1}\left(\omega^{t+1}\right), n_t\left(\omega^t\right), c_t\left(\omega^t\right), L_t\right\}_{t\geq 0}$$

are (19), (20), (22), (23), (21), and (24) respectively while setting $\delta = 0$ in (20) and (21). Use (23) the first order condition with respect to $n_t\left(\omega^t\right)$ becomes

$$\begin{aligned} & n_t\left(\omega^t\right) : \frac{\mu_t}{u'(c_t\left(\omega^t\right))}\Phi_1\left(n_t\left(\omega^t\right), \omega\right) E_t U_{t+1}\left(\omega^{t+1}\right) \\ = & \sum_{\omega^{t+1}|\omega^t} \gamma_{t+1}\left(\omega^{t+1}\right)\pi\left(\omega_{t+1}, \omega_t\right) + \mu_t \eta\left(\omega_t\right) + \kappa_t l\left(\omega_t\right)\lambda_t\left(\omega_t\right) \end{aligned}$$

Get $\theta_0\left(\omega^0\right)$ from (23) and plug it into (19),

$$\frac{\Psi\left(N_0\left(\omega^0\right)\right)}{N_0\left(\omega^0\right)} = \frac{\mu_0}{u'(c_0\left(\omega^0\right))}$$

By assumption $\left\{N_0\left(\omega^0\right)\right\}_{\omega^0\in\Omega}$ are given, we can use this equality to express consumption of $c_0\left(\tilde{\omega}^0\right)$ for all $\tilde{\omega}^0 \in \Omega$ as a function of $c_0\left(\omega^0\right)$, which is

$$\frac{u'(c_0\left(\tilde{\omega}^0\right))}{u'(c_0\left(\omega^0\right))} = \frac{\Psi\left(N_0\left(\omega^0\right)\right)}{N_0\left(\omega^0\right)}\frac{N_0\left(\tilde{\omega}^0\right)}{\Psi\left(N_0\left(\tilde{\omega}^0\right)\right)}, \quad \text{for all } \omega^0, \tilde{\omega}^0 \in \Omega.$$

This set of equations and the first period's resource constraint can be used to solve for $\left\{c_0\left(\omega^0\right)\right\}_{\omega^0\in\Omega}$ in the system. Use the first order condition with respect to $n_t\left(\omega^t\right)$ and L_t to obtain

$$\begin{aligned} & E_t U_{t+1}\left(\omega^{t+1}\right) \\ = & \frac{u'(c_t\left(\omega^t\right))}{\mu_t \Phi_n\left(n_t\left(\omega^t\right), \omega\right)}\left[\begin{array}{c} \sum_{\omega^{t+1}|\omega^t}\gamma_{t+1}\left(\omega^{t+1}\right)\pi\left(\omega_{t+1}, \omega_t\right) \\ +\mu_t\eta\left(\omega_t\right) + \mu_t F_2(K, L_t; A) l\left(\omega_t\right)\lambda_t\left(\omega_t\right) \end{array}\right] \end{aligned}$$

for all $t \geq 0$. Plug the first order condition with respect to $c_t\left(\omega^t\right)$ into that with respect to $U_{t+1}\left(\omega^{t+1}\right)$,

$$\frac{u'(c_t\left(\omega^t\right))}{u'(c_{t+1}\left(\omega^{t+1}\right))} = \frac{\mu_t}{\mu_{t+1}}\frac{\Phi\left(n_t\left(\omega^t\right), \omega_t\right)}{n_t\left(\omega^t\right)} \tag{67}$$

so $n_t\left(\omega^t\right)$ is independent of ω_{t+1} and $E_t U_{t+1}\left(\omega^{t+1}\right)$ can be expressed as

$$\begin{aligned} & E_t U_{t+1}\left(\omega^{t+1}\right) \\ = & \frac{u'(c_{t+1}\left(\omega^{t+1}\right))}{\mu_{t+1}}\frac{1}{\psi}\left[\begin{array}{c} \sum_{\omega^{t+1}|\omega^t}\gamma_{t+1}\left(\omega^{t+1}\right)\pi\left(\omega_{t+1}, \omega_t\right) \\ +\mu_t\eta\left(\omega_t\right) + \mu_t F_2(K, L_t; A) l\left(\omega_t\right)\lambda_t\left(\omega_t\right) \end{array}\right] \end{aligned} \tag{68}$$

for all $t \geq 0$. Move the first order condition with respect to $N_{t+1}\left(\omega^{t+1}\right)$ by one period backward,

$$\begin{aligned} & \sum_{\omega^{t+1}|\omega^t}\gamma_{t+1}\left(\omega^{t+1}\right)\pi\left(\omega_{t+1}, \omega_t\right) + \mu_t\eta\left(\omega_t\right) + \mu_t F_2(K, L_t; A) l\left(\omega_t\right)\lambda_t\left(\omega_t\right) \\ = & \frac{1}{n_t\left(\omega^t\right)}\left[\gamma_t\left(\omega^t\right) + \mu_t F_2(K, L_t; A) l\left(\omega_t\right) - \mu_t c_t\left(\omega^t\right)\right] \quad \text{for all } t \geq 1, \end{aligned}$$

so

$$\begin{aligned} & E_t U_{t+1}\left(\omega^{t+1}\right) \\ = & \frac{u'(c_t\left(\omega^t\right))}{\mu_t}\frac{1}{\Phi_n\left(n_t\left(\omega^t\right), \omega_t\right) n_t\left(\omega^t\right)}\left[\gamma_t\left(\omega^t\right) + \mu_t F_2(K, L_t; A) l\left(\omega_t\right) - \mu_t c_t\left(\omega^t\right)\right] \end{aligned}$$

for all $t \geq 1$, Plug it into the value function for all $t \geq 1$,

$$\begin{aligned} & U_t\left(\omega^t\right) \\ = & u'(c_t\left(\omega^t\right))\left[c_t\left(\omega^t\right)\frac{1}{\xi} + \frac{1}{\mu_t}\frac{1}{\psi}\left[\gamma_t\left(\omega^t\right) + \mu_t F_2(K, L_t; A) l\left(\omega_t\right) - \mu_t c_t\left(\omega^t\right)\right]\right] \end{aligned}$$

Forward one period and take expectations for all $t \geq 0$,

$$E_t U_{t+1}\left(\omega^{t+1}\right) = u'(c_{t+1}\left(\omega^{t+1}\right)) \left[\begin{array}{c} c_{t+1}\left(\omega^{t+1}\right)\frac{1}{\xi} \\ +\frac{1}{\mu_{t+1}}\frac{1}{\psi} \left[\begin{array}{c} E_t\gamma_{t+1}\left(\omega^{t+1}\right) + \mu_{t+1}F_2(K, L_{t+1}; A)E_t l\left(\omega_{t+1}\right) \\ -\mu_{t+1}c_{t+1}\left(\omega^{t+1}\right) \end{array} \right] \end{array} \right]$$

Use this equation and (68) we get

$$\begin{aligned} & \frac{1}{\mu_{t+1}}\frac{1}{\psi}\left[\mu_t\eta\left(\omega_t\right) + \mu_t F_2(K, L_t; A)l\left(\omega_t\right)\lambda_t\left(\omega_t\right)\right] \\ = \;& c_{t+1}\left(\omega^{t+1}\right)\left(\frac{1}{\xi} - \frac{1}{\psi}\right) + \frac{1}{\psi}F_2(K, L_{t+1}; A)E_t l\left(\omega_{t+1}\right) \end{aligned}$$

Hence, $c_{t+1}\left(\omega^{t+1}\right) = c_{t+1}\left(\omega_t\right)$ and the optimal consumption for all $t \geq 0$ is (44). Use it and (67), fertility can be solved as follows:

$$\Phi_n\left(n_t\left(\omega_{t-1}, \omega_t\right)\right) = \psi\frac{\mu_{t+1}}{\mu_t}\frac{u'(c_t\left(\omega^t\right))}{u'(c_{t+1}\left(\omega^{t+1}\right))} \text{ for } t \geq 0.$$

where $n_0\left(\omega_{-1}, \omega_0\right) = n_0\left(\omega_0\right)$. When $\Phi\left(n\right) = \beta n^{\psi}$ as in the Barro and Becker (1988) and (1989),

$$n_t\left(\omega^t\right) = n_t\left(\omega_{t-1}, \omega_t\right) = \left[\beta\frac{\mu_t}{\mu_{t+1}}\frac{u'\left(c_{t+1}\left(\omega_t\right)\right)}{u'(c_t\left(\omega_{t-1}\right))}\right]^{\frac{1}{1-\psi}}.$$

(14), (15), and (16) are the constraints of the social planner's problem and need to be satisfied. Next let us solve the transversality condition about population. In this paper capital is land, which is fixed at the amount K. Given $N_t\left(\omega^t\right)$, the value of $N_{t+1}\left(\omega^{t+1}\right)$ to the value function is

$$\begin{aligned} & \theta_t\left(\omega^t\right)N_t\left(\omega^t\right)\Phi'\left(n_t\left(\omega^t\right)\right)\frac{E_t U_{t+1}\left(\omega^{t+1}\right)}{N_t\left(\omega^t\right)\pi\left(\omega_{t+1}|\omega_t\right)} - \mu_t\frac{\eta\left(\omega_T\right)}{\pi\left(\omega_{t+1}|\omega_t\right)} - \kappa_t\frac{\lambda_T\left(\omega_T\right)l\left(\omega_T\right)}{\pi\left(\omega_{t+1}|\omega_t\right)} \\ = \;& \theta_t\left(\omega^t\right)\Phi'\left(n_t\left(\omega^t\right)\right)\frac{E_t U_{t+1}\left(\omega^{t+1}\right)}{\pi\left(\omega_{t+1}|\omega_t\right)} - \mu_t\frac{\eta\left(\omega_T\right)}{\pi\left(\omega_{t+1}|\omega_t\right)} - \mu_t\frac{F_L(K, L_T; A)\lambda_T\left(\omega_T\right)l\left(\omega_T\right)}{\pi\left(\omega_{t+1}|\omega_t\right)} \end{aligned}$$

The transversality condition becomes

$$\begin{aligned} & \lim_{T\to\infty}\frac{\beta^T}{\pi\left(\omega_{t+1}|\omega_t\right)}\left[\begin{array}{c} \theta_t\left(\omega^t\right)\Phi'\left(n_t\left(\omega^t\right)\right)E_t U_{t+1}\left(\omega^{t+1}\right) \\ -\mu_t\left(\eta\left(\omega_T\right) + F_L(K, L_T; A)\lambda_T\left(\omega_T\right)l\left(\omega_T\right)\right) \end{array} \right] N_{T+1}\left(\omega^{T+1}\right) \\ = \;& 0. \end{aligned}$$

Use (20) to express $\theta_t\left(\omega^t\right)$ and iterate,

$$\begin{aligned} \theta_t\left(\omega^t\right) &= \theta_{t-1}\left(\omega^{t-1}\right)\frac{\Phi\left(n_{t-1}\left(\omega^{t-1}\right)\right)}{n_{t-1}\left(\omega^{t-1}\right)} = \cdots \\ &= \theta_0\left(\omega^0\right)\left(\frac{N_t\left(\omega^t\right)}{N_0\left(\omega^0\right)\pi\left(\omega^t, \omega^0\right)}\right)^{\psi-1}\beta^t \end{aligned}$$

Plug it and (19) into (23), and use the specified functional forms,

$$\begin{aligned} \mu_t &= \theta_0\left(\omega^0\right)\left(\frac{N_t\left(\omega^t\right)}{N_0\left(\omega^0\right)\pi\left(\omega^t, \omega^0\right)}\right)^{\psi-1}\beta^t u'\left(\frac{C_t\left(\omega^t\right)}{N_t\left(\omega^t\right)}\right) \\ &= \frac{\psi\left(N_0\left(\omega^0\right)\right)}{N_0\left(\omega^0\right)}\beta^t\frac{N_t\left(\omega^t\right)^{\psi-\xi}}{\left(N_0\left(\omega^0\right)\pi\left(\omega^t, \omega^0\right)\right)^{\psi-1}}C_t\left(\omega^t\right)^{\xi-1} \end{aligned}$$

$$\begin{aligned} & \lim_{t\to\infty}\theta_t\left(\omega^t\right)\Phi'\left(n_t\left(\omega^t\right)\right)\frac{E_t U_{t+1}\left(\omega^{t+1}\right)}{\pi\left(\omega_{t+1}|\omega_t\right)}N_{t+1}\left(\omega^{t+1}\right) \\ = \;& \lim_{t\to\infty}\beta^t\frac{\psi\left(N_0\left(\omega^0\right)\right)}{N_0\left(\omega^0\right)}\left(\frac{N_t\left(\omega^t\right)}{N_0\left(\omega^0\right)\pi\left(\omega^t, \omega^0\right)}\right)^{\psi-1}\Phi'\left(n_t\left(\omega^t\right)\right)\frac{E_t U_{t+1}\left(\omega^{t+1}\right)}{\pi\left(\omega_{t+1}|\omega_t\right)} \\ = \;& 0. \end{aligned}$$

The transversality condition becomes

$$\lim_{T\to\infty}\beta^T \frac{N_T\left(\omega^T\right)^{\psi-\xi} C_T\left(\omega^T\right)^{\xi-1}}{\pi\left(\omega^T,\omega^0\right)^{\psi-1}\pi\left(\omega_{T+1},\omega_T\right)}\left(\eta\left(\omega_T\right)+\lambda_T\left(\omega_T\right)F_2(K,L_T;A)l\left(\omega_T\right)\right)N_{T+1}\left(\omega^{T+1}\right)=0.$$

Write consumption into per capita term, we obtain the transversality condition in Proposition 10. ■

Proof of Corollary 12. At steady state (44) becomes (45). (46) can be obtained using (42) and the specified functional forms. The law of motion of population (14) becomes (47). Total population is constant and therefore average fertility is equal to 1 as stated by (48). Equations (49) and (50) are steady state versions of (16) and (15). ■

Proof of Proposition 13. The proof is similar to that of Proposition 3 and uses guess and verify. A longer direct proof is also possible. Consider the solution for an initial amount of land, say K_0. Let $\frac{K_0}{N_0}$ and L_0 be the steady state solution of land-population ratio and labor for K_0. Then consider a different amount of land, say K_1. Guess that the solution for the new steady state is identical to the initial solution except for two changes: $N_1=K_1/\frac{K_0}{N_0}$ so that the land-labor ratio is unchanged, and $L_1=N_1\frac{L_0}{N_0}$ so that the labor-population ratio is unchanged. Notice that under the proposed solution the marginal product of labor is unchanged too. One can then use Corollary 12 to verify that under the proposed guess, the solutions for consumption, fertility, the distribution of population and Q that solve for K_0 also solve for K_1. Similarly for land-augmenting technological progress, guess that population responds to keep $\frac{AK}{L}$ unchanged for different levels of A, while labor responds to keep the ratio $\frac{L}{A}$ unchanged, and nothing else changes. One verifies that the proposed solution satisfies all equations in Corollary 12. ■

Proof of Proposition 14. Recall that $\xi\equiv\frac{u'\left(c_{t+1}\left(\omega^{t+1}\right)\right)c_{t+1}\left(\omega^{t+1}\right)}{u\left(c_{t+1}\left(\omega^{t+1}\right)\right)}$ and $\psi\equiv\frac{n_t\left(\omega^t\right)\Phi'\left(n_t\left(\omega^t\right)\right)}{\Phi\left(n_t\left(\omega^t\right)\right)}$. Let us solve the problem by taking first order conditions:

$$\begin{aligned} k_t\left(\omega^t\right) \quad &: \quad \prod_{j=0}^{t-2}\Phi\left(n_j\left(\omega^j\right)\right)u'(c_{t-1}\left(\omega^{t-1}\right))n_{t-1}\left(\omega^{t-1}\right)q_{t-1}\left(\omega_t,\omega_{t-1}\right) \\ &= \quad \prod_{j=0}^{t-1}\Phi\left(n_j\left(\omega^j\right)\right)u'(c_t\left(\omega^t\right))\pi\left(\omega_t,\omega_{t-1}\right)\left(r_t+p_t\right)\end{aligned}$$

for all $t>0$. Simplify it,

$$k_t\left(\omega^t\right):u'(c_{t-1}\left(\omega^{t-1}\right))=\frac{\Phi\left(n_{t-1}\left(\omega^{t-1}\right)\right)}{n_{t-1}\left(\omega^{t-1}\right)}u'(c_t\left(\omega^t\right))\frac{r_t+p_t}{p_{t-1}}\quad\text{for all } t>0. \tag{69}$$

so $c_{t+1}\left(\omega^{t+1}\right)$ depends on ω^t but not on ω_{t+1} for all $t\geq 0$, it can be written as (53).

$$\begin{aligned} n_t\left(\omega^t\right) \quad &: \quad \prod_{j=0}^{t-1}\Phi\left(n_j\left(\omega^j\right)\right)u'(c_t\left(\omega^t\right))\left[\eta\left(\omega_t\right)+\sum_{\omega_{t+1}}q_t\left(\omega_{t+1},\omega_t\right)k_{t+1}\left(\omega^{t+1}\right)+w_t\left(\omega_t\right)\lambda\left(\omega_t\right)\right] \\ &= \quad E_t\sum_{m=t+1}^{\infty}\prod_{j=0}^{m-1}\Phi\left(n_j\left(\omega^j\right)\right)u\left(c_m\left(\omega^m\right)\right)\frac{\Phi'\left(n_t\left(\omega^t\right)\right)}{\Phi\left(n_t\left(\omega^t\right)\right)}\end{aligned}$$

for all $t\geq 0$. Cancel $\prod_{j=0}^{t-1}\Phi\left(n_j\left(\omega^j\right)\right)$,

$$\begin{aligned} n_t\left(\omega^t\right) \quad &: \quad u'(c_t\left(\omega^t\right))\left[\eta\left(\omega_t\right)+\sum_{\omega_{t+1}}q_t\left(\omega_{t+1},\omega_t\right)k_{t+1}\left(\omega^{t+1}\right)+w_t\left(\omega_t\right)\lambda\left(\omega_t\right)\right] \\ &= \quad \Phi'\left(n_t\left(\omega^t\right)\right)E_t\sum_{m=t+1}^{\infty}\prod_{j=t+1}^{m-1}\Phi\left(n_j\left(\omega^j\right)\right)u\left(c_m\left(\omega^m\right)\right)\end{aligned}$$

where $\prod_{j=t+1}^{t} \Phi\left(n_j\left(\omega^j\right)\right) = 1$. Use first order condition with respect to $k_{t+1}\left(\omega^{t+1}\right)$ and forward by one period to substitute out $u'(c_t\left(\omega^t\right))$,

$$\begin{aligned}
&\frac{\Phi\left(n_t\left(\omega^t\right)\right) u'(c_{t+1}\left(\omega^{t+1}\right))}{n_t\left(\omega^t\right)} \frac{r_{t+1}+p_{t+1}}{p_t} \left[\begin{array}{c} \eta\left(\omega_t\right)+\sum_{\omega_{t+1}} q_t\left(\omega_{t+1}, \omega_t\right) k_{t+1}\left(\omega^{t+1}\right) \\ +w_t\left(\omega_t\right) \lambda\left(\omega_t\right) \end{array}\right] \\
=\ & \Phi'\left(n_t\left(\omega^t\right)\right) E_t \sum_{m=t+1}^{\infty} \prod_{j=t+1}^{m-1} \Phi\left(n_j\left(\omega^j\right)\right) u\left(c_m\left(\omega^m\right)\right)
\end{aligned} \tag{70}$$

where

$$\begin{aligned}
& E_t \sum_{m=t+1}^{\infty} \prod_{j=t+1}^{m-1} \Phi\left(n_j\left(\omega^j\right)\right) u\left(c_m\left(\omega^m\right)\right) \\
=\ & E_t\left[u\left(c_{t+1}\left(\omega^{t+1}\right)\right)+\Phi\left(n_{t+1}\left(\omega^{t+1}\right)\right) \sum_{m=t+2}^{\infty} \prod_{j=t+2}^{m-1} \Phi\left(n_j\left(\omega^j\right)\right) u\left(c_m\left(\omega^m\right)\right)\right]
\end{aligned} \tag{71}$$

Forward (70) by one period,

$$\begin{aligned}
&\frac{\Phi\left(n_{t+1}\left(\omega^{t+1}\right)\right)}{n_{t+1}\left(\omega^{t+1}\right)} u'(c_{t+2}\left(\omega^{t+2}\right)) \frac{r_{t+2}+p_{t+2}}{p_{t+1}} \left[\begin{array}{c} \eta\left(\omega_{t+1}\right) \\ +\sum_{\omega_{t+2}} q_{t+1}\left(\omega_{t+2}, \omega_{t+1}\right) k_{t+2}\left(\omega^{t+2}\right) \\ +w_{t+1}\left(\omega_{t+1}\right) \lambda\left(\omega_{t+1}\right) \end{array}\right] \\
=\ & \Phi'\left(n_{t+1}\left(\omega^{t+1}\right)\right) E_{t+1} \sum_{m=t+2}^{\infty} \prod_{j=t+2}^{m-1} \Phi\left(n_j\left(\omega^j\right)\right) u\left(c_m\left(\omega^m\right)\right) \quad \text{for } t \geq 0.
\end{aligned} \tag{72}$$

By (69), (70), (71), (72) and the $(t+1)$-period budget constraint, we have

$$\begin{aligned}
& u'(c_{t+1}\left(\omega^{t+1}\right)) \frac{r_{t+1}+p_{t+1}}{p_t}\left[\eta\left(\omega_t\right)+\sum_{\omega_{t+1}} q_t\left(\omega_{t+1}, \omega_t\right) k_{t+1}\left(\omega^{t+1}\right)+w_t\left(\omega_t\right) \lambda\left(\omega_t\right)\right] \\
=\ & \psi u\left(c_{t+1}\left(\omega^{t+1}\right)\right)+u'\left(c_{t+1}\left(\omega^{t+1}\right)\right) E_t\left[\begin{array}{c} w_{t+1}\left(\omega_{t+1}\right)+\left(r_{t+1}+p_{t+1}\right) k_{t+1}\left(\omega^{t+1}\right) \\ -c_{t+1}\left(\omega^{t+1}\right) \end{array}\right]
\end{aligned}$$

By the actuarially fair price of $q\left(\omega_{t+1}, \omega_t\right)$, we are able to cancel the term associated with

$$E_t\left[\left(r_{t+1}+p_{t+1}\right) k_{t+1}\left(\omega^{t+1}\right)\right]$$

and solve $c_{t+1}\left(\omega^{t+1}\right)$ as (52) for all $t \geq 0$. $n_t\left(\omega^t\right)$ is given by (53) where $n_0\left(\omega_{-1}, \omega_0\right)=n_0\left(\omega_0\right)$. Notice that $c_{t+1}\left(\omega^{t+1}\right)$ depends on ω_t for all t, and hence $n_t\left(\omega^t\right)$ depends on ω_t and ω_{t-1} for $t \geq 1$ while $n_0\left(\omega^0\right)$ depends on ω_0. Given prices $\left\{w_t\left(\omega^t\right)\right\}_{t=0, \omega^t \in \Omega^{t+1}}^{\infty}$, $\left\{r_t\right\}_{t=0}^{\infty}$, $\left\{k_0\left(\omega^0\right)\right\}_{\omega^0 \in \Omega}$, and $\left\{N_0\left(\omega^0\right)\right\}_{\omega^0 \in \Omega}$, in equilibrium $\left\{c_0\left(\omega^0\right)\right\}_{\omega^0 \in \Omega}$, $\left\{c_{t+1}\left(\omega^{t+1}\right)\right\}_{t=0, \omega^{t+1} \in \Omega^{t+2}}^{\infty}$, $\left\{n_t\left(\omega^t\right)\right\}_{t=0, \omega_{t-1}, \omega^t \in \Omega^{t+1}}^{\infty}$, $\left\{k_t\left(\omega^t\right)\right\}_{t=1, \omega^t \in \Omega^{t+1}}^{\infty}$, $\left\{N_{t+1}\left(\omega^{t+1}\right)\right\}_{t=0, \omega^{t+1} \in \Omega^{t+2}}^{\infty}$, and $\left\{L_t\right\}_{t=0}^{\infty}$ are solved by the system of equations consisting of (52), (53), (51), (14) and (15) where $n_0\left(\omega_{-1}, \omega_0\right)=n_0\left(\omega_0\right)$. Next let us solve for the transversality condition associated with population. Rewrite the household's problem as:

$$\max_{\left\{N_{t+1}, K_{t+1}\right\}} \sum_{t=0}^{\infty} \beta^t \sum_{\omega^t \mid \omega^0}\left(\frac{1}{N_0\left(\omega^0\right)^{\psi} \pi\left(\omega^t \mid \omega^0\right)^{\psi-1}}\right) N_t\left(\omega^t\right)^{\psi-\xi} \frac{C_t\left(\omega^t\right)^{\xi}}{\xi}$$

where

$$\begin{aligned}
& C_t\left(\omega^t\right)=c_t\left(\omega^t\right) N_t\left(\omega^t\right) \\
=\ & \left(r_t+p_t\right) K_t\left(\omega^t\right)-\left(\eta\left(\omega_t\right)+\lambda\left(\omega_t\right) w_t\left(\omega_t\right)\right) \frac{N_{t+1}\left(\omega^{t+1}\right)}{\pi\left(\omega^{t+1} \mid \omega^t\right)} \\
& +w_t\left(\omega_t\right) N_t\left(\omega^t\right)-p_t K_{t+1}\left(\left(\omega^t\right)\right)
\end{aligned}$$

the transversality condition of population is

$$\lim_{T\to\infty}\beta^T\frac{N_t\left(\omega^t\right)^{\psi-\xi}}{\pi\left(\omega^t|\omega^0\right)^{\psi-1}}C_t\left(\omega^t\right)^{\xi-1}\left(\eta\left(\omega_t\right)+\lambda\left(\omega_t\right)w_t\left(\omega_t\right)\right)\frac{N_{T+1}\left(\omega^{T+1}\right)}{\pi\left(\omega^{t+1}|\omega^t\right)}=0$$

Assume there is a representative firm in the competitive market who is a price taker. The equilibrium prices are $r_t = F_1(K, L_t; A)$ and $w_t(\omega_t) = F_2(K, L_t; A)\, l(\omega_t)$. $\{p_t\}_{t=0}^{\infty}$ is determined by land market equilibrium: the supply and demand of land equalizes, that is $K = \sum_{\omega^t} N_t(\omega^t)\, k_t(\omega^t)$ for all $t \geq 0$. ■

Proof of Proposition 16. Under the condition $\frac{r_{t+1}+p_{t+1}}{p_t} = \frac{\mu_t}{\mu_{t+1}}$ and the equilibrium price given in Proposition 13, the equivalence of consumption and fertility can be easily seen from the equivalence between the two systems in Proposition 10 and 13. The equivalence of the transversality condition is also obvious. In particular, there must exists an initial land distribution $\left\{k_0\left(\omega^0\right)\right\}_{\omega^0\in\Omega}$ that leads to the same competitive equilibrium level of consumption $\left\{c_0\left(\omega^0\right)\right\}_{\omega^0\in\Omega}$ as that in the social planner's problem given other allocations are simultaneously determined by their respective system. For the rest of the proof it suffices to show that the budget constraints (51) in the competitive equilibrium is consistent with the resource constraint (16) in the social planner's problem. Adding up individuals' budget constraint over different ability history ω^t and weight it by its population $N_t(\omega^t)$,

$$\begin{aligned}
&\sum_{\omega^t}N_t\left(\omega^t\right)c_t\left(\omega^t\right)+\sum_{\omega^t}N_t\left(\omega^t\right)\eta\left(\omega_t\right)n_t\left(\omega^t\right)\\
&+\sum_{\omega^t}N_t\left(\omega^t\right)n_t\left(\omega^t\right)\sum_{\omega_{t+1}}q_t\left(\omega_{t+1},\omega_t\right)k_{t+1}\left(\omega^{t+1}\right)\\
=\;&\sum_{\omega^t}N_t\left(\omega^t\right)w_t\left(\omega_t\right)\left(1-\lambda\left(\omega_t\right)n_t\left(\omega^t\right)\right)+\sum_{\omega^t}N_t\left(\omega^t\right)\left(r_t+p_t\right)k_t\left(\omega^t\right)
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{\omega^t}N_t\left(\omega^t\right)n_t\left(\omega^t\right)\sum_{\omega_{t+1}}q_t\left(\omega_{t+1},\omega_t\right)k_{t+1}\left(\omega^{t+1}\right)\\
=\;&\sum_{\omega^t}\sum_{\omega_{t+1}}p_tN_t\left(\omega^t\right)n_t\left(\omega^t\right)\pi\left(\omega_{t+1},\omega_t\right)k_{t+1}\left(\omega^{t+1}\right)\\
=\;&\sum_{\omega^t}\sum_{\omega_{t+1}}p_tN_{t+1}\left(\omega^{t+1}\right)k_{t+1}\left(\omega^{t+1}\right)\\
=\;&p_tK,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\omega^t}N_t\left(\omega^t\right)w_t\left(\omega_t\right)\left(1-\lambda\left(\omega_t\right)n_t\left(\omega^t\right)\right)\\
=\;&\sum_{\omega^t}N_t\left(\omega^t\right)F_2\left(K,L_t;A\right)l_t\left(\omega_t\right)\left(1-\lambda\left(\omega_t\right)n_t\left(\omega^t\right)\right)\\
=\;&F_2\left(K,L_t;A\right)L_t,
\end{aligned}$$

the budget constraint can be written as

$$\begin{aligned}
&\sum_{\omega^t}N_t\left(\omega^t\right)\left[c_t\left(\omega^t\right)+\eta\left(\omega_t\right)n_t\left(\omega^t\right)\right]\\
=\;&F_2\left(K,L_t;A\right)L_t+r_tK=F_2\left(K,L_t;A\right)L_t+F_1\left(K,L_t;A\right)K\\
=\;&F\left(K,L_t;A\right).
\end{aligned}$$

Therefore

$$\sum_{\omega^t}N_t\left(\omega^t\right)\left[c_t\left(\omega^t\right)+\eta\left(\omega_t\right)n_t\left(\omega^t\right)\right]=F\left(K,L_t;A\right).$$

It indicates that household's budget constraints satisfy the resource constraint faced by the social planner. ■

A.2. Deterministic case with one type.

A.2.1.Case $\delta \geq \beta$.

Lemma 17 *Assume $\eta > 0$ and the production function $F(\cdot,\cdot)$ satisfies Inada condition and*

$$\lim_{L\to\infty} F_{KL}(K,L) = 0.$$

(i) If $\delta > \beta$, a steady state satisfies the following equations:

$$\begin{aligned} & \frac{N\Psi'(N)}{\Psi(N)} \frac{\delta-\beta}{(1-\beta)(\delta-1)} \\ = & \frac{\xi(c)}{c}\left[\Phi'(1)\frac{c}{\xi(c)}\frac{1}{1-\beta} - \eta + F\left(\frac{K}{N}, 1-\lambda, A\right)\left(\frac{\alpha\delta}{\delta-1} - \frac{(1-\alpha)\lambda}{1-\lambda}\right)\right]. \end{aligned}$$

$$F\left(\frac{K}{N}, 1-\lambda, A\right) = c+\eta \tag{73}$$

and

$$\frac{\theta_{t+1}}{\theta_t} = \frac{\mu_{t+1}}{\mu_t} = \frac{\gamma_{t+1}}{\gamma_t} = \delta.$$

(ii) If $\delta = \beta$, the steady state does not exist.

Proof. (i) When $\eta > 0$ and N is finite, one can first show that $\theta_{t+1}/\theta_t = \delta$. Otherwise if $\theta_{t+1}/\theta_t > \delta$, then in the limit, according to (27),

$$\theta_{t+1}/\theta_t = \beta + \frac{\delta^{t+1}\Psi(N)}{N\theta_t} \tag{74}$$

then $\theta_{t+1}/\theta_t = \beta < \delta$, a contradiction. If $\theta_{t+1}/\theta_t < \delta$, then the right hand side of (74) explodes which also leads to a contradiction. (23) then implies that the growth rate of μ_t is the same as that of θ_t, which is δ.

Furthermore, (28) at steady state simplifies to:

$$\Psi'(N)U - \frac{\mu_{t+1}}{\delta^{t+1}}F_K\frac{K}{N} = \frac{\gamma_{t+1}}{\delta^{t+1}}\left(\frac{\gamma_{t+2}}{\gamma_{t+1}} - 1\right).$$

The left hand side of this equality is constant in steady state since the growth rate of μ is δ. Then for the right hand side to converge to a constant we have the following three possibilities: γ_t grows at a rate smaller than δ, γ_t grows at the rate δ, and γ_t keeps constant over time, e.g. $\frac{\gamma_{t+2}}{\gamma_{t+1}} = 1$.

Consider the first possibility when γ_t grows at a rate smaller than δ, then

$$\Psi'(N)U = \frac{\mu_{t+1}}{\delta^{t+1}}F_K\frac{K}{N} \tag{75}$$

Express (27) and (23) at steady state,

$$\theta_t = \frac{\delta^{t+1}\Psi(N)}{N(\delta-\beta)}$$

$$\mu_t = \theta_t u'(c) = \frac{\delta^{t+1}\Psi(N)}{N(\delta-\beta)}u'(c) \tag{76}$$

Plug it into (75) multiplied by $\frac{N}{\Psi(N)}$ and use $U = \frac{u(c)}{1-\beta}$ at steady state,

$$\frac{N\Psi'(N)}{\Psi(N)}c = \frac{\delta(1-\beta)}{\delta-\beta}\xi(c)F_K\frac{K}{N}$$

By the constant return to scale assumption and the definition of α above, it can be written as

$$\frac{N\Psi'(N)}{\Psi(N)}\frac{c}{\xi(c)}=\frac{\delta(1-\beta)}{\delta-\beta}\alpha\frac{F(K,L,A)}{N} \tag{77}$$

which together with (73) and $L=N(1-\lambda)$ can be used to solve (c,N). Given that the growth rate of γ_t is smaller than that of μ_t according to (76), we express (29) at steady state as

$$\Phi'(1)\frac{c}{\xi(c)}\frac{1}{1-\beta}=\eta+(1-\alpha)\frac{F(K,L,A)}{N}\frac{\lambda}{1-\lambda} \tag{78}$$

Use $\Phi'(1)=\psi\beta$ and (73) we solve consumption as

$$c=\frac{\xi(c)\frac{1-\beta}{\beta\psi}\left[1+(1-\alpha)\frac{\lambda}{1-\lambda}\right]\eta}{1-\xi(c)\frac{1-\beta}{\beta\psi}(1-\alpha)\frac{\lambda}{1-\lambda}}.$$

(c,N) solved from (77) and (73) does not satisfy (78) in general. Therefore, in the case of δ bigger than β, the steady state with each multiplier growing at a constant rate, in particular γ_t growing at a constant rate smaller than δ, is not the optimal solution except for a knife-edge condition in which (c,N) satisfies (73), (77), and (78) simultaneously.

Next consider the second possibility when γ_t grows at the rate of δ. Express (27), (29) and (23) at steady state, respectively, as

$$\frac{\delta^{t+1}}{N\theta_t}\Psi(N)+\Phi(1)=\delta\Rightarrow\theta_t=\frac{\delta^{t+1}\Psi(N)}{N(\delta-\beta)}, \tag{79}$$

$$\gamma_t=\frac{1}{\delta}\mu_t\left[\Phi'(1)\frac{U}{u'(c)}-\eta-F_L\lambda\right], \tag{80}$$

and

$$\mu_t=\theta_t u'(c)=\frac{\delta^{t+1}\Psi(N)}{N(\delta-\beta)}u'(c). \tag{81}$$

Plug (80) into the steady state formula of (28), we get

$$\frac{\delta^{t+1}}{\gamma_{t+1}}\Psi'(N)U=\frac{\delta}{\Phi'(1)\frac{U}{u'(c)}-\eta-F_L\lambda}F_K\frac{K}{N}+\delta-1$$

Use (79), (80) and (81) we get

$$\frac{N(\delta-\beta)}{\Psi(N)}\Psi'(N)U=\delta u'(c)F_K\frac{K}{N}+\frac{\delta-1}{\delta}\frac{\mu_{t+1}}{\mu_t}u'(c)\left[\Phi'(1)\frac{U}{u'(c)}-\eta-F_L\lambda\right]$$

We have shown that μ_t grows at the rate δ, then

$$\begin{aligned}&\frac{N}{\Psi(N)}\Psi'(N)\frac{1}{1-\beta}\\=&\frac{\delta}{\delta-\beta}\frac{u'(c)}{u(c)}F_K\frac{K}{N}+\frac{\delta-1}{\delta-\beta}\frac{u'(c)}{u(c)}\left[\Phi'(1)\frac{U}{u'(c)}-\eta-F_L\lambda\right]\\=&\frac{\delta-1}{\delta-\beta}\frac{\xi(c)}{c}\left[\Phi'(1)\frac{U}{u'(c)}-\eta-F_L\lambda+\frac{\delta}{\delta-1}F_K\frac{K}{N}\right]\end{aligned}$$

Note that

$$F_L=\left(\frac{1-\alpha}{1-\lambda}\right)\frac{F}{N}$$

$$F_K\frac{K}{N}=\alpha\frac{F}{N}.$$

So the above equality becomes

$$\begin{aligned}&\frac{N\Psi'(N)}{\Psi(N)}\frac{1}{1-\beta}\\=&\frac{\delta-1}{\delta-\beta}\frac{\xi(c)}{c}\left[\Phi'(1)c\frac{1}{\xi(c)}\frac{1}{1-\beta}-\eta+F\left(\frac{K}{N},1-\lambda,A\right)\left(\frac{\alpha\delta}{\delta-1}-\frac{(1-\alpha)\lambda}{1-\lambda}\right)\right].\end{aligned}$$

which together with (73) solves (c, N).The second case gives the solution.

For the third possibility $\frac{\gamma_{t+1}}{\gamma_t} = 1$ at steady state, which together with $\frac{\mu_{t+1}}{\mu_t} = \delta < 1$ contradicts with equation (29).

(ii) If $\delta = \beta$, (27) becomes

$$\frac{\theta_{t+1}}{\theta_t} = \frac{\beta^{t+1}}{\theta_t}\frac{\Psi(N)}{N} + \beta \geq \beta.$$

If $\lim_{t\to\infty} \frac{\theta_{t+1}}{\theta_t} = \beta$, then $\lim_{t\to\infty} \frac{\beta^{t+1}}{\theta_t}\frac{\Psi(N)}{N} > 0$ and $\frac{\theta_{t+1}}{\theta_t}$ converges to a number strictly bigger than β and a contradiction arises since N is a finite number when $\eta > 0$. If $\lim_{t\to\infty} \frac{\theta_{t+1}}{\theta_t} > \beta$, then $\lim_{t\to\infty} \frac{\theta_{t+1}}{\theta_t} = \lim_{t\to\infty} \frac{\beta^{t+1}}{\theta_t}\frac{\Psi(N)}{N} = 0$, a contradiction. Hence steady state with Lagrange multipliers growing at constant rate over time does not exist in this case. ■

A.2.2. Stability of the steady state

To get some insights about the stability of the steady state, in this section we also focus on the deterministic case with one type and $\delta = 0$. In that case the social planner cares about future generations to the extent that the initial generation does. Furthermore, assume no time costs of raising children, $\lambda = 0$, Barro-Becker's functional forms $\Phi(n) = \beta n^{\psi}$, $u(c) = \frac{c^{\xi}}{\xi}$, a Cobb-Douglas production function $F(K, N_t; A) = AK^{\alpha}N_t^{1-\alpha}$, and $N_0 = 1$. The restriction $\psi > \xi$ is required for concavity.

Initial parent's utility is then given by

$$U_0 = u(c_0) + \beta n_0^{\psi} U_1 = \sum_{t=0}^{\infty} \beta^t \prod_{j=0}^{t-1} n_j^{\psi} u(c_t) = \sum_{t=0}^{\infty} \beta^t N_t^{\psi} \frac{1}{\xi} c_t^{\xi}.$$

The following is the social planner's problem:

$$\max_{\{C_t, N_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t N_t^{\psi} \frac{1}{\xi} c_t^{\xi} \text{ subject to } c_t N_t = F(K, L_t; A) - N_{t+1}\eta.$$

Substitute the budget constraint into the objective function,

$$\max_{\{C_t, N_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t N_t^{\psi} \frac{1}{\xi} \left(\frac{F(K, N_t; A) - N_{t+1}\eta}{N_t}\right)^{\xi}$$

The optimal choice of population in period t is

$$N_t^{\psi-\xi} C_t^{\xi-1}\eta = \beta N_{t+1}^{\psi-\xi} C_{t+1}^{\xi-1} \left[\left(\frac{\psi}{\xi} - \alpha\right) AK^{\alpha} N_{t+1}^{-\alpha} - \eta \frac{\psi - \xi}{\xi}\frac{N_{t+2}}{N_{t+1}}\right]$$

where $C_t = c_t N_t$ is the aggregate consumption of all people of generation t. It is convenient to define the variable $X_t \equiv N_t^{\frac{\xi-\psi}{1-\xi}} C_t$, a mix between aggregate consumption and a factor that depends on population. Then the dynamic system can be described by the following two equations:

$$X_t = N_t^{\frac{\xi-\psi}{1-\xi}} \left[AK^{\alpha} N_t^{1-\alpha} - N_{t+1}\eta\right]$$

$$\left(\frac{X_{t+1}}{X_t}\right)^{1-\xi} \eta = \beta \left[\left(\frac{\psi}{\xi} - \alpha\right) AK^{\alpha} N_{t+1}^{-\alpha} - \eta \frac{\psi - \xi}{\xi}\frac{N_{t+2}}{N_{t+1}}\right]$$

The first equation is the resource constraint and the second is the optimality condition for population. Steady state population, N^*, can be solved as

$$AK^{\alpha} N^{*-\alpha} = \frac{\psi - \xi + \xi/\beta}{\psi - \alpha\xi}\eta. \tag{82}$$

Next we take a first-order Taylor expansion of this system around the steady state to analyze its stability. It is determined by the system of equations (83):

$$W \begin{bmatrix} \frac{dN_{t+2}}{N^*} \\ \frac{dX_{t+1}}{X^*} \end{bmatrix} = G \begin{bmatrix} \frac{dN_{t+1}}{N^*} \\ \frac{dX_t}{X^*} \end{bmatrix} \tag{83}$$

where

$$W = \begin{bmatrix} \eta \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} & 1 \\ \beta \frac{\psi-\xi}{\xi} & 1-\xi \end{bmatrix}$$

and

$$G = \begin{bmatrix} (\xi-\psi)/(1-\xi) + (1-\alpha)\left(1+\eta \frac{N^{*(1-\psi)/(1-\xi)}}{X}\right) & 0 \\ (1-\alpha)\beta \frac{\psi-\xi}{\xi} - \alpha & 1-\xi \end{bmatrix}.$$

The following proposition provides the necessary and sufficient conditions under which the steady state is saddle path stable.

Proposition 18 *The necessary and sufficient conditions for saddle path stability of the steady state are*

$$(2-\alpha)\frac{\frac{\psi-\xi}{\xi}(1-2\xi+(1-\alpha)\beta)+(1-\alpha)2(1-\xi)}{1/\beta-(1-\alpha)} > \alpha(1-2\xi)-2(1-\psi)$$

and

$$\frac{1}{\beta}(\alpha(1-\xi)-1+\psi) \neq (1-\alpha)\frac{\psi-\xi}{\xi}.$$

Proof. Equation (83) can be written as:

$$\begin{bmatrix} \frac{dN_{t+2}}{N^*} \\ \frac{dX_{t+1}}{X^*} \end{bmatrix} = D \begin{bmatrix} \frac{dN_{t+1}}{N^*} \\ \frac{dX_t}{X^*} \end{bmatrix}.$$

where

$$\begin{aligned} & D \equiv W^{-1}G \\ = & \begin{bmatrix} \eta \frac{N^{*(1-\psi)/(1-\xi)}}{X^*} & 1 \\ \beta \frac{\psi-\xi}{\xi} & 1-\xi \end{bmatrix}^{-1} \begin{bmatrix} \frac{\xi-\psi}{1-\xi} + (1-\alpha)\left(1+\eta \frac{N^{(1-\psi)/(1-\xi)}}{X}\right) & 0 \\ (1-\alpha)\beta\frac{\psi-\xi}{\xi} - \alpha & 1-\xi \end{bmatrix} \\ = & \frac{1}{d}\begin{bmatrix} \begin{pmatrix} \xi-\psi+(1-\xi)(1-\alpha)\left(1+\eta\frac{N^{(1-\psi)/(1-\xi)}}{X}\right) \\ +\alpha-(1-\alpha)\beta\frac{\psi-\xi}{\xi} \end{pmatrix} & \xi-1 \\ -\beta\frac{\psi-\xi}{\xi}((1-\psi)/(1-\xi)-\alpha)-\alpha\eta\frac{N^{(1-\psi)/(1-\xi)}}{X} & \eta\frac{N^{*(1-\psi)/(1-\xi)}}{X^*}(1-\xi) \end{bmatrix} \end{aligned}$$

with $d = \eta\frac{N^{*(1-\psi)/(1-\xi)}}{X^*}(1-\xi) - \beta\frac{\psi-\xi}{\xi}$. Let λ_1 and λ_2 denote the eigenvalues of the matrix D. Assume, without loss of generality, that $\lambda_1 > \lambda_2$. They are solved by

$$\begin{aligned} \lambda_1 &= \frac{tr(D)+\sqrt{tr(D)^2-4\det(D)}}{2} \\ \lambda_2 &= \frac{tr(D)-\sqrt{tr(D)^2-4\det(D)}}{2} \end{aligned}$$

where $\det(D)$ and $tr(D)$ can be solved as

$$d^2\det(D) = \left\{ \begin{array}{c} \begin{bmatrix} \xi-\psi+(1-\xi)(1-\alpha)\left(1+\eta\frac{N^{*(1-\psi)/(1-\xi)}}{X^*}\right) \\ +\alpha-(1-\alpha)\beta\frac{\psi-\xi}{\xi} \end{bmatrix} \eta\frac{N^{*(1-\psi)/(1-\xi)}}{X^*}(1-\xi) \\ -(1-\xi)\left[\beta\frac{\psi-\xi}{\xi}((1-\psi)/(1-\xi)-\alpha)+\alpha\eta\frac{N^{*(1-\psi)/(1-\xi)}}{X^*}\right] \end{array} \right\}$$

$$d\cdot tr(D) = \left\{ \begin{array}{c} \xi-\psi+(1-\xi)(1-\alpha)\left(1+\eta\frac{N^{*(1-\psi)/(1-\xi)}}{X^*}\right)+\alpha \\ -(1-\alpha)\beta\frac{\psi-\xi}{\xi}+\eta\frac{N^{*(1-\psi)/(1-\xi)}}{X^*}(1-\xi) \end{array} \right\}$$

For saddle-path stability either $|\lambda_1| < 1$ and $|\lambda_2| > 1$ or $|\lambda_1| > 1$ and $|\lambda_2| < 1$. Since $\lambda_1 > \lambda_2$ by assumption this condition can be divided into two cases: (i) $\lambda_1 > 1$ and $-1 < \lambda_2 < 1$ and (ii)

$\lambda_2 < -1$ and $-1 < \lambda_1 < 1$. Let us first consider case (i) which can be reduced to $1 - tr\,(D) < -\det(D) < 1 + tr\,(D)$, and then to

$$(1 - D_{11})(1 - D_{22}) < D_{12}D_{21} < (1 + D_{11})(1 + D_{22}),$$

where D_{ij} refers to the (i,j) element of matrix D. The condition in case (ii) which can be reduced to $tr\,(D) + 1 < -\det(D) < -tr\,(D) + 1$, and then to

$$(1 + D_{11})(1 + D_{22}) < D_{12}D_{21} < (1 - D_{11})(1 - D_{22}).$$

Now let us derive terms in these conditions.

$$\begin{aligned} & d^2 D_{12}D_{21} \\ = \; & \left(-\beta\frac{\psi-\xi}{\xi}\left((1-\psi)/(1-\xi)-\alpha\right)-\alpha\eta\frac{N^{*(1-\psi)/(1-\xi)}}{X}\right)(\xi-1) \end{aligned}$$

At steady state,

$$\frac{N^{*(1-\psi)/(1-\xi)}}{X^*} = \frac{\psi/\xi-\alpha}{1/\beta-(1-\alpha)} \tag{84}$$

$$d(1-D_{11}) = \psi - (1+\alpha\xi) + \alpha(1-\xi)\eta\frac{N^{(1-\psi)/(1-\xi)}}{X} - \alpha\beta\frac{\psi-\xi}{\xi}$$

$$d(1-D_{22}) = -\beta\frac{\psi-\xi}{\xi}$$

$$\begin{aligned} & d^2(1-D_{11})(1-D_{22}) \\ = \; & -\beta\frac{\psi-\xi}{\xi}\left(\psi-1-\alpha\xi+\alpha(1-\xi)\eta\frac{N^{(1-\psi)/(1-\xi)}}{X}-\alpha\beta\frac{\psi-\xi}{\xi}\right) \end{aligned}$$

Use the above result, derive the condition $(1-D_{11})(1-D_{22}) < D_{12}D_{21}$ in case (i). It holds if and only if

$$\eta(1-\xi)\frac{N^{*(1-\psi)/(1-\xi)}}{X^*} > \beta\frac{\psi-\xi}{\xi}. \tag{85}$$

Substitute out $\frac{N^{*(1-\psi)/(1-\xi)}}{X^*}$ using (84) it becomes

$$\frac{1}{\beta}\left(-1+\psi+\alpha(1-\xi)\right) < \frac{\psi-\xi}{\xi}(1-\alpha). \tag{86}$$

$D_{12}D_{21} < (1+D_{11})(1+D_{22})$ holds if and only if

$$\begin{aligned} & \left(\begin{array}{c} 2(2-\alpha)\eta(1-\xi)\frac{N^{*(1-\psi)/(1-\xi)}}{X^*} \\ -(2-\alpha)\beta\frac{\psi-\xi}{\xi}+2(1-\psi)+\alpha-2\alpha(1-\xi) \end{array}\right)\left(\eta(1-\xi)\frac{N^{*(1-\psi)/(1-\xi)}}{X^*}-\beta\frac{\psi-\xi}{\xi}\right) \\ > \; & 0 \end{aligned}$$

When (85) holds true, this inequality holds if and only if

$$2(2-\alpha)\eta(1-\xi)\frac{N^{*(1-\psi)/(1-\xi)}}{X^*}-(2-\alpha)\beta\frac{\psi-\xi}{\xi} > 2\alpha(1-\xi)-2(1-\psi)-\alpha.$$

Substitute out $\frac{N^{*(1-\psi)/(1-\xi)}}{X^*}$ using (84),

$$(2-\alpha)\frac{\frac{\psi-\xi}{\xi}\left(1-2\xi+(1-\alpha)\beta\right)+(1-\alpha)2(1-\xi)}{1/\beta-(1-\alpha)} > \alpha(1-2\xi)-2(1-\psi) \tag{87}$$

Hence the condition for case (i) is (86) and (87). In the same way we can derive that the condition for case (ii) is the following two inequalities

$$\frac{1}{\beta}\left(\alpha(1-\xi)-1+\psi\right) > (1-\alpha)\frac{\psi-\xi}{\xi}$$

and (87). Summarize case (i) and case (ii), the sufficient and necessary condition for saddle path stability is (87) and

$$\frac{1}{\beta}\left(\alpha\left(1-\xi\right)-1+\psi\right) \neq \left(1-\alpha\right)\frac{\psi-\xi}{\xi}.$$

■

Given $\xi < \psi$, the Barro-Becker's assumption for the concavity of the problem, this condition holds for most sets of parameters. The second condition holds except for a wide range of parameters. In particular, a nice sufficient condition guarantees saddle path stability is $\xi < \frac{1}{2}$ and $\alpha\left(1-2\xi\right) \leq 2\left(1-\psi\right)$. We summarize it in the following Corollary.

Corollary 19 *A sufficient condition for saddle path stability of the steady state are* $\xi < \frac{1}{2}$ *and* $\alpha\left(1-2\xi\right) < 2\left(1-\psi\right)$.

Under this condition, the left hand side $\frac{1}{\beta}\left(\alpha\left(1-\xi\right)-1+\psi\right)$ is positive while its right hand side is non-positive.