# ESTIMATION OF THE BOUNDARY OF AN INCLUSION OF KNOWN MATERIAL FROM SCATTERING DATA

John M. Richardson

Rockwell International Science Center
Thousand Oaks, California 91360

## ABSTRACT

A computationally tractable inversion algorithm has been developed for the case of the scattering of longitudinal elastic waves from an inclusion. It is assumed that the material properties of the inclusion are known *a priori* but that the boundary geometry is unknown -- in fact the boundary could belong to two or more separate inclusions. It is further assumed that the material properties of the inclusion are sufficiently close to those of the host that the Born approximation can be employed.

## INTRODUCTION

The probabilistic approach to the inverse problem associated with the scattering of elastic waves from an unknown flaw involves a stochastic measurement model that contains assumptions concerning the *a priori* statistics of both the measurement error and the possible flaws. Then the output of the inversion procedure is the most probable flaw given the measurments. It is clear that the performance of the inversion procedure for a given set of scattering data improves with the amount of correct *a priori* information restricting the flaw statistics. Here we consider a highly restricted case in which it is assumed that the flaw is an inclusion of known material (isotropic) but with unknown boundary. The *a priori* statistical model of the inclusion is nonparametric in the sense that all possible boundaries are represented -- including the boundaries of two or more separate inclusions. Each of these possible geometries is confined to a specified rectangular localization domain.

The above is an example of non-Gaussian flaw statistics. An investigation of probabilistic inversion involving several types of flaw statistics has been conducted by Richardson and Gysbers (1977) using a relatively conventional approach entailing the minimization of "many-valley" functions, or, alternatively, the maximization of "many-mountain" functions. The computational difficulties arising from this situation were severe. The present treatment, to be discussed in the ensuing sections, obviates the many-valley difficulties by a procedure that leads ultimately to the computational minimization of a convex function which inherently has the "single-valley" property.

## FORMULATION

Thie appropriate measurement model for $L \rightarrow L$ pulse-echo scattering from an inclusion is given by the expression

$$f(t,\vec{e}) = \alpha \sum_{\vec{r}} \delta\vec{r} p''(t - 2c^{-1}\vec{e}\cdot\vec{r})\, \Gamma(\vec{r}) + \nu(t,\vec{e}) \quad , \tag{1}$$

where $f(t,\vec{e})$ is the received waveform at time t with an incident propagation direction $\vec{e}$ and $\nu(t,\vec{e})$ is the associated error. The function $p''(t)$ is the second derivative of $p(t)$ the so-called reference pulse: $\Gamma(\vec{r})$ is the characteristic function of the inclusion at the position $\vec{r}$; c is the velocity of L waves; and $\alpha$ is a parameter dependent on the material properties of the inclusion and the host medium both assumed to be known *a priori*. Here we assume that $t, \vec{e}$, and $\vec{r}$ are discretely valued. In particular, $\vec{r}$ takes vector values on a finite cubic lattice were $\delta\vec{r}$ is the volume of a unit cell.

To complete the description of the measurement model the *a priori* statistical properties of $\nu$ and $\Gamma$ must be defined. We assume that $\nu$ and $\Gamma$ are statistically independent of each other. The error $\nu$ is assumed to be a set of Gaussian random variables with the properties

$$E\nu(t,\vec{e}) = 0 \tag{2a}$$

$$E\nu(t,\vec{e})\, \nu(t',\vec{e}') = \delta_{\vec{e}\vec{e}'}\, \delta_{tt'}\, \sigma^2 \quad . \tag{2b}$$

The values of the characteristic function at two positions are assumed to be statistically independent with $\Gamma(\vec{r})$ taking the values 0 and 1 with probabilities 1-P and P, respectively. In the present treatment we will assume that P is independent of $\vec{r}$ and thus $\Gamma(\vec{r})$ is an example of a stationary non-Gaussian random process.

SOLUTION

We first calculate the joint probability function* $P(\Gamma,\nu)$ for the characteristic function $\Gamma(\vec{r})$ for all points $\vec{r}$ on the lattice and the measurement error $\nu(t,\vec{e})$ for all t and $\vec{e}$ values. The statistical assumptions, discussed in the previous section, imply the relation

$$\log P(\Gamma,\nu) = -\frac{1}{2\sigma^2}\sum_{t,\vec{e}} \nu^2(t,\vec{e}) + \sum_{\vec{r}} \left[\Gamma(\vec{r}) \log P + \left(1 - \Gamma(\vec{r})\right) \log (1 - P) \right] \quad (3)$$

in which an ignorable additive constant has been neglected. Our procedure is to maximize the above expression with respect to $\Gamma$ and $\nu$ while regarding the measurement model as a set of constraints. Using a somewhat nonstandard form of the Lagrange multiplier method for handling constraints we obtain the following variational function

$$\begin{aligned}\phi &\equiv \phi\,(\Gamma,\ \nu,\ w,\ f) \\ &= \log P(\Gamma,\nu) - \sum_{t,\vec{e}} w(t,\vec{e}) \Big[f(t,\vec{e}) \\ &\quad - \alpha \sum_{\vec{r}} \delta\,\vec{r}\ p''\,(t - 2c^{-1}\,\vec{e}\cdot\vec{r})\ \Gamma(\vec{r}) - \nu(t,\vec{e})\Big] \quad , \qquad (4)\end{aligned}$$

where $w = w(t,\vec{e})$ is the Lagrange multiplier vector. It is to be noted that setting the variation of $\phi$ with respect to w equal to zero implies the measurement model (1). Thus, the vanishing of the variations with respect to $\Gamma$, $\nu$, and w implies a maximum with respect to $\Gamma$ and $\nu$ constrained by (1).

Our procedure is first to maximize $\phi$ with respect to $\Gamma$ and $\nu$ keeping w fixed. It is possible to perform this maximization analytically with the result

---

*The term "probability function" is meant to imply an entity that is a probability density with respect to continuous-valued variables and an ordinary probability with respect to discrete-valued variables.

$$\phi(\hat{\Gamma}, \hat{\nu}, w, f) \equiv \phi(w, f)$$

$$= \sum_{t,\vec{e}} \left[\frac{1}{2} \sigma^2 w^2(t,\vec{e}) - f(t,\vec{e})\, w(t,\vec{e})\right]$$

$$+ \sum_{\vec{r}} g\left(\lambda(\vec{r})\right) \quad , \tag{5}$$

where the function $g(\lambda)$ is given by

$$g(\lambda) = \frac{1}{2} \left[\lambda + \log P + \log (1 - P)\right]$$

$$+ \frac{1}{2} \left|\lambda + \log P - \log (1 - P)\right| \tag{6}$$

and $\lambda$ is defined by

$$\lambda(\vec{r}) = \alpha \sum_{t,\vec{e}} \delta\vec{r}p''(t - 2c^{-1}\, \vec{e}\cdot\vec{r})\, w(t,\vec{e}) \quad . \tag{7}$$

It can be readily shown that $\phi$ is a convex function of w, i.e.

$$\phi(\beta_1 w_1 + \beta_2 w_2, f) \leqslant \beta_1 \phi(w_1,f) + \beta_2 \phi(w_2,f) \ , \tag{8}$$

where $\beta_1$ and $\beta_2$ are positive real constants subject to the condition $\beta_1 + \beta_2 = 1$ and where $w_1$ and $w_2$ are any two values of the vector w. Thus, a minimum must exist and relative minima in other locations cannot exist. However, this minimum may not be unique, but then the non-uniqueness must be of a special kind. Informally speaking, if non-unique, the minimum must be like a flat region at the "bottom of the valley" and this region must have a convex boundary.

In any case, if a unique minimum exists, then the minimization $\psi(f,w)$ on w yields a best estimate $\hat{w}$ from which the corresponding estimate of the characteristic function is given by the relation

$$\hat{\Gamma}(\vec{r}) = 1\left(\hat{\lambda}(\vec{r}) + \log P - \log (1 - P)\right) \quad . \tag{9}$$

where $1(\cdot)$ is the unit-step function. In the last expression $\hat{\lambda}$ is given by substituting $\hat{w}$ into (7). This minimization must be carried out by computational means.

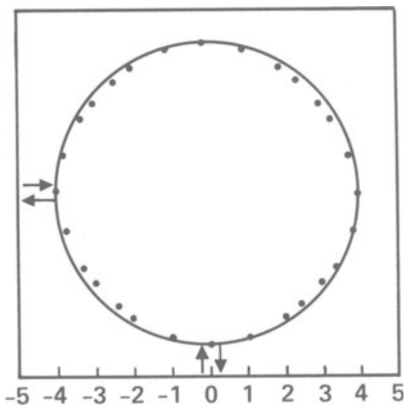


Figure 1. Estimated boundary (·) or the characteristic function versus assumed boundary (-). (Incident and scattered directions of elastic waves indicated by arrows).

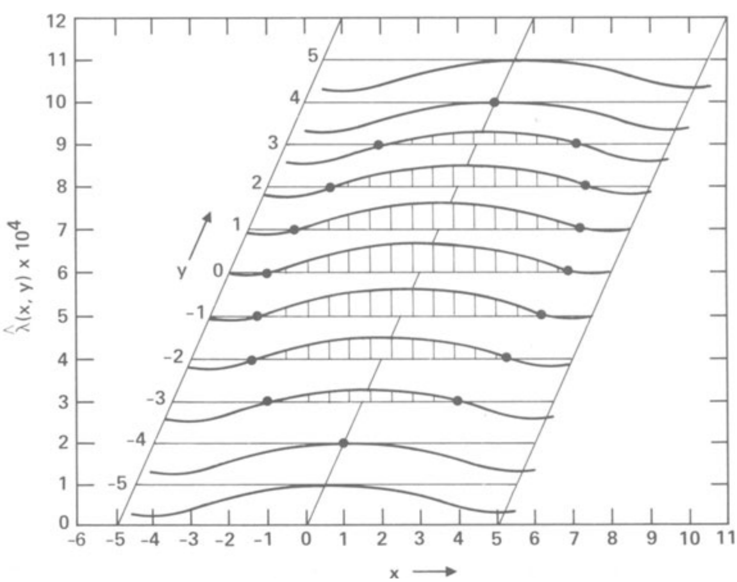


Figure 2. The nature of the function $\lambda(\vec{r}) = \lambda(x,y)$.

## COMPUTATIONAL EXAMPLE

A test run was made for the two-dimensional case using a noiseless set of test waveforms $f(t,\vec{e})$ derived on the assumption that the actual characteristic function corresponds to a circular inclusion of radius 4 (in dimensionless units). The assumed reference waveform was a typical sinusoid modulated by a Gaussian envelope. Two incident directions (each defined by a vector e) were chosen, one orthogonal to the other. The estimated characteristic function is shown in Fig. 1. The boundary of the assumed circular inclusion is indicated by the solid line. Several points on the boundary of the estimated characteristic function are represented by the heavy dots. The agreement is surprisingly good, especially in view of the fact that only two incident directions are involved. The nature of the function $\lambda(\vec{r})$ is shown in Fig. 2.

## ACKNOWLEDGEMENT

This work was supported by Independent Research and Development funds of Rockwell International.

## REFERENCE

J.M. Richardson and J.C. Gysbers, "Application of Estimation Theory to Image Improvement," 1977 Ultrasonics Symposium Proceedings, IEEE Cat. #77 CH 1264-ISU.