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The Enhanced Principal Rank Characteristic Sequence

Abstract

The enhanced principal rank characteristic sequence (epr-sequence) of a symmetric $n \times n$ matrix is a sequence $l_1 l_2 \cdots l_n$ where l_k is A, S, or N according as all, some, or none of its principal minors of order k are nonzero. Such sequences give more information than the (0,1) pr-sequences previously studied (where basically the k th entry is 0 or 1 according as none or at least one of its principal minors of order k is nonzero). Various techniques including the Schur complement are introduced to establish that certain subsequences such as NAN are forbidden in epr-sequences over fields of characteristic not two. Using probabilistic methods over fields of characteristic zero, it is shown that any sequence of As and Ss ending in A is attainable, and any sequence of As and Ss followed by one or more Ns is attainable; additional families of attainable epr-sequences are constructed explicitly by other methods. For real symmetric matrices of orders 2, 3, 4, and 5, all attainable epr-sequences are listed with justifications.

Keywords

Principal rank characteristic sequence; Enhanced principal rank characteristic sequence, Minor, Rank, Symmetric matrix, Hermitian matrix, Schur complement

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The enhanced principal rank characteristic sequence

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March 21, 2015

Abstract

The enhanced principal rank characteristic sequence (epr-sequence) of a symmetric $n \times n$ matrix is a sequence $\ell_1 \ell_2 \cdots \ell_n$ where ℓ_k is A, S, or N according as all, some, or none of its principal minors of order k are nonzero. Such sequences give more information than the (0,1) pr-sequences previously studied (where basically the k th entry is 0 or 1 according as none or at least one of its principal minors of order k is nonzero). Various techniques including the Schur complement are introduced to establish that certain subsequences such as NAN are forbidden in epr-sequences over fields of characteristic not two. Using probabilistic methods over fields of characteristic zero, it is shown that any sequence of As and Ss ending in A is attainable, and any sequence of As and Ss followed by one or more Ns is attainable; additional families of attainable epr-sequences are constructed explicitly by other methods. For real symmetric matrices of orders 2, 3, 4, and 5, all attainable epr-sequences are listed with justifications.

Keywords. Principal rank characteristic sequence, enhanced principal rank characteristic sequence, minor, rank, symmetric matrix, Hermitian matrix, Schur complement

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1 Introduction

For a symmetric matrix over a field F or a complex Hermitian matrix, Brualdi et al. [3] and Barrett et al. [1] considered a principal rank characteristic sequence, which records with a 1 or a 0 whether or not there is a full rank principal submatrix of each order. More precisely, the *principal rank characteristic sequence* of an $n \times n$ symmetric or complex Hermitian matrix B is the sequence $\text{pr}(B) = r_0]r_1r_2 \cdots r_n$, where for $k = 0, 1, \dots, n$, a 1 in the k th position indicates the existence of a principal submatrix of rank k and a 0 indicates no such submatrix exists. To obtain more information, we refine this sequence and instead of considering the presence or absence of such a principal submatrix, we consider three possibilities in the following definition.

Definition 1.1. The *enhanced principal rank characteristic sequence* of a symmetric matrix $B \in F^{n \times n}$ (or Hermitian matrix $B \in \mathbb{C}^{n \times n}$) is the sequence (epr-sequence) $\text{epr}(B) = \ell_1\ell_2 \cdots \ell_n$ where

$$\ell_k = \begin{cases} \mathbf{A} & \text{if all } k \times k \text{ principal minors of the given order are nonzero;} \\ \mathbf{S} & \text{if some but not all } k \times k \text{ principal minors are nonzero;} \\ \mathbf{N} & \text{if none of the } k \times k \text{ principal minors are nonzero, i.e., all are zero.} \end{cases}$$

We are interested in which epr-sequences are *attainable* over a given field F , i.e., can be attained by some (symmetric or Hermitian) matrix over F , and also which sequences are *forbidden* over a given field, i.e., no such matrix has the sequence. We can now drop the convention of having a 0th term given by r_0 in the pr-sequence. In particular the relationship between the old and new naming conventions for the beginning of a sequence is as follows: $1]0 \leftrightarrow \mathbf{N}$, $1]1 \leftrightarrow \mathbf{S}$, and $0]1 \leftrightarrow \mathbf{A}$.

Brualdi et al. [3] introduced the definition of a pr-sequence for a real symmetric matrix as a simplification of the principal minor assignment problem as stated in [5]. The study of epr-sequences provides additional information that may be helpful in work on the principal minor assignment problem, while remaining somewhat tractable. Furthermore, the enhanced principal rank characteristic sequence can be used to answer the following question [6, p. 112]: “For a real symmetric matrix, which lists of sizes, for which there exists a singular principal submatrix, can occur?” (See Corollary 4.7.)

In Section 2, we identify certain forbidden and certain attainable epr-sequences, with some results depending on the field; the Schur complement method for establishing forbidden subsequences is discussed in this section. In Section 3, we focus on epr-sequences attained by adjacency matrices of graphs. For fields of characteristic 0, we use probabilistic methods in Section 4 to establish that any sequence of As and Ss ending in A is attainable, and any sequence of As and Ss followed by one or more Ns is attainable. For real symmetric matrices, in Section 5 we determine which epr-sequences are attainable for orders 2, 3, 4, and 5.

For $B \in F^{n \times n}$, $\alpha, \beta \subseteq \{1, 2, \dots, n\}$, the submatrix of B lying in rows indexed by α and columns indexed by β is denoted by $B[\alpha, \beta]$. Further, the complementary submatrix obtained from B by deleting the rows indexed by α and columns indexed by β is denoted by $B(\alpha, \beta)$. If $\alpha = \beta$, then the principal submatrix $B[\alpha, \alpha]$ is abbreviated to $B[\alpha]$, while the complementary principal submatrix is denoted by $B(\alpha)$. The complement of α is denoted by α^c .

61 Following the notation in [1], we let $\overline{\ell_i \cdots \ell_j}$ indicate that the (complete) sequence may be
62 repeated as many times as desired (or may be omitted entirely). All matrices are symmetric
63 over a field F , or are complex Hermitian.

64 2 The enhanced principal rank characteristic sequence

65 We begin with some simple observations and applications of known results that are valid for
66 epr-sequences of symmetric matrices over any field and for complex Hermitian matrices.

67 **Observation 2.1.** *If $\text{pr}(B) = r_0]r_1 \dots r_n$ and $\text{epr}(B) = \ell_1 \ell_2 \cdots \ell_n$ is the enhanced pr-*
68 *sequence for B , then $\ell_k = \mathbf{N}$ if and only if $r_k = 0$, and $r_k = 1$ if and only if $\ell_k = \mathbf{A}$ or*
69 *\mathbf{S} , for $k = 1, \dots, n$.*

70 There is only one submatrix of full order so it either has full rank or it does not, giving
71 \mathbf{A} or \mathbf{N} as the last term in the epr-sequence. For the classes of matrices considered, the rank
72 of the matrix is equal to the maximum rank of a *principal* submatrix; see, for example, [1,
73 Theorem 1.1]. These statements lead to the following observation.

74 **Observation 2.2.** *An epr-sequence of a symmetric (or complex Hermitian) matrix B must*
75 *end in \mathbf{N} or \mathbf{A} , and $\text{rank } B$ is equal to the index of the last \mathbf{A} or \mathbf{S} in $\text{epr}(B)$.*

76 The next result was proved over the real numbers in [3], and for any field in [1].

77 **Theorem 2.3.** [1, Theorem 2.1] *Suppose $B \in F^{n \times n}$ is symmetric or complex Hermitian,*
78 *$\text{epr}(B) = \ell_1 \cdots \ell_n$, and $\ell_k = \ell_{k+1} = \mathbf{N}$ for some k . Then $\ell_i = \mathbf{N}$ for all $i \geq k$. (That is, if an*
79 *epr-sequence of a matrix ever has \mathbf{NN} , then it must have \mathbf{N} s from that point forward.)*

80 Jacobi's determinantal identity is used to relate the epr-sequence of a nonsingular matrix
81 to that of its inverse. It is valid for symmetric matrices over any field and for complex
82 Hermitian matrices. This implies that most epr-sequences that end in \mathbf{A} now come in natural
83 pairs. The situation for pr-sequences is more complicated, with the 0th term in the pr-
84 sequence of the inverse depending on the existence of some zero principal minor of order
85 $n - 1$ in the original matrix.

86 **Theorem 2.4.** (Inverse Theorem) *If $\text{epr}(B) = \ell_1 \ell_2 \cdots \ell_{n-1} \mathbf{A}$ then $\text{epr}(B^{-1}) = \ell_{n-1} \ell_{n-2} \cdots \ell_1 \mathbf{A}$*
87 *(i.e., simply reverse the sequence except for the last \mathbf{A}).*

88 2.1 Forbidden epr-sequences

89 The next result extends [3, Theorem 4.1] to epr-sequences.

90 **Proposition 2.5.** *The epr-sequence $\mathbf{SN} \cdots \mathbf{A} \cdots$ is forbidden for symmetric matrices over*
91 *any field and for complex Hermitian matrices.*

92 *Proof.* Suppose $\text{epr}(B) = \mathbf{SN} \cdots$. The \mathbf{S} in the first position implies there is some $b_{ii} = 0$.
93 The \mathbf{N} in the second position implies $b_{ij} = 0$ for $j = 1, \dots, n$. Thus every entry in row i is 0,
94 and so for any k there is a singular $k \times k$ submatrix. \square

95 The following theorem, which shows when a portion of an epr-sequence is inherited, is a
 96 useful tool when working with subsequences.

97 **Theorem 2.6.** *Suppose that $B \in F^{n \times n}$ is symmetric or complex Hermitian, $m \leq n$, and*
 98 *$1 \leq i \leq m$.*

- 99 1. *If $[\text{epr}(B)]_i = \mathbf{N}$, then $[\text{epr}(C)]_i = \mathbf{N}$ for all $m \times m$ principal submatrices C .*
- 100 2. *If $[\text{epr}(B)]_i = \mathbf{A}$, then $[\text{epr}(C)]_i = \mathbf{A}$ for all $m \times m$ principal submatrices C .*
- 101 3. *If $[\text{epr}(B)]_m = \mathbf{S}$, then there exist $m \times m$ principal submatrices C_A and C_N of B such*
 102 *that $[\text{epr}(C_A)]_m = \mathbf{A}$ and $[\text{epr}(C_N)]_m = \mathbf{N}$.*
- 103 4. *If $i < m$ and $[\text{epr}(B)]_i = \mathbf{S}$, then there exists an $m \times m$ principal submatrix C_S such*
 104 *that $[\text{epr}(C_S)]_i = \mathbf{S}$.*

105 *Proof.* The inheritance of \mathbf{N} and \mathbf{A} simply follow by noting that a principal submatrix of a
 106 principal submatrix is a principal submatrix. The ability to choose ℓ_m in statement 3 follows
 107 by noting that there is some submatrix of full rank and there is also some submatrix that is
 108 not of full rank, so the appropriate one is chosen.

109 For the final statement, note that there are two submatrices of order i and that one has
 110 full rank and the other does not. Now suppose that the rows/columns of the submatrix with
 111 full rank are p_1, p_2, \dots, p_i and that the rows/columns of the submatrix that does not have
 112 full rank are q_1, q_2, \dots, q_i (and moreover without loss of generality they are ordered so that
 113 any common indices occur in the same spot on the two lists). Now consider the following
 114 possible sets of rows and columns.

$$\begin{array}{c}
 p_1, p_2, p_3, \dots, p_i \\
 q_1, p_2, p_3, \dots, p_i \\
 q_1, q_2, p_3, \dots, p_i \\
 q_1, q_2, q_3, \dots, p_i \\
 \dots \\
 q_1, q_2, q_3, \dots, q_i
 \end{array}$$

115

116 Since the first list corresponds with a submatrix of full rank and the last list does not, then
 117 somewhere in between there are two consecutive rows where one list corresponds with a
 118 submatrix of full rank and the other list does not. The union of these two row index sets
 119 is of cardinality $i + 1$ (since they only differ in one position); thus adding the remaining
 120 $m - i - 1$ indices arbitrarily gives a principal submatrix of the correct order with the desired
 121 epr-sequence. □

122 **Corollary 2.7.** *No symmetric matrix over any field (or complex Hermitian matrix) can have*
 123 *NSA in its epr-sequence. Further, no symmetric matrix over any field (or complex Hermitian*
 124 *matrix) can have the epr-sequence $\dots \mathbf{ASN} \dots \mathbf{A} \dots$.*

125 *Proof.* By Proposition 2.5, no epr-sequence has the form $\mathbf{SN} \dots \mathbf{A}$, and thus by Theorem 2.4,
 126 no epr-sequence can end with \mathbf{NSA} (because if it did then applying the inverse would result

127 in a forbidden epr-sequence). Thus, no epr-sequence can contain NSA (because if it did then
 128 Theorem 2.6 could be applied to find a principal submatrix having epr-sequence ending in
 129 NSA, giving an impossible epr-sequence). The second statement follows by noting that if such
 130 a matrix exists, then there is an appropriate submatrix with an inverse having epr-sequence
 131 containing NSA, which is impossible. \square

132 2.2 Forbidden initial epr-sequences and Schur complements

133 In this section, we rule out certain initial sequences for symmetric matrices over fields of
 134 characteristic not 2, and use a technique involving Schur complements to rule out other
 135 placements of subsequences.

136 **Proposition 2.8.** *No symmetric matrix over a field of characteristic not 2 has an epr-*
 137 *sequence starting NAN... or NAS... .*

138 *Proof.* Suppose $B \in F^{n \times n}$, $\text{char } F \neq 2$, and $\text{epr}(B) = \text{NAL}_3 \dots$. The N in the first position
 139 means that the main diagonal entries are all 0, while the A in the second position forces all
 140 of the off-diagonal entries to be nonzero. Therefore any 3×3 principal minor is $2pqr \neq 0$
 141 where p, q, r are the three off-diagonal entries, so $\ell_3 = \text{A}$. \square

142 The hypotheses that the matrix is symmetric and $\text{char } F \neq 2$ are important, as the
 143 matrices in the next example illustrate.

144 **Example 2.9.** The complex Hermitian matrix $B = \begin{bmatrix} 0 & i & 1 \\ -i & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and the symmetric matrix
 145 $C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \in \mathbb{Z}_2^{n \times n}$ both have epr-sequence NAN.

146 **Corollary 2.10.** *A symmetric matrix over a field of characteristic not 2 cannot have SANA*
 147 *or NANA in its epr-sequence.*

148 *Proof.* If one of these sequences is present, then by Theorems 2.6 and 2.4 there is an appro-
 149 priate submatrix with an inverse that has an epr-sequence beginning with NAS or NAN, which
 150 is impossible by Proposition 2.8. \square

151 **Proposition 2.11.** *Over a field of characteristic not 2 any epr-sequence of a symmetric*
 152 *matrix that starts SAN is of the form SAN \bar{N} .*

153 *Proof.* Let B be a symmetric matrix of order ≥ 4 with $\text{epr}(B) = \text{SAN} \dots$. Since the first letter
 154 of $\text{epr}(B)$ is S, there is at least one 0 term on the diagonal and without loss of generality we
 155 can assume it is in the (1, 1)-position. Since the second letter is A, all the other entries in
 156 the first row/column must be nonzero (or else there is a 2×2 principal submatrix that does
 157 not have full rank). By a diagonal congruence, we may assume that these other entries in
 158 the first row/column are 1.

159 Next note that, since the third letter in $\text{epr}(B)$ is \mathbb{N} , the determinant of the 3×3 principal
 160 submatrix $B[\{1, i, j\}]$ is 0, giving

$$161 \quad 0 = \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & b_{ii} & b_{ij} \\ 1 & b_{ij} & b_{jj} \end{bmatrix} = 2b_{ij} - b_{ii} - b_{jj}.$$

162 So

$$163 \quad b_{ij} = \frac{b_{ii} + b_{jj}}{2},$$

164 and thus every principal submatrix of B is completely determined by its diagonal. For
 165 example, when $n = 4$ such a matrix is of the form

$$166 \quad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & \frac{a+a}{2} & \frac{a+b}{2} & \frac{a+c}{2} \\ 1 & \frac{b+a}{2} & \frac{b+b}{2} & \frac{b+c}{2} \\ 1 & \frac{c+a}{2} & \frac{c+b}{2} & \frac{c+c}{2} \end{bmatrix}$$

167 for some choice of a, b, c . Matrices of this form have rank 2, i.e., every row is a linear
 168 combination of rows one and two. This implies that there is no nonzero minor of order 4 or
 169 larger. \square

170 **Corollary 2.12.** *Suppose B is a symmetric matrix over a field of characteristic not 2 that*
 171 *has $\text{epr}(B) = \text{SAN} \cdots$. Then B has at most two zeros on the diagonal.*

172 *Proof.* If $b_{ii} = b_{jj} = b_{kk} = 0$ then by Theorem 2.6, $\text{epr}(B[\{i, j, k\}]) = \text{NAN}$, which is impossible
 173 by Proposition 2.8. \square

174 Recall that if C is a given $n \times n$ matrix, with a nonsingular principal submatrix $C[\alpha]$, then
 175 the matrix given by $C/C[\alpha] = C[\alpha^c] - C[\alpha^c, \alpha](C[\alpha])^{-1}C[\alpha, \alpha^c]$ is the *Schur complement* of
 176 $C[\alpha]$ in C . Schur complements of symmetric matrices have the following properties.

177 **Proposition 2.13.** *Suppose F is a field of characteristic not 2 and $C \in F^{n \times n}$ is a symmetric*
 178 *matrix of rank m . Let $C[\alpha]$ be a nonsingular principal submatrix of C with $|\alpha| = k \leq m$,*
 179 *and let $B = C/C[\alpha]$. Then the following results hold.*

- 180 1. B is an $(n - k) \times (n - k)$ symmetric matrix.
- 181 2. Assuming the indexing of B is inherited from C , any principal minor of B is given by
 182 $\det B[\gamma] = \det C[\gamma \cup \alpha] / \det C[\alpha]$.

183 3. $\text{rank } B = m - k$.

184 4. Any nonsingular principal submatrix of C of order at most m is contained in a non-
 185 singular principal submatrix of order m .

186 *Proof.* The first three statements are all basic properties involving Schur complements, so
 187 we omit their verification. For the fourth statement, which is surely known, we offer the
 188 following short argument. Suppose $C[\alpha]$ is a full rank principal submatrix of order $k \leq m$.
 189 Then by property (3), the rank of B is $m - k$. Hence there exists an $(m - k) \times (m - k)$
 190 principal submatrix of B of full rank. In this case, using property (2), it follows that there
 191 exists an $m \times m$ principal submatrix of C of full rank that contains α , as desired. \square

192 We note that results of Johnson et al. [6, Section 5 (S)] imply that for a real symmetric
 193 matrix, no epr-sequence can end in NAN. Here we present a brief independent proof of a more
 194 general result.

195 **Theorem 2.14.** *Neither the epr-sequences NAN nor NAS can occur as a subsequence of the*
 196 *epr-sequence of a symmetric matrix over a field of characteristic not 2.*

197 *Proof.* Suppose that there exists a real symmetric matrix C with subsequence NAN occurring
 198 in its epr-sequence in positions $k+1, k+2, k+3$, respectively. By Proposition 2.8, NAN is not
 199 at the start of the epr-sequence for C , and by Theorem 2.3, in the place directly to the left
 200 of the first N there is either an A or an S. Then there exists a $k \times k$ principal submatrix of C ,
 201 say $C[\alpha]$, that is nonsingular. Let $B = C/C[\alpha]$. In this case, using Proposition 2.13.2, the
 202 epr-sequence associated with B starts with NAN, which contradicts Proposition 2.8. Hence
 203 no such C exists.

204 If a sequence contained NAS, where the S entry is in the k th position of the epr-sequence
 205 for C , then C must have a singular $k \times k$ principal submatrix and further this matrix has
 206 nonsingular $(k-1) \times (k-1)$ principal submatrices and only singular $(k-2) \times (k-2)$ principal
 207 matrices. Therefore it contains a principal submatrix that contains the subsequence NAN.
 208 But this is impossible by the above argument. \square

209 **Theorem 2.15.** *In the epr-sequence of a symmetric matrix over a field of characteristic not*
 210 *2, the subsequence ANS can occur only as the initial subsequence.*

211 *Proof.* Suppose that a symmetric matrix B has ANS occurring in positions $k, k+1$, and
 212 $k+2$. By Theorem 2.6, B contains some principal submatrix C of order $k+3$ whose epr-
 213 sequence $\ell_1 \cdots \ell_{k+3}$ also has $\ell_k \ell_{k+1} \ell_{k+2} = \text{ANS}$. Since Corollary 2.7 excludes NSA, and S is
 214 not allowed as the last entry of any epr-sequence, $\ell_{k+3} = \text{N}$. Because C is singular and
 215 contains a nonsingular $(k+2) \times (k+2)$ principal submatrix, the rank of C is $k+2$, and
 216 hence by Proposition 2.13(4), every order k principal submatrix is contained in an order
 217 $k+2$ nonsingular principal submatrix of C .

218 Since $\ell_k = \text{A}$, any $k \times k$ principal submatrix $C[\alpha]$ of C is nonsingular, so we can take its
 219 3×3 Schur complement $C/C[\alpha]$. Consider $\text{epr}(C/C[\alpha]) = \ell'_1 \ell'_2 \ell'_3$. By Proposition 2.13(3),
 220 $\text{rank}(C/C[\alpha]) = k+2 - k = 2$ so ℓ'_2 is S or A, and $\ell'_3 = \text{N}$. Choose a single index i
 221 of $C/C[\alpha]$. By Proposition 2.13(2), $\det((C/C[\alpha])[\{i\}]) = \det C[\alpha \cup \{i\}] / \det C[\alpha]$ Since
 222 $C[\alpha \cup \{i\}]$ is $(k+1) \times (k+1)$ and $\ell_{k+1} = \text{N}$, $\det((C/C[\alpha])[\{i\}]) = 0$, i.e., $\ell'_1 = \text{N}$. Thus
 223 $\text{epr}(C/C[\alpha]) = \text{NAN}$ or NSN , but NAN is prohibited by Theorem 2.14, so $\text{epr}(C/C[\alpha]) = \text{NSN}$.
 224 So we can choose $\{i, j\}$ such that $(C/C[\alpha])[\{i, j\}]$ is singular. Then by Proposition 2.13(2)
 225 $\det((C/C[\alpha])[\{i, j\}]) = \det C[\alpha \cup \{i, j\}] / \det C[\alpha]$ so $C[\alpha \cup \{i, j\}]$ is singular. Thus $C[\alpha]$ is also
 226 contained in a singular $(k+2) \times (k+2)$ principal submatrix of C .

227 Partition the index set $\{1, \dots, k+3\}$ into a pair of sets $X = \{i : C(\{i\}) \text{ is singular}\}$ and
 228 $Y = \{i : C(\{i\}) \text{ is nonsingular}\}$. If either X or Y had a three-element subset U , then $C(U)$
 229 would be an order k principal submatrix of C that was not contained in both a nonsingular
 230 and a singular principal submatrix of order $k+2$. It follows that $|X| = |Y| = 2$, and so
 231 $k = 1$. \square

2.3 Attainable epr-sequences

We now consider methods for constructing families of matrices attaining given epr-sequences. For order n , the identity matrix is denoted by I_n and the all 1s matrix by J_n , while $\mathbf{1}_n$ denotes the all 1s vector of length n . For a field F , $a \in F$ and $n \geq 2$, define the matrices

$$L_n(a) := \begin{bmatrix} I_{n-1} & \mathbf{1}_{n-1} \\ \mathbf{1}_{n-1}^T & a \end{bmatrix}.$$

Observation 2.16. *For any field:*

- $\text{epr}(I_n) = \mathbf{A}\bar{\mathbf{A}}$.
- For $n \geq 2$, $\text{epr}(J_n) = \mathbf{A}\bar{\mathbf{N}}$.
- For $n \geq 2$, $\text{epr}(I_{n-2} \oplus J_2) = \mathbf{A}\bar{\mathbf{S}}\bar{\mathbf{N}}$.
- $\text{epr}(0_n) = \mathbf{N}\bar{\mathbf{N}}$.
- For $n \geq 2$, $\text{epr}(I_1 \oplus 0_{n-1}) = \mathbf{S}\bar{\mathbf{N}}$.
- For $n \geq 2$, $\text{epr}(I_{n-2} \oplus L_2(0)) = \mathbf{S}\bar{\mathbf{S}}\bar{\mathbf{A}}$.
- For $n \geq 2$, $\text{epr}(I_{n-1} \oplus 0_1) = \mathbf{S}\bar{\mathbf{S}}\bar{\mathbf{N}}$.

The next result follows from [3, Theorem 2.2] and symmetry.

Proposition 2.17. *For a field of characteristic 0, $n \geq 2$ and $1 \leq k \leq n$, $\text{epr}(J_n - kI_n) = \bar{\mathbf{A}}\mathbf{N}\bar{\mathbf{A}}$ with the \mathbf{N} in the k th position.*

Proposition 2.18. *For a field F of characteristic 0, $n \geq 2$, $\text{epr}(L_n(k-1)) = \bar{\mathbf{A}}\mathbf{S}\bar{\mathbf{A}}\mathbf{A}$ for $1 \leq k < n$, with the \mathbf{S} in the k th position.*

Proof. Suppose $\text{epr}(L_n(k-1)) = \ell_1 \cdots \ell_n$. For $1 \leq m \leq n$, every $m \times m$ principal submatrix is of the form $L_m(k-1)$ or I_m , and note that $\det L_m(k-1) = (k-1) - (m-1) = k-m$. Thus $\ell_m = \mathbf{A}$ for $m \neq k$ and $m = n$, and $\ell_k = \mathbf{S}$. \square

It was observed in [3] that given a matrix and its pr-sequence, a matrix that has this pr-sequence extended by an additional 0 can be found by doing a simple copy of the last row and column [3, Theorem 2.6]. However, it *cannot* be guaranteed that \mathbf{N} can be added to an attainable epr-sequence to obtain another attainable epr-sequence. Over a field of characteristic not 2, any epr-sequence ending $\mathbf{N}\mathbf{A}$ cannot be extended by adding \mathbf{N} because $\mathbf{N}\mathbf{A}\mathbf{N}$ is prohibited. The problem is that a simple row and column copy may destroy the delicate property of having all minors of order $i > 1$ be nonsingular. But singularity can be preserved.

Observation 2.19. *Let $B \in F^{n \times n}$ have epr-sequence $\ell_1 \ell_2 \cdots \ell_n$.*

1. *Form a matrix B' from B by copying the last row down and then the last column across. Then the epr-sequence of B' is $\ell_1 \ell'_2 \cdots \ell'_n \mathbf{N}$ with $\ell'_i = \mathbf{N}$ if $\ell_i = \mathbf{N}$ and $\ell'_i = \mathbf{S}$ otherwise for $2 \leq i \leq n$.*

265 2. Form a matrix B'' from B by taking the direct sum with $[0]$. Then the epr-sequence of
 266 B'' is $\ell''_1 \ell''_2 \cdots \ell''_n \mathbf{N}$ with $\ell''_i = \mathbf{N}$ if $\ell_i = \mathbf{N}$ and $\ell''_i = \mathbf{S}$ otherwise for $1 \leq i \leq n$.

267 The attainability of the following sequences is established by applying Observation 2.19
 268 to the sequences $\mathbf{S}\bar{\mathbf{S}}\mathbf{N}$ in Observation 2.16 and $\bar{\mathbf{A}}\mathbf{N}\bar{\mathbf{A}}\mathbf{A}$ in Proposition 2.17.

269 **Corollary 2.20.**

- 270 1. For any field, the epr-sequence $\mathbf{S}\bar{\mathbf{S}}\mathbf{N}\bar{\mathbf{N}}$ is attainable.
 271 2. For a field of characteristic 0, $n \geq 3$ and $1 \leq k \leq n$, the epr-sequence $\bar{\mathbf{S}}\mathbf{N}\bar{\mathbf{S}}\mathbf{N}\bar{\mathbf{N}}$ with the
 272 first \mathbf{N} in the k th position is attainable.

273 In [1, Theorem 2.3] it was shown that

$$274 \quad \text{supp}(B_1 \oplus B_2) = (\text{supp}(B_1) + \text{supp}(B_2)) \cup \text{supp}(B_1) \cup \text{supp}(B_2) \quad (1)$$

275 where $\text{supp}(B) = \{i : (\text{pr}(B))_i = 1\}$, and for sets X and Y , $X + Y = \{x + y : x \in X, y \in Y\}$.
 276 Here we define $\mathbf{AS}(B) := \{i : (\text{epr}(B))_i = \mathbf{A} \text{ or } (\text{epr}(B))_i = \mathbf{S}\}$, $\mathbf{AS}_0(B) := \mathbf{AS}(B) \cup \{0\}$,
 277 and $\mathbf{NS}(B) := \{i : (\text{epr}(B))_i = \mathbf{N} \text{ or } (\text{epr}(B))_i = \mathbf{S}\}$. With this notation, (1) becomes
 278 $\mathbf{AS}_0(B_1 \oplus B_2) = \mathbf{AS}_0(B_1) + \mathbf{AS}_0(B_2)$ (adding a zero into the set avoids the need to take the
 279 union, because $\mathbf{AS}(B_i) \subseteq \mathbf{AS}_0(B_1) + \mathbf{AS}_0(B_2), i = 1, 2$). The next theorem extends this to
 280 obtain the epr-sequence of a direct sum of two matrices. Define $[m] := \{0, 1, \dots, m\}$, and
 281 note that for any set S , $S + \emptyset = \emptyset$.

282 **Theorem 2.21.** (Reducible Matrix Theorem) *Let $B_i \in F^{n_i \times n_i}, i = 1, 2$ be symmetric matri-*
 283 *ces over a field F or complex Hermitian matrices and let $\text{epr}(B_1 \oplus B_2) = \ell_1 \ell_2 \cdots \ell_n$. Then*

$$284 \quad \mathbf{AS}_0(B_1 \oplus B_2) = \mathbf{AS}_0(B_1) + \mathbf{AS}_0(B_2), \quad \mathbf{AS}(B_1 \oplus B_2) = \mathbf{AS}_0(B_1 \oplus B_2) \setminus \{0\}. \quad (2)$$

$$285 \quad \mathbf{NS}(B_1 \oplus B_2) = (\mathbf{NS}(B_1) + [n_2]) \cup ([n_1] + \mathbf{NS}(B_2)) \quad (3)$$

$$286 \quad \ell_i = \begin{cases} \mathbf{A} & \text{if } i \in \mathbf{AS}(B_1 \oplus B_2) \setminus \mathbf{NS}(B_1 \oplus B_2); \\ \mathbf{S} & \text{if } i \in \mathbf{AS}(B_1 \oplus B_2) \cap \mathbf{NS}(B_1 \oplus B_2); \\ \mathbf{N} & \text{if } i \in \mathbf{NS}(B_1 \oplus B_2) \setminus \mathbf{AS}(B_1 \oplus B_2). \end{cases} \quad (4)$$

287 *Proof.* As noted earlier, (2) follows from [1, Theorem 2.3]. For (3): $\mathbf{NS}(B)$ is the set of indices
 288 k such that B has a singular $k \times k$ principal submatrix. A singular $k \times k$ principal submatrix
 289 of $B_1 \oplus B_2$ can be obtained by taking a singular piece in B_1 and the rest in B_2 (and by
 290 including $0 \in [m]$, the piece in B_2 may be empty) or vice versa. Then (4) follows from (2)
 291 and (3). \square

294 **Corollary 2.22.**

- 295 • Over any field, $n \geq 3$ and $1 \leq s \leq n - 2$, $\text{epr}(I_{s-1} \oplus L_{n-s+1}(1)) = \mathbf{A}\bar{\mathbf{S}}\bar{\mathbf{A}}\mathbf{A}$ where there
 296 are $s \geq 1$ copies of \mathbf{S} .
 297 • Over a field of characteristic 0, $n \geq 2$, $1 \leq k$, and $1 \leq s \leq n - 1$, $\text{epr}(I_{s-1} \oplus (J_{n-s+1} -$
 298 $kI_{n-s+1})) = \bar{\mathbf{A}}\bar{\mathbf{S}}\bar{\mathbf{A}}\mathbf{A}$ where the first \mathbf{S} is in position k and there are s copies of \mathbf{S} .

299 Over a field of characteristic 0, we can use Hankel matrices of binomial coefficients to
 300 generate epr-sequences of the form $\mathbf{S}\bar{\mathbf{A}}\bar{\mathbf{N}}$ (another proof that $\mathbf{S}\bar{\mathbf{A}}\bar{\mathbf{N}}$ is attainable over any field
 301 of characteristic 0 is given in Section 4). Define $H_n^{(k)} = [h_{ij}^{(k)}]$ where $h_{i,j}^{(k)} = \binom{i+j+k-3}{k}$.

302 **Example 2.23.** Observe that

$$303 \quad H_5^{(1)} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}, \quad H_5^{(2)} = \begin{bmatrix} 0 & 1 & 3 & 6 & 10 \\ 1 & 3 & 6 & 10 & 15 \\ 3 & 6 & 10 & 15 & 21 \\ 6 & 10 & 15 & 21 & 28 \\ 10 & 15 & 21 & 28 & 36 \end{bmatrix}, \quad H_5^{(3)} = \begin{bmatrix} 0 & 1 & 4 & 10 & 20 \\ 1 & 4 & 10 & 20 & 35 \\ 4 & 10 & 20 & 35 & 56 \\ 10 & 20 & 35 & 56 & 84 \\ 20 & 35 & 56 & 84 & 120 \end{bmatrix}.$$

304 It is straightforward to verify that $\text{epr}(H_5^{(1)}) = \mathbf{SANNN}$, $\text{epr}(H_5^{(2)}) = \mathbf{SAANN}$ and $\text{epr}(H_5^{(3)}) =$
 305 \mathbf{SAAAN} . Then $\text{epr}(H_4^{(1)}) = \mathbf{SANN}$, $\text{epr}(H_4^{(2)}) = \mathbf{SAAN}$, and $\text{epr}(H_3^{(1)}) = \mathbf{SAN}$ follow from Theorem
 306 2.6 (or by direct verification).

307 **3 Graphs and epr-sequences**

308 The $(0, 1)$ adjacency matrix of a graph G is denoted by $A(G)$. The complete graph, the star
 309 graph (centered at 1), the path graph and the cycle graph, all on n vertices, are denoted by
 310 $K_n, K_{1,n}, P_n$ and C_n , respectively. Adjacency matrices of graphs provide numerous examples
 311 of attainable epr-sequences. Note that the epr-sequence of $A(G)$ for any graph G always
 312 begins with \mathbf{N} .

313 **Observation 3.1.**

- 314 • Over a field of characteristic 0, $\text{epr}(A(K_n)) = \bar{\mathbf{N}}\bar{\mathbf{A}}$.
- 315 • For $n \geq 3$, $\text{epr}(A(K_{1,n-1})) = \mathbf{NS}\bar{\mathbf{N}}$.
- 316 • For $n \geq 3$, $\text{epr}(A(P_n)) = \bar{\mathbf{N}}\bar{\mathbf{S}}\bar{\mathbf{N}}$ if n is odd; $\text{epr}(A(P_n)) = \bar{\mathbf{N}}\bar{\mathbf{S}}\bar{\mathbf{N}}\bar{\mathbf{A}}$ if n is even.
- 317 • Over a field of characteristic not 2, with $n \geq 4$, $\text{epr}(A(C_n)) = \bar{\mathbf{N}}\bar{\mathbf{S}}\bar{\mathbf{N}}\bar{\mathbf{N}}$ if $n = 4k$;
 318 $\text{epr}(A(C_n)) = \bar{\mathbf{N}}\bar{\mathbf{S}}\bar{\mathbf{N}}\bar{\mathbf{A}}\bar{\mathbf{A}}$ if $n = 4k + 1$ or $n = 4k + 3$; $\text{epr}(A(C_n)) = \bar{\mathbf{N}}\bar{\mathbf{S}}\bar{\mathbf{N}}\bar{\mathbf{A}}$ if $n = 4k + 2$.

319 The next corollary follows from Observation 3.1 and Observation 2.19.

320 **Corollary 3.2.** For $n \geq 3$ and $1 \leq k \leq n$, the epr-sequence $\bar{\mathbf{N}}\bar{\mathbf{S}}\bar{\mathbf{N}}\bar{\mathbf{N}}$ is attainable over a field
 321 of characteristic 0.

322 For symmetric matrices with zero diagonal we can view the matrix as a weighted adja-
 323 cency matrix, and associate a (simple) graph to the matrix. The graph $\mathcal{G}(B)$ of a symmetric
 324 matrix $B \in F^{n \times n}$ with zero diagonal is the simple graph with vertices $\{1, \dots, n\}$ and edges
 325 $\{\{i, j\} : b_{ij} \neq 0 \text{ and } i \neq j\}$. Here are some general observations relating terms in an epr-
 326 sequence of a symmetric matrix $B \in F^{n \times n}$ with zero diagonal and its associated graph
 327 $\mathcal{G}(B)$.

328 **Observation 3.3.** Suppose $B \in F^{n \times n}$ is a symmetric matrix with zero diagonal, and let
 329 $\text{epr}(B) = \mathbb{N}\ell_2 \cdots \ell_n$.

330 1. If $\ell_2 = \mathbf{A}$, then $\mathcal{G}(B) = K_n$.

331 2. If $\ell_k = \mathbf{A}$ for some k , then $\mathcal{G}(B)$ has no isolated vertex.

332 3. If $\ell_3 = \mathbb{N}$ and $\text{char } F \neq 2$, then $\mathcal{G}(B)$ is triangle-free (otherwise, B has a 3×3 principal

333 minor equal to $\det \begin{bmatrix} 0 & p & q \\ p & 0 & r \\ q & r & 0 \end{bmatrix} = 2pqr \neq 0$).

334 4 Probabilistic techniques for fields of characteristic 335 zero

336 In this section we use probabilistic methods to establish that over a field of characteristic
 337 0, any epr-sequence that does not contain an \mathbb{N} is attainable (Theorem 4.4), as is any epr-
 338 sequence that has all copies of \mathbb{N} consecutively at the end of the sequence (Theorem 4.6).

339 **Proposition 4.1.** Suppose $\text{char } F = 0$ and let $B \in F^{n \times n}$ be symmetric. Assume $\text{epr}(B) =$
 340 $\ell_1 \ell_2 \cdots \ell_n$ and $r = \text{rank } B$. Construct a matrix B' from B by adjoining a new last row formed
 341 by taking a random linear combination of r independent rows. Then adjoin the same linear
 342 combination of the columns. Denote the epr-sequence of B' by $\ell'_1 \ell'_2 \cdots \ell'_n \ell'_{n+1}$. Then $\ell'_i = \mathbb{N}$
 343 for $i = r + 1, \dots, n + 1$, and for $1 \leq i \leq r$ with high probability $\ell'_i = \mathbf{A}$ if $\ell_i = \mathbf{A}$ and $\ell'_i = \mathbf{S}$ if
 344 $\ell_i = \mathbf{S}$ or $\ell_i = \mathbb{N}$.

345 *Proof.* Since the maximum number of linearly independent rows in B' is r , $\ell'_i = \mathbb{N}$ for
 346 $i = r + 1, \dots, n + 1$. Suppose $k \leq r$. If $\ell_k = \mathbf{S}$, then clearly $\ell'_k = \mathbf{S}$. So it remains to show
 347 that with high probability $\ell_k = \mathbf{A}$ implies $\ell'_k = \mathbf{A}$ and $\ell_k = \mathbb{N}$ implies $\ell'_k = \mathbf{S}$.

348 Let $C = B[T]$ be a $(k - 1) \times (k - 1)$ principal submatrix of B with $\text{rank } C \geq k - 2$. Define
 349 $C' := B'[T \cup \{n + 1\}]$. We claim that with high probability $\text{rank } C' = k$. If $\text{rank } C = k - 1$,
 350 then the new row restricted to the first $k - 1$ entries is in the span of the rows of C , but
 351 with high probability the (k, k) diagonal entry is wrong for adding a new row and column
 352 without increasing the rank, so $\text{rank } C' = k$. If $\text{rank } C = k - 2$, with high probability the
 353 new row is not in the span of the rows of C , so $\text{rank } C' = \text{rank } C + 2 = k$.

354 Suppose $\ell_k = \mathbf{A}$. Then for every $(k - 1) \times (k - 1)$ principal submatrix C , it is possible
 355 to add a row and column from B and obtain a nonsingular matrix, so $\text{rank } C \geq k - 2$, and
 356 with high probability $\text{rank } C' = k$. Thus, with high probability $\ell'_k = \mathbf{A}$.

357 Suppose $\ell_k = \mathbb{N}$. Then $\ell_{k-1} \neq \mathbb{N}$ by Theorem 2.3, so there exists a $(k - 1) \times (k - 1)$
 358 principal submatrix $C = B[T]$ with $\text{rank } C = k - 1$. Then with high probability $\text{rank } C' = k$,
 359 and so with high probability $\ell'_k = \mathbf{S}$. \square

360 **Lemma 4.2.** Suppose $\text{char } F = 0$ and let $B \in F^{n \times n}$ be symmetric. Assume $\text{epr}(B) =$
 361 $\ell_1 \ell_2 \cdots \ell_{n-1} \mathbf{A}$ with $\ell_k \in \{\mathbf{A}, \mathbf{S}\}$ for $k = 1, \dots, n - 1$. Then there exists a matrix $B' \in$
 362 $F^{(n+1) \times (n+1)}$ such that $\text{epr}(B') = \ell_1 \cdots \ell_{n-1} \mathbf{A} \mathbf{A}$.

363 *Proof.* Form a symmetric matrix B' from B by appending a last row and column of entries
364 chosen as random rational numbers. Let $\text{epr}(B') = \ell_1 \ell'_2 \cdots \ell'_n \ell'_{n+1}$. Clearly if $\ell_k = \mathbf{S}$ then
365 $\ell'_k = \mathbf{S}$. We show that if $\ell_k = \mathbf{A}$ then $\ell'_k = \mathbf{A}$, and $\ell'_{n+1} = \mathbf{A}$, both with high probability.
366 Suppose $\ell_k = \mathbf{A}$. Then for every $(k-1) \times (k-1)$ principal submatrix, it is possible to
367 append a row and column and obtain a nonsingular matrix. Then with high probability
368 for any $(k-1) \times (k-1)$ principal submatrix of B , appending the relevant part of row and
369 column $n+1$ in B' results in a nonsingular matrix. Similarly, with high probability, B' is
370 nonsingular. \square

371 **Lemma 4.3.** *Suppose $\text{char } F = 0$ and let $B \in F^{n \times n}$ be symmetric. Assume $\text{epr}(B) =$
372 $\ell_1 \ell_2 \cdots \ell_{n-1} \mathbf{A}$ with $\ell_k \in \{\mathbf{A}, \mathbf{S}\}$ for $k = 1, \dots, n$. Then there exists a matrix $B' \in F^{(n+1) \times (n+1)}$
373 such that $\text{epr}(B') = \ell_1 \cdots \ell_{n-1} \mathbf{S} \mathbf{A}$.*

374 *Proof.* Since $\ell_{n-1} \in \{\mathbf{A}, \mathbf{S}\}$, there exists an index i such that $B(\{i\})$ is nonsingular; without
375 loss of generality $i = 1$, and we abbreviate this submatrix by $B(1)$. Then because any
376 $(n-1)$ -vector is in the range of $B(1)$, B can be partitioned as $B = \begin{bmatrix} c & \mathbf{v}^T B(1) \\ B(1)\mathbf{v} & B(1) \end{bmatrix}$ with
377 $c \neq \mathbf{v}^T B(1)\mathbf{v}$ because B is nonsingular. Define

$$378 \quad B' := \begin{bmatrix} c & \mathbf{v}^T B(1) & c' \\ B(1)\mathbf{v} & B(1) & B(1)\mathbf{w} \\ c' & \mathbf{w}^T B(1) & \mathbf{w}^T B(1)\mathbf{w} \end{bmatrix}$$

379 where $B(1)\mathbf{w}$ is a random vector and c' is random. Then with high probability B' is nonsin-
380 gular and no ℓ_k has been altered for $i \leq n-1$. Observe that $B'(1)$ is singular. \square

381 **Theorem 4.4.** *Any epr-sequence that does not contain \mathbf{N} and ends in \mathbf{A} is attainable over*
382 *every field of characteristic 0.*

383 *Proof.* The proof is by induction. The sequences \mathbf{A} , \mathbf{AA} , and \mathbf{SA} are all attainable. Assume all
384 epr-sequences of length $\leq n$ consisting of \mathbf{A} and \mathbf{S} and ending in \mathbf{A} are attainable. Consider
385 the sequence $\ell_1 \cdots \ell_n \mathbf{A}$. The sequence $\mathbf{S}\bar{\mathbf{S}}\mathbf{A}$ is attainable (Observation 2.16), so assume there
386 exists $i \leq n$ such that $\ell_i = \mathbf{A}$ and let k be the largest index such that $\ell_k = \mathbf{A}$. By the
387 induction hypothesis there is a symmetric matrix B such that $\text{epr}(B) = \ell_1 \cdots \ell_k$.

388 If $k = n$ then $\ell_1 \cdots \ell_n \mathbf{A}$ is attainable by Lemma 4.2. So assume $k < n$ and $\ell_{k+1} = \cdots =$
389 $\ell_n = \mathbf{S}$. By the induction hypothesis $\ell_1 \cdots \ell_{k-1} \mathbf{A}$ is attainable. Then by applying Lemma 4.2
390 followed by Lemma 4.3 $n-k$ times, $\ell_1 \cdots \ell_{k-1} \mathbf{A} \mathbf{S} \cdots \mathbf{S} \mathbf{A} = \ell_1 \cdots \ell_n \mathbf{A}$ is attained. \square

391 **Lemma 4.5.** *Suppose $\text{char } F = 0$ and let $B \in F^{n \times n}$ be symmetric. Assume $\text{epr}(B) =$
392 $\ell_1 \ell_2 \cdots \ell_{n-1} \mathbf{A}$ with $\ell_k \in \{\mathbf{A}, \mathbf{S}\}$ for $k = 1, \dots, n$. Then there exists a matrix $B' \in F^{(n+1) \times (n+1)}$
393 such that $\text{epr}(B') = \ell_1 \cdots \ell_{n-1} \mathbf{S} \mathbf{N}$.*

394 *Proof.* Since $\ell_{n-1} \in \{\mathbf{A}, \mathbf{S}\}$, there exists an index i such that $B(\{i\})$ is nonsingular; without
 395 loss of generality $i = 1$, and we abbreviate this submatrix by $B(1)$. Then because any
 396 $(n-1)$ -vector is in the range of $B(1)$, B can be partitioned as $B = \begin{bmatrix} c & \mathbf{v}^T B(1) \\ B(1)\mathbf{v} & B(1) \end{bmatrix}$ with
 397 $c \neq \mathbf{v}^T B(1)\mathbf{v}$ because B is nonsingular. Define

$$398 \quad B' := \begin{bmatrix} c & \mathbf{v}^T B(1) & \mathbf{v}^T B(1)\mathbf{w} \\ B(1)\mathbf{v} & B(1) & B(1)\mathbf{w} \\ \mathbf{w}^T B(1)\mathbf{v} & \mathbf{w}^T B(1) & \mathbf{w}^T B(1)\mathbf{w} \end{bmatrix}$$

399 where $B(1)\mathbf{w}$ is a random vector. Then with high probability no ℓ_k has been altered for
 400 $i \leq n-1$. Observe that B' and $B'(1)$ are singular. \square

401 **Theorem 4.6.** *Any epr-sequence $\ell_1 \ell_2 \cdots \ell_n \mathbf{N} \cdots \mathbf{N}$ with $\ell_k \in \{\mathbf{A}, \mathbf{S}\}$ for $k = 1, \dots, n$ and $t \geq 1$
 402 copies of \mathbf{N} is attainable over a field of characteristic 0.*

403 *Proof.* By Theorem 4.4 $\ell_1 \cdots \ell_{n-1} \mathbf{A}$ is attainable. If $\ell_n = \mathbf{A}$, then apply Proposition 4.1
 404 t times to $\ell_1 \cdots \ell_{n-1} \mathbf{A}$ to obtain $\ell_1 \cdots \ell_{n-1} \mathbf{A} \mathbf{N} \cdots \mathbf{N} = \ell_1 \cdots \ell_n \mathbf{N} \cdots \mathbf{N}$. If $\ell_n = \mathbf{S}$, then apply
 405 Lemma 4.5 to $\ell_1 \cdots \ell_{n-1} \mathbf{A}$ to obtain $\ell_1 \cdots \ell_{n-1} \mathbf{S} \mathbf{N} = \ell_1 \cdots \ell_n \mathbf{N}$, and then apply Proposition 4.1
 406 $t-1$ times to $\ell_1 \cdots \ell_n \mathbf{N}$ to obtain $\ell_1 \cdots \ell_n \mathbf{N} \cdots \mathbf{N}$. \square

407 Theorems 4.4 and 4.6 can be used to answer the following question of Johnson et al. [6,
 408 p. 112]: Which subsets T of $\{1, \dots, n\}$ can occur as the list of sizes k for which there exists
 409 a $k \times k$ singular principal submatrix of B ?

410 **Corollary 4.7.** *For any subset T of $\{1, \dots, n\}$ there exists an $n \times n$ real symmetric matrix
 411 such that T is the list of sizes k for which there exists a $k \times k$ singular principal submatrix
 412 of B .*

413 *Proof.* A real symmetric matrix B realizes such a list of sizes T if and only if $\text{epr}(B) = \ell_1 \cdots \ell_n$
 414 and for $k = 1, \dots, n$, $k \in T$ if and only if $\ell_k = \mathbf{N}$ or \mathbf{S} . So given T , use Theorem 4.4 or 4.6 to
 415 construct matrix B with $\text{epr}(B) = \ell_1 \cdots \ell_n$ having the following properties:

- 416 • $\ell_k = \mathbf{S}$ if $k \in T$ and $\ell_k = \mathbf{A}$ if $k \notin T$ for $k = 1, \dots, n-1$.
- 417 • $\ell_n = \mathbf{N}$ if $n \in T$ and $\ell_n = \mathbf{A}$ if $n \notin T$. \square

418 The need to prove Theorems 4.4 and 4.6 also illustrates the additional information pro-
 419 vided by the enhanced principal rank characteristic sequence, as opposed to the principal
 420 rank characteristic sequence (the only pr-sequences covered by these theorems are of the
 421 form $\bar{1}\bar{0}$, which is attained by $I_k \oplus 0_{n-k}$).

Table 2: All epr-sequences for order 2 that can be attained by real symmetric matrices. The sequences are listed in lexicographic order.

| epr-Sequence | Real matrix | Result |
|--------------|----------------------|------------------|
| AA | I_2 | Observation 2.16 |
| AN | J_2 | Observation 2.16 |
| NA | $A(K_2) = J_2 - I_2$ | Proposition 2.17 |
| NN | 0_2 | Observation 2.16 |
| SA | $L_2(0)$ | Proposition 2.18 |
| SN | $I_1 \oplus 0_1$ | Observation 2.16 |

Table 3: All epr-sequences for order 3 that can be attained by real symmetric matrices. The sequences are listed in lexicographic order.

| epr-Sequence | Real matrix | Result |
|--------------|----------------------|------------------|
| AAA | I_3 | Observation 2.16 |
| AAN | $J_3 - 3I_3$ | Proposition 2.17 |
| ANA | $J_3 - 2I_3$ | Proposition 2.17 |
| ANN | J_3 | Observation 2.16 |
| ASA | $L_3(1)$ | Proposition 2.18 |
| ASN | $I_1 \oplus J_2$ | Observation 2.16 |
| NAA | $A(K_3) = J_3 - I_3$ | Proposition 2.17 |
| NNN | 0_3 | Observation 2.16 |
| NSN | $A(K_{1,2})$ | Observation 3.1 |
| SAA | $L_3(0)$ | Proposition 2.18 |
| SAN | $H_3^{(1)}$ | Example 2.23 |
| SNN | $I_1 \oplus 0_2$ | Observation 2.16 |
| SSA | $I_1 \oplus L_2(0)$ | Observation 2.16 |
| SSN | $I_2 \oplus 0_1$ | Observation 2.16 |

438 We now determine all attainable sequences of order 4 over \mathbb{R} . The next result follows
 439 from Theorem 5.1 and Table 1.

440 **Corollary 5.2.** *The only attainable order 4 epr-sequences that begin AN are ANAA, ANNN,*
 441 *and ANSN.*

442 There are 54 order 4 sequences that end in N or A. Of these 54 sequences, we eliminate
 443 those that contain NNS or NNA (Theorem 2.3), leaving 47 possible sequences. The subse-
 444 quences NSA, NAS, and NAN are ruled out by Corollary 2.7 and Theorem 2.14, leaving 37
 445 possible sequences. The sequence SANA is ruled out by Proposition 2.11, the sequence ASNA
 446 is ruled out by Corollary 2.7, and the sequence SNAA is ruled out by Proposition 2.5. The
 447 remaining 34 epr-sequences are all attainable over \mathbb{R} ; see Table 4. The next example gives
 448 a realizing matrix for the particular sequence NAAN. For a given epr-sequence, the notation
 449 M_{epr} denotes a specific matrix realizing this epr-sequence.

450 **Example 5.3.** For $M_{\text{NAAN}} := \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 4 \\ 1 & 1 & 4 & 0 \end{bmatrix}$, $\text{epr}(M_{\text{NAAN}}) = \text{NAAN}$.

Table 4: All epr-sequences for order 4 that can be attained by real symmetric matrices. The sequences are listed in lexicographic order.

| epr-Sequence | Real matrix | Result |
|--------------|---------------------------|---------------------------------------|
| AAAA | I_4 | Observation 2.16 |
| AAAN | $J_4 - 4I_4$ | Proposition 2.17 |
| AANA | $J_4 - 3I_4$ | Proposition 2.17 |
| AANN | | Theorem 4.6 |
| AASA | $L_4(0)^{-1}$ | Theorem 2.4, Proposition 2.18 |
| AASN | | Theorem 4.6 |
| ANAA | $J_4 - 2I_4$ | Proposition 2.17 |
| ANNN | J_4 | Observation 2.16 |
| ANSN | | Corollary 5.2 |
| ASAA | $L_4(1)$ | Proposition 2.18 |
| ASAN | | Theorem 4.6 |
| ASNN | | Theorem 4.6 |
| ASSA | $I_1 \oplus L_3(1)$ | Corollary 2.22 |
| ASSN | | Theorem 4.6 |
| NAAA | $A(K_4) = J_4 - I_4$ | Proposition 2.17 |
| NAAN | M_{NAAN} | Example 5.3 |
| NNNN | 0_4 | Observation 2.16 |
| NSNA | $A(P_4)$ | Observation 3.1 |
| NSNN | $A(K_{1,3})$ | Observation 3.1 |
| NSSA | $A(G_1)$ | $G_1 = \text{paw graph}$ |
| NSSN | $(J_3 - I_3) \oplus 0_1$ | Corollary 2.20 |
| SAAA | $L_4(0)$ | Proposition 2.18 |
| SAAN | $H_4^{(2)}$ | Example 2.23 |
| SANN | $H_4^{(1)}$ | Example 2.23 |
| SASA | | Theorem 4.4 |
| SASN | | Theorem 4.6 |
| SNNN | $I_1 \oplus 0_3$ | Observation 2.16 |
| SNSN | $(J_3 - 2I_3) \oplus 0_1$ | Corollary 2.20 |
| SSAA | $I_1 \oplus A(K_3)$ | Corollary 2.22 |
| SSAN | | Theorem 4.6 |
| SSNA | $A(G_1)^{-1}$ | Theorem 2.4, $G_1 = \text{paw graph}$ |
| SSNN | $I_2 \oplus 0_2$ | Corollary 2.20 |
| SSSA | $I_2 \oplus L_2(0)$ | Observation 2.16 |
| SSSN | $I_3 \oplus 0_1$ | Observation 2.16 |

451 The list of reasons that epr-sequences are unattainable (and thus not listed in Table 4)
452 is summarized in [4], and similarly for order 5. Note that the justifications for attainability
453 given in Table 4 do not provide explicit matrices for some of these sequences. In each case
454 where no matrix is listed, we have constructed such matrices using essentially the method
455 cited¹ and the documentation is available in [4].

456 5.2 Epr-sequences of order 5 over \mathbb{R}

457 The next result follows from Theorem 5.1 and Table 1.

458 **Corollary 5.4.** *The only attainable order 5 epr-sequences that begin AN are ANAAA, ANNNN,*
459 *ANSNA, ANSNN, ANSSA, and ANSSN.*

¹In the case of random linear combinations, nonrandom combinations with the same independence properties were used.

460 **Proposition 5.5.** *The epr-sequences NSSNA and NAANA are forbidden as initial subsequences*
 461 *of epr-sequences for real symmetric matrices.*

462 *Proof.* We use X to denote A or S , for the purpose of connecting epr-sequences to pr-sequences.
 463 Because 1]01101 is forbidden as an initial sequence of a pr-sequence for real symmetric
 464 matrices [3, Theorem 6.4], all epr-sequences of the form $NXXNA$ are forbidden as the initial
 465 sequence of an epr-sequence for real symmetric matrices. In particular, $NSSNA$ and $NAANA$
 466 are forbidden. \square

467 There are 162 order 5 sequences that end in N or A . Of these 162 sequences, we eliminate
 468 those that contain NNS or NNA (Theorem 2.3); there are 33 such sequences, leaving 129
 469 remaining possible sequences. Of these, those containing the subsequences NSA , NAS , and
 470 NAN are ruled out by Corollary 2.7 and Theorem 2.14; there are 39 additional such sequences,
 471 leaving 90 possible sequences. The subsequences $SANA$ and $SANS$ are ruled out by Proposition
 472 2.11; there are 4 additional such sequences, leaving 86 possible sequences. The sequences
 473 $AASNA$, $ASNAA$, and $SASNA$ are ruled out by Corollary 2.7, the sequences $SNAAAA$, $SNAAN$,
 474 $SNSNA$, and $SNSSA$ are ruled out by Proposition 2.5, and the sequence $AANSN$ is ruled out
 475 by Theorem 2.15, leaving 78 remaining possible sequences. The sequences $NSSNA$ and $NAANA$
 476 are eliminated by Proposition 5.5, leaving 76 remaining possible sequences. Finally, $ANAAN$
 477 is ruled out by exhaustive search (Corollary 5.4). The remaining 75 epr-sequences are all
 478 attainable over \mathbb{R} ; see Table 5. The next example gives some of the realizing matrices used
 479 to establish that these epr-sequences are attainable.

Example 5.6.

$$480 \quad M_{ASNSN} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 & 2 \end{bmatrix}, \quad M_{ASSNA} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \end{bmatrix}, \quad M_{NAAAN} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & -1 & 1 \\ 1 & 1 & -1 & 0 & -1 \\ 1 & -1 & 1 & -1 & 0 \end{bmatrix},$$

$$481 \quad M_{NAANN} = \begin{bmatrix} 0 & 1 & 1 & 3 & 3 \\ 1 & 0 & -3 & -4 & -1 \\ 1 & -3 & 0 & -1 & -4 \\ 3 & -4 & -1 & 0 & -3 \\ 3 & -1 & -4 & -3 & 0 \end{bmatrix}, \quad M_{NAASA} = \begin{bmatrix} 0 & 2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & -1 & 2 & 0 \end{bmatrix}, \quad M_{NAASN} = \begin{bmatrix} 0 & 2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 2 & 0 \end{bmatrix},$$

$$482 \quad M_{NSSAN} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & -1 & 0 \end{bmatrix}, \quad M_{SSNAA} = \begin{bmatrix} 0 & 3 & 3 & 0 & 0 \\ 3 & 2 & 1 & 3 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 & 0 \end{bmatrix}, \quad M_{SSSNA} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Table 5: All epr-sequences for order 5 that can be attained by real symmetric matrices. The sequences are listed in lexicographic order.

| epr-Sequence | Real matrix | Result |
|--------------|---------------------------|---------------------------------------|
| AAAAA | I_5 | Observation 2.16 |
| AAAAAN | $J_5 - 5I_5$ | Proposition 2.17 |
| AAANA | $J_5 - 4I_5$ | Proposition 2.17 |
| AAANN | | Theorem 4.6 |
| AAASA | $L_5(3)$ | Proposition 2.18 |
| AAASN | | Theorem 4.6 |
| AANAA | $J_5 - 3I_5$ | Proposition 2.17 |
| AANNN | | Theorem 4.6 |
| AASAA | $L_5(2)$ | Proposition 2.18 |
| AASAN | | Theorem 4.6 |
| AASNN | | Theorem 4.6 |
| AASSA | $I_1 \oplus (J_4 - 3I_4)$ | Corollary 2.22 |
| AASSN | | Theorem 4.6 |
| ANAAA | $J_5 - 2I_5$ | Proposition 2.17 |
| ANNNN | J_5 | Observation 2.16 |
| ANSNA | | Corollary 5.4 |
| ANSNN | | Corollary 5.4 |
| ANSSA | | Corollary 5.4 |
| ANSSN | | Corollary 5.4 |
| ASAAA | $L_5(1)$ | Proposition 2.18 |
| ASAAAN | | Theorem 4.6 |
| ASANN | | Theorem 4.6 |
| ASASA | | Theorem 4.4 |
| ASASN | | Theorem 4.6 |
| ASNNN | | Theorem 4.6 |
| ASNSN | M_{ASNSN} | Example 5.6 |
| ASSAA | $I_1 \oplus L_4(1)$ | Corollary 2.22 |
| ASSAN | | Theorem 4.6 |
| ASSNA | M_{ASSNA} | Example 5.6 |
| ASSNN | | Theorem 4.6 |
| ASSSA | $I_2 \oplus (J_3 - 2I_3)$ | Corollary 2.22 |
| ASSSN | $I_3 \oplus J_2$ | Observation 2.16 |
| NAAAA | $A(K_5) = J_5 - I_5$ | Proposition 2.17 |
| NAAAAN | M_{NAAAAN} | Example 5.6 |
| NAANN | M_{NAANN} | Example 5.6 |
| NAASA | M_{NAASA} | Example 5.6 |
| NAASN | M_{NAASN} | Example 5.6 |
| NNNNN | 0_5 | Observation 2.16 |
| NSNAA | $A(C_5)$ | Observation 3.1 |
| NSNNN | $A(K_{1,4})$ | Observation 3.1 |
| NSNSN | $A(P_5)$ | Observation 3.1 |
| NSSAA | $A(G_2)$ | G_2 where G_2 is the bowtie graph |
| NSSAN | M_{NSSAN} | Example 5.6 |
| NSSNN | $(J_3 - I_3) \oplus 0_2$ | Corollary 2.20 |
| NSSSA | $A(G_3)$ | G_3 is the house graph |
| NSSSN | $(J_4 - I_4) \oplus 0_1$ | Corollary 2.20 |
| SAAAA | $L_5(0)$ | Proposition 2.18 |
| SAAAAN | $H_5^{(3)}$ | Example 2.23 |
| SAANA | M_{NAASA}^{-1} | Example 5.6 and Theorem 2.4 |
| SAANN | $H_5^{(2)}$ | Example 2.23 |
| SAASA | | Theorem 4.4 |
| SAASN | | Theorem 4.6 |
| SANNN | $H_5^{(1)}$ | Example 2.23 |

485 Table 5 (continued): All epr-sequences for order 5 that can be attained by real symmetric
 486 matrices.

| epr-Sequence | Real matrix | Result |
|--------------|---------------------------|------------------|
| SASAA | | Theorem 4.4 |
| SASAN | | Theorem 4.6 |
| SASNN | | Theorem 4.6 |
| SASSA | | Theorem 4.4 |
| SASSN | | Theorem 4.6 |
| SNNNN | $I_1 \oplus 0_4$ | Observation 2.16 |
| SNSNN | $(J_3 - 2I_3) \oplus 0_2$ | Corollary 2.20 |
| SNSSN | $(J_4 - 2I_4) \oplus 0_1$ | Corollary 2.20 |
| SSAAA | $I_1 \oplus (J_4 - I_4)$ | Corollary 2.22 |
| SSAAN | | Theorem 4.6 |
| SSANN | | Theorem 4.6 |
| SSASA | | Theorem 4.4 |
| SSASN | | Theorem 4.6 |
| SSNAA | M_{SSNAA} | Example 5.6 |
| SSNNN | $I_2 \oplus 0_3$ | Corollary 2.20 |
| SSNSN | $(J_4 - 3I_4) \oplus 0_1$ | Corollary 2.20 |
| SSSAA | $I_2 \oplus (J_3 - I_3)$ | Corollary 2.22 |
| SSSAN | | Theorem 4.6 |
| SSSNA | M_{SSSNA} | Example 5.6 |
| SSSNN | $I_3 \oplus 0_2$ | Corollary 2.20 |
| SSSSA | $I_3 \oplus L_2(0)$ | Observation 2.16 |
| SSSSN | $I_4 \oplus 0_1$ | Observation 2.16 |

487

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