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Keywords
Pareto improvements, Hicks-Kaldor criterion, transitional dynamics, public debt, public education, overlapping-generations models

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INTERGENERATIONAL DEBT DYNAMICS WITHOUT TEARS*

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Governments, motivated by a desire to improve upon long-run laissez faire, routinely undertake enduring, productive expenditures, say, in public education, that generate positive externalities across cohorts but require investments be made up front. If everyone after the policy is initiated is at least as happy as before and there are some outstanding resources, the Hicks-Kaldor efficiency rule suggests that the present value of these resources could, *hypothetically*, be distributed to future generations creating the *potential* for generational Pareto improvement. The literature recognizes the challenge in constructing a policy that is *actually* Pareto-improving since the policy itself may generate general-equilibrium gains and losses spread across generations. The paper takes on this task. In a dynamically-efficient economy with an intergenerational human capital externality, it constructs an equilibrium path with public education financed by non-explosive debt and taxes that truly improves upon laissez faire, yet no generation is harmed along the transition, not even the current ones.

JEL classification: D91; E21; O41
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1 Introduction

Left on their own, markets typically underinvest in areas such as infrastructure, health, human capital and R&D, activities that generate positive externalities across generations but require up-front investments. Governments routinely get involved in such areas motivated by a desire to improve upon laissez faire. Such intervention, though, may create intergenerational conflict since many of the benefits arrive in the future to generations that did not incur its cost in the present: there may emerge long-run winners and short-term losers.[1]

A standard approach to checking whether a proposed policy intervention is desirable is to employ the Kaldor-Hicks criterion and verify if the policy generates an efficiency gain – a hypothetical potential for the winners to compensate the losers (and leave the latter as happy as before the policy was initiated). If everyone post policy is at least at their pre-policy utility and there are still some resources left standing, then following Auerbach and Kotlikoff (1987), the present value of these new, net resources could, *in principle*, be distributed in some fashion to future generations; this is how the *potential* for Pareto improvement is demonstrated. An important, recent paper that takes this approach is Nishiyama and Smetters (2007) where a hypothetical LSRA (lump sum redistributive authority – Auerbach and Kotlikoff, 1987) compensates the losers from social security privatization. In their words, "the LSRA is not being proposed as an actual government institution. Instead, it is simply a hypothetical mechanism that allows us to measure the standard Hicksian efficiency gains in general equilibrium associated with privatization." They go on to write, "[C]onstructing a policy that is *actually* Pareto improving from a policy that improves Hicksian efficiency is a tougher task." [*emphasis ours*] Why is it tougher? Because an actual policy, once implemented, would release its own dynamics and produce general-equilibrium gains and losses spread across generations. All such gains and losses would have to be properly accounted for if the policy is to be deemed genuinely Pareto-improving, generation by generation. It is this "tougher task" we take on.

To that end, we study a standard overlapping-generations model of a closed economy – similar in many aspects to the models in Azariadis and Drazen (1990) and Boldrin and Montes (2005) – with three generations (young, middle-aged, and old), no population growth, and endogenous physical and human capital accumulation. There is no intra-generational heterogeneity, and importantly, no altruism. Human capital accumulation is subject to a standard intergenerational externality: loosely speaking, it is easier for agents to acquire more education if their parents are already well-educated. When young, agents may borrow from perfect capital markets to pay for their educational expenses. When middle aged, they pay off these loans, work, consume, and save for old age. All markets, education-loan, labor and capital, are complete. Just to be clear, define a laissez-faire (LF) economy as the market economy, just described, with no government presence. Assume the LF economy is in a steady state and that it is

[1] A popular approach to slicing through such conflict is to postulate an intergenerational social welfare function with weights on all generations – see, for example, Blanchard and Fischer (1989; Ch. 3); an optimal policy may be constructed by maximizing such a function. Of course, as is well-known, this requires the researcher to take a stand on the weights; also, just because some measure of overall weighted welfare is higher doesn't mean every generation along the way is happier.

dynamically efficient.

In the presence of the human capital externality, education spending in the LF is too low and a government could intervene to raise education spending levels *beyond* LF hoping to improve long-run welfare.[2] To fix ideas, start from the LF steady state and consider an *incremental* but permanent increase in public education spending above its LF level.[3] We work with a setting in which private and public spending on education are perfect substitutes which means public spending crowds out private spending one for one. As a result, all education spending becomes publicly financed, relieving the young of the cost of private spending. To finance public education, the government sells bonds to the current middle-aged (who, recall, were schooled under laissez faire). The resulting debt is, therefore, productive – targeted at public education – and is to be serviced at market rates. The debt must be serviced using taxes and, if necessary, new debt may be issued. But how much taxes can the government impose? For us, at all points in time, the government's ability to tax is restrained by the Pareto criterion so as to ensure all generations, including the currently-alive ones, are at least as well off after the intervention as before. Evidently, this sort of public education-debt-tax policy releases multiple general-equilibrium effects downstream. Under dynamic efficiency, it is possible that debt explodes unless enough taxes are raised to put on the brakes.[4] Our question, then, becomes, is there a sequence of minimum-required, incremental government debt (and associated lump-sum taxes) that leave all generations as well off as under LF *and* ensures the path of debt is non-exploding?

What sort of welfare gains are available to be taxed? Under the policy initiated at some date t, the current young accumulate higher levels of human capital (than they would have under LF), a possible source of direct welfare gain to them at $t+1$, during *their* middle age. The inaugural middle-aged, schooled under LF, do not directly benefit from the policy; indirectly, they do. If, on impact, debt crowds out capital – Diamond (1965) – the wage rate falls and the rate of return rises. The former harms the middle-aged at $t+1$, the latter benefits the old, the middle-aged at t who took on the debt. This rate-of-return effect helps bring down the opportunity cost of financing education to the middle-aged at t. On net, there may not be enough welfare gains to tax (Pareto-constrained) and repay the debt entirely in one generation. New debt would have to be issued to pay for past-debt service, debt

[2] Indeed, even in the absence of the externality, education spending in a dynamically-efficient LF is too low because private spending has an attached opportunity cost, say R, the market interest rate on loans, while public education, intermediated across generations, has a cost of 1, where $R > 1$ (dynamic efficiency). This point has been fleshed out in Andersen and Bhattacharya (2017).

[3] The discussion in the introduction is kept somewhat loose for expositional clarity. In Section 5.3 below, we prove the equivalence of the LF steady state with, what we call, the LF package steady state, one where policy exactly replicates the LF steady state equilibrium. The transition studied in the paper starts when the government considers a small but permanent increase in education spending above its LF *package* steady state level.

[4] Anytime a debt is (partly) serviced by means of another debt, the issue arises, is the debt on an explosive path? As is well known, this issue does not arise if the gross return on bonds lies below unity (assuming zero population growth) thereby allowing for Ponzi debt as well as bond seigniorage (Bullard and Russell, 1999), the ability of the government to raise net revenue from the sale of bonds itself. In our setting, debt must bear the same return as capital and *if* we focus on the case where the gross return on bonds exceeds unity (dynamic efficiency), then the path of debt we construct may explode: "debt will grow so large that the government will be unable to find buyers for all of it, facing either default or a tax increase" (Elmendorf and Mankiw, 1999).

retirement, and current education spending; this process is repeated here on releasing indirect, general-equilibrium gains and losses over time.[5] Thankfully, the education externality gains vigor as the stock of human capital rises and produces direct income gains downstream that may be taxed to support debt repayment. The challenge is to keep track of all these direct and indirect gains and losses unleashed by public education, debt and taxes – each constrained by the generation-by-generation Pareto criterion – and ensure debt does not explode.

We prove that all competitive equilibria are summarized by a two-dimensional, first-order, non-linear dynamical system in debt and capital. These are linearized near the LF steady state and the dynamics of debt is studied with special focus on the possibility of debt retirement in finite time.[6] The main analytical finding is a strong, possibility result. Starting from a laissez-faire steady state, it is possible for the government to implement an incremental debt-financed, public education scheme that generates more human capital and higher welfare in the long run relative to laissez faire, and most importantly, along the trajectory no transitional generation is harmed relative to laissez faire; additionally, the path of debt does not explode.[7] Sufficient conditions for this local result to hold are reasonable: if the LF steady state is locally stable, the policy perturbation is incremental, and public debt increases on policy impact, then debt levels will fall right away for any dynamically-efficient economy with a human capital externality, even when private and public education are perfect substitutes. Our insistence that debt levels go to zero over time has an important implication. Once the debt is entirely paid off (but welfare gains from higher human capital linger) taxes may be cut relative to their levels under the Pareto criterion, making subsequent generations *strictly* better off relative to LF. Put differently, we prove the existence of a Pareto-*superior* debt/tax trajectory; for that, it is sufficient (*not necessary*) that debt levels go to zero.

The findings clearly indicate the main argument developed in the paper does not require the policy transition to start from the LF steady state; indeed, it can start any point along the transition to the LF steady state, anywhere private spending on education is inefficiently low. Similarly, while the availability of lump-sum taxes make the analytics more tractable, they are not necessary for the main intuition to proceed; after all, since there is no within-generation heterogeneity, and hence, no possibility of redistribution (an issue of some significance for research in the Mirleesian tradition, such as

[5] Absent the education externality, intuition inspired by Ricardian equivalence may lead one to think that the swap (more human capital for more taxes) would be one-for-one with no real effects especially if interest rates were held fixed, as in a small open economy. This logic is faulty partly because Ricardian equivalence does not hold in our no-altruism, overlapping-generations economy. But more importantly, the debt here is productive.

[6] Debt dynamics is not of much interest in an infinitely-lived agent framework because the same agent pays for and receives benefits from the intervention. There, under standard settings, Ricardian equivalence would hold and hence all that is needed for an intervention to be financed is that the present value of taxes (net of transfers) equals the present value of the cost of the intervention. Any path of debt that satisfies this would do the job. The issue of integenerational conflict due to timing differences between costs and benefits therefore cannot arise in an infinitely-lived agent framework. And it is precisely this conflict that makes debt dynamics worthy of attention in our setup.

[7] In a sense, the paper proves the existence of the economic equivalent of a *perpetual-motion machine*: a nation borrows from its own people to fuel a higher level of long-run prosperity and eventually retires that debt without sacrificing anyone's happiness – no tears – along the way.

Hosseini and Shourideh, 2017) distorting taxes add a tax rate-change in addition to the tax base-change already present with lump-sum taxes, not much else.

Finally, as discussed above, the analytics rely heavily on the fact that policy changes are incremental and can therefore be studied using linearization techniques. Clearly, the incremental approach developed in the paper is limited in what it can teach us about how to design Pareto-improving transitions to steady states that are not in a small neighborhood of the original equilibrium. Not much is known about global properties of generic, non-linear systems which is why analytical progress with discrete policy perturbations is stalled.[8] In numerical simulations, we explore the possibility of engineering a discrete policy move, allowing distortionary taxes on wage income, and allowing the start date for the reform to be away from the LF steady state. To that end, we reformulate the model to allow for an additional labor-leisure choice so as to give distorting taxes more to work with. The expanded model economy in the LF steady state is loosely disciplined to replicate some features of the U.S. economy. We restrict attention to an education policy package that delivers ****% higher education spending per pupil in perpetutiy over its starting value and ***% higher utility (than its value in the LF steady state) for every current and subsequent generation. The numerical findings help reassure us that the main line of logic remains intact even when policy changes are not incremental, taxes are distortionary, and policy reform starts along the transition to the LF steady state. In the model economy, the debt to GDP ratio rises to about 1.8% on impact and then starts to fall. It takes longer for the debt levels to fall to zero if the human capital externality is weak or the initial policy is quite ambitious.

The rest of the paper is organized as follows. Section 2 offers a quick review of the surrounding literature. Section 3 lays out the main issues and intuition by studying a no-frills, two-period version of our main model. The three-period model itself is laid out in Section 4, and the definition of equilibrium and its stability properties are clarified in Section 5. The possibility of implementing an incremental education-debt-taxes package under the Pareto-criterion is analyzed in Section 6, while Section 7 studies the debt dynamics. Section 8 presents the results from a numerical example that allows for distorting taxes and an endogenous labor-leisure choice. Section 9 offers few concluding remarks. The appendix contains proofs of all major results.

2 Literature review

At first glance, it may appear that our research question lacks an element of surprise; after all, it is well understood that in the presence of generational spillovers, a planner can improve upon laissez faire, and with access to debt and non-distorting taxes a policymaker may, in principle, exploit future tax payers (who benefit from the inaugural investment) and compensate the original financiers (who do not). No doubt, the potential for such winners-pay-losers, debt-tax policies to exist is clearly there – see

[8] We thank the Associate Editor for suggesting that our approach indicates a transition to a discretely-better steady state may be implemented as a sequence of transitions to incrementally-better steady states.

Hendren, 2014. What is by no means obvious is if a policy package that is *actually* Pareto-improving exists, our entry point into this literature.

At this juncture, it is instructive to compare our goals to the debt-tax-equivalence/decentralization results in the literature. It is well known that in the textbook Diamond model with two period-lived generations – see de la Croix and Michel (2002; Ch. 4) for an excellent exposition – any feasible path of debt, more so a Pareto-optimal one *if it exists*, may be decentralized by "an adequate mix of lump-sum taxes" that can be levied on both the young and the old generations. We wish to point out that these sorts of equivalence/decentralization results are somewhat tangential to our focus.[9] Our aim is not to demonstrate the equivalence of debt and taxes nor is it to show that *if* an optimal trajectory involving debt exists, it can be decentralized via taxes – that may well be true. Our goal is to construct and prove the very existence of a Pareto-superior debt/tax trajectory in the first place, one that improves on laissez faire without hurting any transitional generation (indeed, making some generations better off); our goal is *not* to assume the existence of such a path and demonstrate a way to decentralize it with taxes alone.

Public debt (with accompanying taxes) is the set of instruments studied in this paper but it is one, among many, fiscal instruments we could have studied that are available to governments for that purpose. Indeed, there is a large literature – see Gale (1990) and de la Croix and Michel (2002; Ch 3,4) – that studies equivalence between various fiscal instruments showing how a certain equilibrium trajectory under one instrument may be exactly replicated by another. Such equivalence is not our interest here. Our foremost concern is whether the financing instrument we study – debt – can help rationalize a genuine welfare-improving government intervention without hurting any transitional generation.

Our use of the Pareto criterion is in line with a large literature – prominent examples include the seminal paper by Breyer (1989), Breyer and Straub (1989), Belan et.al (1998), Pemberton (1999), Köthenbürger and Poutvaara (2006) – that studies, in a dynamically efficient economy, the possibility of a Pareto-improving transition from a return-dominated, pay-as-you-go pension system to a fully funded one. Our paper takes a first step at fleshing out the dynamics of *productive* debt under an intergenerational Pareto criterion. In fact, the worthiness of our exercise is clear from the following quote from Ball and Mankiw (1995):

> "...Thus, the winners from budget deficits are current taxpayers and future owners of capital, while the losers are future taxpayers and future workers. Because these gains and losses balance, a policy of running budget deficits cannot be judged by appealing to the Pareto criterion or other notions of economic efficiency" (Ball and Mankiw, 1995, p. 108).

In a recent, influential paper, Hosseini and Shourideh (2017) consider another institution grappling with intergenerational conflict – social security – and study the possibility of Pareto-improving reforms

[9] For starters, ours is not the textbook setting: debt here finances a productive activity which means it can influence the budget sets of agents and even crowd out private spending. Also, our government scheme requires only the middle-aged be taxed. These differences imply that the textbook decentralization result may not be automatic in our setup.

of the same. They use a quantitative, overlapping generations economy consistent with aggregate and distributional features of the U.S. economy. Their model allows for individuals of each cohort to be heterogeneous in their earning ability, mortality and patience – all private information – which allows them to study the role of reforms in reducing within-generation distortions and the balancing of equity-efficiency concerns. Similar to us, they stress the importance of accounting for the downstream, general-equilibrium effects of such reforms. Like us and Werning (2007), the seminal paper in this area, they eschew the use of a social welfare function. They find that, when lump sum taxes are not available (say, due to the unobserved heterogeneity), the existence of Pareto-improving policy reforms in their environment is not routine; but, as they show, sufficient conditions do exist.

Greulich, Laczo and Marcet (2016) study Pareto-improving tax reforms to distorting labor and capital taxes in a Ramsey model with agents who are heterogeneous in their labor productivity and wealth. By insisting that the planner not be allowed to drive consumption below some minimum, they find that even though the long-run tax rate on capital should be zero, Pareto-improving reforms must set labor taxes initially low and capital taxes should remain high "for a very long time". This sort of "gradualism" is important for achieving Pareto improvement. The latter result is reminiscent of our result about the path of debt hitting zero in finite time and its importance for generating Pareto improvements.

The broader literature on debt and deficits is voluminous – Elmendorf and Mankiw (1999), though dated, is a good survey for our purposes. Much of the literature starting from Diamond (1965) studies *un*productive government debt, and is hence, not directly relevant to this paper. O'Connell and Zeldes (1988) study environments with unproductive debt in which the sale of new debt finances all of the interest payments on outstanding debt as well as some transfers to the young. In that world, the government can cut current taxes without ever raising future taxes to finance the increased interest payments. Ball et. al (1998) study whether such a Ponzi scheme can ever lead to Pareto improvements, especially in stochastic environments. Dynamic *in*efficiency is key to these results. Bullard and Russell (1999) and Chalk (2000) study issues relating to sustainability of permanent bond-financed deficits in reasonably-calibrated, dynamically-inefficient OG models. There is a line of work – Turnovsky (1997), Greiner and Semmler (2000), Yakita (2008) – that studies the financing of public capital via debt finance. Greiner (2008) studies a continuous-time, infinitely-lived agent economy wherein the government aids human capital formation by financing public education via debt finance. Under a fiscal rule ensuing a sustainable path of public debt, he studies the growth effects and associated dynamics of public debt. Given his model environment, implementation problems of the kind we encounter are not his concern.[10]

[10] There is also a large body of work – see Cooley and Soares (1999) or more recently, Kass (2003), Kaganovich and Zilcha (2012), Bishnu and Wang (2017), and many others – in the political-economy domain, somewhat tangential to our focus, that studies public education policies, absent debt. In a sense, our use of the Pareto criterion establishes the set of policies that are unlikely to face opposition as they negotiate their way through standard democratic processes. It bears emphasis that these papers do not focus on the implementation hurdles and do not impose the Pareto criterion generation by generation.

3 Simplified exposition and intuition

To set the scene for what is to come, we start by expositing a simpler, no-frills version of the model economy we study further below. This will help fix ideas and intuition. Consider an overlapping-generations economy wherein at each date $t = 1, 2, .., \infty$, a continuum of unit mass of identical *two* period-lived agents is born. When young, agents get some income ω_t, consume, and save for old age. When old, they consume and die. Saving in young age earns a gross return of R_{t+1} between t and $t+1$. For the present, we are agnostic as to how ω and R are determined. The generation born in t has consumption $c_{y,t}$ as young in t, and $c_{o,t+1}$ as old in $t+1$. All generations have standard preferences representable by the utility function $U_t\left(c_{y,t}, c_{o,t+1}\right) \equiv u(c_{y,t}) + \beta u(c_{o,t+1}), \beta \in (0,1)$. The lifetime budget constraint of an agent is $c_{y,t} + \frac{c_{o,t+1}}{R_{t+1}} = \omega_t + y_t$,where y_j is some to-be-determined income taken as given by the agent. The corresponding expenditure function is given by

$$\mathcal{E}(\omega_t, R_{t+1}, U_t) = \min_{c_{y,t}, c_{o,t+1}} \left\{ c_{y,t} + \frac{c_{o,t+1}}{R_{t+1}} - (\omega_t + y_t) \mid U\left(c_{y,t}, c_{o,t+1}\right) \geq U_t \right\}$$

Consider an equilibrium trajectory in the absence of policy – call it laissez-faire, LF – with associated factor prices $\left\{\omega_t^{LF}, R_{t+1}^{LF}\right\}_{t=0}^{\infty}$ delivering utility $\left\{U_t^{LF}\right\}_{t=0}^{\infty}$. Then, by definition

$$\mathcal{E}\left(\omega_t^{LF}, R_{t+1}^{LF}, U_t^{LF}\right) = 0$$

i.e., the exogenous income which would deliver utility U_t^{LF} at factor prices ω_t^{LF} and R_{t+1}^{LF} is 0. At these equilibrium prices, agents attain lifetime utility U_t^{LF}.

3.1 Intertemporal Pareto criterion

Next, consider a public intervention at time t (fix the time index at $\hat{t}$ and start the j index) which will generate an equilibrium trajectory (over $j \geq 0$) with factor prices $\left\{\hat{\omega}_j, \hat{R}_{j+1}\right\}_{j=0}^{\infty}$ and associated utility $\left\{U_j\right\}_{j=0}^{\infty}$. For now, we remain agnostic as to the nature of the intervention or why it is needed as long as it can satisfy the intertemporal Pareto criterion,

$$U_j \geq U_{\hat{t}}^{LF} \text{ for } j \geq 0,$$

in which case the public intervention will leave everyone at least as well off as before the policy inauguration. To go further, let us take a stand on both the cost of the intervention and its financing. Suppose the intervention requires the government to spend $\left\{G_j\right\}_{j=0}^{\infty}$. Define $\mathcal{T}_j$ as a lump-sum tax/transfer. Imposing the constraint $U_j \geq U_{\hat{t}}^{LF}$ for $j \geq 0$ implies that the path $\mathcal{T}_j\left(\cdot\right)$ has to satisfy

$$\mathcal{T}_j\left(\cdot\right) \equiv \mathcal{E}\left(\hat{\omega}_j, \hat{R}_{j+1}, U_{\hat{t}}^{LF}\right) = \begin{cases} > 0, \text{ a tax} \\ < 0, \text{ a subsidy/compensation} \end{cases} \quad \forall j \geq 0$$

where $\hat{\omega}_j, \hat{R}_{j+1}$ are the equilibrium prices under the path $\mathcal{T}_j(\cdot)$: cohorts along the "intervention trajectory" that generate the same (or higher) utility compared to the LF. It is noteworthy that $\mathcal{T}_j(\cdot)$ captures the maximum taxation capacity alluded to in the introduction. This is the compensating variation. Of course, the entire $\left\{G_j, \mathcal{T}_j\right\}_{j=0}^{\infty}$ package needs to satisfy the intertemporal budget constraint of the government. This raises the question, can the government raise enough taxes ($\mathcal{T}_j(\cdot)$, under the Pareto restriction) to cover its expenses $\left\{G_j\right\}_{j=0}^{\infty}$, especially under dynamic efficiency, $\hat{R}_j > 1$.

3.2 Kaldor-Hicks

For that, we would need to verify if the NPV criterion

$$(1) \quad \sum_{j=0}^{\infty}\left(\frac{1}{\hat{R}_j}\right)^j\left[\mathcal{T}_j\left(\hat{\omega}_j, \hat{R}_{j+1}, U_{\hat{t}}^{LF}\right) - G_j\right] \geq 0$$

holds. A constraint like (1) may, on first glance, look like the standard Kaldor-Hicks criterion but it is not. The latter would compute the hypothetical potential for winners under the intervention to compensate the losers, i.e., it would calculate the present value of hypothetical compensating variations (taxes and transfers) required to leave all current and future cohorts as happy as before the policy was introduced – hypothetical, because it would ignore the general-equilibrium effects arising from *actual* implementation of said taxes and transfers. The point is that the standard Kaldor-Hicks criterion checks

$$(2) \quad \sum_{j=0}^{\infty}\left(\frac{1}{R_j^h}\right)^j\left[\mathcal{T}_j^h\left(\omega_j^h, R_{j+1}^h, U_{\hat{t}}^{LF}\right) - G_j\right] \geq 0$$

(not (1)) where ω_j^h and R_{j+1}^h are *hypothetical* wages and interest rates igoring the fact that equilibrium prices are themselves affected by the compensating taxes and transfers the NPV criterion proposes. This is precisely what Nishiyama and Smetters (2007) do when they consider a hypothetical LSRA (lump sum redistributive authority – Auerbach and Kotlikoff, 1987). [11]

The challenge, of course, is that, with endogenous $\left(\hat{\omega}_j, \hat{R}_{j+1}\right)$, condition (1) is daunting, if not impossible, to verify. This is the entry point of our paper. We deliver an alternative, intuitive way to check whether implementation of a $\left\{G_j, \mathcal{T}_j\right\}_{j=0}^{\infty}$ package is feasible without needing to verify (1).

3.3 Role of debt

To begin with, there is no requirement $\mathcal{T}_j$ must equal G_j at every date – the government's budget need not be balanced per by period. In that case, it is possible $\mathcal{T}_j < G_j$ holds for some j and the government

[11] Of course, it is by no means obvious if a policy that satisfies (2) necessarily satisfies (1) or not.

would have to issue debt, B_j, and service it at the market interest rate $\hat{R}_{j+1}$. It follows

$$\begin{aligned} & B_j = G_j - \mathcal{T}_j + \hat{R}_j B_{j-1} \\ \iff & B_j - B_{j-1} = \left[G_j - G_{j-1}\right] - \left[\mathcal{T}_j - \mathcal{T}_{j-1}\right] + \hat{R}_j B_{j-1} - \hat{R}_{j-1} B_{j-2} \end{aligned}$$

Again, to fix ideas, for now hold $\hat{R}_{j+1} = \hat{R} > 1 \forall j$ and $G_j = G \; \forall j$. Then

$$B_j - B_{j-1} = \underbrace{\hat{R}\left[B_{j-1} - B_{j-2}\right]}_{\text{debt service}} - \underbrace{\left[\mathcal{T}_j - \mathcal{T}_{j-1}\right]}_{\text{taxation capacity}}$$

If the up-front cost at $j = 0$, G_t, exceeds $\mathcal{T}_t$, an initial deficit, there will be some initial debt of amount $B_t = G_t - \mathcal{T}_t > 0$. Debt dynamics (dynamics in B) is thus driven by two forces i) the cost of debt service: since $\hat{R} > 1$, this part tends to grow/explode (the Ponzi part),[12] ii) changes in the government's taxation capacity, $\left[\mathcal{T}_j - \mathcal{T}_{j-1}\right]$, which is to be constrained by the Pareto criterion. Consider the plausible case where $\mathcal{T}_j \geq \mathcal{T}_{j-1}$ i.e., the welfare gains from the intervention build up over time allowing taxation capacity to ramp up. If we had a handle on the growth of taxation capacity, $\left[\mathcal{T}_j - \mathcal{T}_{j-1}\right]$, we could check whether it was growing fast enough relative to debt service costs (which is growing at rate, $\hat{R}$) so that the debt level, B_j, would not explode. Unfortunately, questions about the growth rate of taxation capacity under the Pareto criterion are as daunting as checking whether (1) is satisfied. Our tack is to, instead, seek a turning point, k, in the path of debt such that $B_{t+k} - B_{t+k-1} < 0$. If such a point exists, then a tail wind will emerge — debt is falling, the cost of debt service is declining, and since $\mathcal{T}$ is growing, debt will fall even further.[13]

Having exposited the underlying intuition, we lay out a quick road map of where the paper is headed. First, we study a model in which the presence of an intergenerational externality – a textbook human capital externality – makes it desirable for the government to intervene. We show, under certain conditions, an *incremental* policy move beyond the steady state LF regime is possible under the intertemporal Pareto criterion (using lump-sum taxes and debt) in a dynamically efficient economy. Also, the associated path of debt does not explode. This is a strong possibility result, but it is a "local" result – it does not indicate how "ambitious" the policy can be. Since the analytical challenges associated with the latter issue are insurmountable, we go on to present numerical illustrations showing that the general tenor of our results holds i) if distortionary taxes are used, ii) if the policy is implemented starting outside of LF steady-state, and iii) non-incremental policy moves are undertaken.

[12]This issue of debt explosion does not arise in models that specify a debt policy rule, such as the Maastricht criterion – see Ghosh and Mourmouras (2004) and Futagami et. al (2008) – that forces the government to keep the amount of debt in check.

[13]After some point in time, debt levels may converge to a positive level or they may become negative. Here on, taxes may be reduced since the path has passed a feasibility test, no cohorts are worse off, and public debt can be eliminated leaving future cohorts *strictly* better off.

4 An overlapping-generations model with a human capital externality

4.1 Primitives

We start by describing the market economy in the absence of any government presence. Consider a closed economy, in the tradition of Diamond (1965), wherein, at each date $t = 1, 2, .., \infty$, a continuum of identical *three* period-lived agents is born.[14] There is no population growth and the size of a cohort of newborns at any date is held fixed at 1. When young, agents can access perfect capital markets to secure education loans (L). When middle aged, they pay off these loans, work in competitive labor markets at a wage ω, consume, and save (S) for old age. When old, they are retired: they consume whatever they have and die. Saving in middle-age is in the form of education loans to the young and physical capital.

The generation born in $t-1$ has consumption $c_{m,t}$ as middle-aged in t, and $c_{o,t+1}$ as old in $t+1$. All generations have preferences representable by the utility function $U(c_{m,t}) + \beta U(c_{o,t+1}), \beta \in (0, 1]$ where U is twice-continuously differentiable, strictly increasing, strictly concave in its arguments and the standard Inada conditions are assumed to hold.

Output is produced using a constant-returns-to-scale, production function $F(K_t, H_t)$ where K_t denotes the physical capital input and H_t denotes the human capital input at t. Let $k_t \equiv K_t/H_t$ denote the physical to human capital ratio – called 'capital ratio' for short – at date t and $f(k_t) \equiv F(K_t/H_t, 1)$. We assume $f(0) = 0$, $f_k > 0 > f_{kk}$, and the usual Inada conditions hold (note, hereafter, function subscripts denote derivatives, such as $f_k \equiv \frac{\partial f(\cdot)}{\partial k}$). The final output can either be consumed in the period it is produced, or stored to yield physical capital the following period. Without loss of generality, assume physical capital depreciates 100% between periods.

Human capital in period t depends on educational spending (E_{t-1}) in period $t-1$, and the human capital of the previous generation:

(3) $$H_t = h(E_{t-1}, H_{t-1}),$$

The h-function is assumed to have the following standard properties:

Assumption 1

$$h(0,0) > 0; \; \frac{\partial H_t}{\partial E_{t-1}} > 0, \; \frac{\partial^2 H_t}{\partial E_{t-1}^2} < 0, \; \frac{\partial H_t}{\partial H_{t-1}} \in (0, 1), \; \frac{\partial^2 H_t}{\partial H_{t-1}^2} < 0, \; \frac{\partial H_t}{\partial E_{t-1} \partial H_{t-1}} \geq 0.$$

If $h_H = 0$, no education externality is present. If $h_H > 0$, an intergenerational human capital externality is present; current educational spending increases the human capital of the next and all future

[14]Below, in Section 6, we introduce policy action at $t = \infty$, the steady state of this economy; there on, the time index is j and $j = 0, 1, 2, ..., \infty$. The policy is initiated at $j = 0$.

generations. It will be clear below that $h_H \in (0,1)$ is a necessary condition for ensuring the h function is strictly concave which, in turn, rules out the possibility of perpetual endogenous growth. In the absence of policy, all spending on education is private, and hence, $E = L$.

4.2 Decision rules

The constraints faced by an agent are

$$(4) \quad \text{lifetime:} \quad c_{m,t} + \frac{c_{o,t+1}}{R_{t+1}} = \omega_t H_t - R_t L_{t-1},$$

$$(5) \quad \text{middle-age:} \quad c_{m,t} = \omega_t H_t - R_t L_{t-1} - S_t,$$

$$(6) \quad \text{old-age:} \quad c_{o,t+1} = R_{t+1} S_t.$$

where ω_t is the wage rate at date t and R_{t+1} is the market return on saving between t and $t+1$.[15]

The decision problem for the representative agent at any date is: choose education spending and saving by solving

$$\max_{S_t > 0,\, 0 \leq L_{t-1} < \frac{\omega_t H_t}{R_t}} \quad U\left(\omega_t\, h\left(E_{t-1}, H_{t-1}\right) - R_t L_{t-1} - S_t\right) + \beta U(R_{t+1} S_t)$$

subject to (5)-(6) and the usual non-negativity constraints on consumption, taking all prices and government policy variables as given. Taking H_{t-1} as given, the first order condition characterizing private education choice is

$$(7) \quad \begin{cases} \omega_t\, h_{E_{t-1}}(E_{t-1}, H_{t-1}) = R_t \text{ if } E_{t-1} > 0 \\ \omega_t\, h_{E_{t-1}}(E_{t-1}, H_{t-1}) < R_t \text{ if } E_{t-1} = 0 \end{cases},$$

and it is easily verified that the second order condition is fulfilled, since $\omega h_{EE}(\cdot) < 0$. Of course, since private agents take the human capital externality as given and do not internalize the effect of their own choices on the stock of human capital available to future generations, they *underinvest* in education relative to what a planner would. To foreshadow, a government that internalizes this externality could improve matters by raising E beyond what is privately optimal – see also Andersen and Bhattacharya (2017).

The first order condition characterizing the savings decision is

$$(8) \quad U_{c_{m,t}}(\cdot) = \beta R_{t+1} U_{c_{o,t+1}}(\cdot);$$

again, the second order condition is fulfilled, since $U_{c_m c_m}(\cdot) + R^2 U_{c_o c_o}(\cdot) < 0$. Eq. (8) defines an implicit

[15]Usual non-negativity conditions on consumption apply. There is also a borrowing constraint: $0 \leq L_{t-1} \leq \frac{\omega_t H_t}{R_t}$ which does not bind under the assumed Inada conditions on utility.

savings function $S_t = S(Y_t, R_{t+1})$ where $Y_t \equiv \omega_t H_t - R_t L_{t-1}$. The response of saving to the rate of return is important; we note

$$(9) \quad S_R \equiv \frac{\partial S_t}{\partial R_{t+1}} = -\frac{U_{c_o c_o}(\cdot)\beta R_{t+1} S_t + \beta U_{c_o}(\cdot)}{U_{c_m c_m}(\cdot) + \beta R_{t+1}^2 U_{c_o c_o}(\cdot)}.$$

We restrict attention to utility functions that satisfy the following assumption

Assumption 2

$$(10) \quad U_{c_o}(\cdot) + U_{c_o c_o}(\cdot) R_{t+1} S_t \geq 0.$$

This implies the substitution effect of an interest rate change dominates the income effect which, in turn, implies $\frac{\partial S_t}{\partial R_{t+1}} > 0$, the standard case considered in the literature. For later reference, note

$$(11) \quad S_Y \equiv \frac{\partial S_t}{\partial Y_t} = \frac{U_{c_m c_m}(\cdot)}{U_{c_m c_m}(\cdot) + \beta R_{t+1}^2 U_{c_o c_o}(\cdot)} \in (0,1)$$

i.e., an increase in period t income induces the middle-aged to increase saving.

5 Equilibrium

5.1 Laissez faire equilibrium

Physical capital accumulation is described by $K_{t+1} = S_t - L_t$. Then, the capital-ratio is defined by

$$(12) \quad k_{t+1} \equiv \frac{K_{t+1}}{H_{t+1}} = \frac{S_t - L_t}{H_{t+1}}.$$

Factor prices are determined by standard factor-pricing relationships:

$$(13) \quad R_t \equiv R(k_t) = f_k(k_t), \quad \omega_t \equiv \omega(k_t) = f(k_t) - f_k(k_t)\, k_t.$$

Also note $R_k(k) < 0$ and $\omega_k(k) = -kR_k(\cdot) > 0$ implying a higher capital-ratio reduces the return on capital and raises the wage rate. These return effects will be important in the discussion below.

A competitive equilibrium is defined as sequences for capital and human capital with savings determined from (8) and private education from (7) and human capital from (3) and factor prices determined from (13). Consumptions follow from (5) and (6).

A particularly important competitive equilibrium, the *laissez faire* (**LF**) is one with *no public intervention*. Private education in LF is described by $\omega_t^{LF} h_E\left(E_{t-1}^{LF}, H_{t-1}^{LF}\right) = R_t^{LF}$. Private saving in LF is described by $S_t^{LF} = S\left(Y_t^{LF}, R_{t+1}^{LF}\right)$ where $Y_t^{LF} \equiv \omega_t^{LF}\, h\left(E_{t-1}^{LF}, H_{t-1}^{LF}\right) - R_t^{LF} L_{t-1}^{LF}$ and $S_t^{LF} = K_{t+1}^{LF} + L_t^{LF}$.

Note $\frac{\partial H_t}{\partial H_{t-1}} \in (0,1)$ and $\frac{\partial^2 H_t}{\partial H_{t-1}^2} < 0$ (Assumption 1) guarantee concavity of $h(., H_{t-1})$ which, along with $h(0,0) > 0$, ensures the existence of an unique steady state H for given E – call it $H(E)$ where $H(E) \equiv h(E,H)$. It follows from Assumption 1 that $\frac{\partial H(E)}{\partial E} \equiv \frac{h_E(E,H)}{1-h_H(E,H)} > 0$; further, assume $\frac{\partial^2 H(E)}{\partial E^2} < 0$.

A LF steady-state is a time-invariant LF competitive equilibrium with $E^{LF} = L^{LF}$. Let $\omega^{LF} \equiv \omega(k^{LF})$, $R^{LF} \equiv R(k^{LF})$ and $H^{LF} \equiv H(E^{LF})$ denote the wage rate, the interest rate and the human capital associated with a LF steady state. Such a steady state is characterized by

$$(14) \quad k^{LF} = \frac{S\left(\omega\left(k^{LF}\right) H\left(E^{LF}\right) - R\left(k^{LF}\right) L^{LF}, R\left(k^{LF}\right)\right) - L^{LF}}{H\left(E^{LF}\right)}$$

where E^{LF} is described by

$$(15) \quad \omega\left(k^{LF}\right) h_E(E^{LF}, H\left(E^{LF}\right)) = R\left(k^{LF}\right).$$

and

$$(16) \quad H\left(E^{LF}\right) = h(E^{LF}, H^{LF}).$$

Formally, a LF steady state is a 3-tuple fixed point $\left(k^{LF}, L^{LF}, H^{LF}\right)$ to equations (14)-(16); recall, in equilibrium, $E^{LF} = L^{LF}$. We assume the existence and uniqueness of such a fixed point.[16]

For future reference, using (19), the lifetime utility associated with the LF steady state is $V^{LF} \equiv V\left(\omega^{LF} H^{LF} - R^{LF} L^{LF}, R^{LF}\right)$. A policy is said to satisfy the *intergenerational Pareto criterion* if indirect utility under the policy delivers at least as much utility as V^{LF}.

5.2 Laissez faire package equilibrium

We move on to descrive government intervention in this economy. Our goal is to describe a policy package that will exactly replicate the LF steady state, hence ensuring every generation living under the policy has utility, V^{LF}. The policy transition we will study will be a perturbation around the policy package steady state.

The government offers public education, $G_t \geq 0$, to the young (as a possible top-up on their own education spending, L_t). Following Buiter and Kletzer (1995) and many others, the two types of educational inputs, private and public, are assumed to be perfect substitutes in (3): $E_{t-1} \equiv L_{t-1} + G_{t-1}$.[17] Note, when $L_{t-1} > 0$, increases in public education crowds out private education one-for-one, i.e.,

$$(17) \quad \frac{\partial L_{t-1}}{\partial G_{t-1}} = -1 \text{ for } L_{t-1} > 0.$$

[16]Our description of the laissez faire economy is virtually identical to those described in Del Rey and Lopez-Garcia (2013) and Bishnu (2013). Their arguments for existence and uniqueness of a steady state, suitably modified, also apply here.

[17]Since G and L are both educational inputs, they can, in general, be complements or substitutes. It is hardly a challenge to argue for public spending if it acts as a complement to private spending.

It follows, there exists a path of G_t, high enough, such that private education expenses are optimally driven to the zero corner. In the subsequent analysis, starting Section 5.3, $E_t = G \Rightarrow L_t = 0 \; \forall t$.[18]

We posit public education is financed by selling riskless, one-period government bonds (B) at market rates: the gross return on bonds between t and $t+1$ is R_{t+1}, the same as the return to physical capital (or saving, in general). The government is not allowed to save. The government also imposes a lump-sum tax $(\mathcal{T}_t)$ on the middle-aged but not on the old. Let B_t be the number of one-period bonds issued at the start of period t. Then, the government budget constraint evolves as

$$(18) \quad B_t = R_t B_{t-1} + G - \mathcal{T}_t$$

where $R_t B_{t-1}$ is the interest cost on past debt, (B_{t-1}). Taxes on the middle-aged $(\mathcal{T}_t)$ as well as the sale of new debt (also to the middle-aged) finances the cost of debt service to their parents' generation and public education for their childrens'. Below, the path of taxes $\mathcal{T}_t$ will be constrained by the Pareto criterion.[19]

For future use, the indirect utility for a generation-t agent under the government's package is given by

$$(19) \quad V(Y_t, R_{t+1}) \equiv \begin{cases} \underset{c_{m,t}, c_{o,t+1}}{\arg\max} \; [U(c_{m,t}) + \beta U(c_{o,t+1})] \\ \text{s.t.} \;\; c_{m,t} + \frac{c_{o,t+1}}{R_{t+1}} = Y_t \end{cases} .$$

where Y_t is defined as $Y_t \equiv \omega_t H_t - \mathcal{T}_t - R_t L_{t-1}$, $S_t = [K_{t+1} + B_t]$, $c_{m,t} = \omega_t H_t - R_t L_{t-1} - \mathcal{T}_t - S_t$, and $c_{o,t+1} = R_{t+1} S_t$. Since $V_R(\cdot) = U_{c_{m,t}} \frac{c_{o,t+1}}{R_{t+1}^2} > 0$ and $V_Y(\cdot) = U_{c_{m,t}} > 0$, indirect utility is increasing in life-time income, Y_t, and the rate of return, R_{t+1}.[20]

5.3 Replicating the LF steady state with the LF package steady state

In a LF steady state equilibrium, public education, taxes and debt are absent ($G_t = \mathcal{T}_t = B_t = 0$), and private education is at L^{LF}. We now replicate this equilibrium with one that has government intervention – call the latter, a **LF package** equilibrium.

In a LF package equilibrium, henceforth denoted with a "*", noting private and public spending

[18] In effect, this removes education loans from the economy. In Wigger (2005), public debt crowds out physical capital and reduces future wage rates (and raises interest rates) and thereby, may reduce the *private* incentive to invest in human capital. In a sense, we go to the extreme by eliminating any incentive for private education spending.

[19] Unlike Diamond (1965) or Chalk (2000), here, as in Greiner (2008), government expenditure is productive in the sense it contributes to human capital formation. Also, unlike much of the literature studying debt dynamics – see Chalk (2000) – the government has no target level of debt-to-GDP ratio in mind. If debt is unproductive, then as discussed in Diamond (1965), under dynamic efficiency, debt benefits only the current generation and hurts all future generations. The fact that debt is productive in our setting is, hence, crucial for satisfying the intergenerational Pareto criterion.

[20] If the tax, instead, was distortionary, say a fraction τ_t of income, $\omega_t H_t$, then via (7) and (8) it is clear that both the education spending decision – hence, the tax base – and the saving decision would be distorted. Presumably, under standard preferences, a higher tax on labor income would depress education spending via the substitution effect, and reduce saving via the income effect. This analytical tougher case will be studied in the numerical section below.

on education are perfect substitutes, there exists a level of public spending (G) that drives private spending to zero yet total spending on education remains at its LF level, L^{LF}, i.e., $G = E^{LF} = L^{LF}$. In a steady state, in any period, the government floats debt $B^* = L^{LF}$ and uses it to finance education $G = L^{LF}$. This debt is held by the middle-aged. In the following period, the government pays $R^{LF}L^{LF}$ as interest payment on that debt to the *then* old by taxing the *then* middle-aged an amount equal to $R^{LF}L^{LF}$ (the same amount they would have paid under private loan servicing in the LF world). This leaves everyone at the same utility level as in the LF steady state, except the government has perfectly swapped private spending, L^{LF}, with public spending G, and private loan repayment $\left(R^{LF}L^{LF}\right)$ with a lump-sum tax, $\mathcal{T}^* = R^{LF}L^{LF}$. Except for the reclassification of bonds, from private loans to public bonds, everything is the same. Henceforth, call the steady state that replicates the LF steady state, $G = L^{LF}$, $B^* = L^{LF}$ and $\mathcal{T}^* = R^*L^{LF}$, the '**LF package** steady state or LFPSS for short. Specifically,

middle-age LF steady state: $c_m = \omega^{LF}H^{LF} - R^{LF}L^{LF} - [K^{LF} + L^{LF}]$,
old-age LF steady state: $c_o = R^{LF}\left(K^{LF} + L^{LF}\right)$
lifetime: $c_m + \dfrac{c_o}{R^{LF}} = \omega^{LF}H^{LF} - R^{LF}L^{LF}$

middle-age LFPSS: $c_m = \omega^*H^* - [K^* + B^*] - \mathcal{T}^*$,
old-age LFPSS: $c_o = R^*[K^* + B^*]$
lifetime: $c_m + \dfrac{c_o}{R^*} = \omega^*H^* - \mathcal{T}^*$

where $S^{LF} \equiv [K^{LF} + L^{LF}] = S^* \equiv [K^* + B^*]$, $K^{LF} = K^*$, $L^{LF} = B^*$, and $\mathcal{T}^* = R^*G = R^{LF}L^{LF}$. The lemma below formalizes this idea.[21]

Lemma 1 *The LF steady state with private education level, L^{LF}, and return, R^{LF}, is equivalent to a LF package steady state with zero private education spending ($L = 0$) and publicly-funded education $G = L^{LF}$, debt $B^* = G$ and a tax $\mathcal{T}^* = R^*G$ levied on the middle-aged such that*

(20) $$V^* \equiv V\left(\omega^*H^* - \mathcal{T}^*, R^*\right) = V^{LF} \equiv V\left(\omega^{LF}H^{LF} - R^{LF}L^{LF}, R^{LF}\right).$$

The proof follows by noting that under the policy, the budget set of agents and all relative prices are unchanged, and hence all decisions and the equilibrium allocation is the same. Note, this result is stated for a steady state equilibrium. It does *not* hold outside of steady state because debt crowds out capital at the initial date causing the interest rate to change. Specifically, at the initial date, $c_{m,t} = \omega_t^*H_t^* - R_t^*L_{t-1}^* - S_t, S_t = K_{t+1} + B_t, c_{o,t+1} = R_{t+1}K_{t+1} + R_{t+1}B_t \neq R_{t+1}^*S_t$. This issue will feature prominently below. Importantly, for us, factor prices in the LFPSS are the same as they are in the LF

[21] This lemma is, at its heart, a variant of the equivalence between debt and lump-sum taxes shown in de la Croix and Michel (2002; Ch. 4).

steady state; moreover, lifetime utility across the two are identical.

Assumption 3 $R_k(\cdot)\left[S_R(\cdot) - S_Y(\cdot) S^*\right] \frac{1}{H^*} < 1$

For future use, the following lemma provides a condition under which the LF package steady-state is locally stable.

Lemma 2 *The LFPSS is locally stable provided Assumption 3 holds.*
Proof. *See Appendix A.* ■

Thus far, we have set up our version of a standard Diamond (1965) economy augmented to include a human capital accumulation process subject to an intergenerational externality. This is the laissez faire economy, a world with complete markets. Therein, private spending on education is too low. We replicated that economy, in steady state, with one involving government intervention where education spending is inefficiently low at L^* – see eq. (7) – even though it is fully nationalized but still delivers V^*; the expenditure is paid for by debt and taxes. It is apparent that a marginal increase in government spending over L^* can improve long-run welfare. But can it be done in a Pareto-improving manner?

In Section 6 below, the idea is for the government to intervene in the LFPSS – this is where the time index t ends and is replaced by j. At $j = 0$, the government ushers in a permanent level of public education spending, call it G_n, where $G_n > L^*$ and continues to fully crowd out private education spending for all j thereafter.[22] The question is, can G_n be implemented so that no transitional generation will have welfare less than V^*?

6 Introducing debt-financed, public education under the Pareto criterion

We start with the LFPSS characterized by government policy, $G = L^*$, $B^* = G$ and $\mathcal{T}^* = R^* G$. At $j = 0$, a new level of G – call it G_n – is introduced where $G_n = L^* + \widetilde{G}$, and $\widetilde{G} > 0$ is a "small" increase over L^*. (Hereon, the subscript n is added to emphasize the "new" government program). Assume the new program is entirely unanticipated which means, at $j = 0$, all state variables are at their LFPSS levels. We ask whether G_n can be implemented under the Pareto criterion. The exact steps of the argument, here on, are as follows:

1. suppose the economy is at LFPSS with government spending, $G = L^*$, and associated indirect utility, V^*

2. at $j = 0$, increase government expenses to G_n permanently (for all $j \geq 0$) where $G_n = L^* + \widetilde{G}$

[22] What if the government intervenes at some point along the transition to the LF equilibrium, a point associated with inefficient private spending? Lemma 1 is no longer useful in that case. We take this issue up in numerical examples below.

3. find the sequence of minimum-required lump-sum taxes $\{\mathcal{T}_{n,j}\}_{j=0}^{\infty}$ that leave all generations as well off as V^*
4. show that the associated path of government debt, $B_{n,j}$, does not explode.

It is instructive to lay bare the action in the first two periods after the policy initiation. To that end, notice, for the inaugural middle-aged at $j = 0$, $E_{t-1} = L^* = G$ and $H_0 = H^*$ (their human capital is at its LFPSS level). At $j = 0$, the government's expense on education becomes $G_n > G$. The direct beneficiaries of this extra spending are the young at $j = 0$; since $E = G_n > G$, these initial young will see income gains at $j = 1$ via h_E. But, as yet, the externality has not kicked in – their parents' human capital is still at H^* and so the human capital of the middle-aged at date $j = 1$ (those young at $j = 0$) is $H_{n,1} = h(G_n, H_0) = h(G_n, H^*) > h(L^*, H^*)$.

What of the initial middle-aged? These agents, schooled under LFPSS, do not see any direct benefit from the government's new program at $j = 0$. They hold debt of amount $B_{n,0} = G_n$, more debt compared to B^*. This crowds out their capital holdings, causing R to increase (from R^* to $R_{n,1}$) where $R_{n,1}$ is the gross real return on bonds between $j = 0$ and $j = 1$. (This is formally show in Section 6.2 below.) Specifically, under the new policy, consumptions are given by

$$
\begin{aligned}
&\text{middle-age at } j = 0: \quad c_{m,0} = \omega^* H^* - \mathcal{T}_{n,0} - \left[K_{n,1} + \underbrace{B_{n,0}}_{B^*+\widetilde{G}} \right], \\
&\text{old-age at } j = 1: \quad c_{o,1} = R_{n,1}\left[K_{n,1} + B^*\right] + R_{n,1}\widetilde{G} \\
&c_{m,0} + \frac{c_{o,1}}{R_{n,1}} = \omega^* H^* - \mathcal{T}_{n,0}.
\end{aligned}
$$

Had life under the LFPSS continued, the consumptions would have been defined as in Section 5.3:

$$
\begin{aligned}
&\text{middle-age at } j = 0: \quad c_{m,0} = \omega^* H^* - \mathcal{T}^* - \left[K^* + B^*\right], \\
&\text{old-age at } j = 1: \quad c_{o,1} = R^*\left[K^* + B^*\right] \\
&c_{m,0} + \frac{c_{o,1}}{R^*} = \omega^* H^* - \mathcal{T}^*
\end{aligned}
$$

Suppose, for an instant, S does not respond to R and R does not respond to capital, as in, say, a small open economy. Since $B_{n,0} > B^* \Longrightarrow K_{n,1} < K^*$ with one-for-one crowding out, it is clear the new policy would leave the inaugural middle-aged at utility V^*. More generally, though, when $S_R(R)$ and $R_k(k)$ are not equal to zero, the extra debt, $\widetilde{G}$, will partially crowd out capital $(K_{n,1} < K^*)$ which will change R from R^* to $R_{n,1}$ and S from $K^* + B^*$ to $K_{n,1} + B_{n,0}$. On net, it is not clear if the inaugural middle-aged at $j = 0$ are better off relative to V^* or not. Below, we will show they indeed are. This means the welfare gain accruable to the inaugural middle-aged at $j = 0$ may be taxed away (by changing $\mathcal{T}_{n,0}$ relative to $\mathcal{T}^*$) so as to leave them at V^*, and hence, indifferent to the new policy. If the tax on them is enough to retire the extra debt completely, the story ends *within* a generation. Otherwise, as we study in Section 6.2, new debt would have to be issued and held by the middle-aged at $j = 1$, and intergenerational debt dynamics would be unleashed for $j > 1$. Every generation $j > 1$ benefits (relative to life in the LFPSS) directly from the new program and *also* indirectly, via the intergenerational externality, from having more educated parents. (As before, indirect, general-equilibrium gains from changes in the rate of return may also arise.) Some of these benefits can be taxed away $\left(\mathcal{T}_{n,j}\right)$ to help bring down the debt yet leave no generation with welfare less than V^* – the Pareto criterion. The big question is, can this be done? Is it consistent with the debt not exploding?

6.1 Path of taxes, debt and capital-ratio

We exploit the equivalence in eq. (20) – in Lemma 1 – which equates the utility level in the LF steady state with that achieved under the LFPSS. Note that $\omega_{n,0} = \omega^*$ since the capital-ratio at start of $j = 0$ is predetermined at its LFPSS value. The entire path of taxes $\left\{\mathcal{T}_{n,j}\right\}_{j=0}^{\infty}$ is thus determined by the Pareto "equal payoff" conditions

$$V\left(\omega_{n,0}\underbrace{h\left(G, H_{t-1}\right)}_{H^*}-\mathcal{T}_{n,0},\ R_{n,1}\right) = V\left(\omega^*\underbrace{h\left(G, H_{t-1}\right)}_{H^*}-\mathcal{T}_{n,0},\ R_{n,1}\right) = V^* \text{ for } j=0$$

$$V\left(\omega_{n,1}\ \underbrace{h\left(G_n, H^*\right)}_{H_{n.1}}-\mathcal{T}_{n,1},\ R_{n,2}\right) = V^* \text{ for } j=1$$

$$V\left(\omega_{n,j}\ \underbrace{h\left(G_n, H_{j-1}\right)}_{H_{n,j}}-\mathcal{T}_{n,j},\ R_{n,j+1}\right) = V^* \text{ for } j>1$$

which, for compactness sake, is written as

(21) $V\left(\omega^* H^* - \mathcal{T}_{n,0},\ R_{n,1}\right) = V^* \text{ for } j = 0$

(22) $V\left(\omega_{n,j}\, h\left(G_n, H_{n,j-1}\right) - \mathcal{T}_{n,j},\ R_{n,j+1}\right) = V^* \text{ for } j > 0,$

where the condition (21) on $\mathcal{T}_{n,0}$ gives the initial condition for the tax on the inaugural middle-aged, and (22) gives the dynamic path for future taxes, $\mathcal{T}_{n,j}$, that must satisfy the Pareto-criterion for all subsequent generations. Along with (18), (3), (13) and (12), eqns. (21)-(22) describe the entire joint dynamics of taxes and debt in this economy. The resulting system of difference equations is multidimensional, non-linear, and analytically daunting in general.

Define a variable

$$\widetilde{X}_{n,j} \equiv X_{n,j} - X^*$$

as the *deviation* in X under the government's "new" program relative to its value in the LFPSS, X^*. In general, an "$\sim$" indicates variables are measured in deviations from the LFPSS. By taking a first-order Taylor-approximation of the dynamical system – eqs. (21)-(22) – around the LFPSS, we get

$$V\left(\omega^* H^* - \mathcal{T}_{n,o}, R_{n,1}\right) \simeq V^* - V_Y(\cdot)\widetilde{\mathcal{T}}_{n,0} + V_R(\cdot)\widetilde{R}_{n,1}.$$

Define

$$\text{(23)} \quad \phi_j \equiv \phi\left(Y_j, R_{j+1}\right) \equiv \frac{V_R(Y_j, R_{j+1})}{V_Y(Y_j, R_{j+1})} = \frac{c_{o,n,j+1}}{R^2_{n,j+1}} > 0.$$

Using (6), we have

$$\text{(24)} \quad \phi_j = \frac{S_{n,j}}{R_{n,j+1}}.$$

Also $\phi^* \equiv \frac{S^*}{R^*}$. We can analytically show (see Appendix B)

$$\begin{aligned} \text{(25)} \quad \widetilde{\mathcal{T}}_{n,0} &= \phi^* \widetilde{R}_{n,1} \text{ for } j = 0 \\ \text{(26)} \quad \widetilde{\mathcal{T}}_{n,j} &= H^* \widetilde{\omega}_j + \omega^* \widetilde{H}_{n,j} + \phi^* \widetilde{R}_{n,j+1} \gtreqless 0 \text{ for } j > 0 \end{aligned}$$

Recall, $\widetilde{\mathcal{T}}_{n,j} \equiv \mathcal{T}_{n,j} - \mathcal{T}^*$ refers to the deviation in taxes (or subsidies) under the government's "new" program relative to its value in the LFPSS, $\mathcal{T}^*$. If the policy on impact causes the return, R, to increase (eq. 25), then the inaugural middle-aged (as savers) are made better-off, and the equal pay-off condition implies they can be taxed an amount equal to the value of the change in the return. For subsequent periods, the tax deviation to be paid under the Pareto-criterion, $\widetilde{\mathcal{T}}_{n,j}$, reflects the direct gain from more human capital $\left(\omega^* \widetilde{H}_{n,j} > 0\right)$, the indirect effect from a change in the wage rate $\left(H^* \widetilde{\omega}_{n,j}\right)$ brought about by a change in the capital-ratio, and possible changes in the rate of return $\left(\phi^* \widetilde{R}_{n,j+1}\right)$. Each of these general-equilibrium components bring about changes in welfare. As for the wage and return effects, we are unsure, as yet, as to whether they help or hurt. Once the effect on the capital stock is clarified, these will be pinned down. The point is that if the government's policy generates *net* welfare gains then these can be taxed under the Pareto criterion to bring down the debt.

The path of debt follows

$$\begin{aligned} B_{n,0} &= R^* B^* + G_n - \mathcal{T}_{n,0} \text{ for } j = 0 \\ B_{n,1} &= R_{n,2} B_{n,0} + G_n - \mathcal{T}_{n,1} \text{ for } j = 1 \\ B_{n,j} &= R_{n,j} B_{n,j-1} + G_n - \mathcal{T}_{n,j} \text{ for } j > 1 \end{aligned}$$

Since $B^* = R^* B^* + G - \mathcal{T}^*$ (from the LFPSS), we have $\widetilde{B}_{n,0} \equiv B_{n,0} - B^* = G_n - G - (\mathcal{T}_{n,0} - \mathcal{T}^*) = \widetilde{G} - \widetilde{\mathcal{T}}_{n,0}$; this means, relative to the LFPSS, an extra spending of $\widetilde{G}$ may be financed by sale of new bonds $\left(\widetilde{B}_{n,0}\right)$ and new taxes $\left(\widetilde{\mathcal{T}}_{n,0}\right)$ relative to their levels in LFPSS. What, then, does the path of Pareto-neutral tax deviations – eq. (25)-(26) – imply for the associated debt-deviation dynamics? The entire path of public debt deviations may be written as (see Appendix B):

(27) $\widetilde{B}_{n,0} = \widetilde{G} + \varphi_1 \widetilde{k}_{n,1}$ for $j = 0$

(28) $\widetilde{B}_{n,j} = R^* \widetilde{B}_{n,j-1} + \varphi_1 \widetilde{k}_{n,j+1} - \varphi_2 \widetilde{k}_{n,j} - \varphi_3 \widetilde{H}_{n,j} + \widetilde{G}$ for $j > 0$

where

(29) $\varphi_1 \equiv -\phi^* R_k > 0;\ \varphi_2 \equiv -S^* R_k > 0;\ \varphi_3 \equiv \omega^* > 0.$

First, note that for all j, φ_1, φ_2, and φ_3 are fixed parameters because they are evaluated at their LFPSS values. Next, note that the different signs for the coefficients of $\widetilde{k}_{n,j+1}$ and $\widetilde{k}_{n,j}$ refer to the different roles played by the rate of return and the wage rate. A higher capital-ratio lowers the rate of return, and thus requires a lower tax; a higher capital-ratio increases the wage requiring the tax to increase.

Turn now to the dynamics of saving. Notice factor returns (ω, R) appearing in (21)-(22), are endogenous, depending both on the level of education offered and taxes. The key issue, then, becomes the endogenous determination of saving, and thus the capital ratio, via the previously-derived optimality conditions:

(30) $S_{n,0} : U_{c_m}(\omega^* H^* - R^* L^* - \mathcal{T}_{n,0} - S_{n,0}) = \beta R_{n,1} U_{c_o}(R_{n,1} S_{n,0})$ for $j = 0$

(31) $S_{n,j} : U_{c_m}\left(\omega_{n,j}\, h\left(G_n, H_{n,j-1}\right) - \mathcal{T}_{n,j} - S_{n,j}\right) = \beta R_{n,j+1} U_{c_o}(R_{n,j+1} S_{n,j})$ for $j > 0$

where

(32) $k_{n,1} = \dfrac{K_{n,1}}{H^*} = \dfrac{S_{n,0} - B_{n,0}}{H^*}$ for $j = 0$ and $k_{n,j+1} = \dfrac{K_{n,j+1}}{h\left(G_n, H_{n,j-1}\right)} = \dfrac{S_{n,j} - B_{n,j}}{h\left(G_n, H_{n,j-1}\right)}$ for $j > 0$.

Taking a first-order Taylor-approximation around the LFPSS, we can show (see Appendix B), for $j = 0$,´

(33) $\widetilde{k}_{n,1} = -\kappa_1\, \widetilde{B}_{n,0} - \kappa_3 \widetilde{G}$ where

(34) $\kappa_1 \equiv \dfrac{\beta \left(R^*\right)^2 U_{c_o c_o}}{\varkappa} > 0$

(35) $\kappa_3 \equiv \dfrac{\left\{U_{c_m c_m} + \left[U_{c_m c_m} + \beta \left(R^*\right)^2 U_{c_o c_o}\right] k^* h_E(\cdot)\right\}}{\varkappa} > 0$

(36) $\varkappa \equiv \left\{\left(U_{c_m c_m} + \beta \left(R^*\right)^2 U_{c_o c_o}\right) H^* + \beta \left[U_{c_o} + S^* R^* U_{c_o c_o}\right] R_k\right\} < 0$

where $\varkappa < 0$ using Assumption (2). Further down the transition, the path of k-deviations is given by

$$(37)\quad \widetilde{k}_{n,j+1} = \kappa_0 \widetilde{k}_{n,j} - \kappa_1 \widetilde{B}_{n,j} - \kappa_2 \widetilde{B}_{n,j-1} - \kappa_3 \widetilde{G} + \kappa_4 \widetilde{H}_{n,j}$$

where κ_1 and κ_3 are defined in (34)-(35) and

$$\begin{aligned} \kappa_0 &\equiv \frac{-U_{c_m c_m} R_k S^*}{\varkappa} > 0,\ \kappa_2 \equiv \frac{R^* U_{c_m c_m}}{\varkappa} > 0 \\ \kappa_4 &\equiv \frac{\omega^* U_{c_m c_m} - \left[U_{c_m c_m} + \beta (R^*)^2 U_{c_o c_o}\right] k^* h_H(\cdot)}{\varkappa} = \frac{\omega^* S_Y(\cdot) - k^* h_H(\cdot)}{H^* - S_R(\cdot) R_k(\cdot)} \lesseqgtr 0. \end{aligned}$$

Eq. (37) is the equilibrium linearized law of motion involving k and B deviations for all $j > 0$. The same for $j = 0$ is given by (33). Eqs. (27)-(28) is another set of equilibrium linearized laws of motion involving k and B deviations (the latter subsumes the tax dynamics). They are each derived from the equal payoff conditions governed by the Pareto criterion. Together, they describe the capital-ratio and debt deviation dynamics in this economy. We study these in detail further below.

6.2 Impact effects

It follows from (27) and (33) that

$$(38)\quad \widetilde{k}_{n,1} = \Gamma_G \widetilde{G}; \quad \Gamma_G \equiv -\frac{\kappa_3 + \kappa_1}{1 + \varphi_1 \kappa_1} < 0.$$

Since $\varphi_1 > 0, \kappa_1 > 0$,and $\kappa_3 > 0$, it follows $\Gamma_G < 0 \Rightarrow \widetilde{k}_{n,1} < 0$ – the policy, on impact, induces a decrease in the capital stock in period 1. This sort of crowding-out happens via two channels. First, ceteris paribus, the higher educational spending increases human capital which reduces the capital ratio (and hence, wages). Second, the tax payment reduces income, and therefore, saving. The net result of all this is a higher return on capital, which the initial middle-aged benefit from.[23] As a result, taxes go up on impact: $\widetilde{\mathcal{T}}_{n,0} = \phi^* \widetilde{R}_{n,1} = \frac{S^*}{R^*} \widetilde{R}_{n,1} = \frac{S^*}{R^*} R_k \Gamma_G \widetilde{G} > 0$: the initial middle-aged, who do not directly benefit from the higher educational spending, can be taxed under the Pareto criterion.

The question is, is the taxation capacity of the initial middle-aged able to finance the entire extra education spending, i.e., is $\widetilde{\mathcal{T}}_{n,0} = \widetilde{G}$ possible? That is, is the aforediscussed return-benefit enough to tax and finance $\widetilde{G}$ *and* leave the inaugural middle-aged as happy as in the LF? If that were possible, $\widetilde{B}_{n,0} = \widetilde{G} - \widetilde{\mathcal{T}}_{n,0}$ would imply no change in debt relative to B^*. This means the problem at hand would

[23] The general-equilibrium return effect that permits raising taxes to finance the higher educational level is important because it helps the middle-aged and hence they can be taxed more. The return effect is often used in work that attempts to generate political support for welfare programs that benefit the old. Poutvaara (2003) and Köthenbürger and Poutvaara (2006) argue it can be in the interest of the old to finance activities for the young since it may indirectly benefit the old via changes in asset prices or the tax base. Note an important difference with our analysis though; this line of work, in effect, uses the return effect to show how, in a political equilibrium, short-term gains (via return effects, say) can help support policies that are *welfare-reducing* in the long run. Here, we present the opposite perspective; we ask whether short-term gains – a higher interest rate – can help usher in policies that come with long-run gains.

be limited to the one initial generation; there would not arise any *intergenerational* conflict. This happens if

$$(39) \quad \chi_G \equiv \frac{1-\varphi_1\kappa_3}{1+\varphi_1\kappa_1} < 0.$$

As our paper is largely focused on intergenerational conflict and the role of debt in amelioration of the conflict, we proceed by assuming the *opposite* of (39) , the more anaytically-challenging case where taxes (under the Pareto criterion) in the initial period alone are insuffficent to pay for $\widetilde{G}$. This means, the government runs a deficit in the inaugural period and public debt *increases* on impact at the first date $\left(\widetilde{\mathcal{T}}_{n,0} < \widetilde{G} \Leftrightarrow \widetilde{B}_{n,0} > 0\right)$. The presence of intergenerational conflict, therefore, arises under the following assumption.

Assumption 4 $\chi_G \equiv \frac{1-\varphi_1\kappa_3}{1+\varphi_1\kappa_1} > 0.$

When does that happen? Consider the impact effect on the public debt in period t. It follows from (27) and (33) that

$$(40) \quad \widetilde{B}_{n,0} = \chi_G \widetilde{G}; \quad \chi_G \equiv \frac{1-\varphi_1\kappa_3}{1+\varphi_1\kappa_1}; \ \varphi_1 > 0, \kappa_1 > 0, \kappa_3 > 0$$

where χ_G – how the initial debt deviation responds to a marginal change in G – is of ambiguous sign. If $\chi_G > 0$ holds, it follows that $\widetilde{B}_{n,0} > 0$ or equivalently, $\widetilde{\mathcal{T}}_{n,0} < \widetilde{G}$.

Consider the taxation capacity of the initial middle-aged. On impact, more education decreases the capital ratio, which in turn raises the return to capital which benefits the middle-aged once they turn old raising their taxation capacity. Clearly, the more they can be taxed, the less is the impact effect on debt. How much they can be taxed depends on three factors: i) the size of the existing human capital stock which partly determines how much the capital-ratio reacts to a higher G, ii) how strongly the rate of return responds to a change in the capital ratio, and iii) by how much would agents have to be compensated (or taxed) for a given change in the rate of return. All of this is captured in the expression $\varphi_1\kappa_3$. How much the return is affected is measured by κ_3 and by how much the taxation capacity is affected by the return is measured by φ_1.

For debt to increase on impact, we need $\chi_G > 0 \Leftrightarrow \varphi_1\kappa_3 < 1$. Focus, for now on φ_1 where, recall, $\varphi_1 \equiv -\phi^* R_k = -\frac{S^*}{R^*} R_k$. Immediately, the following is clear: were $R_k = 0$, as in, say, a small open economy, then $\widetilde{B}_{n,0} = \widetilde{G}$ and $\widetilde{\mathcal{T}}_{n,0} = 0$ meaning debt or taxes would not budge on impact: $\widetilde{G} > 0$ cannot be funded. This suggests that the return effect is necessary to generate a rise in taxation capacity – but not too much – so that debt rises on impact. This requires the return effect to not be too weak or too strong.

Using S_Y and S_R as defined in (11) and (9), one can check

$$\varphi_1\kappa_3 < 1 \Longleftrightarrow \underbrace{1 + \frac{S^*}{R^*}R_k\frac{S_Y}{H^* - S_R R_k}}_{>0} + \frac{\frac{S^*}{R^*}k^*}{H^* - S_R R_k}\left[R_k h_E(\cdot)\right] > 0$$

where the underscored term is positive via Assumption 3 (stability). Notice the importance of the last term on the r.h.s in square parentheses: both R_k (as we just saw) and $h_E(\cdot)$ are important. The upshot is that intergenerational conflict can only arise when debt increases on policy initiation, and for that to happen, both the return effect and the human capital externality are crucial.

7 Debt dynamics under the Pareto criterion

The underlying $j > 0$ dynamics of the economy in a neighborhood of the LFPSS is entirely captured by (33) and (37) and (27)-(28), a two-dimensional, first-order linear dynamical system with a time-evolving forcing term involving human capital:

$$\begin{array}{r} \widetilde{B}_{n,j} = R^*\widetilde{B}_{n,j-1} + \varphi_1\widetilde{k}_{n,j+1} - \varphi_2\widetilde{k}_{n,j} - \varphi_3\widetilde{H}_{n,j} + \widetilde{G} \\ \widetilde{k}_{n,j+1} = \kappa_0\widetilde{k}_{n,j} - \kappa_1\widetilde{B}_{n,j} - \kappa_2\widetilde{B}_{n,j-1} - \kappa_3\widetilde{G} + \kappa_4\widetilde{H}_{n,j} \end{array} \quad \forall j > 0$$

Additionally, there are two initial conditions for $j = 0$ given by

$$\begin{bmatrix} \widetilde{B}_{n,0} \\ \widetilde{k}_{n,1} \end{bmatrix} = \begin{bmatrix} \chi_G\widetilde{G} \\ \Gamma_G\,\widetilde{G} \end{bmatrix};\ \Gamma_G < 0, \chi_G > 0 \text{ (by Assumption 4)}$$

which are pinned down by the model, i.e., not imposed exogenously. In matrix form, the dynamical system can be time-updated and compactly summarized as

$$(41)\quad \begin{bmatrix} \widetilde{B}_{n,j+1} \\ \widetilde{k}_{n,j+2} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix}\begin{bmatrix} \widetilde{B}_{n,j} \\ \widetilde{k}_{n,j+1} \end{bmatrix} + \begin{bmatrix} A_3 & A_4 \\ B_3 & B_4 \end{bmatrix}\begin{bmatrix} \widetilde{G} \\ \widetilde{H}_{n,j+1} \end{bmatrix};\ j \geq 1$$

where

$$\begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix} \equiv \begin{bmatrix} \frac{R^*-\kappa_2\varphi_1}{1+\varphi_1\kappa_1} & \frac{-\varphi_2+\varphi_1\kappa_0}{1+\varphi_1\kappa_1} \\ \frac{-R^*\kappa_1-\kappa_2}{1+\varphi_1\kappa_1} & \frac{\varphi_2\kappa_1+\kappa_0}{1+\varphi_1\kappa_1} \end{bmatrix},\ \begin{bmatrix} A_3 & A_4 \\ B_3 & B_4 \end{bmatrix} \equiv \begin{bmatrix} \frac{1-\varphi_1\kappa_3}{1+\varphi_1\kappa_1} & \frac{-\varphi_3+\varphi_1\kappa_4}{1+\varphi_1\kappa_1} \\ \frac{-\kappa_1-\kappa_3}{1+\varphi_1\kappa_1} & \frac{\varphi_3\kappa_1+\kappa_4}{1+\varphi_1\kappa_1} \end{bmatrix}.$$

Note, the determinant

$$\det\begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix} = -\frac{1}{\kappa_1\varphi_1 + 1}\left(\kappa_2\varphi_2 - \kappa_0 R^*\right) = 0,$$

implying (41) is a system of linearly-dependent equations. The implication is, the dynamics of k fully mirror the dynamics of B; also, $\widetilde{B}_{n,j+1}$ does not involve $\widetilde{k}_{n,j+1}$ and $\widetilde{k}_{n,j+2}$ does not involve $\widetilde{B}_{n,j}$. That the dynamics depends on the education externality is apparent from A_4 and B_4.

We are now ready to state our *main* results. To that end, briefly recall what Assumptions 1-4 imply. Respectively, they guarantee existence of a LF steady state, saving increasing in rate of return, stability of the LF steady state, and that the debt level rises relative to its LF level upon policy impact.

Proposition 1 *Suppose Assumptions 1-4 hold. Consider an incremental increase in public spending on education beyond the LFPSS level in period $j = 0$. Under the Pareto criterion, the inaugural debt deviation (relative to its value at the LF) is $\widetilde{B}_{n,0} = \chi_G \widetilde{G} > 0$. In subsequent periods, the path of debt deviations evolves as*

(42) $\widetilde{B}_{n,j+1} = \psi_B \widetilde{B}_{n,j} + \psi_G \widetilde{G} + \psi_H \widetilde{H}_{n,j+1} \;\forall\, j \geq 1.$

where ψ_B, ψ_G and ψ_H are determined by LF quantities and involve properties of the utility, production, and human capital functions. Also,

$$\psi_B = R^*, \quad \psi_H < 0$$

and

$$\psi_G < 0$$

holds if $R^* \geq 1$ (dynamic efficiency).
Proof. *See Appendix D.* ■

Increasing public educational spending beyond the LF package level raises the tax bill on the initial middle-aged. They benefit because the interest on their past saving is higher due to the crowding out of capital. In subsequent periods, the issue is more nuanced. Future cohorts benefit directly from the higher education levels in terms of higher labor income induced by the externality. This goes in the direction of increasing the tax that can be levied on those generations ($j \geq 1$) ($\psi_H < 0$) helping to bring debt levels down. In addition, there may be lagged gains due to the education spending ($\psi_G < 0$).

To understand further, note from Appendix D, ψ_H depends crucially on h_H; if $h_H = 0$, then $\psi_H = 0$ and for ψ_H to be negative, it is necessary that $h_H > 0$. Consider what happens at the initial date if $h_H = 0$. Then (42) implies $\widetilde{B}_{n,1} = \psi_B \widetilde{B}_{n,0} + \psi_G \widetilde{G} \Leftrightarrow \widetilde{B}_{n,1} - \widetilde{B}_{n,0} = (\psi_B - 1)\, \widetilde{B}_{n,0} + \psi_G \widetilde{G} \Leftrightarrow \widetilde{B}_{n,1} - \widetilde{B}_{n,0} = \widetilde{G}\, [(\psi_B - 1)\, \chi_G + \psi_G]$. But $(\psi_B - 1)\, \chi_G + \psi_G = 0$ if $h_H = 0$. This means when the human capital externality is absent, $\widetilde{B}_{n,1} = \widetilde{B}_{n,0}$, and hence, debt levels do not budge. The implication is, absent the externality, debt levels cannot come down. This explains why even a small externality $(h_H > 0 \Leftrightarrow \psi_H < 0)$ may be sufficient to start the process of debt reduction as in (42).

The critical question is whether a debt explosion can be avoided if $R^* \geq 1$. It follows from (42) that

(43) $$\widetilde{B}_{n,j+2} - \widetilde{B}_{n,j+1} = R^* \left[\widetilde{B}_{n,j+1} - \widetilde{B}_{n,j}\right] + \psi_H \left[\widetilde{H}_{n,j+2} - \widetilde{H}_{n,j+1}\right], \; j > 0$$

Recall, two things: a) $\widetilde{B}_{n,j}$ refers to the deviation of the debt at date j from its value in the LFPSS, $B^* = L^*$, and b) $\widetilde{B}_{n,0} > 0$ implying the initial debt deviation is positive (which holds under Assumption 4). From (43), we see two immediate implications: i) in the absence of the education externality, i.e., if $h_H = 0 \Leftrightarrow \psi_H = 0$, it follows if the debt deviation does not shrink rightaway (i.e., between dates 0 and 1) it *never* will, ii) since human capital deviations are non-decreasing $\widetilde{H}_{n,j+2} - \widetilde{H}_{n,j+1} \geq 0$ for $h_H > 0$, then given $\psi_H < 0$ holds, the externality works in the direction of reducing future debt deviations (which are positive), i.e., $\widetilde{B}_{n,j+2} - \widetilde{B}_{n,j+1} < \psi_B \left(\widetilde{B}_{n,j+1} - \widetilde{B}_{n,j}\right)$. Hence, *if* $\left(\widetilde{B}_{n,j+1} - \widetilde{B}_{n,j}\right) < 0$, i.e., if there is a "turning point" in the debt deviation sequence, it is clear that $\widetilde{B}_{n,j+2} - \widetilde{B}_{n,j+1} < 0$ holds thereafter. The implication is that if the debt deviation is decreasing in any period j ($j \geq 2$), it will keep decreasing *forever* after. As we will see below, at some point, the debt deviation (not the debt) will hit zero (i.e., the debt level will coincide with B^*); thereafter, debt levels will fall further. This is also the point where the tax rate may be brought down (relative to its level determined by the Pareto criterion) allowing generations to come to experience welfare *improvements* over the LFPSS level.[24]

In Proposition 2, we establish conditions under which the debt deviation starts to fall right from the period after the policy was ushered in, i.e., $\widetilde{B}_{n,1} < \widetilde{B}_{n,0}$.which means if $\psi_G < (1 - \psi_B)\,\chi_G$ holds, debt deviations will start to fall immediately, right from the period after the policy was ushered in.

Proposition 2 *Public debt starts to fall one period after the initiation date, i.e.,* $\widetilde{B}_{n,1} < \widetilde{B}_{n,0}$*, iff*

(44) $$\psi_G < (1 - R^*)\,\chi_G$$

holds. Sufficient conditions for condition (44) to hold are $R^* > 1$ *and* $h_H > 0$.
Proof. *See Appendix E.* ■

The implication is that it is possible to implement an education level above the LFPSS level ($\widetilde{G} > 0$) under the Pareto criterion where the first-contributing generation pays a tax ($\widetilde{G} > \widetilde{\mathcal{T}}_{n,0} > 0$, $\widetilde{B}_{n,0} > 0$), and all subsequent generations are taxed so that the level of public debt can be brought down starting from the period after the policy was inaugurated (and forever after). Importantly, no cohorts are worse off (some are potentially better off) compared to their lifetime utilities under the LFPSS.

In the presence of an intergenerational education externality, possibly a weak one – all that (44) requires is $h_H > 0$ – enough welfare gains are generated by public education in a dynamically efficient ($R^* > 1$) economy. Needless to say, public debt does not *have* to fall right away. Proposition 2 simply notes a *sufficient* condition for which it does.

[24] The dynamics of debt beyond such a point are not studied in this paper. The government can, if it so wishes, decide to stop taxing agents under the Pareto criterion and switch to taxing them under a balanced budget constraint. Alternatively, if it didn't, as debt levels fall below B^*, capital holdings rise, causing R to fall, possibly below 1 – violating dynamic efficiency.

Proposition 3 *Under Assumptions 1-4 and the conditions specified in Proposition 2, the education policy ($\widetilde{G}$ > 0) with associated debt deviation paths $\widetilde{B}_{n,j}$ – eq. (42) – leaves some generations potentially better off and no generation worse off in utility terms relative to V^*, the utility under the laissez faire package steady state.*

As soon as debt levels have been brought below B^*, its level under the LFPSS, the government is raising more revenue than is needed to finance the education level $G > L^*$. From this point on, a Pareto improvement is possible since welfare gains from higher human capital levels linger and taxes can be reduced below the level dictated by the Pareto-criterion. It follows that all generations born after such a time may be *better off* than in the LF. [25]

8 Numerics

The analysis above has several limitations that have to do with concerns of analytical tractability. First, labor supply was exogenously specified; second, the perturbation of public education, $\widetilde{G}$, was incremental; and finally, the tax rate was lump-sum. It was also the case that the starting point of the policy was the LFPSS. Below, we sketch out the equations that would determine equilibrium paths under endogenous labor supply, Q, non-incremental change in G and a distorting tax rate on labor income. We then analyze this expanded economy using numerical techniques.

Assume all is same as before except agents now have a labor-leisure choice when middle-aged and are retired when old. Using otherwise the same notation as before, one can write lifetime utility in the LF for the cohort young in $t+j-1$ as

(45)
$$\Omega(\tau_{t+j}, \omega_{t+j}, R_{t+j}, R_{t+j+1}) = U\left(\left[1-\tau_{t+j}\right]\omega_{t+j}\, h\left(E_{t+j-1}, H_{t+j-1}\right) Q_{t+j} - R_{t+j}L_{t+j-1} - S_{t+j}, Q_{t+j}\right) + \beta U(R_{t+j+}$$

where

$$\begin{aligned} (46) \qquad c_{m,t+j} &= \left[1-\tau_{t+j}\right]\omega_{t+j}\, h\left(E_{t+j-1}, H_{t+j-1}\right) Q_{t+j} - R_{t+j}L_{t+j-1} - S_{t+j} \\ (47) \qquad c_{o,t+j+1} &= R_{t+j+1}S_{t+j} \end{aligned}$$

Given a pre-determined education decision $\left(L_{t+j-1}\right)$ from young age, the middle-aged choose labor supply and saving by maximizing (45) subject to (46)-(47). The first order conditions are:

$$\begin{aligned} (48) \qquad S_{t+j} &: \; -U_{c_m}\left(c_{m,t+j}, Q_{t+j}\right) + \beta R_{t+j+1} U_{c_o}\left(c_{o,t+j+1}\right) = 0 \\ (49) \qquad Q_{t+j} &: \; U_{c_m}\left(c_{m,t+j}, Q_{t+j}\right)\left[1-\tau_{t+j}\right]\omega_{t+j} h\left(E_{t+j-1}, H_{t+j-1}\right) + U_{Q_{t+j}}\left(c_{m,t+j}, Q_{t+j}\right) = 0 \end{aligned}$$

[25] It is not necessary that only generations born after the debt is eliminated be made better off. It may be possible to release some of these gains to earlier generations, thereby making *every* generation post policy strictly better off. We allow for this in the numerical example in Section 8.

The latter is a one-period decision. Notice optimal labor supply depends on the human capital investment made as young. The other one-period decision, educational decision as young is

$$\max_{L_{t+j-1}} \; U\left(\left[1-\tau_{t+j}\right]\omega_{t+j}\, h\left(E_{t+j-1}, H_{t+j-1}\right) Q_{t+j} - R_{t+j}L_{t+j-1} - S_{t+j}, L_{t+j}\right) + \beta U(R_{t+j+1}S_{t+j}),$$

given (48) and (49), is described by

$$\left[1-\tau_{t+j}\right]\omega_{t+j}\, h_E\left(E_{t+j-1}, H_{t+j-1}\right) Q_{t+j} - R_{t+j} = 0 \text{ if } Q_{t+j-1} > 0.$$

In implicit form, these optimality conditions may be summarized as

$$\begin{aligned} Q_{t+j-1} &= N(\tau_{t+j}, \omega_{t+j}, R_{t+j}, R_{t+j+1}) \\ L_{t+j} &= L(\tau_{t+j}, \omega_{t+j}, R_{t+j}, R_{t+j+1}) \\ S_{t+j} &= S(\tau_{t+j}, \omega_{t+j}, R_{t+j}, R_{t+j+1}) \end{aligned}$$

The capital to effective labor ratio evolves according as

$$k_{t+j+1} \equiv \frac{K_{t+j+1}}{H_{t+j+1}Q_{t+j}} = \frac{S_{t+j} - \left[B_{t+j} + L_{t+j}\right]}{H_{t+j+1}Q_{t+j}},$$

and the public debt is governed by

$$B_{n+j} + \tau_{n+j}\omega_{n+j}h\left(L_{n+j-1}, H_{n+j-1}\right) Q_{n+j} = R_{n+j}B_{nt+j-1} + G_{n+j}.$$

The idea is to introduce a G quite a bit bigger than the L chosen in the LF steady state (such that private agents set L to 0). Post intervention, we restrict utility for all current and future cohorts to be better than utility under the LF by a factor $(1+\kappa)$, $\kappa > 0$. That is, we pick the tax rate sequence, τ_{n+j}, to satisfy the following condition:

$$\Omega_{n+j}(\cdot) = (1+\kappa)\,\Omega^{LF}_{t+j}(\cdot)$$

If we find such a sequence and the associated path of B_{n+j} shows a turning point, then we will have found an example of a policy package that delivers strictly higher utility to all current and future cohorts.

8.1 Parameter choices

Ours is not a full-blown calibration exercise since the three-period overlapping-generations model we study is ill-suited for serious calibration. Nevertheless, we attempt to set up the parameters so that the baseline/LF model roughly captures some features of a economy like the U.S. If a period in the model is roughly 20 years of calendar time, then, assuming an annual (after-tax) gross interest rate of 1.03, our LF steady state generates $R \approx 1.8$ which is roughly the same as $(1.03)^{20}$. The capital-to-output ratio in the data is roughly 3; our LF steady state has it at 0.15. Since K is a stock and Y a flow, an appropriate measure of the capital-to-output ratio in the model is ..... In the LF, there is no public education. Estimates of private education spending to GDP for the U.S. are just under 2%; our LF steady state has it at ..... Estimates of public education spending to GDP for the U.S. are near 5%; our LF steady state has it at .....

We assume the production function to be Cobb-Douglas,

$$Y_{t+j} = AK_{t+j}^{1-\alpha}\left(Q_{t+j}H_{t+j}\right)^{\alpha}$$

and assume preferences of the CES form:

$$\Omega_{t+j}(c_{m,t+j}, Q_{t+j}, c_{o,t+j+1}) = \chi_c \frac{\left[c_{m,t+j}\right]^{1-\sigma} - 1}{1-\sigma} + \frac{1}{1-\gamma}\chi_l\left[1 - Q_{t+j}\right]^{1-\gamma} + \frac{1}{1+\delta}\chi_o \frac{\left[c_{o,t+j+1}\right]^{1-\sigma} - 1}{1-\sigma}.$$

The human capital function is given by

$$H_{t+j} = \chi_h\left(F + E_{t+j-1}\right)^{\epsilon} H_{t+j-1}^{\eta}.$$

The baseline parameters are:

$$\begin{aligned}
&A = 2.5;\ \alpha = \frac{2}{3};\ F = 0.001;\ \epsilon = 0.63;\ \eta = 0.3\\
&\gamma = 10;\ \sigma = 2;\ \delta = 2\\
&\chi_c = 1;\ \chi_l = 1;\ \chi_o = 1;\ \chi_h = 2
\end{aligned}$$

Our choice of δ implies a 30-year discount factor of $1/2$. Here α is the labor-share of output which roughly matches with 66% in U.S. data. $\sigma = 2$ is commonly used. The following scale parameters are shut down: $\chi_c = 1$; $\chi_l = 1$; $\chi_o = 1$. ϵ, η and F conform to parameter choices made in de la Croix and Doepke (1993) and Cavalcanti and Giannitsarou (2017); indeed our choice of F is lower than theirs so as to not bias the results in our favor. A is a scale parameter that was chosen to deliver $R \approx 1.6$. The value of γ was chosen to...

8.2 Results

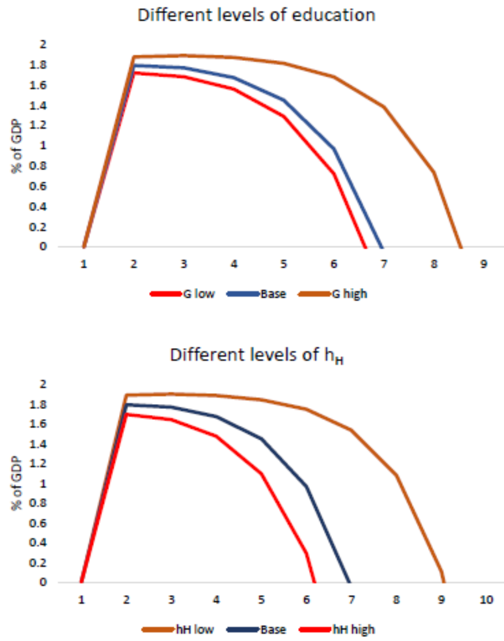

Figure 1: Debt-to-GDP ratio across time

In base case, the value of G is set at 2.5 times the level of education at the LF steady state. Clearly, this is not an incremental change from the LF. In the top panel of Figure 1, we also report on G-high (2.75 times) and G-low (2.25 times) the level of education at the LF steady state. In the LF steady state, debt is zero, $R \approx 1.9$. On impact, debt rises to about 1.8% of GDP; thereafter, it starts to fall, and is eliminated by seven periods. For G-high (G-low), the time to debt elimination is longer (shorter), that is, more ambitious packages require more debt at the inaugural stage and take longer to pay off. All along, though, no generation is hurt relative to the LF steady state – indeed, by construction every generation post policy is $\kappa = ...$ times better off.

How does the strength of the human capital externality influence the results? In the bottom panel of Figure 1, we report on three values of the externality, $\eta = 0.3$ (base), $\eta = 0.295$ (low) and $\eta = 0.305$ (high). In each case, the value of G is set at 2.5 times the level of education at the LF steady state, the same as in the baseline case above. The case with a strong externality requires a shorter time for debt to disappear.

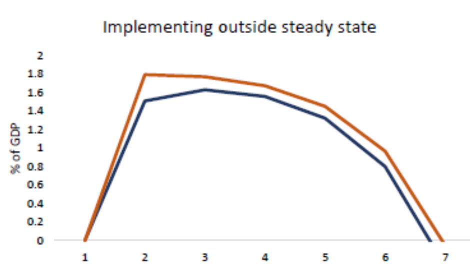

Figure 2: Policy reform away from the LF steady state

Finally, in Figure 2, we report on a policy reform that starts at a date well before the economy has reached the LF steady state. Before the policy is introduced, agents receive the same utility as one would along the LF transition. However, by construction, starting from the inaugural date of the policy, all agents receive the same utility as they would have in the LF steady state, a *much higher* level of utility than they were used to before the reform. As Figure 2 illustrates, not much additional insight is generated except that trying to generate long-run utility levels in the short-run is ambitious, and as such, debt lingers on a bit longer.

9 Concluding remarks

In this paper, we study a government policy that internalizes an intergenerational human capital externality and attempts to introduce public education spending at levels above laissez faire. The government uses a mix of public debt and lump-sum taxes on the working generation (those who stand to gain directly from the government's program) to help finance its education policy. Public education generates long-run income gains by increasing the stock of accumulated human capital via the education externality. Other general-equilibrium welfare gains arise from the crowding out of capital by debt. These produce additional changes to income at various points in the lifecycle via changes in factor prices. These gains are taxable and may be used to bring down the debt. Whether (and how) this is possible is technically challenging given the various, confounding general-equilibrium effects. Most strikingly, we are able to establish that it is possible to implement the government's policy under a generational Pareto criterion, meaning no generation in the transition is hurt relative to the no-policy world. This holds in a dynamically efficient economy under the sufficient condition that there is a human capital externality even when private and public education are perfect substitutes.

It bears mention here that while the entire analysis was conducted using debt and lump-sum taxes on the middle-aged, it is quite possible that other instrument combinations (such as pensions – Boldrin and Montes (2005) and Andersen and Bhattacharya, 2017) could have been used to overcome the implementation problem that lies at the heart of this paper although the technical challenge of working with a closed economy would nevertheless remain.

The bigger message is that intergenerational conflict – short-run costs become a barrier to the introduction of policies which have long-term gains – may be overcome without hurting any transitional generation and debt finance can make it so. The techniques used in this paper may therefore be used to study the implementation of a whole host of policies – health, infrastructure, environment, gains from trade, and so on – that routinely create the scope for intergenerational conflict. Parenthetically, our analysis offers an improved Kaldor-Hicks criterion, one that no longer verifies a hypothetical net PDV but works with the actual PDV of costs versus benefits (including all general equilibrium effects) that a policy would bring about. This widens the scope of use of the Kaldor-Hicks criterion which has suffered from limited applicability because of the hypothetical nature of the compensations it studied.

Appendix

A Proof of Lemma 2

Consider the stability properties of the LFPSS equilibrium. Since $E = L^* = G$, we have $H(G) = H^*$. The first order condition determining saving reads

$$U_{c_m}\left(\omega\left(k_t\right)H^* - R(k_t)G - S_t\right) - \beta R\left(k_{t+1}\right)U_{c_o}\left(R(k_{t+1})S_t\right) = 0.$$

Recalling $S_t = K_{t+1} + B_t$ and $b_t \equiv \frac{B_t}{H_t}$, this can be rewritten as

$$U_{c_m}\left(\omega\left(k_t\right)H^* - R\left(k_t\right)G - H\left(k_{t+1} + b_t\right)\right) - \beta R\left(k_{t+1}\right)U_{c_o}\left(R(k_{t+1})H^*\left[k_{t+1} + b_t\right]\right) = 0.$$

Taking a first-order linearization around the LFPSS equilibrium, and defining $b^* \equiv \frac{B^*}{H^*}$ we have

$$\begin{aligned}
&U_{c_m c_m}(\cdot)\left[\omega_k(\cdot)H^*\widetilde{k}_t - R_k(\cdot)G\widetilde{k}_t - H^*\widetilde{k}_{t+1}\right] - \beta R_k(\cdot)U_{c_o}(\cdot)\widetilde{k}_{t+1} \\
&-\beta R^* U_{c_o c_o}(\cdot)\left[R^* H^* + R_k H^*\left[k^* + b^*\right]\right)\right]\widetilde{k}_{t+1} = 0
\end{aligned}$$

or

$$U_{c_m c_m}(\cdot)\left[\omega_k(\cdot)H^* - R_k(\cdot)G\right]\widetilde{k}_t = \left[H^* U_{c_m c_m}(\cdot) + \beta R_k(\cdot)U_{c_o}(\cdot) + \beta R^* U_{c_o c_o}(\cdot)\left[R^* H^* + R_k H^*\left[k^* + b^*\right]\right]\right]\widetilde{k}_{t+1}$$

and further as

$$\widetilde{k}_{t+1} = \frac{U_{c_m c_m}(\cdot)\left[\omega_k(\cdot)H^* - R_k(\cdot)G\right]}{H^* U_{c_m c_m}(\cdot) + \beta R_k(\cdot)U_{c_o}(\cdot) + \beta R^* U_{c_o c_o}(\cdot)\left[R^* H^* + R_k(\cdot)H^*\left[k^* + b^*\right]\right]}\widetilde{k}_t$$

or

$$\widetilde{k}_{t+1} = \frac{U_{c_m c_m}(\cdot)\left[\omega_k(\cdot)H^* - R_k G\right]}{H^*\left[U_{c_m c_m}(\cdot) + \beta\left[R^*\right]^2 U_{c_o c_o}(\cdot)\right] + \beta R_k(\cdot)\left[U_{c_o}(\cdot) + R^* U_{c_o c_o}(\cdot)H^*\left[k^* + b^*\right]\right]}\widetilde{k}_t.$$

Routine steps follow:

$$\begin{aligned}
\widetilde{k}_{t+1} &= \frac{\frac{U_{c_m c_m}(\cdot)}{U_{c_m c_m}(\cdot) + \beta\left[R^*\right]^2 U_{c_o c_o}(\cdot)}\left[\omega_k(\cdot)H^* - R_k(\cdot)G\right]}{H^* + \beta R_k(\cdot)\frac{U_{c_o}(\cdot) + R^* U_{c_o c_o}(\cdot)H^*\left[k^* + b^*\right]}{U_{c_m c_m}(\cdot) + \beta\left[R^*\right]^2 U_{c_o c_o}(\cdot)}}\widetilde{k}_t = \frac{S_Y(\cdot)\left[\omega_k(\cdot)H^* - R_k(\cdot)G\right]}{H^* - R_k(\cdot)S_R(\cdot)}\widetilde{k}_t \\
&= \frac{S_Y(\cdot)\left[-k^* R_k(\cdot)H^* - R_k(\cdot)G\right]}{H^* - R_k(\cdot)S_R(\cdot)}\widetilde{k}_t = -\frac{S_Y(\cdot)\left[k^* H^* + G\right]R_k(\cdot)}{H^* - R_k(\cdot)S_R(\cdot)}\widetilde{k}_t \\
&= -\frac{S_Y(\cdot)\left[K^* + G\right]R_k(\cdot)}{H^* - R_k(\cdot)S_R(\cdot)}\widetilde{k}_t
\end{aligned}$$

where $S_R \equiv \frac{\partial S}{\partial R}$ and $S_Y \equiv \frac{\partial S}{\partial Y}$. This yields

$$\widetilde{k}_{t+1} = -\frac{S_Y(\cdot)\, S^* R_k(\cdot)}{H^* - R_k(\cdot)\, S_R(\cdot)}\widetilde{k}_t$$

where $S^* = K^* + G$ and $k^* H^* = K^*$. Note $-\frac{S_Y(\cdot)S^*R_k(\cdot)}{H^*-R_k(\cdot)S_R(\cdot)} > 0$. For stability, we require $-\frac{S_Y(\cdot)S^*R_k(\cdot)}{H^*-R_k(\cdot)S_R(\cdot)} < 1$ which can be rewritten as $R_k(\cdot)\,[S_R(\cdot) - S_Y(\cdot)\, S^*]\,\frac{1}{H^*} < 1$.

B Linearized model

To analyze the dynamics, the model is linearized around the LF steady state. We exploit the equivalence in eq. (20) – in Lemma 1 – which equates the utility level in the LF with that achieved under the LF package.

Life time utility for cohort j can, by a first-order Taylor-approximation, be written

$$V(\omega^* H^* - R^* L^* - \mathcal{T}_{n,0}, R_{n,1}) \simeq V^* - V_Y(\cdot)\widetilde{\mathcal{T}}_{n,0} + V_R(\cdot)\widetilde{R}_{n,1}$$

implying

$$\widetilde{\mathcal{T}}_{n,0} = \phi^* \widetilde{R}_{n,1} \text{ where } \phi^* = \frac{V_R(\cdot)}{V_Y(\cdot)}$$

For subsequent periods, the "equal pay-off condition" reads

$$V\left(\omega_{n,j} H_{n,j}\left(G_n, H_{n,j-1}\right) - \mathcal{T}_{n,j}, R_{n,j+1}\right) = V^*.$$

Taking a first-order Taylor approximation around the LFPSS, we get

$$V\left(\omega_{n,j} H_{n,j}\left(G_n, H_{n,j-1}\right) - \mathcal{T}_{n,j}, R_{n,j+1}\right) \cong V^* + V_Y(\cdot)\left[H^*\widetilde{\omega}_{n,j} + \omega^*\widetilde{H}_{n,j} - \widetilde{\mathcal{T}}_{n,j}\right] + V_R(\cdot)\widetilde{R}_{n,j+1}$$

It follows

(50) $\widetilde{\mathcal{T}}_{n,j} = H^*\widetilde{\omega}_{n,j} + \omega^*\widetilde{H}_{n,j} + \phi^*\widetilde{R}_{n,j+1} \gtrless 0$ for $j > 0$.

Using $\widetilde{\mathcal{T}}_{n,0} = \phi^*\widetilde{R}_{n,1}$ and noting that for a small perturbation from the steady state, $\widetilde{R}_{n,1} = R_k\widetilde{k}_{n,1}$, we have $\widetilde{B}_{n,0} = \widetilde{G} - \phi^*\widetilde{R}_{n,1} \Rightarrow \widetilde{B}_{n,0} = \widetilde{G} - \phi^* R_k\widetilde{k}_{n,1} \Rightarrow \widetilde{B}_{n,0} = \widetilde{G} + \varphi_1\widetilde{k}_{n,1}$ where $\varphi_1 \equiv -\phi^* R_k = -\frac{S^*}{R^*}R_k > 0$. Future debt evolves according to

$$B_{n,j} = R_{n,j}B_{n,j-1} + G_n - \mathcal{T}_{n,j}.$$

A Taylor approximation yields

$$\widetilde{B}_{n,j} = R^*\widetilde{B}_{n,j-1} + \widetilde{R}_{n,j}B_{n,j-1} + \widetilde{G} - \widehat{\mathcal{T}}_{n,j}$$

where $\widehat{\mathcal{T}}_{n,j}$ is given in (50). In more compact form, the debt equations can be written

$$\widetilde{B}_{n,0} = \widetilde{G} + \varphi_1 \widetilde{k}_{n,1} \text{ for } j = 0$$
$$\widetilde{B}_{n,j} = \widetilde{R}_{n,j} B_{n,j-1} + R^* \widetilde{B}_{n,j-1} - H^* \widetilde{\omega}_{n,j} - \omega^* \widetilde{H}_{n,j} - \phi^* \widetilde{R}_{n,j+1} + \widetilde{G} \text{ for } j > 0$$

Saving in period 0, and thus the period 1 capital stock, is determined from eq. (30). Taking a first-order Taylor-approximation around the LF equilibrium, we get

$$U_{c_m c_m}(\cdot)\left[-\widetilde{S}_{n,0} - \widetilde{\mathcal{T}}_{n,0}\right] = \beta \widetilde{R}_{n,1} U_{c_o}(\cdot) + \beta R^* U_{c_o c_o}(\cdot)\left[\widetilde{R}_{n,1} S^* + R^* \widetilde{S}_{n,0}\right]$$
$$\Leftrightarrow \left[U_{c_m c_m}(\cdot) + \beta \left[R^*\right]^2 U_{c_o c_o}(\cdot)\right] \widetilde{S}_{n,o} = -\beta \widetilde{R}_{n,1} U_{c_o}(\cdot) - \beta R^* U_{c_o c_o}(\cdot) S^* \widetilde{R}_{n,1} - U_{c_m c_m}(\cdot) \widetilde{\mathcal{T}}_{n,0}.$$

Using (32), we have $k_{n,1} = \frac{K_1}{H_1} \Longrightarrow \widetilde{k}_{n,1} = \frac{\widetilde{K}_{n,1}}{H^*} - \frac{K^*}{H^{*2}} h_E(\cdot)\widetilde{G}$, since $\widetilde{H}_{n,1} = h_E(\cdot)\widetilde{G}$. Using $\widetilde{K}_{n,1} = \widetilde{S}_{n,0} - \widetilde{B}_{n,0}$, we have $\widetilde{S}_{n,0} = H^* \widetilde{k}_{n,1} + k^* h_E(\cdot)\widetilde{G} + \widetilde{B}_{n,0}$ and using $\widetilde{B}_{n,0} = \widetilde{G} - \widetilde{\mathcal{T}}_{n,0}$, we can rewrite the above equation as

$$\left[U_{c_m c_m}(\cdot) + \beta \left[R^*\right]^2 U_{c_o c_o}(\cdot)\right]\left[H^* \widetilde{k}_{n,1} + k^* h_E(\cdot)\widetilde{G} + \widetilde{B}_{n,0}\right] = -\beta \widetilde{R}_{n,1} U_{c_o}(\cdot) - \beta R^* U_{c_o c_o}(\cdot) S^* \widetilde{R}_{n,1}$$
$$-U_{c_m c_m}(\cdot)\left[\widetilde{G} - \widetilde{B}_{n,0}\right].$$

Some routine manipulation yields (33)

$$\widetilde{k}_{n,1} = \frac{-\beta \left[R^*\right]^2 U_{c_o c_o}(\cdot) \widetilde{B}_{n,0} - \left\{U_{c_m c_m}(\cdot) + \left[U_{c_m c_m}(\cdot) + \beta \left[R^*\right]^2 U_{c_o c_o}(\cdot)\right] k^* h_E(\cdot)\right\} \widetilde{G}}{\left[U_{c_m c_m}(\cdot) + \beta \left[R^*\right]^2 U_{c_o c_o}(\cdot)\right] H^* + \beta R_k \left[U_{c_o}(\cdot) + R^* S^* U_{c_o c_o}(\cdot)\right]}$$

which may, further, be compactly written as

$$\widetilde{k}_{n,1} = -\kappa_1 \widetilde{B}_{n,0} - \kappa_3 \widetilde{G}.$$

For all periods $j > 0$: From (31), a similar first-order Taylor approximation of

$$S_{n,j} : U_{c_m}\left(\omega_{n,j} h\left(G_n, H_{n,j-1}\right) - \mathcal{T}_{n,j} - S_{n,j}\right) = \beta R_{n,j+1} U_{c_o}(R_{n,j+1} S_{n,j}) \text{ for } j > 0$$

yields

$$U_{c_m c_m}(\cdot)\left[\widetilde{\omega}_{n,j} H^* + \omega^* \widetilde{H}_{n,j} - \widetilde{\mathcal{T}}_{n,j} - \widetilde{S}_{n,j}\right] = \beta \widetilde{R}_{n,j+1} U_{c_o}(\cdot) + \beta R^* U_{c_o c_o}(\cdot)\left[\widetilde{R}_{n,j+1} S^* + R^* \widetilde{S}_{n,j}\right]$$
$$\Longleftrightarrow \left[U_{c_m c_m}(\cdot) + \beta \left[R^*\right]^2 U_{c_o c_o}(\cdot)\right] \widetilde{S}_{n,j} = U_{c_m c_m}(\cdot)\left[\widetilde{\omega}_{n,j} H^* + \omega^* \widetilde{H}_{n,j} - \widetilde{\mathcal{T}}_{n,j}\right] - \beta \widetilde{R}_{n,j+1}\left[U_{c_o}(\cdot) + S^* R^* U_{c_o c_o}$$

or using $\widetilde{B}_{n,j} = \widetilde{R}_{n,j} B^* + R^* \widetilde{B}_{n,j-1} + \widetilde{G} - \widetilde{\mathcal{T}}_{n,j}$, and $B^* = G$, we can restate the savings equation in terms of public debt as follows

$$\left[U_{c_m c_m} + R^2 U_{c_o c_o}\right] \widetilde{S}_{n,j} = U_{c_m c_m}\left[\widetilde{\omega}_{n,j}H + \omega \widetilde{H}_{n,j} + \widetilde{B}_{n,j} - \widetilde{R}_{n,j}G - R^*\widetilde{B}_{n,j-1} - \widetilde{G}\right] - \left[U_{c_o} + S^* R^* U_{c_o c_o}\right]\widetilde{R}_{n,j+1}$$

Using $\widetilde{k}_{n,j+1} = \frac{\widetilde{K}_{n,j+1}}{H^*} - \frac{K}{H^{*2}}\widetilde{H}_{n,j+1} \Leftrightarrow H^*\widetilde{k}_{n,j+1} + k^*\left[h_E(\cdot)\widetilde{G} + h_H(\cdot)\widetilde{H}_{n,j}\right] = \widetilde{K}_{n,j+1}$ and $\widetilde{H}_{n,j+1} = h_E(\cdot)\widetilde{G} + h_H(\cdot)\widetilde{H}_{n,j}$ and $\widetilde{K}_{n,j+1} = \widetilde{S}_{n,j} - \widetilde{B}_{n,j}$,we have

$$\widetilde{S}_{n,j} = H^*\widetilde{k}_{n,j+1} + k^* h_E(\cdot)\widetilde{G} + k^* h_H(\cdot)\widetilde{H}_{n,j} + \widetilde{B}_{n,j}$$

and so

$$\begin{aligned}
&\left[U_{c_m c_m} + R^{*2} U_{c_o c_o}\right]\left[H^*\widetilde{k}_{n,j+1} + k^* h_E(\cdot)\widetilde{G} + k^* h_H(\cdot)\widetilde{H}_{n,j} + \widetilde{B}_{n,j}\right] \\
&= U_{c_m c_m}\left[\widetilde{\omega}_{n,j}H + \omega \widetilde{H}_{n,j} + \widetilde{B}_{n,j} - \widetilde{R}_{n,j}G - R^*\widetilde{B}_{n,j-1} - \widetilde{G}\right] - \left[U_{c_o} + SRU_{c_o c_o}\right]\widetilde{R}_{n,j+1} \\
&\left[U_{c_m c_m} + R^{*2} U_{c_o c_o}\right] H^*\widetilde{k}_{n,j+1} + \left[U_{c_m c_m} + R^2 U_{c_o c_o}\right]\left[k^* h_E(\cdot)\widetilde{G} + k^* h_H(\cdot)\widetilde{H}_{n,j} + \widetilde{B}_{n,j}\right] \\
&= U_{c_m c_m}\left[\widetilde{\omega}_{n,j}H + \omega \widetilde{H}_{n,j} + \widetilde{B}_{n,j} - \widetilde{R}_{n,j}G - R^*\widetilde{B}_{n,j-1} - \widetilde{G}\right] - R_k\left[U_{c_o} + SRU_{c_o c_o}\right]\widetilde{k}_{n,j+1} \\
&\left[\left[U_{c_m c_m} + R^{*2} U_{c_o c_o}\right] H^* + R_k\left[U_{c_o} + SRU_{c_o c_o}\right]\right]\widetilde{k}_{n,j+1} \\
&= U_{c_m c_m}\left[\widetilde{\omega}_{n,j}H + \omega \widetilde{H}_{n,j} + \widetilde{B}_{n,j} - R_k\widetilde{k}_{n,j}G - R^*\widetilde{B}_{n,j-1} - \widetilde{G}\right] - \left[U_{c_m c_m} + R^2 U_{c_o c_o}\right]\left[k^* h_E(\cdot)\widetilde{G} + k^* h_H(\cdot)\widetilde{H}_{n,j}\right.
\end{aligned}$$

or,

$$\varkappa\widetilde{k}_{n,j+1} = U_{c_m c_m}\left[\widetilde{\omega}_{n,j}H + \omega \widetilde{H}_{n,j} + \widetilde{B}_{n,j} - R^*\widetilde{B}_{n,j-1} - \widetilde{G}\right] - \left(U_{c_m c_m} + R^{*2} U_{c_o c_o}\right)\left[kh_E(\cdot)\widetilde{G} + kh_H(\cdot)\widetilde{H}_{n,j} + \widetilde{B}_{n,j}\right] -$$

where $\varkappa \equiv \left\{\left(U_{c_m c_m} + R^{*2} U_{c_o c_o}\right) H + \left[U_{c_o} + SRU_{c_o c_o}\right] R_k\right\} < 0$. Collecting terms etc., we get

$$\begin{aligned}
\varkappa\widetilde{k}_{n,j+1} \;=\; & -U_{c_m c_m} R_k \widetilde{k}_{n,j}\left[K + G\right] + \left[U_{c_m c_m}\omega - \left(U_{c_m c_m} + R^{*2} U_{c_o c_o}\right) kh_H(\cdot)\right]\widetilde{H}_{n,j} - R^{*2} U_{c_o c_o}\widetilde{B}_{n,j} \\
& -U_{c_m c_m} R^*\widetilde{B}_{n,j-1} - \left[U_{c_m c_m} + \left(U_{c_m c_m} + R^{*2} U_{c_o c_o}\right) kh_E(\cdot)\right]\widetilde{G}
\end{aligned}$$

or

$$\begin{aligned}
&\varkappa\widetilde{k}_{n,j+1} = -U_{c_m c_m}(\cdot)\left[G + K^*\right] R_k \widetilde{k}_j + \beta\left[R^*\right]^2 U_{c_o c_o}(\cdot)\,\widetilde{B}_{n,j} - R^* U_{c_m c_m}(\cdot)\,\widetilde{B}_{n,j-1} \\
&-\left[U_{c_m c_m}(\cdot) + \left(U_{c_m c_m}(\cdot) + \beta\left[R^*\right]^2 U_{c_o c_o}(\cdot)\right) k^* h_E(\cdot)\right]\widetilde{G} + \left[\omega^* U_{c_m c_m}(\cdot) - \left(U_{c_m c_m}(\cdot) + \beta\left[R^*\right]^2 U_{c_o c_o}(\cdot)\right.\right.
\end{aligned}$$

implying (37).

$$\widetilde{k}_{n,j+1} = \kappa_0\widetilde{k}_{n,j} - \kappa_1\widetilde{B}_{n,j} - \kappa_2\widetilde{B}_{n,j-1} - \kappa_3\widetilde{G} + \kappa_4\widetilde{H}_{n,j}$$

In sum, for period $j = 0$, we have

$$\begin{aligned}\widetilde{B}_{n,0} &= \widetilde{G} + \varphi_1 \widetilde{k}_{n,1} \\ \widetilde{k}_{n,1} &= -\kappa_1 \widetilde{B}_{n,0} - \kappa_3 \widetilde{G}\end{aligned}$$

implying

$$\begin{aligned}\widetilde{B}_{n,0} &= \frac{1-\varphi_1\kappa_3}{1+\varphi_1\kappa_1}\widetilde{G} = \chi_G \widetilde{G}; \quad \chi_G \equiv \frac{1-\varphi_1\kappa_3}{1+\varphi_1\kappa_1} \\ \widetilde{k}_{n,1} &= -\frac{\kappa_3+\kappa_1}{1+\varphi_1\kappa_1} = \Gamma_G \widetilde{G}; \quad \Gamma_G \equiv -\frac{\kappa_3+\kappa_1}{1+\varphi_1\kappa_1} < 0\end{aligned}$$

The underlying dynamics of the economy for $j > 0$ is given by

$$\left.\begin{aligned}\widetilde{B}_{n,j} &= R^* \widetilde{B}_{n,j-1} + \varphi_1 \widetilde{k}_{n,j+1} - \varphi_2 \widetilde{k}_{n,j} - \varphi_3 \widetilde{H}_{n,j} + \widetilde{G} \\ \widetilde{k}_{n,j+1} &= \kappa_0 \widetilde{k}_{n,j} - \kappa_1 \widetilde{B}_{n,j} - \kappa_2 \widetilde{B}_{n,j-1} - \kappa_3 \widetilde{G} + \kappa_4 \widetilde{H}_{n,j}\end{aligned}\right. \quad \forall j > 0$$

or, rearranging,

$$\begin{aligned}\widetilde{B}_{n,j} - \varphi_1 \widetilde{k}_{n,j+1} &= R^* \widetilde{B}_{n,j-1} - \varphi_2 \widetilde{k}_{n,j} - \varphi_3 \widetilde{H}_{n,j} + \widetilde{G} \\ \kappa_1 \widetilde{B}_{n,j} + \widetilde{k}_{n,j+1} &= -\kappa_2 \widetilde{B}_{n,j-1} + \kappa_0 \widetilde{k}_{n,j} - \kappa_3 \widetilde{G} + \kappa_4 \widetilde{H}_{n,j}\end{aligned}$$

a two-dimensional first-order linear dynamical system. In matrix form, this system can be compactly written as

$$\begin{bmatrix} 1 & -\varphi_1 \\ \kappa_1 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{B}_{n,j} \\ \widetilde{k}_{n,j+1} \end{bmatrix} = \begin{bmatrix} R^* & -\varphi_2 \\ -\kappa_2 & \kappa_0 \end{bmatrix} \begin{bmatrix} \widetilde{B}_{n,j-1} \\ \widetilde{k}_{n,j} \end{bmatrix} + \begin{bmatrix} 1 & -\varphi_3 \\ -\kappa_3 & \kappa_4 \end{bmatrix} \begin{bmatrix} \widetilde{G} \\ \widetilde{H}_{n,j} \end{bmatrix},$$

or

$$\begin{aligned}\begin{bmatrix} \widetilde{B}_{n,j} \\ \widetilde{k}_{n,j+1} \end{bmatrix} &= \begin{bmatrix} 1 & -\varphi_1 \\ \kappa_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} R^* & -\varphi_2 \\ -\kappa_2 & \kappa_0 \end{bmatrix} \begin{bmatrix} \widetilde{B}_{n,j-1} \\ \widetilde{k}_{n,j} \end{bmatrix} \\ &+ \begin{bmatrix} 1 & -\varphi_1 \\ \kappa_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -\varphi_3 \\ -\kappa_3 & \kappa_4 \end{bmatrix} \begin{bmatrix} \widetilde{G} \\ \widetilde{H}_{n,j} \end{bmatrix}\end{aligned}$$

which can be evaluated as

$$\begin{bmatrix} \widetilde{B}_{n,j} \\ \widetilde{k}_{n,j+1} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix} \begin{bmatrix} \widetilde{B}_{n,j-1} \\ \widetilde{k}_{n,j} \end{bmatrix} + \begin{bmatrix} A_3 & A_4 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} \widetilde{G} \\ \widetilde{H}_{n,j} \end{bmatrix}$$

the same as (41) once time updated.

C Proof of Lemma 3

Lemma 3

$$\det \begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix} = -\frac{1}{\kappa_1\varphi_1 + 1}\left(\kappa_2\varphi_2 - \kappa_0 R^*\right) = 0,$$

Proof.

$$\begin{aligned} A_1 B_2 &= B_1 A_2 \Leftrightarrow \frac{R^* - \kappa_2\varphi_1}{1+\varphi_1\kappa_1}\frac{\varphi_2\kappa_1 + \kappa_0}{1+\varphi_1\kappa_1} - \left(\frac{-\varphi_2 + \varphi_1\kappa_0}{1+\varphi_1\kappa_1}\frac{-R^*\kappa_1 - \kappa_2}{1+\varphi_1\kappa_1}\right) = 0 \\ &\Leftrightarrow \frac{R^*\kappa_0 - \kappa_2\varphi_2}{\kappa_1\varphi_1 + 1} = 0 \end{aligned}$$

which holds since

$$R^*\kappa_0 = \frac{-R^* U_{c_m c_m}(\cdot) R_k(\cdot)(G + K^*)}{\varkappa}, \text{ and } \kappa_2\varphi_2 = \frac{-R^* U_{c_m c_m}(\cdot) S^* R_k(\cdot)}{\varkappa} \text{ with } S^* = G + K^*.$$

This proves the determinant is zero. ■

D Proof of Proposition 1

We start by back timing and rewriting (41) as

$$\begin{aligned} &(51) \quad \widetilde{B}_{n,j} = A_1\widetilde{B}_{n,j-1} + A_2\widetilde{k}_{n,j} + A_3\widetilde{G} + A_4\widetilde{H}_{n,j} \\ &(52) \quad \widetilde{k}_{n,j+1} = B_1\widetilde{B}_{n,j-1} + B_2\widetilde{k}_{n,j} + B_3\widetilde{G} + B_4\widetilde{H}_{n,j} \end{aligned}$$

Using (52), solve for $\widetilde{B}_{n,j-1}$ to get $\widetilde{B}_{n,j-1} = \frac{1}{B_1}\widetilde{k}_{n,j+1} - \frac{1}{B_1}B_2\widetilde{k}_{n,j} - \frac{1}{B_1}B_3\widetilde{G} - \frac{1}{B_1}B_4\widetilde{H}_{n,j}$, and then substitute it back into (51) to get

$$\widetilde{B}_j = \frac{A_1}{B_1}\widetilde{k}_{n,j+1} + \underbrace{\left[A_2 - \frac{A_1}{B_1}B_2\right]}_{=0}\widetilde{k}_{n,j} + \left[A_4 - \frac{A_1}{B_1}B_4\right]\widetilde{H}_{n,j} + \left[A_3 - \frac{A_1}{B_1}B_3\right]\widetilde{G}$$

or

$$\widetilde{B}_{n,j} = \frac{A_1}{B_1}\widetilde{k}_{n,j+1} + \left[A_4 - \frac{A_1}{B_1}B_4\right]\widetilde{H}_{n,j} + \left[A_3 - \frac{A_1}{B_1}B_3\right]\widetilde{G}$$

Define $\pi_1 \equiv \frac{A_1}{B_1}$, $\pi_0 \equiv [A_3 - \pi_1 B_3]$ and $\pi_2 \equiv A_4 - \pi_1 B_4$. Then,

$$(53) \quad \widetilde{B}_{n,j} = \pi_o\widetilde{G} + \pi_1\widetilde{k}_{n,j+1} + \pi_2\widetilde{H}_{n,j}$$

Similarly, one can write

$$\widetilde{k}_{n,j+1} = B_1 \frac{1}{A_1} \widetilde{B}_{n,j} + \underbrace{\left[B_2 - B_1 \frac{A_2}{A_1} \right]}_{=0} \widetilde{k}_{n,j} - B_1 \frac{1}{A_1} \left[A_3 \widetilde{G} + A_4 \widetilde{H}_{n,j} \right] + B_3 \widetilde{G} + B_4 \widetilde{H}_{n,j}$$

or simplifying,

$$\widetilde{k}_{n,j+1} = \frac{B_1}{A_1} \widetilde{B}_{n,j} + \left(B_4 - \frac{B_1}{A_1} A_4 \right) \widetilde{H}_{n,j} + \left(B_3 - \frac{B_1}{A_1} A_3 \right) \widetilde{G}$$

Using (53) and bringing back a period, we get $\widetilde{k}_{n,j} = \frac{1}{\pi_1} \left[\widetilde{B}_{n,j-1} - \pi_0 \widetilde{G} - \pi_2 \widetilde{H}_{n,j-1} \right]$. Hence it follows from (51) that

$$\begin{aligned} \widetilde{B}_{n,j} &= A_1 \widetilde{B}_{n,j-1} + \frac{A_2}{\pi_1} \left[\widetilde{B}_{n,j-1} - \pi_0 \widetilde{G} - \pi_2 \widetilde{H}_{n,j-1} \right] + A_3 \widetilde{G} + A_4 \widetilde{H}_{n,j} \\ &= \left[A_1 + \frac{A_2}{\pi_1} \right] \widetilde{B}_{n,j-1} + \left[A_3 - \frac{A_2}{\pi_1} \pi_0 \right] \widetilde{G} + A_4 \widetilde{H}_{n,j} - \frac{A_2}{\pi_1} \pi_2 \widetilde{H}_{n,j-1} \end{aligned}$$

which, using $\widetilde{H}_{n,j} = h_E(\cdot) \widetilde{G} + h_H(\cdot) \widetilde{H}_{n,j-1}$ yields,

$$\widetilde{B}_j = \left[A_1 + \frac{A_2}{\pi_1} \right] \widetilde{B}_{n,j-1} + \left[A_3 - \frac{A_2}{\pi_1} \pi_0 \right] \widetilde{G} + A_4 \left[h_E(\cdot) \widetilde{G} + h_H(\cdot) \widetilde{H}_{n,j-1} \right] - \frac{A_2}{\pi_1} \pi_2 \widetilde{H}_{n,j-1}$$

or,

$$\widetilde{B}_{n,j} = \left[A_1 + \frac{A_2}{\pi_1} \right] \widetilde{B}_{n,j-1} + \left[A_3 - \frac{A_2}{\pi_1} \pi_0 + A_4 h_E(\cdot) \right] \widetilde{G} + \left[A_4 h_H(\cdot) - \pi_2 \frac{A_2}{\pi_1} \right] \widetilde{H}_{n,j-1}$$

or

$$(54) \quad \widetilde{B}_{n,j} = \psi_B \widetilde{B}_{n,j-1} + \psi_G \widetilde{G} + \psi_H \widetilde{H}_{n,j-1},$$

where

$$\begin{aligned} \psi_B &\equiv A_1 + \frac{A_2}{\pi_1} = A_1 + B_2, \\ \psi_G &\equiv A_3 - \frac{\pi_0}{\pi_1} A_2 + h_E(\cdot) A_4, \\ \psi_H &= h_H(\cdot) A_4 - \frac{A_2}{\pi_1} \pi_2. \end{aligned}$$

We now prove $\psi_B \equiv A_1 + B_2 = R^* > 1$. To see this, use the coefficient definitions to see that $A_1 + B_2$ can be written as

$$A_1 + B_2 = \frac{R^* - \kappa_2 \varphi_1}{1 + \varphi_1 \kappa_1} + \frac{\varphi_2 \kappa_1 + \kappa_0}{1 + \varphi_1 \kappa_1}.$$

Since $\kappa_0 = \varphi_1 \kappa_2$ (Note that $\kappa_0 = \varphi_1 \kappa_2$ since $K = S - G$) and since $R^* \kappa_0 = \varphi_2 \kappa_2$, it follows that $R^* \varphi_1 \kappa_2 = \varphi_2 \kappa_2$ or $R^* \varphi_1 = \varphi_2$ we have

$$\psi_B \equiv A_1 + B_2 = \frac{R^* - \kappa_2 \varphi_1}{1 + \varphi_1 \kappa_1} + \frac{\varphi_2 \kappa_1 + \kappa_0}{1 + \varphi_1 \kappa_1} = \frac{R^* + \varphi_2 \kappa_1}{1 + \varphi_1 \kappa_1} = \frac{R^* \left[1 + \varphi_1 \kappa_1\right]}{1 + \varphi_1 \kappa_1} = R^* > 1$$

Next, we establish the sign of ψ_G. Using $\pi_0 = A_3 - \pi_1 B_3$, ψ_G reduces to

$$\psi_G = A_3 - A_3 B_2 + B_3 A_2 + h_E(\cdot) A_4 = \left[1 - \kappa_3 \varphi_1 + \kappa_3 \varphi_2 - \kappa_0 + h_E(\cdot)\left[-\varphi_3 + \varphi_1 \kappa_4\right]\right] \frac{1}{1 + \varphi_1 \kappa_1}.$$

Then, $\psi_G < 0$ if

$$1 - \kappa_3 \varphi_1 + \kappa_3 \varphi_2 - \kappa_0 + h_E(\cdot)\left[-\varphi_3 + \varphi_1 \kappa_4\right] < 0.$$

Using $\varphi_3 h_E(\cdot) = R^*$, and $R^* \varphi_1 = \varphi_2$, routine algebra verifies

$$\psi_G < 0 \Rightarrow 1 - \kappa_3 \varphi_1 + \kappa_3 \varphi_2 - \kappa_0 + h_E(\cdot)\left[-\varphi_3 + \varphi_1 \kappa_4\right] = \left[1 - R^*\right]\left[1 - \kappa_3 \varphi_1\right] - \phi_a < 0$$

where (using $\omega^* h_E(\cdot) = R^*$),

$$(55) \quad \phi_a \equiv -\frac{1}{\varkappa} \frac{S^*}{R^*} R_k(\cdot) \left[U_{c_m c_m}(\cdot) + \beta \left[R^*\right]^2 U_{c_o c_o}(\cdot)\right] k^* h_H(\cdot) h_E(\cdot) > 0 \text{ given } h_H(\cdot) > 0$$

For $R^* > 1$ and since $\kappa_3 \varphi_1 - 1 < 0$ (by Assumption 4), $\psi_G < 0$ holds.

Finally, consider $\psi_H < 0$. Routine algebra verifies

$$\begin{aligned}
\psi_H &= h_H(\cdot) A_4 - B_2 A_4 + A_2 B_4 \\
&= h_H(\cdot) \frac{-\varphi_3 + \varphi_1 \kappa_4}{1 + \varphi_1 \kappa_1} - \frac{\varphi_2 \kappa_1 + \kappa_0}{1 + \varphi_1 \kappa_1} \left(\frac{-\varphi_3 + \varphi_1 \kappa_4}{1 + \varphi_1 \kappa_1}\right) + \left(\frac{-\varphi_2 + \varphi_1 \kappa_0}{1 + \varphi_1 \kappa_1}\right) \frac{\varphi_3 \kappa_1 + \kappa_4}{1 + \varphi_1 \kappa_1} \\
&= h_H(\cdot) \left[\frac{-\varphi_3 + \varphi_1 \kappa_4}{1 + \varphi_1 \kappa_1}\right] + \frac{\varphi_3 \kappa_0 - \kappa_4 \varphi_2}{1 + \varphi_1 \kappa_1},
\end{aligned}$$

and hence

$$sign\ \psi_H = sign\ \Theta$$

where

$$\Theta \equiv h_H(\cdot)\left[-\varphi_3 + \varphi_1 \kappa_4\right] + \varphi_3 \kappa_0 - \kappa_4 \varphi_2$$

Inserting φ and κ expressions, we get

$$\begin{aligned}\Theta &= h_H(\cdot)\left[-\omega^* - \phi^* R_k \frac{\omega^* S_Y(\cdot) - k^* h_H(\cdot)}{H - S_R(\cdot) R_k(\cdot)}\right] - \omega^* \frac{S_Y(\cdot) S^* R_k}{H - S_R(\cdot) R_k(\cdot)} + \frac{\omega^* S_Y(\cdot) - k^* h_H(\cdot)}{H - S_R(\cdot) R_k(\cdot)} S^* R_k \\ &= h_H(\cdot)\left[-\omega^* - \phi^* R_k \frac{\omega^* S_Y(\cdot) - k h_H(\cdot)}{H - S_R(\cdot) R_k(\cdot)}\right] - \frac{k^* h_H(\cdot)}{H - S_R(\cdot) R_k(\cdot)} S^* R_k\end{aligned}$$

Assumption 4 implies that $1 - \varphi_1 \kappa_3 > 0$ or

$$R^* + S^* R_k \frac{S_Y}{H^* - S_R R_k} + \frac{S^* k^*}{H^* - S_R R_k}\left[R_k h_E(\cdot)\right] > 0$$

We have that

$$\begin{aligned}\Theta &= h_H(\cdot)\left[-\omega^* - \phi^* R_k \frac{\omega^* S_Y(\cdot) - k h_H(\cdot)}{H - S_R(\cdot) R_k(\cdot)} - \frac{1}{h_E(\cdot)} \frac{k^* h_E(\cdot)}{H - S_R(\cdot) R_k(\cdot)} S^* R_k\right] \\ &= h_H(\cdot)\left[\frac{S^*}{R^*} R_k \frac{k h_H(\cdot)}{H - S_R(\cdot) R_k(\cdot)} - \frac{1}{h_E(\cdot)}\left[R^* + S^* R_k \frac{S_Y}{H^* - S_R R_k} + \frac{k^* h_E(\cdot)}{H - S_R(\cdot) R_k(\cdot)} S^* R_k\right]\right] < 0\end{aligned}$$

and hence under Assumption 4, $\psi_H < 0$ holds for $h_H(\cdot) > 0$.

E Proof of Proposition 2

From (54), we have $\widetilde{B}_{n,j} = \psi_B \widetilde{B}_{n,j-1} + \psi_G \widetilde{G} + \psi_H \widetilde{H}_{n,j-1}$ for $j \geq 1$. For $j = 1$,

$$\widetilde{B}_{n,1} = \psi_B \widetilde{B}_{n,0} + \psi_G \widetilde{G} + \psi_H \widetilde{H}_{n,0}$$

we know

$$\widetilde{B}_{n,0} = \frac{1 - \varphi_1 \kappa_3}{1 + \varphi_1 \kappa_1} \widetilde{G} = \chi_G \widetilde{G}; \quad \widetilde{H}_{n,0} = 0$$

then,

$$\begin{aligned}
&\widetilde{B}_{n,1} = \psi_B \widetilde{B}_{n,0} + \psi_G \widetilde{G} < \widetilde{B}_{n,0} \\
&(\psi_B - 1)\, \widetilde{B}_{n,0} + \psi_G \widetilde{G} < 0 \\
&(\psi_B - 1)\, \frac{1 - \varphi_1 \kappa_3}{1 + \varphi_1 \kappa_1} \widetilde{G} + \psi_G \widetilde{G} < 0
\end{aligned}$$

question is, does $\psi_G < \chi_G (1 - \psi_B) - \psi_H h_E(\cdot)$ hold? For this, it is sufficient that $\psi_G < \chi_G (1 - \psi_B)$ holds since $\psi_H h_E(\cdot) < 0$. Using

$$\psi_G = \frac{1}{1 + \varphi_1 \kappa_1} \left[1 - \kappa_3 \varphi_1 + \kappa_3 \varphi_2 - \kappa_0 + h_E(\cdot) \left[-\varphi_3 + \varphi_1 \kappa_4\right]\right]; \; \psi_B = R^*; \; \chi_G = \frac{1 - \varphi_1 \kappa_3}{1 + \varphi_1 \kappa_1}$$

the condition $\psi_G < \chi_G (1 - \psi_B)$ can be written as

$$\frac{1}{1 + \varphi_1 \kappa_1} \left[1 - \kappa_3 \varphi_1 + \kappa_3 \varphi_2 - \kappa_0 + h_E(\cdot) \left[-\varphi_3 + \varphi_1 \kappa_4\right]\right] < \left[1 - R^*\right] \frac{1 - \varphi_1 \kappa_3}{1 + \varphi_1 \kappa_1} \tag{56}$$

Since $\kappa_0 = \varphi_1 \kappa_2$ (Note that $\kappa_0 = \varphi_1 \kappa_2$ since $K = S - G$) and since $R^* \kappa_0 = \varphi_2 \kappa_2$, it follows that $R^* \varphi_1 \kappa_2 = \varphi_2 \kappa_2$ or $R^* \varphi_1 = \varphi_2$. Routine algebra verifies (56) reduces to

$$1 - \kappa_0 + h_E(\cdot) \left[-\varphi_3 + \varphi_1 \kappa_4\right] < \left[1 - R^*\right]$$

and using $\varphi_3 h_E(\cdot) = R^*$ and $\varphi_1 \kappa_4 h_E(\cdot) = \kappa_0 - \phi_a$ it reduces further to

$$1 - \kappa_0 - R^* + \kappa_0 - \phi_a < 1 - R^* \Leftrightarrow 1 - R^* - \phi_a < 1 - R^* \Leftrightarrow -\phi_a < 0$$

It follows from (55) the above inequality is fulfilled for $h_H(\cdot) > 0$. Note that $\phi_a = 0$ for $h_H(\cdot) = 0$ implying a knife-edge case where $\widetilde{B}_{n,j+1} = \widetilde{B}_{n,j}$.

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