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Abstract

We establish the bounds $\frac{4}{3} \leqslant b_\nu \leqslant b_\xi \leqslant \sqrt{2}$, where b_ν and b_ξ are the Nordhaus-Gaddum sum upper bound multipliers, i.e., $\nu(G)+\nu(\overline{G}) \leqslant b_\nu|G|$ and $\xi(G)+\xi(\overline{G}) \leqslant b_\xi|G|$ for all graphs G, and ν and ξ are Colin de Verdière type graph parameters. The Nordhaus-Gaddum sum lower bound for ν and ξ is conjectured to be $|G|-2$, and if these parameters are replaced by the maximum nullity $\mathrm{M}(G)$, this bound is called the Graph Complement Conjecture in the study of minimum rank/maximum nullity problems.

Keywords: Nordhaus-Gaddum; Colin de Verdière type parameter; Graph Complement Conjecture; maximum nullity; minimum rank; graph complement

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1 Introduction

Nordhaus-Gaddum problems have been studied for many different graph parameters, including chromatic number, independence number, domination number, Hadwiger number, etc. (see, for example, [6] and the references therein). In this note we discuss the Nordhaus-Gaddum sum upper bounds for the Colin de Verdière type parameters ν, ξ, and μ. The Graph Complement Conjecture in the study of minimum rank/maximum nullity problems is a conjectured lower bound for related Nordhaus-Gaddum problems.

All graphs in this paper are simple, undirected, and finite. The *complement* of a graph $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$, where $\overline{E}$ consists of all possible edges between vertices in V that are not in E. Let G be a graph with vertices $\{1, \dots, n\}$ and let S_n denote the set of symmetric $n \times n$ real matrices. For $A = [a_{ij}] \in S_n$, the *graph* of A, denoted by $\mathcal{G}(A)$, is the graph with vertices $\{1, \dots, n\}$ and edges $\{ij : a_{ij} \neq 0 \text{ and } i \neq j\}$. The *set of symmetric matrices associated with* G is $\mathcal{S}(G) = \{A \in S_n : \mathcal{G}(A) = G\}$. The *maximum nullity of* G is $\mathrm{M}(G) = \max\{\operatorname{null} A : A \in \mathcal{S}(G)\}$, and the *maximum positive semidefinite nullity of* G is $\mathrm{M}_+(G) = \max\{\operatorname{null} A : A \in \mathcal{S}(G) \text{ and } A \text{ is positive semidefinite}\}$.

In [7] and [8], Colin de Verdière introduced the parameters $\mu(G)$ and $\nu(G)$, defined to be the maximum nullity among matrices $A \in \mathcal{S}(G)$ that satisfy the Strong Arnold Hypothesis and additional conditions. In [4] the parameter $\xi(G)$ was defined to be the maximum nullity among matrices $A \in \mathcal{S}(G)$ that satisfy the Strong Arnold Hypothesis. A real symmetric matrix A satisfies the *Strong Arnold Hypothesis* provided there does not exist a nonzero real symmetric matrix X satisfying $AX = 0$, $A \circ X = 0$, and $I \circ X = 0$, where $\circ$ denotes the Hadamard (entry-wise) product and I is the identity matrix. The Strong Arnold Hypothesis is equivalent to the requirement that certain manifolds intersect transversally (see [13]). For ν, the only additional condition (besides the Strong Arnold Hypothesis) is that the matrix must be positive semidefinite. For μ, the additional conditions are that the matrix must be a generalized Laplacian (i.e., have nonpositive off-diagonal entries) and have exactly one negative eigenvalue. Clearly $\nu(G) \leqslant \mathrm{M}_+(G) \leqslant \mathrm{M}(G)$, $\nu(G) \leqslant \xi(G) \leqslant \mathrm{M}(G)$, and $\mu(G) \leqslant \xi(G)$, and each of these inequalities can be strict (see [2]).

An important property of Colin de Verdière type parameters is minor monotonicity. The *contraction* of edge $e = uv$ of G is obtained by identifying the vertices u and v, deleting any loops that arise in this process, and replacing any multiple edges by a single edge. A *minor* of G arises by performing a sequence of deletions of edges, deletions of isolated vertices, and/or contractions of edges. The notation $H \preceq G$ means that H is a minor of G. A graph parameter β is *minor monotone* if for any minor H of G, $\beta(H) \leqslant \beta(G)$. In [7], [8], and [4] (respectively) it is shown that μ, ν, and ξ are minor monotone.

For any graph G, the *Hadwiger number* $\mathrm{h}(G)$ is the maximum order of a clique minor of G. It was shown in [7] and [8] that $\mu(K_n) = n - 1$ and $\nu(K_n) = n - 1$ (where K_n denotes the complete graph on n vertices), so by minor monotonicity $\mathrm{h}(G) - 1 \leqslant \mu(G)$ and $\mathrm{h}(G) - 1 \leqslant \nu(G)$.

Let $\kappa(G)$ denote the *vertex connectivity of* G, i.e., if G is not complete, the smallest number k such that there is a set of vertices S, with $|S| = k$, for which the graph obtained by deleting the vertices in S and all edges incident with a vertex in S, denoted by $G-S$, is disconnected (by convention, $\kappa(K_n) = n-1$). It is proved in [15, 16] that $\kappa(G) \leqslant \mathrm{M}_+(G)$ for every graph G. It was noted in [12] that the proof in [15] establishes $\kappa(G) \leqslant \nu(G)$ for all G. As defined in [2], the *minor monotone ceiling* of κ is $\lceil\kappa\rceil(G) = \max\{\kappa(H) : H \preceq G\}$. It follows from the definition that $\mathrm{h}(G) - 1 \leqslant \lceil\kappa\rceil(G)$, since the $K_{\mathrm{h}(G)}$ minor of G implies $\kappa(K_{\mathrm{h}(G)}) \leqslant \lceil\kappa\rceil(G)$, and $\lceil\kappa\rceil(G) \leqslant \nu(G)$, since $\kappa(G) \leqslant \nu(G)$ and ν is minor monotone (see [2] for more detail).

A Nordhaus-Gaddum type result is a (sharp) lower or upper bound on the sum or product of a parameter of a graph and of its complement. The Graph Complement Conjecture for ν [3] is a Nordhaus-Gaddum sum lower bound.

Conjecture 1.1 (GCC_ν)**.** *For any graph G,*

$$\nu(G) + \nu(\overline{G}) \geqslant |G| - 2. \tag{1}$$

It is not possible to raise the lower bound $|G| - 2$ since equality is attained for any tree that includes a P_4: For such a tree, it is shown in [1] that $\mathrm{M}_+(\overline{T}) = |T| - 3$. Since $\mathrm{M}_+(T) = 1$, $\mathrm{M}_+(T) + \mathrm{M}_+(\overline{T}) = |T| - 2$. It is shown in [17] that GCC_ν is true for graphs with tree-width at most 3, and thus for trees. Thus GCC_ν conjectures that $|G| - 2$ is a tight Nordhaus-Gaddum sum lower bound for ν. This conjecture is studied in [3], where it is established for certain graphs. Various other forms of this conjecture have appeared, including: GCC_+, i.e., $\mathrm{M}_+(G) + \mathrm{M}_+(\overline{G}) \geqslant |G| - 2$, [3]; GCC, i.e., $\mathrm{M}(G) + \mathrm{M}(\overline{G}) \geqslant |G| - 2$, [5]; and GCC_ξ, i.e., $\xi(G) + \xi(\overline{G}) \geqslant |G| - 2$, [9]. Of course GCC_ν implies GCC_+ implies GCC, and GCC_ν implies GCC_ξ implies GCC. The graph complement conjecture for μ, i.e., $\mu(G) + \mu(\overline{G}) \geqslant |G| - 2$, appeared in [14].

Here we discuss values of the multiplier b for a Nordhaus-Gaddum sum upper bound for the parameter β where β is one of $\mathrm{h}, \lceil\kappa\rceil, \nu, \xi$, or μ. We denote by b_β the least value of b making

$$\beta(G) + \beta(\overline{G}) \leqslant b\,|G|$$

true for every graph of order at least two, and call b_β the *NG upper multiplier for* β. Stiebitz [18] has shown that

$$\mathrm{h}(G) + \mathrm{h}(\overline{G}) \leqslant \frac{6}{5}|G|$$

and there exist graphs achieving $\mathrm{h}(G) + \mathrm{h}(\overline{G}) = \frac{6}{5}(1-\varepsilon)|G|$ for arbitrarily small ε, so $b_{\mathrm{h}} = \frac{6}{5}$. We establish bounds for $b_{\lceil\kappa\rceil}, b_\nu$, and b_ξ. Clearly $b_{\mathrm{h}} \leqslant b_{\lceil\kappa\rceil} \leqslant b_\nu \leqslant b_\xi$, and $b_{\mathrm{h}} \leqslant b_\mu \leqslant b_\xi$. In Section 2 we construct a family of graphs to show that $b_{\lceil\kappa\rceil} \geqslant \frac{4}{3}$. In Section 3 we show that $b_\xi \leqslant \sqrt{2}$. In Section 4 we summarize our conclusions. Note that the Nordhaus-Gaddum sum upper bound for the parameters M and M_+ is not interesting because it is trivially $2|G| - 1$:

$$\mathrm{M}(K_n) + \mathrm{M}(\overline{K_n}) = \mathrm{M}_+(K_n) + \mathrm{M}_+(\overline{K_n}) = (n-1) + n = 2|K_n| - 1.$$

2 Lower bound for NG upper multiplier for $\lceil\kappa\rceil$

In this section we construct a self-complementary graph $H(a)$ on $12a-4$ vertices for $a \geqslant 2$, and show that $H(a)$ has a minor $\widehat{H(a)}$ with $\delta(\widehat{H(a)}) = \kappa(\widehat{H(a)}) = 8a-4$, where $\delta(G)$ denotes the minimum degree of a vertex of G. It is shown that this example implies that $b_{\lceil\kappa\rceil} \geqslant 4/3$.

Example 2.1. Construct the graph $H(a) = (V, E)$ as follows (see Figure 1): The $12a-4$ vertices of $H(a)$ are partitioned into four sets $V_i, i = 1, 2, 3, 4$ of $r = 3a-1$ vertices each. The sets V_1 and V_2 are the "sparse part" of $H(a)$, with $H(a)[V_i] = \overline{K_r}, i = 1, 2$ (where $G[W]$ denotes the subgraph of G induced by the subset W of the vertices of G). The sets V_3 and V_4 are the "dense part" of $H(a)$, with $H(a)[V_i] = K_r, i = 3, 4$. Every edge between a vertex in V_1 and a vertex in V_3 is in the edge set E, and likewise for V_2 and V_4. There are no edges between V_1 and V_4, nor between V_2 and V_3. Regarding the edges between V_1 and V_2, number the vertices of V_1 as $u_{2i-1}, i = 1, \ldots, r$ and the vertices of V_2 as $u_{2i}, i = 1, \ldots, r$. Then vertex $u_s \in V_1$ is adjacent to the a vertices $u_{s+j} \in V_2,\ j = 1, 3, \ldots, 2a-3, 2a-1$ (where for $k = 6a-1, \ldots, 8a-4$, u_k is interpreted as u_ℓ with $\ell \equiv k \mod (6a-2)$ and $1 \leqslant \ell \leqslant 2a-2$). If $u \in V_1 \cup V_2$, then $\deg_{H(a)} u = r + a = 4a-1$ (where $\deg_G w$ denotes the degree of w in G). Regarding the edges between V_3 and V_4, number the vertices of V_3 as $v_{2i-1}, i = 1, \ldots, r$ and the vertices of V_4 as $v_{2i}, i = 1, \ldots, r$. Then vertex $v_s \in V_3$ is adjacent to all vertices $v_p \in V_4$ except for $p = s + j, j = 1, 3, \ldots, 2a-3, 2a-1$. If $v \in V_3 \cup V_4$, then $\deg_{H(a)} v = (r-1) + r + (r-a) = 8a-4$. It is clear from the construction that $\overline{H(a)} = H(a)$.

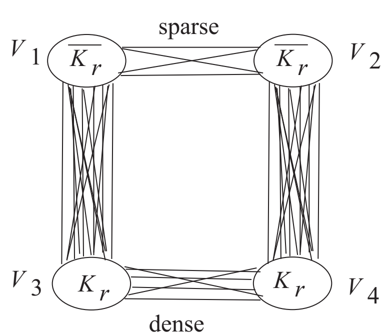

Figure 1: Schematic diagram for the construction of $H(a)$

Construct the minor $\widehat{H(a)}$ by contracting the edges $u_{2i-1}u_{2i}, i = 1, \ldots, r$, and denote the set of these r vertices by $V_{1,2}$. If $v \in V_3 \cup V_4$, then $\deg_{\widehat{H(a)}} v = \deg_{H(a)} v = 8a-4$. Note that each of the new vertices in $V_{1,2}$ has degree equal to $2((4a-1)-1) = 8a-4$, so $\widehat{H(a)}$ is $(8a-4)$-regular. Furthermore, if $w \in V_{1,2}$, w is adjacent to all $2r = 6a-2$ vertices in $V_3 \cup V_4$ so $\widehat{H(a)}[V_{1,2}]$ is $(2a-2)$-regular. Since each vertex in $V_3 \cup V_4$ is adjacent to $r = 3a-1$ vertices in $V_{1,2}$, $\widehat{H(a)}[V_3 \cup V_4]$ is $(5a-3)$-regular.

To establish that $\kappa(\widehat{H(a)}) = \delta(\widehat{H(a)})$, we use the property that for certain circulants C, $\kappa(C) = \delta(C)$, establish a method for computing κ, and examine parts of $\widehat{H(a)}$ separately. For $1 \leqslant t \leqslant \lfloor \frac{n-1}{2} \rfloor$, the *consecutive circulant* $\mathrm{Circ}_n\langle 1, \dots, t\rangle$ is the graph on the vertices $\{0, 1, \dots, n-1\}$ with vertex i adjacent to vertices $i+j$ and $i-j$ for $j = 1, \dots, t$ (with arithmetic mod n). We will use the fact that for a consecutive circulant the vertex connectivity is equal to the (common) degree; Harary [11] gave the consecutive circulant as an example of a graph having maximum vertex connectivity $\frac{2m}{n}$ among graphs having n vertices and m edges (when $\frac{2m}{n}$ is an integer), and this result is now well known.

Theorem 2.2. [19, Theorem 4.1.5 (Harary)] *For* $1 \leqslant t \leqslant \lfloor \frac{n-1}{2} \rfloor$,

$$\kappa(\mathrm{Circ}_n\langle 1, \dots, t\rangle) = \delta(\mathrm{Circ}_n\langle 1, \dots, t\rangle) = 2t.$$

Theorem 2.3. *Let G be a connected graph on n vertices with $G \neq K_n$ and let $1 \leqslant t \leqslant n - 1 - \delta(G)$. Define*

$$f(t) = \max\{s \in \mathbb{Z}^+ : K_{t,s} \textit{ is a subgraph of } \overline{G}\},$$

where $K_{s,t}$ denotes the complete bipartite graph on s and t vertices. Then

$$\kappa(G) = \min\{n - (t + f(t)) : 1 \leqslant t \leqslant n - 1 - \delta(G)\}.$$

Proof. For every t such that $1 \leqslant t \leqslant n-1-\delta(G)$, $\overline{G}$ contains a $K_{t,1}$ (by choosing a vertex v of degree $\delta(G)$ as the partite set of 1 vertex, and t of its non-neighbors as the other partite set), so $f(t)$ is defined.

Choose t such that $1 \leqslant t \leqslant n - 1 - \delta(G)$. Let U be a set of t vertices and let W be a set of $f(t)$ vertices such that $\overline{G}[U \cup W]$ contains a $K_{t,f(t)}$ subgraph. Then $G[U \cup W]$ is disconnected, so $\kappa(G) \leqslant |V \setminus (U \cup W)| = n - (t + f(t))$. Since this is true for every t such that $1 \leqslant t \leqslant n - 1 - \delta(G)$,

$$\kappa(G) \leqslant \min\{n - (t + f(t)) : 1 \leqslant t \leqslant n - 1 - \delta(G)\}.$$

Choose a set S such that $|S| = \kappa(G)$ and $G - S$ is disconnected. Let U be the set of vertices of one connected component, let $t_0 = |U|$, and let $W = V \setminus (U \cup S)$; note $|W| = n - (t_0 + \kappa(G))$. Then $\overline{G}$ contains $K_{t_0, n-(t_0+\kappa(G))}$ with bipartition U, W. Thus $f(t_0) \geqslant n - (t_0 + \kappa(G))$, so

$$\kappa(G) \geqslant n - (t_0 + f(t_0)) \geqslant \min\{n - (t + f(t)) : 1 \leqslant t \leqslant n - 1 - \delta(G)\}. \qquad \square$$

We now return to establishing the properties of one of the graphs constructed in Example 2.1.

Observation 2.4. *Let G be a graph whose vertex set V can be partitioned as $V = X \dot\cup Y$ such that each vertex in X is adjacent to each vertex in Y. Let $G_X = G[X]$ and $G_Y = G[Y]$. Then*

- $\delta(G) = \min\{\delta(G_X) + |Y|,\, \delta(G_Y) + |X|\}$,
- $\kappa(G) = \min\{\kappa(G_X) + |Y|,\, \kappa(G_Y) + |X|\}$.

Theorem 2.5. *For $\widehat{H(a)}$ as in Example 2.1,*

$$\kappa(\widehat{H(a)}) = \delta(\widehat{H(a)}) = 8a - 4.$$

Proof. Let $X = V_{1,2}$ and $Y = V_3 \cup V_4$. Then $V(\widehat{H(a)}) = X \dot{\cup} Y$ and every vertex in X is adjacent to every vertex in Y. By Observation 2.4, if we show that $\kappa(\widehat{H(a)}[V_{1,2}]) = \delta(\widehat{H(a)}[V_{1,2}])$ and $\kappa(\widehat{H(a)}[V_3 \cup V_4]) = \delta(\widehat{H(a)}[V_3 \cup V_4])$, it follows that $\kappa(\widehat{H(a)}) = \delta(\widehat{H(a)})$.

Since $\widehat{H(a)}[V_{1,2}] = \mathrm{Circ}_r\langle 1, \ldots, a-1\rangle$, we have $\kappa(\widehat{H(a)}[V_{1,2}]) = \delta(\widehat{H(a)}[V_{1,2}])$ by Theorem 2.2.

Recall that in $\widehat{H(a)}[V_3 \cup V_4] = H(a)[V_3 \cup V_4]$, the vertices of V_3 are numbered as $v_{2i-1}, i = 1, \ldots, r$ and the vertices of V_4 as $v_{2i}, i = 1, \ldots, r$, and vertex $v_s \in V_3$ is adjacent to all vertices $v_p \in V_4$ except for $p = s+j, j = 1, 3, \ldots, 2a-3, 2a-1$ (where for $p > 6a-2$, v_p means $v_{p-(6a-2)}$). Thus the vertex v_s in $\overline{H(a)[V_3 \cup V_4]}$ is adjacent to precisely the vertices $v_p,\ p = s+j, j = 1, 3, \ldots, 2a-3, 2a-1$. This is a bipartite regular graph with a great deal of symmetry, so in determining the shared neighborhood of two vertices, no generality is lost by considering the vertices 1 and $1 + 2d$ (with $d \leqslant \lceil \frac{3a-1}{2} \rceil$). The size of the shared neighborhood is $\max(a - d, 0)$. For $1 \leqslant t \leqslant a = 2r - 1 - (5a - 3) = |\widehat{H(a)}[V_3 \cup V_4]| - 1 - \delta(\widehat{H(a)}[V_3 \cup V_4])$, the maximum neighborhood intersection of a set of t vertices happens when those vertices are consecutive in the same bipartition set, and it follows that in this case $f(t) = a + 1 - t$. So for all $t \in \{1, \ldots, a\}$, $|\widehat{H(a)}[V_3 \cup V_4]| - (t + f(t)) = 5a - 3$. Thus $\kappa(\widehat{H(a)}[V_3 \cup V_4]) = 5a - 3 = \delta(\widehat{H(a)}[V_3 \cup V_4])$. □

Corollary 2.6. *For the graph $H(a)$ in Example 2.1,*

$$\lceil \kappa \rceil(H(a)) \geqslant 8a - 4$$

and

$$\lceil \kappa \rceil(H(a)) + \lceil \kappa \rceil(\overline{H(a)}) \geqslant \frac{4}{3}(1 - \frac{1}{6a-2})|H(a)|.$$

Thus

$$b_{\lceil \kappa \rceil} \geqslant \frac{4}{3}.$$

Proof. By Theorem 2.5, $\lceil \kappa \rceil(H(a)) \geqslant \kappa(\widehat{H(a)}) = 8a-4$. Since $H(a)$ is self-complementary, $\lceil \kappa \rceil(\overline{H(a)}) \geqslant 8a-4$ also, and thus $\lceil \kappa \rceil(H(a)) + \lceil \kappa \rceil(\overline{H(a)}) \geqslant 16a-8$. The second statement can then be established by arithmetic. Since $b_{\lceil \kappa \rceil} \geqslant \frac{1}{|H(a)|}(\lceil \kappa \rceil(H(a)) + \lceil \kappa \rceil(\overline{H(a)}))$, by taking the limit as $a \to \infty$ we see that $b_{\lceil \kappa \rceil} \geqslant \frac{4}{3}$. □

3 Upper bound for NG upper multiplier for ξ

In this section we show that the NG upper multiplier b_ξ is at most $\sqrt{2}$.

Theorem 3.1. [10] *Let $G = (V_G, E_G)$ be a connected graph. Then*

$$|E_G| + a \geqslant \frac{\xi(G)(\xi(G)+1)}{2} \tag{2}$$

where $a = 1$ if G is bipartite and every optimal matrix for $\xi(G)$ has zero diagonal, and $a = 0$ otherwise.

Since $\xi(G)$ is the maximum of $\xi(G_i)$ taken over the connected components G_i of G, the hypothesis that G is connected is unnecessary.

Corollary 3.2. *Let $G = (V_G, E_G)$ be a graph. Then*

$$|E_G| + 1 \geqslant \frac{\xi(G)(\xi(G)+1)}{2}. \tag{3}$$

Corollary 3.3. *Let $G = (V_G, E_G)$ be a graph with at least one edge. Then*

$$\xi(G) \leqslant \sqrt{2|E_G|}. \tag{4}$$

Proof. Algebraic manipulation of (3) gives $\xi(G) \leqslant \frac{1}{2}(-1+\sqrt{8|E_G|+9})$. Further manipulation shows that the inequality $\frac{1}{2}(-1+\sqrt{8|E_G|+9}) \leqslant \sqrt{2|E_G|}$ is equivalent to $2 \leqslant |E_G|$, so (4) is true if G has at least two edges. If G has exactly one edge then G has components K_2 and possibly some K_1's, and thus $\xi(G) = 1 < \sqrt{2}$. □

Nordhaus-Gaddum bounds usually take one of two forms: additive or multiplicative. The form of inequality (4) suggests a third category of Nordhaus-Gaddum bound: Pythagorean.

Corollary 3.4. *Let $G = (V_G, E_G)$ be a graph of order at least two. Then*

$$\xi(G)^2 + \xi(\overline{G})^2 \leqslant |G|^2 - |G|.$$

Proof. Let $|G| = n \geqslant 2$. In the case where either G has no edges or $\overline{G}$ has no edges, ξ will take the value 1 for one of the two graphs and the value $n-1$ for the other, in which case the result holds. In any other case inequality (4) applies both to G and to $\overline{G}$, giving us two inequalities the sum of whose squares is

$$\xi(G)^2 + \xi(\overline{G})^2 \leqslant 2|E_G| + 2|E_{\overline{G}}| = |G|^2 - |G|.$$ □

Corollary 3.5. *Let $G = (V_G, E_G)$ be a graph of order at least two. Then*

$$\xi(G) + \xi(\overline{G}) \leqslant \sqrt{2}|G|,$$

and thus $b_\xi \leqslant \sqrt{2}$.

Proof. Let $|G| = n \geqslant 2$, and by Corollary 3.4 choose $x \geqslant \xi(G)$ and $y \geqslant \xi(\overline{G})$ such that x and y lie on the circle $x^2+y^2 = n^2$. The maximum value of $x+y$ on this circle is $\sqrt{2}n$. □

4 Conclusions

In summary, we have established

$$1.333 < \frac{4}{3} \leqslant b_{\lceil\kappa\rceil} \leqslant b_\nu \leqslant b_\xi \leqslant \sqrt{2} < 1.415.$$

We have no evidence that the construction in Section 2 is tight, even for $b_{\lceil\kappa\rceil}$. On the other hand, the inequality (2) with $a = 0$ is known to be tight for some small examples and for complete graphs (it is tight with $a = 1$ for $K_{3,3}$). For ν, since a diagonal entry for a vertex of degree at least one cannot be zero, $a = 0$ and the inequality (2) becomes $|E_G| \geqslant \frac{\nu(G)(\nu(G)+1)}{2}$ for graphs with at least one edge; again this is tight for some small graphs and complete graphs. This leaves open the possibility that Corollaries 3.3 — 3.5 may be asymptotically tight.

Question 4.1. *Given x and y positive with $x^2 + y^2 = 1$, does there exist an increasing sequence of graphs G_i on n_i vertices such that $\nu(G_i)/n_i$ approaches x and $\nu(\overline{G_i})/n_i$ approaches y? Or such that $\xi(G_i)/n_i$ approaches x and $\xi(\overline{G_i})/n_i$ approaches y?*

The particular case of $x = y = \frac{\sqrt{2}}{2}$ suggests the next question.

Question 4.2. *Do b_ν and b_ξ take the maximum possible value of $\sqrt{2}$?*

On the other hand it seems more difficult to construct examples for b_μ, and the only bounds we know are those from h (due to Stiebitz [18]) and ξ. i.e.,

$$1.2 = \frac{6}{5} = b_{\mathrm{h}} \leqslant b_\mu \leqslant b_\xi \leqslant \sqrt{2} < 1.415.$$

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