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A Tool for Evaluating Time-Varying-Stress Accelerated Life Test Plans with Log-Location-Scale Distributions

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Abstract

Accelerated life tests (ALTs) are often used to make timely assessments of the life time distribution of materials and components. The goal of many ALTs is estimation of a quantile of a log-location failure time distribution. Much of the previous work on planning accelerated life tests has focused on deriving testplanning methods under a specific log-location distribution. This paper presents a new approach for computing approximate large-sample variances of maximum likelihood estimators of a quantile of general log-location distribution with censoring and time-varying stress. The approach is based on a cumulative exposure model. Using sample data from a published paper describing optimum ramp-stress test plans, we show that our approach and the one used in the previous work give the same variance-covariance matrix of the quantile estimator from the two different approaches. Then, as an application of this approach, we extend the previous work to a new optimum ramp-stress test plan obtained by simultaneously adjusting the ramp rate and the lower start level of stress. We find that the new optimum test plan can have a smaller variance than that of the optimum ramp-stress test plan previously obtained by adjusting only the ramp rate. We also compare optimum ramp-stress test plans with the more commonly used constant-stress accelerated life test plans. We also conduct simulations to provide insight and to check the adequacy of the large-sample approximate results obtained by the approach.

Keywords

cumulative exposure model, Weibull, large-sample approximate variance, lognormal, maximum likelihood, ramp-stress, step-stress

Disciplines

Statistics and Probability

Comments

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Abstract

Accelerated life tests (ALTs) are often used to make timely assessments of the lifetime distribution of materials and components. The goal of many ALTs is estimation of a quantile of a log-location-scale failure time distribution. Much of the previous work on planning accelerated life tests has focused on deriving test-planning methods under a specific log-location-scale distribution. This paper presents a new approach for computing approximate large-sample variances of maximum likelihood estimators of a quantile of general a log-location-scale distribution with censoring and time-varying stress. The approach is based on a cumulative exposure model. Using sample data from a published paper describing optimum ramp-stress test plans, we show that our approach and the one used in the previous work give the same variance-covariance matrix of the quantile estimator from the two different approaches. Then, as an application of this approach, we extend the previous work to a new optimum ramp-stress test plan obtained by simultaneously adjusting the ramp rate and the lower start level of stress. We find that the new optimum test plan can have a smaller variance than that of the optimum ramp-stress test plan previously obtained by adjusting only the ramp rate. We compare optimum ramp-stress test plans with the more commonly used constant-stress accelerated life test plans. We also conduct simulations to provide insight and to check the adequacy of the large-sample approximate results obtained by the approach.

Index Terms

Cumulative exposure model; Large-sample approximate variance; Lognormal, Maximum likelihood, Ramp-stress, Step-stress, Weibull.

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ACRONYMS

ALT	Accelerated life tests
cdf	cumulative distribution function
pdf	probability density function
ML	maximum likelihood
NOR	s-normal distribution
SEV	smallest extreme value distribution
LOGIS	logistic distribution
LEV	largest extreme value distribution

NOTATION

V_L, V_U, V_H	Initial low level, pre-specified use level, and highest possible level of the original stress
s	Transformed stress. When voltage is the ramp-stress $s = \log(V_H/V)$
ξ	Standardized stress $\xi = (s - s_U) / (s_H - s_U)$
k	Ramp rate
τ	Time to reach the highest possible level of stress $\tau = (V_H - V_L)/k$
t, η	Failure time and censoring time
μ, σ	Location and scale parameters of a location-scale distribution
γ_0, γ_1	Parameters of the log linear regression model when $\mu = \gamma_0 + \gamma_1 \xi$
γ_0', γ_1'	Parameters of the log linear regression model when $\mu = \gamma_0' + \gamma_1' s$
$w(t)$	Cumulative exposure at time t.
α	V_H/V_U
$\phi(\cdot)$, $\Phi(\cdot)$	pdf and cdf of a location-scale distribution
p_U, p_H	Probabilities that a unit will fail by time η at use and the highest stress levels, respectively
z_p	p quantile of a standard location-scale distribution
$y_p = y_p(\xi)$	p quantile of a location-scale distribution at stress level ξ
n	Total number of test units
NOR(μ, σ)	s-normal distribution with mean μ and standard deviation σ
a^T	Transpose of vector a

I. INTRODUCTION

A. Accelerated testing background

Accelerated life tests (ALT) are commonly used in product design processes. Because there is limited time to launch new products, engineers use accelerated tests to obtain needed information on the reliability by raising the levels of certain acceleration variables like temperature, voltage, humidity, stress, and pressure. Techniques for performing an ALT include constant stress, step stress, and ramp stress, among others. Evaluation of the variance of an estimator of a log-location-scale distribution quantile (e.g., Weibull or lognormal) with varying stress has many practical applications. Statisticians help engineers design statistically efficient ALT plans and assess estimation precision as a function of sample size. Reviews of the research work in this area can be found in, for example, Nelson [12-15], Meeker and Hahn [7], and Meeker and Escobar [8].

Most previously developed methods to calculate the large-sample approximate variance of Maximum Likelihood (ML) estimators of distribution quantiles treated only a particular log-location-scale distribution and a particular type of test plan (e.g., constant-stress, monotone step-stress, or monotone ramp-stress). For example, Miller and Nelson [10] presented a theory for optimum step-stress test plans for an exponential distribution. Bai and Kim [4] developed a step-stress theory of the large-sample approximate variance for the Weibull distribution. Alhadeed and Yang [1] give an optimum simple step-stress plan for the lognormal distribution.

To consolidate and extend this previous work, it is useful to develop an approach for computing the large-sample approximate variance for a general log-location-scale distribution and a given general form for stress variation. Besides the elegance of having just one algorithm, the generalization allows evaluations for log-location-scale distributions beyond the more commonly used exponential, Weibull, and lognormal distributions, including the loglogistic and Fréchet distributions (e.g., Meeker and Escobar [8, Chapter 4]). Recently, Ma and Meeker [6] provided an approach to calculate the large-sample approximate variance in step-stress test plans for a general log-location-scale distribution. In this paper, we adapt the results of Ma and Meeker [6] to the case of a test plans with the stress varying continuously over time and show how to use the results to find ramp-stress test plans that have good statistical properties.

B. Ramp-stress accelerated test background

Ramp-stress ALTs have been used in practice (e.g., Nelson [13, Chapter 10]). Yin and Sheng [16] studied the properties of the ML estimator of the exponential life distribution parameter from a ramp-stress test plan. Nelson [10, Chapter 10] presented theory to calculate the ML estimator of a quantile associated with a ramp-stress ALT using a Weibull distribution and a particular cumulative exposure model. Bai, Cha and Chung [2] considered ramp-stress ALTs with two ramp rates for the Weibull distribution under Type-I censoring. Bai, Chun and Cha [3] developed a method for finding an optimum ramp-stress ALT test plan for a Weibull distribution by choosing a ramp rate to minimize a large-sample approximate variance.

In a ramp-stress ALT, units begin the test at a low level of stress and the level of stress increases at a constant rate. The test plan can be adjusted by the starting stress level and the rate, given the use and highest levels of stress, and a censoring time. There are two ways to specify test termination. One is that the level of stress increases linearly until a specific censoring time. The other is that initially the level of stress increases linearly. After a time fraction of the censoring time, the level of stress reaches the pre-specified highest level. Then the highest level of stress will be applied to the surviving test units until the specific censoring time.

A ramp-stress ALT can be viewed as a limit of a multiple step-up stress ALT when the number of steps approaches infinity while the change in stress level at each of step approaches zero and the sum of the step jumps is held constant. Like step-up stress ALTs, ramp-stress ALTs can result in failures happening more quickly by increasing the ramp rate and such tests only require a single temperature-controlled chamber for testing.

Different choices of the stress start level and the ramp rate can affect estimation precision of a quantity of interest such as a quantile of the life distribution at use conditions. Our goal is to find test plans with the optimum (i.e., the smallest) variance of ML estimators of such quantiles.

C. Overview

The rest of this paper is organized as follows. Section II describes the ramp-stress test plans used in this paper. Section III presents the general time-varying stress model and likelihood for a general log-location-scale life distribution. This section also shows how to compute the large-sample approximate variance of the ML estimator of a specified quantile of the life distribution under the general time-varying model and log-location-scale distributions.

Section IV obtains optimum ramp-stress plans that minimize the large-sample approximate variance of the ML estimator of a specified quantile and compares these optimum plans with previously-suggested optimum ramp-stress plans and optimum constant-stress plans in terms of the approximate scaled large-sample variance. This section also gives the results of a simulation to check the adequacy of the large-sample approximate variances. Section V states some conclusions and outlines areas for future research.

II. Test Plans

In most ALT models there is an original stress scale (e.g., voltage) and an implied one-to-one transformation of stress (e.g., log of voltage). We use V and s to denote the original and transformed stresses, respectively. The stress applied to test specimens in a ramp-stress test will always be between an initial lower level of stress V_L and the highest possible level of stress V_H. V_L can be either higher or lower than the use level of stress V_U. As in Bai, Chun and Cha [3], in the ramp stress test plans we consider V as a voltage and use a transformed stress, $s = \log\left(V_H/V\right)$. Sometimes, for convenience, a standardized stress $\xi = \left(s - s_U\right) / \left(s_H - s_U\right)$ is used. This standardized stress has the nice properties that $\xi_U = 0$ and $\xi_H = 1$. The maximum test length is η time units (e.g., hours). If we define the time for the stress level to reach V_H to be τ, then $\tau = \left(V_H - V_L\right)/k$ and $\tau/\eta = k_0/k$, where $k_0 = \left(V_H - V_L\right)/\eta$ is the constant rate of change in V (e.g., volts per hour).

Figure 1 shows the two possible test schemes. (a) $\tau \le \eta$, i.e., $k \ge k_0$. (b) $\tau > \eta$, i.e., $k < k_0$. By simultaneously optimizing V_L and k, one can usually obtain a smaller variance of the quantile estimator by using scheme (a), relative to scheme (b).

III. The Model and Log Likelihood

A. Model

This section shows how to compute the large-sample approximate variance of the ML estimator of a quantile of a general log-location-scale distribution at use conditions from continuously time-varying stress accelerated life tests. We begin with a multiple step-stress test. Then we let the number of steps approach infinity as the change of stress level at each step approaches zero, holding the sum of the step jumps constant.

We assume that at any level of stress the failure time T follows a log-location-scale distribution with cdf

$$\Pr(T \le t) = \Phi\left[\frac{\log(t) - \mu}{\sigma}\right],$$

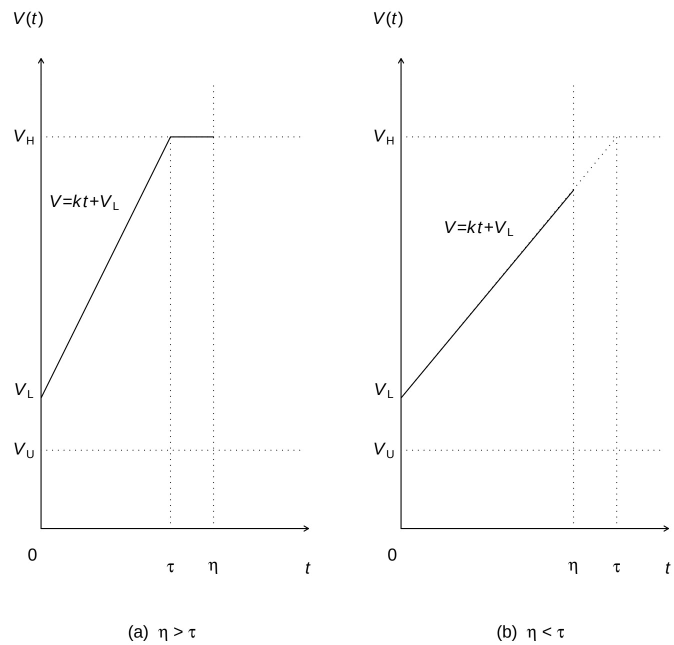

Fig. 1. Ramp-stress test schemes considered in this paper.

where the location parameter is $\mu = \gamma_0 + \gamma_1\xi$ and the scale parameter σ is constant. We also assume that Nelson's [13] cumulative exposure model holds. This model implies that the distribution of remaining life of a test unit depends only on the cumulative exposure it has received no matter how it was exposed.

In a multiple step-stress test, we define the standardized log time at stress level i, to be $z_i = \left[\log\left(t-\delta_{i-1}\right) - \mu\left(\xi_i\right)\right]/\sigma$, $i = 2, 3, \ldots, h$, and $z_\eta = \left[\log\left(\eta-\delta_{h-1}\right) - \mu\left(\xi_h\right)\right]/\sigma$, where $\mu(\xi) = \gamma_0 + \gamma_1\xi$, $\tau_{i-1} \le t \le \tau_i$, τ_i is the time at which the stress level change from s_i to s_{i+1}, $\tau_0 = 0$, and h is the total number of stress levels in the experiment. The time shift induced by the change in stress levels from the beginning of the test to the beginning of step i is

$$\delta_{i-1} = \tau_{i-1} - \sum_{j=1}^{i-1}\left(\tau_j - \tau_{j-1}\right)\ e^{\gamma_1(\xi_i-\xi_j)}.$$

Under the cumulative exposure model, we have $z_i\left(\tau_i\right) = z_{i+1}\left(\tau_i\right)$, $i = 2, 3, \ldots, h$. When $h \to \infty$, under the limiting process described at the beginning of this section, we have $z(t) = \left\{\log[w(t)] - \gamma_0\right\}/\sigma$, where

$$w(t) = \int_0^t \exp\left[-\gamma_1\xi(x)\right]dx$$

is the cumulative exposure at time t. Note that $\xi(t)$ must be integrable. This is a weak condition that is met in both step-stress and ramp-stress test plans, whether the changes in stress are monotone or not. The cdf and pdf of T when $h \to \infty$ are

$$\begin{aligned} F(t;\gamma_0,\gamma_1,\sigma) =& \Phi\left(\frac{\log[w(t)]-\gamma_0}{\sigma}\right) \text{ and} \\ f(t;\gamma_0,\gamma_1,\sigma) =& \frac{\exp\left[-\gamma_1\xi(t)\right]}{w(t)\sigma}\phi\left(\frac{\log[w(t)]-\gamma_0}{\sigma}\right), \end{aligned}$$

respectively. The failure probabilities at the highest level of stress ($\xi_H = 1$) and the use level of stress ($\xi_U = 0$) can be expressed as $p_H = \Phi\left\{[\log(\eta) - \gamma_0 - \gamma_1]/\sigma\right\}$ and $p_U = \Phi\left\{[\log(\eta) - \gamma_0]/\sigma\right\}$, respectively.

B. Log likelihood

The log likelihood for a single test unit having a log-location-scale failure time distribution under stress $\xi(t)$ is

$$l = U_1(t)\left\{-\log(\sigma) - \gamma_1\xi(t) - \log[w(t)] + \log\left[\phi(z)\right]\right\} + \left[1 - U_1(t)\right]\log\left[1 - \Phi(z_\eta)\right], \quad (1)$$

where $U_1(t) = 1$ if $t \leq \eta$ and $U_1(t) = 0$ otherwise. Here $z = \left\{\log[w(t)] - \gamma_0\right\}/\sigma$ and $z_\eta = \left\{\log[w(\eta)] - \gamma_0\right\}/\sigma$. The total log likelihood is obtained by summing (1) over all test units. The ML estimators $\hat{\gamma}_0$, $\hat{\gamma}_1$, and $\hat{\sigma}$ are those values that maximize the total log likelihood. The first and the second partial derivatives of (1) with respect to the model parameters are given in the appendix.

C. The large-sample approximate variance

Under the standard regularity conditions (which hold for log-location-scale distributions), the large-sample approximate variance-covariance matrix of $\hat{\gamma}_0$, $\hat{\gamma}_1$ and $\hat{\sigma}$, denoted by $\Sigma_{\hat{\gamma}_0,\hat{\gamma}_1,\hat{\sigma}}$, is the inverse of the Fisher information matrix (FIM). The FIM is the expectation, with respect to the data, of the negative Hessian matrix (the second derivatives of log likelihood in (1) with respect to the model parameters), evaluated at the model parameters. The appendix provides expressions for the elements of the FIM.

The ML estimator of the p quantile at standardized stress ξ is $\hat{y}_p = \hat{\gamma}_0 + \hat{\gamma}_1\xi + z_p\hat{\sigma}$. The large-sample approximate variance of $\hat{y}_p$ is $\text{Avar}(\hat{y}_p) = (1, \xi, z_p)\,\Sigma_{\hat{\gamma}_0,\hat{\gamma}_1,\hat{\sigma}}\,(1, \xi, z_p)^T$. Our goal is to minimize $\text{Avar}(\hat{y}_p)$ at $\xi = 0$ (i.e., at the use conditions). The test plan properties to be optimized are V_L and k. We did the optimization by using the function `optim()` in R [11].

To compare the relative efficiency of test plans with different sample sizes we use the scaled large-sample approximate variance defined as $(n/\sigma^2)\,\mathrm{Avar}\,(\hat{y}_p)$. The large-sample approximate variance-covariance matrix and properties of optimum test plans depend on the planning values of the parameters p_H, p_U and σ. As usual when dealing with locally optimum designs (i.e., when the optimum depends on the model parameters), such planning values are obtained from some combination of previous experience and engineering judgment.

In the case of the ramp-stress test plans we consider, $s(t) = \log[V_H/(kt + V_L)]$ when $kt+V_L \leq V_H$ and $s\,(t) = 0$ otherwise. Note that ramp stress refers to a test in which $V\,(t)$ is linear in t. $s\,(t)$ will not, in general, be linear in t. Simple deduction shows that $\alpha = V_H/V_U$ defines the shape of time dependence of $s\,(t)$ and thus $\xi\,(t)$. Therefore, in addition to p_H, p_U and σ, the large-sample approximate variance-covariance matrix and properties of the optimum test plans will also depend on α.

IV. Optimum Ramp-Stress Plans

A. Comparison with existing optimum ramp stress plans

In Bai, Chun and Cha [3], a one-dimensional optimum ramp-stress test plan is proposed. This plan is similar to that given in Figure 1 except that V_L = 0 and the ramp rate is optimized. The location parameter is expressed as $\mu = \gamma_0^{'} + \gamma_1^{'}\,s$, which is a little different from our notation described in Section III. It is easy to convert from one parametrization to the other. Bai, Chun and Cha [3] considered a case where $\gamma_0^{'}$ = 6.0, $\gamma_1^{'}$ = 9.0, σ = 0.5, V_L = 0 kV, V_U = 20 kV, V_H = 40 kV and η = 2400 seconds. The quantile of interest is p = 0.1. Bai, Chun and Cha [3] also provided a simulated sample data yielding estimates $\hat{\gamma}_0^{'}$ = 6.13, $\hat{\gamma}_1^{'}$ = 8.73, $\hat{\sigma}$ = 0.451. Our approach gives the same variance-covariance with respect for ($\hat{\gamma}_0^{'}$, $\hat{\gamma}_1^{'}$, $\hat{\sigma}$) as that given in [3], where the authors used a different theoretical approach that is only valid for the Weibull distribution.

Using γ_0' = 6.0, γ_1' = 9.0 and σ = 0.5, as input to the algorithm proposed in this paper we find that the one-dimensional optimum $(n/\sigma^2)\,\mathrm{Avar}\,(\hat{y}_{0.1})$ is 1635.1 at k = 24.0 V/sec when V_L is fixed at 0 kV. We extend the one-dimensional optimization of the ramp-stress test plan in [3] into a two-dimensional optimization. By selecting k = 18.9 V/sec and V_L = 13.9 kV, we obtain the optimum $(n/\sigma^2)\,\mathrm{Avar}\,(\hat{y}_{0.1})$ = 1493.8, a nearly 8.6% reduction in variance with respect to that of the one-dimensional optimization. We also obtained the results for the lognormal case. To make the results comparable, we use $\sigma = 0.5, \alpha = 2, p_U = 0.000135, p_H = 0.9999, V_U = 20$ and obtained γ_0' = 5.9, γ_1' = 5.3 for lognormal case. For

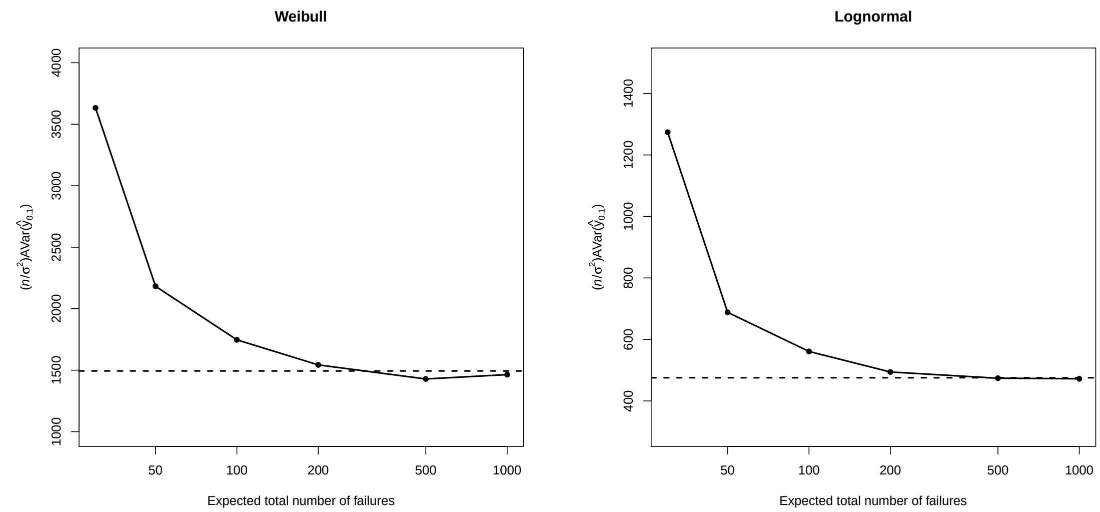

Fig. 2. Simulated scaled variance as a function of the expected total number of failures under the two-dimensional optimum ramp-stress test plan for the Weibull and lognormal distributions as compared to the corresponding large-sample approximate variance (horizontal dashed line). The number of simulations at each point is 5,000. The optimum two-dimensional test plan has the following conditions for the Weibull: $\gamma_0' = 6.0, \gamma_1' = 9.0, \sigma = 0.5$, V_L = 13.9 kV, V_U = 20 kV, V_H = 40 kV, η = 2400 sec, k = 18.9 V/sec and p= 0.1, and for lognormal γ_0' = 5.9, γ_1' = 5.3, σ = 0.5, V_L = 10.7 kV, V_U = 20 kV, V_H = 40 kV, η = 2400 sec, k = 23.9 V/sec and p = 0.1.

the one dimensional case, we have $k = 26.4$ and the scaled optimum variance is 509.6. For the two dimensional case, we have $k = 23.9, V_L = 10.7$, and the scaled optimum variance is 475.5. There is also a nearly 6.7% improvement. In the rest of this section, we will only work with the optimum test plan from our two-dimensional optimization.

B. Comparison of the large-sample approximate variance a variance obtained with simulations

To assess the adequacy of the large-sample approximate variances used in this paper, we conducted simulations to compare with the actual variance as a function of expected total numbers of failures during the test. Figure 2 shows the simulated scaled variance as a function of the expected total number of failures under the two-dimensional optimum ramp-stress test plan. As expected, when the expected total number of failures gets large, the simulated variance approaches the large-sample approximate variance. In this test plan, the probability that a test unit will fail during the test is 99.99%. Thus the expected total number of failures is almost exactly the same as the sample size.

C. Comparison of optimum ramp stress plans with optimum constant-stress plans

Ma and Meeker [6] compared the optimum step-stress and constant-stress test plans in terms of the large-sample approximate variance of the ML estimators of a quantile of interest. In this paper we extend the comparison to optimum ramp-stress tests. For the constant-stress tests the two-dimensional optimization is done by adjusting the lower level of stress as well as the allocations of test units to the constant stress levels. For step-stress tests, the optimization is done by adjusting the lower level of stress as well as the fraction of test time spent at each level. Note that a ramp-stress test plan is a limit of a multiple step-stress test plan under the limiting process described in Section III.A. Note also that the optimum two-level step-stress has the smallest variance for reasonable planning values without any constraint on the fraction of time at each of stress levels. Thus, the variance of a ramp-stress test plan is not expected to be smaller than the variance of the optimum two-level step-stress test plan given the same planning values. Therefore, we compare the ramp-stress test plans with the commonly used constant-stress test plans with the same planning values used in [6].

Figure 3 shows $(n/\sigma^2)\,\mathrm{Avar}\,(\hat{y}_p)$ as a function of σ under the optimum ramp-stress test plans with α = 2 (left) and α = 10 (right) for p = 0.01, 0.10 and 0.50, respectively, at p_H = 0.9 and p_U = 0.001 for the Weibull distribution. Figure 4 shows similar results for the lognormal distribution. The scaled approximate variance $(n/\sigma^2)\,\mathrm{Avar}\,(\hat{y}_p)$ decreases as σ increases. The scaled variance at p = 0.5 is the largest, followed by the variance for p = 0.1 and then p= 0.01. These results are similar to those that were observed in [6] for step-up-stress test plans. Figures 3 and 4 also show that the optimum variance with a larger α is slightly larger than that with a smaller α under the same planning values.

Our major interest is variance comparison. In [6] the optimum two-level constant-stress $(n/\sigma^2)\,\mathrm{Avar}\,(\hat{y}_p)$ are 95, 120 and 149 for p = 0.01, 0.10 and 0.50, respectively with p_H = 0.9 and p_U = 0.001 under the Weibull distribution. Compared with the values of $(n/\sigma^2)\,\mathrm{Avar}\,(\hat{y}_p)$ on the left of Figure 3, we find an optimum two-level constant-stress test plan has a smaller $(n/\sigma^2)\,\mathrm{Avar}\,(\hat{y}_p)$ than that of an optimum ramp stress test plans in this particular situation. The conclusion is similar for the lognormal distribution. Our other results (not shown here) indicate that this conclusion also holds for the three other pairs of planning values (p_H = 0.9 and p_U = 0.00001), (p_H = 0.1 and p_U = 0.00001) and (p_H = 0.1 and p_U = 0.001) when σ is between 0.5 and 1.5.

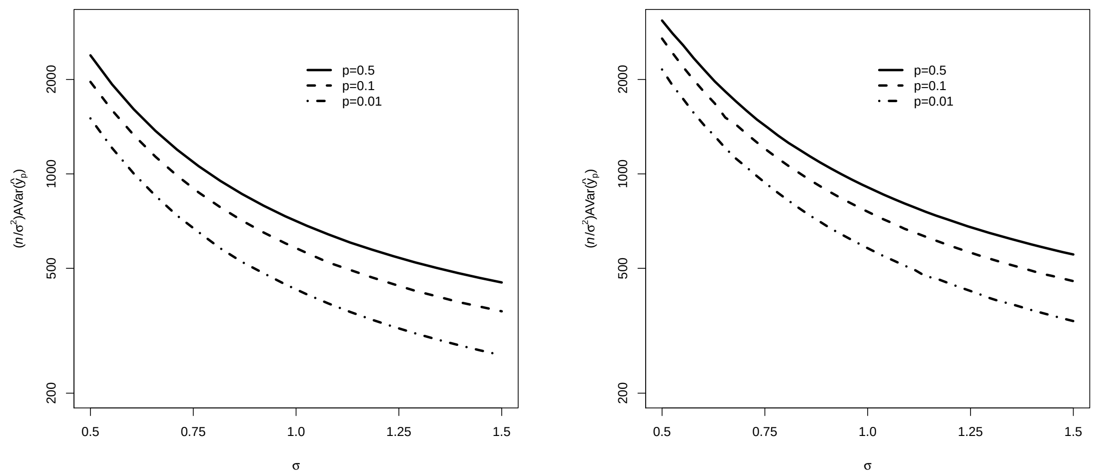

Fig. 3. Scaled optimum variance as a function of σ for the Weibull distribution at three possible quantiles of interest for p_H = 0.9 and $p_U = 0.001$ with α= 2 (left) and α = 10 (right).

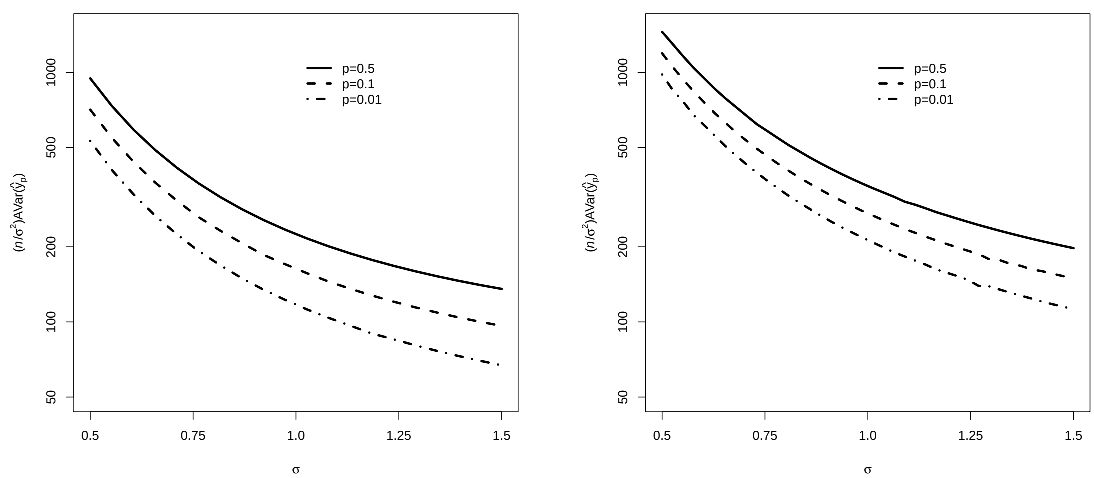

Fig. 4. Scaled optimum variance as a function of σ for lognormal distribution at three possible quantiles of interest for p_H = 0.9 and $p_U = 0.001$ with α= 2 (left) and α = 10 (right).

V. Concluding Remarks

In this paper we propose an approach for computing the large-sample approximate variance of ML estimators of quantiles of the widely-used log-location-scale family of distributions with continuous time-varying stress accelerated life tests and censoring. We applied this approach to a situation that corresponds to the ramp-stress sample data described in [3]. The results obtained by our more general method of computing large-sample approximate variances agree with the special-case results reported in [3]. We also extend previously-developed one-dimensional optimum ramp-stress test plans to the two-dimensional optimum ramp-stress test plans. The variance of the ML estimator of a quantile at the planning values was investigated for the Weibull and lognormal distribution and was found to be larger for the two-dimensional optimum ramp-stress test plans than for the simple optimum two-level constant-stress test plans.

One may raise the point that it is difficult to distinguish between a Weibull and a lognormal distribution when sample sizes are small. In real applications of accelerated testing, engineers conducting the test often have a good idea of the distribution from previous experience with the failure mechanism under study. When such information is not available, analyses are often done with both the lognormal and Weibull distributions and perhaps other distributions that are consistent with the data. Then it is often deemed prudent to use the more conservative answer (generally provided by the Weibull distribution if there is extrapolation in time). The practical purpose of optimum reliability tests is to provide insight into how to plan actual tests that have good statistical properties, but also meet practical constraints (such as robustness to departures from inputs). This is described further in Nelson [13, Chapter 6] and Meeker and Escobar [8, Chapter 20].

There are a number of possible extensions of the work presented here. These include

- The formulas in the appendix could be extended to multiple-stress situations (e.g., temperature and voltage) where either or both could be increasing in the test.
- We have considered only log-location-scale distributions. It would be possible to derive similar results for other non-log-location-scale distributions (e.g., the gamma distribution).
- The response in an accelerated life test is time to failure. In some accelerated tests, the response is degradation. In some applications degradation can be monitored or measured periodically (see, for example, Meeker, Escobar, and Lu [9]). In other applications degradation is a destructive measurement (see, for example, Nelson [12] and Escobar, Meeker, Kugler and Kramer [5]). It would be interesting to develop methods parallel to ours where the degradation measure follows a log-location-scale distribution.

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APPENDIX

This appendix provides the derivatives of the log likelihood given in Section III and the details of the approach that we used to calculate needed expectations and the large-sample approximate variance-covariance matrix of the ML estimators of γ_0, γ_1 and σ.

The log likelihood for a single test unit having a log-location-scale failure time distribution under stress $\xi(t)$ is given in (1). The first partial derivatives of the log likelihood with respect to parameters for a single observation can be expressed as

$$\begin{aligned}
\frac{\partial l}{\partial \gamma_0} &= -\frac{d}{\sigma}\frac{\phi'(z)}{\phi(z)} + \frac{(1-d)}{\sigma}\frac{\phi(z_\eta)}{1-\Phi(z_\eta)} \\
\frac{\partial l}{\partial \gamma_1} &= -d\left[\xi(t) + \frac{w'(t)}{w(t)} - \frac{1}{\sigma}\frac{w'(t)}{w(t)}\frac{\phi'(z)}{\phi(z)}\right] - \frac{(1-d)}{\sigma}\frac{w'(\eta)}{w(\eta)}\frac{\phi(z_\eta)}{1-\Phi(z_\eta)} \\
\frac{\partial l}{\partial \sigma} &= -\frac{d}{\sigma}\left[1 + \frac{\phi'(z)z}{\phi(z)}\right] + \frac{(1-d)}{\sigma}\frac{\phi(z_\eta)z_\eta}{1-\Phi(z_\eta)}.
\end{aligned}$$

Here $d = U_1(t)$ is the failure indicator, and $w'(t) = \partial w(t)/\partial\gamma_1 = \int_0^t -\xi(x)\exp[-\gamma_1\xi(x)]dx$. Note that $w'(t)$ here denotes the derivative with respect to γ_1 instead of t.

The second derivatives are

$$\begin{aligned}
\frac{\partial^2 l}{\partial \gamma_0^2} =& \frac{d}{\sigma^2}\left[\frac{\phi'(z)}{\phi(z)}\right]' - \frac{(1-d)}{\sigma^2}\left[\frac{\phi(z_\eta)}{1-\Phi(z_\eta)}\right]' \\
\frac{\partial^2 l}{\partial \gamma_1^2} =& d\left(-\left[\frac{w'(t)}{w(t)}\right]' + \frac{1}{\sigma}\left[\frac{w'(t)}{w(t)}\right]'\frac{\phi'(z)}{\phi(z)} + \frac{1}{\sigma^2}\left[\frac{w'(t)}{w(t)}\right]^2\left[\frac{\phi'(z)}{\phi(z)}\right]'\right) \\
&- (1-d)\left(\frac{1}{\sigma}\left[\frac{w'(\eta)}{w(\eta)}\right]'\frac{\phi(z_\eta)}{1-\Phi(z_\eta)} + \frac{1}{\sigma^2}\left[\frac{w'(\eta)}{w(\eta)}\right]^2\left[\frac{\phi(z_\eta)}{1-\Phi(z_\eta)}\right]'\right) \\
\frac{\partial^2 l}{\partial \sigma^2} =& \frac{d}{\sigma^2}\left(1 + \frac{2\phi'(z)z}{\phi(z)} + \left[\frac{\phi'(z)}{\phi(z)}\right]'z^2\right) - \frac{(1-d)}{\sigma^2}\left(\frac{2\phi(z_\eta)z_\eta}{1-\Phi(z_\eta)} + \left[\frac{\phi(z_\eta)}{1-\Phi(z_\eta)}\right]'z_\eta^2\right) \\
\frac{\partial^2 l}{\partial \gamma_0\gamma_1} =& -\frac{d}{\sigma^2}\left(\frac{w'(t)}{w(t)}\left[\frac{\phi'(z)}{\phi(z)}\right]'\right) + \frac{(1-d)}{\sigma^2}\left(\frac{w'(\eta)}{w(\eta)}\left[\frac{\phi(z_\eta)}{1-\Phi(z_\eta)}\right]'\right) \\
\frac{\partial^2 l}{\partial \gamma_0\sigma} =& \frac{d}{\sigma^2}\left(\frac{\phi'(z)}{\phi(z)} + \left[\frac{\phi'(z)}{\phi(z)}\right]'z\right) - \frac{(1-d)}{\sigma^2}\left(\frac{\phi(z_\eta)}{1-\Phi(z_\eta)} + \left[\frac{\phi(z_\eta)}{1-\Phi(z_\eta)}\right]'z_\eta\right) \\
\frac{\partial^2 l}{\partial \gamma_1\sigma} =& -\frac{d}{\sigma^2}\left(\frac{w'(t)}{w(t)}\frac{\phi'(z)}{\phi(z)} + \frac{w'(t)}{w(t)}\left[\frac{\phi'(z)}{\phi(z)}\right]'z\right) \\
&+ \frac{(1-d)}{\sigma^2}\left(\frac{w'(\eta)}{w(\eta)}\frac{\phi(z_\eta)}{1-\Phi(z_\eta)} + \frac{w'(\eta)}{w(\eta)}\left[\frac{\phi(z_\eta)}{1-\Phi(z_\eta)}\right]'z_\eta\right).
\end{aligned}$$

Here

$$\left[\frac{w'(t)}{w(t)}\right]' = \frac{\partial}{\partial\gamma_1}\left[\frac{w'(t)}{w(t)}\right] = \frac{w''(t)}{w(t)} - \left[\frac{w'(t)}{w(t)}\right]^2$$

where $w''(t) = \partial^2 w(t)/\partial\gamma_1^2 = \int_0^t [\xi(x)]^2 \exp[-\gamma_1 \xi(x)]dx$. Note that $w''(t)$ here denotes the second derivative with respect to γ_1 instead of t. Also,

$$\left[\frac{\phi'(z)}{\phi(z)}\right]' = \frac{\partial}{\partial z}\left[\frac{\phi'(z)}{\phi(z)}\right] \text{ and } \left[\frac{\phi(z)}{1-\Phi(z)}\right]' = \frac{\partial}{\partial z}\left[\frac{\phi(z)}{1-\Phi(z)}\right].$$

Table I gives the detailed formulae for commonly-used location-scale distributions (normal, smallest extreme value, largest extreme value, and logistic, corresponding to the lognormal, Weibull, Fréchet, and loglogistic log-location-scale distributions). The scaled expectations of the second derivatives are

$$\sigma^2 \mathbf{E}\left[-\frac{\partial^2 l}{\partial \gamma_0^2}\right] = -\int_{-\infty}^{z_\eta} \left[\frac{\phi'(z)}{\phi(z)}\right]' \phi(z)dz + \left[\frac{\phi(z_\eta)}{1-\Phi(z_\eta)}\right]' [1-\Phi(z_\eta)] \tag{2}$$

$$\begin{aligned}\sigma^2 \mathbf{E}\left[-\frac{\partial^2 l}{\partial \gamma_1^2}\right] = & \int_0^\eta \left(\sigma^2 \left[\frac{w'(t)}{w(t)}\right]' - \sigma \left[\frac{w'(t)}{w(t)}\right]' \frac{\phi'(z)}{\phi(z)} - \left[\frac{w'(t)}{w(t)}\right]^2 \left[\frac{\phi'(z)}{\phi(z)}\right]'\right) f(t;\gamma_0,\gamma_1,\sigma)dt \\ & + \left(\sigma \left[\frac{w'(\eta)}{w(\eta)}\right]' \frac{\phi(z_\eta)}{1-\Phi(z_\eta)} + \left[\frac{w'(\eta)}{w(\eta)}\right]^2 \left[\frac{\phi(z_\eta)}{1-\Phi(z_\eta)}\right]'\right) [1-\Phi(z_\eta)]\end{aligned} \tag{3}$$

$$\begin{aligned}\sigma^2 \mathbf{E}\left[-\frac{\partial^2 l}{\partial \sigma^2}\right] = & -\int_{-\infty}^{z_\eta} \left(1 + \frac{2\phi'(z)z}{\phi(z)} + \left[\frac{\phi'(z)}{\phi(z)}\right]' z^2\right) \phi(z)dz \\ & + \left(\frac{2\phi(z_\eta)z_\eta}{1-\Phi(z_\eta)} + \left[\frac{\phi(z_\eta)}{1-\Phi(z_\eta)}\right]' z_\eta^2\right) [1-\Phi(z_\eta)]\end{aligned} \tag{4}$$

$$\begin{aligned}\sigma^2 \mathbf{E}\left[-\frac{\partial^2 l}{\partial \gamma_0 \gamma_1}\right] = & \int_0^\eta \left(\frac{w'(t)}{w(t)} \left[\frac{\phi'(z)}{\phi(z)}\right]'\right) f(t;\gamma_0,\gamma_1,\sigma)dt \\ & - \left(\frac{w'(\eta)}{w(\eta)} \left[\frac{\phi(z_\eta)}{1-\Phi(z_\eta)}\right]'\right) [1-\Phi(z_\eta)]\end{aligned} \tag{5}$$

$$\begin{aligned}\sigma^2 \mathbf{E}\left[-\frac{\partial^2 l}{\partial \gamma_0 \sigma}\right] = & -\int_{-\infty}^{z_\eta} \left(\frac{\phi'(z)}{\phi(z)} + \left[\frac{\phi'(z)}{\phi(z)}\right]' z\right) \phi(z)dz \\ & + \left(\frac{\phi(z_\eta)}{1-\Phi(z_\eta)} + \left[\frac{\phi(z_\eta)}{1-\Phi(z_\eta)}\right]' z_\eta\right) [1-\Phi(z_\eta)]\end{aligned} \tag{6}$$

$$\begin{aligned}\sigma^2 \mathbf{E}\left[-\frac{\partial^2 l}{\partial \gamma_1 \sigma}\right] = & \int_0^\eta \left(\frac{w'(t)}{w(t)} \frac{\phi'(z)}{\phi(z)} + \frac{w'(t)}{w(t)} \left[\frac{\phi'(z)}{\phi(z)}\right]' z\right) f(t;\gamma_0,\gamma_1,\sigma)dt \\ & - \left(\frac{w'(\eta)}{w(\eta)} \frac{\phi(z_\eta)}{1-\Phi(z_\eta)} + \frac{w'(\eta)}{w(\eta)} \left[\frac{\phi(z_\eta)}{1-\Phi(z_\eta)}\right]' z_\eta\right) [1-\Phi(z_\eta)].\end{aligned} \tag{7}$$

Let $\theta = (\gamma_0, \gamma_1, \sigma)^T$. The Fisher information matrix of a time-varying stress ALT can be computed as

$$I_\theta = n\mathbf{E}\left[-\frac{\partial^2 l}{\partial\theta\partial\theta^T}\right].$$

After factoring out the common factor of $1/\sigma^2$, the Fisher information matrix still depends on σ through $\partial^2 l/\partial\gamma_1^2$, $\partial^2 l\partial\gamma_0\partial\gamma_1$ and $\partial^2 l\partial\gamma_1\partial\sigma$. Under the standard regularity conditions,

TABLE I

DETAILED FORMULAE FOR COMMONLY-USED LOCATION-SCALE DISTRIBUTIONS.

	NOR	SEV	LEV	LOGIS
Φ	$\int_{-\infty}^{x}\phi(t)dt$	$1-e^{-e^{x}}$	$e^{-e^{-x}}$	$\frac{e^x}{1+e^x}$
ϕ	$\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$	$e^{x-e^{x}}$	$e^{-x-e^{-x}}$	$\frac{e^x}{(1+e^x)^2}$
$\frac{\phi}{1-\Phi}$	$\frac{\phi}{1-\Phi}$	e^x	$\frac{e^{-x}}{e^{e^{-x}}-1}$	$\frac{e^x}{1+e^x}$
$\left[\frac{\phi}{1-\Phi}\right]'$	$\frac{-x\phi(1-\Phi)+\phi^2}{(1-\Phi)^2}$	e^x	$\frac{(e^{-2x}-e^{-x})e^{e^{-x}}+e^{-x}}{(e^{e^{-x}}-1)^2}$	$\frac{e^x}{(1+e^x)^2}$
$\frac{\phi'}{\phi}$	$-x$	$1-e^x$	$-1+e^{-x}$	$\frac{1-e^x}{1+e^x}$
$\left[\frac{\phi'}{\phi}\right]'$	-1	$-e^x$	$-e^{-x}$	$\frac{-2e^x}{(1+e^x)^2}$

the large-sample approximate variance-covariance matrix of the ML estimators of γ_0, γ_1 and σ is

$$
\begin{aligned}
\Sigma_{\hat{\gamma}_0,\hat{\gamma}_1,\hat{\sigma}} &= \begin{bmatrix} \mathrm{Avar}\,(\hat{\gamma}_0) & \mathrm{Acov}\,(\hat{\gamma}_0,\hat{\gamma}_1) & \mathrm{Acov}\,(\hat{\gamma}_0,\hat{\sigma}) \\ \mathrm{Acov}\,(\hat{\gamma}_0,\hat{\gamma}_1) & \mathrm{Avar}\,(\hat{\gamma}_1) & \mathrm{Acov}\,(\hat{\gamma}_1,\hat{\sigma}) \\ \mathrm{Acov}\,(\hat{\gamma}_0,\hat{\sigma}) & \mathrm{Acov}\,(\hat{\gamma}_1,\hat{\sigma}) & \mathrm{Avar}\,(\hat{\sigma}) \end{bmatrix} \\
&= \frac{1}{n}\begin{pmatrix} \mathbf{E}\left[-\frac{\partial^2 l}{\partial\gamma_0^2}\right] & \mathbf{E}\left[-\frac{\partial^2 l}{\partial\gamma_0\partial\gamma_1}\right] & \mathbf{E}\left[-\frac{\partial^2 l}{\partial\gamma_0\partial\sigma}\right] \\ \mathbf{E}\left[-\frac{\partial^2 l}{\partial\gamma_0\partial\gamma_1}\right] & \mathbf{E}\left[-\frac{\partial^2 l}{\partial\gamma_1^2}\right] & \mathbf{E}\left[-\frac{\partial^2 l}{\partial\gamma_1\partial\sigma}\right] \\ \mathbf{E}\left[-\frac{\partial^2 l}{\partial\gamma_0\partial\sigma}\right] & \mathbf{E}\left[-\frac{\partial^2 l}{\partial\gamma_1\partial\sigma}\right] & \mathbf{E}\left[-\frac{\partial^2 l}{\partial\sigma^2}\right] \end{pmatrix}^{-1}.
\end{aligned}
$$

To compute the $\Sigma_{\hat{\gamma}_0,\hat{\gamma}_1,\hat{\sigma}}$, one needs to compute the scaled expectations of the second derivatives in equations (2) through (7). With the specified distribution and the stress function $\xi(t)$, one needs to substitute the corresponding formulae in Table I into equations (2) through (7) and do the integrations. Numerical integration is needed for the computations.

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