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The Long-Term Structure of Commodity Futures

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Disciplines
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Abstract

Futures markets on agricultural commodities typically trade with maximum maturity dates of less than four years. If these markets did trade with maturities eight or ten years distant, futures prices would have value as price forecasts and as a way to structure long-term swaps and insurance contracts. Agricultural commodity markets generally exhibit mean reversion in spot prices and convenience yields. Spot markets also exhibit seasonality. This study develops and implements a procedure to generate long-term futures curves from existing futures prices. Data on lean hogs and soybeans are used to show that the method provides plausible results.

Key words: Bayesian statistics, commodity markets, futures, mean reversion, seasonality.

Futures contracts on agricultural commodities have a limited number of maturity dates. For example, the most distant maturity date for corn, soybeans and wheat is at most four years. For futures contracts on livestock products, the furthest maturity date is about two years. This situation is unfortunate for two reasons. First, futures markets have long been known to be more accurate in predicting future prices than large-scale econometric models (Just and Rausser (1981)).[1] This suggests that longer-maturity contracts would have public value as predictors of future prices. Second, the agricultural sector has not participated in the development of swap contracts to the extent that is common in other markets and sectors. We hypothesize that some market participants might be willing to use these contracts if there was an inexpensive way to find the fair value of the long-term contracts given the information implicit in the short-term contracts that do trade.

One key piece of information needed to successfully construct a swap is the long-term futures curve. For crude oil and Eurodollars, maturity dates as far as ten years in the future are available. For other markets, such as gold, stock indices and exchange rates, the futures curve can be determined by simple arbitrage formulae (e.g., cost of carry for gold, interest rate minus dividend for stocks, and the interest rate differential for currencies). However, the long-term futures curve cannot be obtained from current futures contracts in agriculture due to the lack of long-term maturities.

To see why it might be useful to introduce long-term swaps in agriculture, consider the circumstances faced by a farmer who is about to purchase land or build a livestock facility, or a soybean processor who plans to construct a new crushing plant. These investments will typically not provide a return that covers costs for a decade or more. We are not aware of any long-term swaps or forward contracts that are routinely used in agriculture to mitigate these long-term risks. Firms making these investments might be willing to forgo the benefits associated with price volatility and instead sign long-term swaps or forward contracts to ensure a return on investment, but they cannot do this because the long-term futures curve is not available.[2] It is also likely that the interest paid on funds that are borrowed to make these long-term investments would be lower if exposure to long-term price risk could be mitigated.

One challenge in estimating the long-term futures curve in agriculture is that commodity supply will typically respond to prices if producers are given enough time. This means that long-term futures contracts, if they did exist, would exhibit a trend toward expected production costs in the absence of risk premia.[3] Equivalently, the market would exhibit mean reversion. The speed of mean reversion will depend on the commodity in question, as well as on particular market circumstances, such as the distance of current spot price from production costs, expected production costs for future periods, the level of carryover stocks, current and expected weather patterns, livestock productivity, or the level of convenience yield. These circumstances will be known to market participants and will be used by them in buying and selling the futures and options contracts that do trade. But these relationships are complex and far more difficult to understand than the simple no-arbitrage relationships that exist for investment commodities such as gold, stock indices, or currency.

This paper develops and implements a procedure for extracting the commodity- and time-specific parameters required to construct long-term futures curves where mean reversion exists. A number of studies report evidence of mean reversion in commodity cash prices (e.g., Peterson, Ma, and Ritchey (1992); Allen, Ma, and Pace (1994); Walburger and Foster (1995)). Our model builds on an influential paper by Schwartz (1997). In an out-of-sample forecasting exercise, Bernard *et al.* (2008) show that Schwartz or Schwartz and Smith (2000) type state-space models greatly outperform other models according to an RMSE criterion.

Schwartz recognized that periods of temporary scarcity in commodity markets, as indicated by a positive convenience yield, would eventually be resolved by market forces. He constructed a model where convenience yield exhibits mean reversion and he used it to create a futures curve for crude oil. The spot price in Schwartz's two-factor model is assumed to be trending rather than mean reverting. When convenience yield is a constant, the spot price in Schwartz's model exhibits geometric Brownian motion. Our problem is more complex because we expect mean reversion both in the convenience yield and the price level. In our setup, the spot price is allowed to exhibit mean reversion in both the historical and risk-neutral measures.[4] For example, if lean hog supplies are plentiful and prices are below production costs, the market might show a very normal convenience yield, but we will expect a contraction in supply and a reversion in the price level to production costs.

A second feature of our model is that we recognize that agricultural markets exhibit seasonality, and that these seasonal patterns will be evident in the futures contracts that do trade and in the long-term futures curve that we want to estimate. Sørensen (2002) modeled seasonality in agricultural commodity futures by adding a deterministic seasonal component to the commodity spot price. He derived a closed-form futures pricing formula based on his one-factor model with seasonality. Richter and Sørensen (2002) proposed a three-factor model to explore the seasonality patterns in both spot price level and volatility in commodity markets. However, closed-form solutions for futures pricing formulas are not available for their model setup.

Seasonality is introduced into our model by allowing the parameters in the drift terms of the two factors (spot price and convenience yield) to be a periodical function of calendar time. The evaluation of futures pricing expressions can be reduced to the problem of solving ordinary differential equations (Duffie, Pan, and Singleton (2000)). Adding seasonality into the model makes the solution more involved, because the corresponding stochastic differential equations are inhomogeneous in time as the drift coefficients are functions of calendar time. However, we are able to derive closed-form expressions for futures formulas, which greatly facilitate the empirical work.

As Schwartz recognized, a negative relationship between supply/inventories and convenience yields is predicted by the theory of storage. Thus, when inventory is low and supply is scarce the convenience yield from marginal storage is high, and the opposite is true when inventory is high and supply is large. Since commodity supply exhibits seasonality, the convenience yield is also assumed to behave as a mean-reverting process with seasonality. The present empirical work suggests that the speed of mean reversion is higher in the lean hog market than in the soybean market. Seasonal patterns are clear in the estimation results for both agricultural commodities. The impact of our two modeling innovations (mean-reverting spot prices and seasonality) is shown by comparing Schwartz's model to ours.

Similar to the partially overlapping time series (POTS) model introduced by Smith (2005), our estimation relies on data for all of the futures contracts being traded on a particular date. However, our study differs from Smith's in a number of important aspects. In particular, Smith focused on capturing the volatility dynamics of commodity futures, whereas our main interest is in estimating the long-term futures curve. Hence, even though the POTS model may prove quite useful for pricing options on futures contracts, it cannot be employed to estimate the futures price of long-term non-traded contracts, which is essential for the present exercise. Another important difference between the POTS model and ours is that our theoretical framework prices the entire futures curve by imposing no-arbitrage restrictions across all contracts.[5] In contrast, the POTS model does not impose any theory-based restriction among the prices of futures contracts for different maturities.

The rest of this paper is organized as follows. In the next section, we generalize Schwartz's two-factor model, and seasonality is introduced into the proposed model. In the third section, futures pricing formulas are derived. Section four describes the empirical specification, the data set, and the estimation method. The econometric results and their analysis are discussed in section five. The last section concludes the paper.

Schwartz's Model and A Generalization

Schwartz advanced a path-breaking model of commodity prices, by incorporating Kaldor's (1939) fundamental insight that commodity markets are characterized by convenience yields. Schwartz postulated that the convenience yield net of storage cost (net convenience yield), c_t, follows the Ornstein-Uhlenbeck stochastic process

$$dc_t = (u_c - k_c c_t)dt + \sigma_c dw_c(t), \tag{1}$$

where u_c/k_c is the long-term mean of the net convenience yield, $k_c > 0$ is the net convenience yield's speed of mean reversion, and $dw_c(t)$ is a Wiener process. However, Schwartz assumed that the process of the commodity spot price, S_t, is not mean reverting. Instead, he assumed it to behave as a geometric Brownian motion when net convenience yield (c_t) is a constant,

$$dS_t = (u_s - c_t)S_t dt + \sigma_s S_t dw_s(t), \tag{2}$$

where $dw_s(t)$ is a Wiener process, and $dw_c(t)dw_s(t) = \rho_{sc}dt$. By defining $x_t \equiv \ln(S_t)$, application of Ito's Lemma yields the stochastic process for x_t,

$$dx_t = (u_x - c_t)dt + \sigma_x dw_x(t), \tag{3}$$

where $u_x \equiv u_s - \sigma_s^2/2$, $\sigma_x \equiv \sigma_s$, $dw_x(t) \equiv dw_s(t)$, and $\rho_{xc} \equiv \rho_{sc}$.

The expected total rate of return to the commodity holder consists of the expected relative price change ($E(dS_t/S_t) = u_s - c_t$) plus the net convenience yield (c_t). In equilibrium, the expected rate of return to the commodity holder must equal the risk-free rate (r) plus the risk premium associated with the stochastic process dx_t (λ_x), i.e., $u_s - c_t + c_t = r + \lambda_x$. Therefore, the corresponding risk-neutral processes are

$$dc_t = (u_c - k_c c_t - \lambda_c)dt + \sigma_c dw_c^Q(t), \tag{4}$$

$$dS_t = (r - c_t)S_t dt + \sigma_s S_t dw_s^Q(t), \tag{5}$$

where λ_c is the market price for the risk associated with the stochastic process of c_t, and $dw_c^Q(t)$ and $dw_s^Q(t)$ are the Wiener processes under the equivalent martingale measure. By application of Ito's lemma, the risk-neutral process of dx_t can be shown to be

$$dx_t = (r - \sigma_x^2/2 - c_t)dt + \sigma_x dw_x^Q(t). \tag{6}$$

Note that $dw_x^Q(t) = dw_s^Q(t)$ and $dw_c^Q(t)dw_x^Q(t) = \rho_{xc}dt$. For convenience, this model is labeled Model 1.

Price Mean Reversion

A stylized fact of commodity markets is that convenience yields are positively associated with spot prices. Typically, when a commodity is in relatively short supply, its price is high and its convenience yield is high as well. Therefore, the net convenience yield is postulated to consist of a linear function of the logarithm of the spot price ($k_x x_t$) plus a stochastic component (y_t):

$$c_t = y_t + k_x x_t. \tag{7}$$

The dynamics of y_t is given by the Ornstein-Uhlenbeck stochastic process

$$dy_t = (u_y - k_y y_t)dt + \sigma_y dw_y(t), \tag{8}$$

with $dw_x(t)dw_y(t) = \rho_{xy}dt$. Hence, the corresponding spot price stochastic process is

$$dS_t = [u_s - y_t - k_x \ln(S_t)]S_t dt + \sigma_s S_t dw_s(t), \tag{9}$$

and Ito's Lemma yields the Ornstein-Uhlenbeck stochastic process for the logarithm of the spot price

$$dx_t \;\; = \;\; (u_x - y_t - k_x x_t)dt + \sigma_x dw_x(t). \tag{10}$$

In equilibrium, the instantaneous expected total return to commodity holders must equal the risk-free rate plus the associated market price of risk:

$$r + \lambda_x = (u_s(t) - y_t - k_x x_t) + (y_t + k_x x_t) \tag{11}$$

$$\Rightarrow (u_s(t) - y_t - k_x x_t) - \lambda_x = r - (y_t + k_x x_t). \tag{12}$$

Therefore, the risk-neutral process of dS_t may be written as:

$$dS_t = [r - (y_t + k_x x_t)]S_t dt + \sigma_s S_t dw_s^Q(t). \tag{13}$$

Then, application of Ito's lemma yields

$$dx_t = [r - \sigma_x^2/2 - (y_t + k_x x_t)]dt + \sigma_x dw_x^Q(t). \tag{14}$$

Denoting the market price for the y_t risk as λ_y, the risk-neutral process of dy_t is

$$dy_t = (u_y - k_y y_t - \lambda_y)dt + \sigma_y dw_y^Q(t), \tag{15}$$

where $dw_x^Q(t)dw_y^Q(t) = \rho_{xy}dt$.

This generalized model is referred to as Model 2. It is clear that Model 1 is a special case of Model 2, because the two models are identical if k_x is restricted to equal zero, in which case $c_t = y_t$. The key difference between Models 1 and 2 is that, when c_t is a constant, the logarithm of the spot price in Model 1 behaves like a Geometric Brownian motion. In contrast, when y_t is a constant, the logarithm of the spot price in Model 2 satisfies an Ornstein-Uhlenbeck stochastic process. Empirically, testing whether k_x is equal to zero or not allows us to determine whether the spot prices are mean reverting in a given market.

Seasonality

The models considered so far assume that all parameters are constant throughout the year. Most commodity markets differ from the markets for stocks, bonds, and other conventional financial assets, in that they typically exhibit seasonal patterns. For example, prices for annual crops are high in the pre-harvest season and low at peak-harvest, and pork prices are usually high during the barbecue months. To capture this feature, the periodicity in the corresponding parameters is represented by a truncated Fourier series. Seasonality is added into the model by setting u_x in equation (10) to be a periodic deterministic function of time:

$$u_x(t) = u_{x,0} + \sum_{h=1}^{H} \left[u_{x,h,\cos} Cos(2\pi ht) + u_{x,h,\sin} Sin(2\pi ht) \right], \tag{16}$$

where H determines the number of terms in the sum, and $u_{x,0}$, $u_{x,h,\cos}$ and $u_{x,h,\sin}$ are constant seasonality parameters. Based on the Akaike Information Criterion (AIC) (see, e.g., Harvey 1981), H is selected to be equal to 2. Note that if $u_{x,h,\cos} = u_{x,h,\sin} = 0$, for $\forall h \geq 1$, $u_x(t) = u_{x,0}$, then the model does not exhibit seasonality.

The long-term mean parameter of the first component of the net convenience yield in equation (15), $u_y(t)$, is similarly generalized to allow for seasonality by assigning to it a functional form analogous to (16). In addition, the risk premia λ_x and λ_y in the previous section are also assumed to be analogous periodic function of calendar time,

$$\lambda_i(t) \equiv \lambda_{i,0} + \sum_{h=1}^{H} \left[\lambda_{i,h,\cos} Cos(2\pi ht) + \lambda_{i,h,\sin} Sin(2\pi ht) \right], \tag{17}$$

for $i = x, y$. For simplicity, Model 2 augmented with seasonality is referred to as Model 3. The risk-neutral processes (14) and (15) incorporating seasonality provide us the basic foundations for pricing futures contracts on commodity markets, which is done in the next section.

Futures Pricing

Commodity spot prices and net convenience yields are modeled in continuous time as a system of stochastic differential equations in an affine term structure class. The key advantage of affine models is that they are tractable for asset pricing purposes. We rely on the traditional no-arbitrage approach to price commodity derivatives. The seasonality component makes the derivation more complicated. However, closed-form solutions for the futures pricing formula can still be obtained. The following paragraphs show the process of valuation of commodity futures contracts in the presence of mean-reversion and seasonality.

The risk-neutral process of the two latent variables defined in the previous section for the advocated model can be written as

$$\begin{bmatrix} dx_t \\ dy_t \end{bmatrix} \sim N\left(\begin{bmatrix} r - \sigma_x^2/2 - k_x x_t - y_t \\ u_y(t) - \lambda_y(t) - k_y y_t \end{bmatrix} dt, \begin{bmatrix} \sigma_x^2 & \rho_{xy}\sigma_x\sigma_y \\ \rho_{xy}\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix} dt \right). \tag{18}$$

This may be expressed more compactly as

$$d\mu_t = (\kappa_0(t) - \kappa_1 \mu_t)dt + V dw^Q(t), \tag{19}$$

by defining $\mu_t \equiv [x_t, y_t]'$, $\kappa_0(t) \equiv [r - \sigma_x^2/2, \psi(t)]'$, $\psi(t) \equiv u_y(t) - \lambda_y(t) \equiv \psi_0 + \sum_{h=1}^{H} [\psi_{h,\cos} Cos(2\pi ht)$ $+ \psi_{h,\sin} Sin(2\pi ht)]$, $\psi_0 \equiv u_{y,0} - \lambda_{y,0}$, $\psi_{h,\cos} \equiv u_{y,h,\cos} - \lambda_{y,h,\cos}$, $\psi_{h,\sin} \equiv u_{y,h,\sin} - \lambda_{y,h,\sin}$,

$\kappa_1 \equiv \begin{bmatrix} k_x & 1 \\ 0 & k_y \end{bmatrix}$, $V \equiv \begin{bmatrix} \sigma_x^2 & \rho_{xy}\sigma_x\sigma_y \\ \rho_{xy}\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}$, and ' is the transpose operator.

Duffie, Pan, and Singleton (2000) analyzed a set of stochastic processes that includes processes like (19). By applying the method they proposed, a closed-form solution for the futures price at date t maturing at time T can be obtained as follows:

$$\begin{aligned} F(t,T) &= E_t^Q[S(T)] \\ &= E_t^Q\{\exp[\phi_0 + \phi\mu(T)]\} \qquad (20) \\ &= \exp[\alpha(t,T) + \beta(t,T)\mu(t)] \end{aligned}$$

$$\Rightarrow \quad f_{t,T} \equiv \ln(F(t,T)) = \alpha(t,T) + \beta(t,T)\mu(t), \qquad (21)$$

where $E_t^Q[.]$ is the expectation operation under the risk-neutral probability measure. Since the first factor is defined to be the logarithm of the spot price ($x_t \equiv \ln(S(t))$), it must be the case that $\phi_0 = 0$ and $\phi^{'} = [1,0]$.[6] To prevent arbitrage, coefficients $\alpha(t,T)$ and $\beta(t,T)$ need to satisfy the following ordinary differential equations (ODEs)

$$\frac{\partial \beta(t,T)}{\partial t} = \kappa_1^{'}\beta(t,T) \text{ and} \qquad (22)$$

$$\frac{\partial \alpha(t,T)}{\partial t} = -\kappa_0(t)\beta(t,T) - \frac{1}{2}\beta^{'}(t,T)V\beta(t,T), \qquad (23)$$

with boundary conditions $\beta(T,T) = \phi$ and $\alpha(T,T) = \phi_0$. Closed-form solutions for $\alpha(t,T)$ and $\beta(t,T)$ are shown in Appendix A.

Empirical Analysis

In the advocated model, we employ the logarithm of the spot price and the net convenience yield as the two latent state variables. Recall equation (19) and define $\Lambda(t) \equiv [\lambda_x(t), \lambda_y(t)]'$. Then, the historical process of the two latent variables can be written in matrix form as

$$d\mu_t \sim N((\kappa_0(t) - \kappa_1\mu_t + \Lambda(t))dt, Vdt). \tag{24}$$

We apply the first-order Euler discretized version of the continuous time model (24) with discretization interval $\Delta = \frac{1}{12}$ to reflect monthly data. The discretized empirical model is

$$\mu_{t+\Delta} = \mu_t + (\kappa_0(t) - \kappa_1\mu_t + \Lambda(t))\Delta + \sqrt{\Delta}\varepsilon_t,\ \varepsilon_t \sim N(\underline{0}_{(2\times1)}, V). \tag{25}$$

The likelihood of observing the latent factors can be calculated from equation (25). In addition, we also observe futures prices from the markets, and the likelihood of observing the market prices can be inferred from the following empirical futures models.

According to equation (21), $f_{t,T} \equiv \ln(F(t,T)) = \alpha(t,T) + \beta(t,T)\mu(t)$. Following Chen and Scott (1993), we assume that all but two futures contract prices are observed with measurement error. Suppose we have a historical data set consisting of $M > 2$ series of (logarithms of) futures prices with M different times to maturity. Assume that among the M futures contracts with distinct maturity dates, two of the prices are perfectly correlated with the state variables μ_t, and the remaining $(M-2)$ prices are observed with normally distributed errors e_t. Denote the vector with the two perfectly correlated futures prices as $f_t^\circ \equiv [f^\circ_{t,T_1^\circ}, f^\circ_{t,T_2^\circ}]'$ and their maturity dates $[T_1^\circ, T_2^\circ]'$. Similarly, let $f_t^\bullet \equiv$

$[f^{\bullet}_{t,T^{\bullet}_1}, f^{\bullet}_{t,T^{\bullet}_2}, \cdots, f^{\bullet}_{t,T^{\bullet}_{M-2}}]^{'}$ represent the $(M-2)$ imperfectly correlated futures and $[T^{\bullet}_1, \cdots, T^{\bullet}_{M-2}]^{'}$ be their maturity dates, respectively. Then,

$$f^{\circ}_t = \alpha^{\circ}(t) + \beta^{\circ}(t)\mu_t, \tag{26}$$

$$f^{\bullet}_t = \alpha^{\bullet}(t) + \beta^{\bullet}(t)\mu_t + e_t, \tag{27}$$

where $\alpha^{\circ}(t) \equiv [\alpha(t, T^{\circ}_1), \alpha(t, T^{\circ}_2)]^{'}$, $\beta^{\circ}(t) \equiv \begin{bmatrix} \beta_1(t, T^{\circ}_1) & \beta_2(t, T^{\circ}_1) \\ \beta_1(t, T^{\circ}_2) & \beta_2(t, T^{\circ}_2) \end{bmatrix}$, $\alpha^{\bullet}(t) \equiv [\alpha(t, T^{\bullet}_1), \alpha(t, T^{\bullet}_2)]^{'}$, and $\beta^{\bullet}(t) \equiv \begin{bmatrix} \beta_1(t, T^{\bullet}_1) & \beta_2(t, T^{\bullet}_1) \\ \beta_1(t, T^{\bullet}_2) & \beta_2(t, T^{\bullet}_2) \end{bmatrix}$. The vector of errors associated with the log-futures not perfectly correlated with the state variables is assumed to be multivariate normally distributed, i.e., $e_t \sim N(\underline{0}_{((M-2)\times 1)}, \sigma^2_e \Omega)$, where $\underline{0}_{((M-2)\times 1)}$ is an $(M-2)$ vector of zeros, $\sigma^2_e > 0$ is a scalar, Ω is an $(M-2) \times (M-2)$ matrix with the i, jth element equal to $\rho^{|i-j|}$ for $\rho \in (-1, 1)$, and $\alpha(t, T)$ and $\beta(t, T)$ are defined in Appendix A.

Since the two latent factors are not observed, direct estimation of the historical evolution equation (25) is not feasible. However, given equation (26), the factors can be solved for as $\mu_t = [\beta^{\circ}(t)]^{-1}[f^{\circ}_t - \alpha^{\circ}(t)]$, provided the (2×2) matrix $\beta^{\circ}(t)$ is invertible. In this way, the value of the state variables can be exactly filtered out at each sample date, by inversion based on the two contract prices observed without errors.

Description of the Data

Futures prices for two agricultural commodities, soybeans and lean hogs, are employed to estimate the models. The futures prices involved are the settlement prices at the Chicago Mercantile Exchange (CME) for the 15th calendar day of each month from January 1978 through January 2010, for a total of 385 observation dates.[7] If the 15th of the month is a holiday, the nearest trading day's settlement price is used. The settlement prices observed on days with zero trading volume are discarded, because they are set by the CME administration for the purpose of calculating margins. In other words, these prices are not actual trading prices. The price units are cents/bushel and cents/pound for soybean futures and lean hogs futures, respectively.

Since the longest maturity in the soybean (lean hog) futures sample is 34 (19) months, the ideal data set would consist of a panel of $385 \times 34 = 13,090$ ($385 \times 19 = 7,315$) observations. However, futures for some maturities are not traded. Soybean futures currently have only seven maturity months: January, March, May, July, August, September, and November. Lean hog futures have eight maturity months: February, April, May, June, July, August, October, and December. In addition, data with far-away maturities are often missing because they are not traded. For example, for January 1980 only seven prices are observed for soybean futures. They are the 2nd, 4th, 6th, 7th, 8th, 10th, and 12th elements of the 25th row of our data set, which correspond to the expiration dates of March, May, July, August, September, and November of 1980 and January of 1981. Letting the i, jth element of our data set be the price of the futures contract that expires j months after date i, this means that all of the elements in the 25th row of our soybean data set are missing except the 2nd, 4th, 6th, 7th, 8th, 10th, and 12th columns. All of the other elements for this data row are recorded as unobserved in our data set. Hence, given the futures contract specifications, the total number of observations available for soybean (lean hog) futures prices is 3,157 (3,032).

Empirical Method

Bayesian Markov chain Monte Carlo (MCMC) methods are employed to estimate the model parameters. Bayesian techniques have been used quite often over the past decade to analyze state-space models (see Hore *et al.* (2010) and Durbin and Koopman (2000)). A recent article by Harvey and Koopman (2009) highlights that the two main approaches to estimate state-space models are maximum likelihood and Bayesian methods. Given that one of the key issues in the development of longer-term futures is the confidence market players have in constructing long-term futures curves, we chose to develop an estimation procedure that would allow us to separate the variation in projections between the parameter uncertainty (model uncertainty) and observational errors. The Bayesian framework does this more naturally than the maximum likelihood approach. Also, the proposed framework provides a streamlined way to produce credible intervals for nonlinear functions of the estimated parameters,[8] such as the projections for the futures curves.

The empirical method described below is designed for Model 3. Models 1 and 2 can be easily retrieved by imposing the corresponding parameter restrictions into the procedure. We assume a constant risk-free rate of r = 5 percent.[9] We adopt non-informative priors for $\vec{u}_x, \vec{u}_y, k_x, k_y, V, \rho, \vec{\lambda}_x$, and $\vec{\lambda}_y$, where $\vec{u}_i \equiv [u_{i,0}, u_{i,1,cos}, u_{i,1,sin}, u_{i,2,cos}, u_{i,2,sin}]$ and $\vec{\lambda}_i \equiv [\lambda_{i,0}, \lambda_{i,1,cos}, \lambda_{i,1,sin}, \lambda_{i,2,cos}, \lambda_{i,2,sin}]$ for $i = x, y$. As such, the posterior distributions for these parameters are effectively the likelihoods for the parameters under the model specification. The exception is σ_e^2, for which we impose the conjugate prior $\sigma_e^2 \sim Inv\text{-}\chi^2(\overline{\nu}_e, \overline{\sigma}_e^2)$. This prior is equivalent to the addition of $\overline{\nu}_e$ data points with a sample variance of $\overline{\sigma}_e^2$. For this study, $\overline{\nu}_e$ is set at 4 and $\overline{\sigma}_e^2$ is set at 0.0005. For several of the parameters, explicit posterior distributions cannot be derived. For those cases, algorithms have been derived to sample from the unspecified distributions based on their proportionality with the model likelihoods.

Defining the set of parameters for the jth iteration as $\Phi^{(j)} \equiv \{V^{(j)}, \vec{\lambda}_y^{(j)}, \vec{\lambda}_x^{(j)}, \vec{\psi}^{(j)}, k_x^{(j)}, k_y^{(j)}, \rho^{(j)}, \sigma_e^{2(j)}\}$, and letting $\Phi_{-z}^{(j)}$ denote all of the components of $\Phi^{(j)}$ except for z, the advocated MCMC iteration steps are as follows.

Step 1. Specify starting values for parameter and missing observations $\Phi^{(0)}$.

Step 2. Given $[V^{(j)}, \vec{\lambda}_y^{(j)}, \vec{\lambda}_x^{(j)}, \sigma_e^{(j)}]$, estimate $[\vec{\psi}^{(j+1)}, k_x^{(j+1)}, k_y^{(j+1)}, \rho^{(j+1)}]$ by means of an effective adaptive, general purpose MCMC algorithm called t-walk developed by Christen and Fox (2010). The t-walk compares the likelihood of observing futures prices and state variables given (i.1) and (i.2) (i.e., the likelihood given existing parameter values) with (ii.1) and (ii.2), (i.e., the likelihood given proposed parameter values).

$$
\begin{aligned}
&\text{(i.1) } [f_t^{\bullet(j)} - \alpha^{\bullet(j)} - \beta^{\bullet(j)}\mu_t^{(j)}] \sim N(\underline{0}_{((M-2)\times 1)}, \sigma_e^{2(j)}\Omega^{(j)}),\\
&\text{(i.2) } \left[\mu_{t+\Delta}^{(j)} - \mu_t^{(j)} - (\kappa_0(t)^{(j)} - \kappa_1^{(j)}\mu_t^{(j)} + \Lambda(t)^{(j)})/\Delta\right]/\sqrt{\Delta} \sim N(\underline{0}_{(2\times 1)}, V^{(j)}),\\
&\text{(ii.1) } [f_t^{\bullet(j)} - \alpha^{\bullet(prop)} - \beta^{\bullet(prop)}\mu_t^{(prop)}] \sim N(\underline{0}_{((M-2)\times 1)}\sigma_e^{2(j)}\Omega^{(prop)}),\\
&\text{(ii.2) } [\mu_{t+\Delta}^{(prop)} - \mu_t^{(prop)} - (\kappa_0^{(prop)}(t) - \kappa_1^{(prop)}\mu_t^{(prop)} + \Lambda(t)^{(j)})/\Delta]/\sqrt{\Delta}\\
&\qquad\qquad \sim N(\underline{0}_{(2\times 1)}, V^{(j)}),
\end{aligned}
$$

where $\mu_t^{(\cdot)}$ is computed from equation (26) using $\vec{\psi}^{(\cdot)}$, $k_x^{(\cdot)}$, and $k_y^{(\cdot)}$.[10]

Step 3. Given $[\vec{\psi}^{(j+1)}, k_x^{(j+1)}, k_y^{(j+1)}, \rho^{(j+1)}, \vec{\lambda}_y^{(j)}, \vec{\lambda}_x^{(j)}, \sigma_e^{(j)}]$, use the Metropolis-Hastings algorithm to generate $V^{(j+1)}$, as follows:

(a) Draw $V^{(prop)} \sim \mathit{Inv\text{-}Wishart}_{Nobs-3}((Nobs-3)V^{(j)})$ where $Nobs$ is the number of observations.

(b) Calculate the acceptance ratio, $R \equiv \frac{Prob(V^{(prop)}|\Phi_{-V}^{j}, f_t^{(j)})Prob(V^{(prop)}|V^{(j)})}{Prob(V^{(j)}|\Phi_{-V}^{j}, f_t^{(j)})Prob(V^{(j)}|V^{(prop)})}$.[11]

(c) Draw a random variable ζ from a standard uniform distribution and set $V^{(j+1)} = V^{(prop)}$ if $\zeta < R$. Otherwise, set $V^{(j+1)} = V^{(j)}$.

Step 4. Given $[V^{(j+1)}, \vec{\psi}^{(j+1)}, k_x^{(j+1)}, k_y^{(j+1)}]$, update the unobserved futures prices to get $f_t^{(j+1)}$. In this step, we first compute $\mu_t^{(j+1)}$ from the estimated risk-neutral parameters and the futures observed with no errors using equation (26). Then, we update the unobserved futures from the estimated factors and the other corresponding parameters by means of equation (21).

Step 5. Given $[V^{(j+1)}, \vec{\psi}^{(j+1)}, k_x^{(j+1)}, k_y^{(j+1)}, \rho^{(j+1)}, \sigma_e^{(j)}]$, draw $\vec{\lambda}_y^{(j+1)}$ and $\vec{\lambda}_x^{(j+1)}$ from a multivariate normal distribution (see Appendix B for details).

Step 6. Given $[\vec{\lambda}_y^{(j+1)}, \vec{\lambda}_x^{(j+1)}, V^{(j+1)}, \vec{\psi}^{(j+1)}, k_x^{(j+1)}, k_y^{(j+1)}, \rho^{(j+1)}]$, draw $\sigma_e^{2(j+1)} \mid f_t^{(j+1)}, \Phi_{-\sigma_e}^{(j+1)} \sim Inv\text{-}\chi^2(\overline{\nu}_e + n_f, \frac{\overline{\nu}_e \overline{\sigma}_e^2 + n_f s_e^2}{\overline{\nu}_e + n_f})$, where n_f is the total number of observed futures prices and s_e^2 is the mean squared error of the observed futures prices.

Step 7. Set $j = j+1$.

Step 8. If the maximum iteration is reached, stop. Otherwise, go to Step 2.

Estimation Results

The advocated Bayesian MCMC procedure is performed with four chains for each model and market. Each chain is started at a different initial value and run for two million iterations. The first one million iterations are discarded as a burn-in period, and the remaining one million iterations are tested for convergence by means of Gelman and Rubin (1992) tests. As evinced by the Gelman-Rubin test statistics reported in Appendix C, all of the chains converge adequately for the three models in both markets.

Lean Hog Market

Parameter estimates for the lean hog market are shown in table 1.[12] The posterior probability of parameter k_x being positive is estimated to be greater than 97.5% for both Models 2 and 3, which supports the postulation that the spot price in the lean hog market is mean reverting. This also implies that the convenience yield is positively related to the spot price. Comparing Model 1 with Model 2, the lower bound of the credible interval for the correlation coefficient between the two factors in Models 1 is larger than the upper bound of the corresponding credible interval estimated by Model 2, after we set the convenience yield to be a function of the logarithm of the spot price. The total expected return on the spot price (u_x), the long-term mean of net convenience yield, and the market prices of convenience yield risk are all negative at the median, but their 95% posterior density region contains zero except for λ_c in Models 1 and 2.

All of the seasonality parameters in Model 3 are estimated precisely enough to determine the sign with high probability. This indicates that the lean hog market exhibits a strong seasonal pattern. If the data exhibit seasonality but the model fails to incorporate it, the seasonal variability in the factors will be captured by the instantaneous volatility term. Hence, given a data set exhibiting seasonality, models not allowing for seasonality will estimate a significantly higher value of σ_x and σ_y than models allowing for it, which is confirmed by the values reported in table 1.

Seasonality may also significantly affect the model's ability to fit the market data. The estimates of σ_e describe the inferred standard deviation on the noise terms that allow for deviations between theoretical and observed log-futures prices. One source of this noise in our specific data set may be that the settlement prices are established by the CME administrators, which may not exactly match the market prices. Errors in data registration, price limits and handling of bid-ask spreads may also contribute to the noise term. As can be seen from table 1, the upper bound of the credible interval for σ_e in Model 3 is smaller than the lower bound of the corresponding credible interval estimated by Model 1 and 2, which signals a better fit of the observed data.

Figures 1 and 2 show the term structure of median lean hog futures prices implied by the three models on January 15, 2010 and December 16, 2002, respectively. On January 15, 2010, the spot price in the lean hog market was high relative to production costs. For the futures curve with a short time to maturity, the curvature depends on the relative value of the net convenience yield. However, in the long run, the futures curve implied by Schwartz's model (Model 1) depends on the risk-neutral drift of the spot price process. If we evaluate the drift at the risk-neutral long-term mean of the net convenience yield using posterior medians, it is negative. So, in the long run, the slope of the futures curve predicted by Model 1 is negative. Model 2 incorporates mean reversion in the spot price. So when the spot price is relatively high, the futures curve implied by Model 2 initially decreases at a faster rate than the futures curve implied by Model 1, and then flattens out as prices approach the market's estimate of production costs. This long-term futures price ($F(t,\infty)$) is independent of the current spot price and the net convenience yield. The futures curve implied by Model 3 follows the trend of Model 2, but with seasonality. It is clear that futures prices implied by Model 3 fit the observed prices more precisely compared to the models which ignore seasonality. A local maximum is observed when time to maturity is six months, which corresponds to a July maturity date. July is the traditional barbecue season in the U.S. and the demand for lean hogs is the highest over the year, which is consistent with historical futures price patterns.

In contrast to January 15, 2010, the spot price for lean hogs was relatively low on

December 16, 2002. With mean reversion embedded, Model 2 predicts that the futures curve will increase at a decreasing rate and will converge to the long-term futures price ($F(t,\infty)$). The curvature of the futures curve implied by Schwartz's model also depends on the relative level of the spot price and the net convenience yield on that date for short time maturities. However, for longer-term maturities the futures curve is predicted to be decreasing regardless of the fact that the spot price may have already been well below production costs.

Soybean Market

Model estimates for the soybean market are shown in table 2. The posterior probability of parameter k_x being positive is estimated to be greater than 97.5%, which provides empirical support for the postulation that the soybean spot price process is also mean reverting. Parameter u_x, which in Schwartz's model describes the expected appreciation rate of the non-stationary state variable (the logarithm of the spot price), also has a posterior probability of being positive greater than 97.5%. The estimates in table 2 indicate that the net convenience yield mean-reversion parameters k_c and k_y are estimated to have a high probability of being positive in Model 1 and Model 2; hence, the state variable c_t in Schwartz's model and y_t in Model 2 are stationary for soybean. The median of the estimated k_c and k_y is about 1.06, corresponding to half-lives of 7.7 months.[13] Compared to the lean hog market, the soybean market exhibits lower speeds of adjustment in the spot price (k_x) and the net convenience yield (k_y). One possible reason explaining this result is that lean hogs have a shorter production cycle, which allows producers to adjust supply faster. In Model 1, parameter u_c has an estimated posterior probability of being positive in excess of 97.5% , which implies that the long-term mean of the net convenience yield in the soybean market is positive. Parameter u_y is estimated to be negative with high probability in Models 2 and 3. However, net convenience yield in Models 2 and 3 is defined as $c_t = y_t + k_x x_t$. If we take the long-term mean of y_t and x_t to evaluate c_t, the latter is also positive.

All of the three models report similar instantaneous volatilities and instantaneous correlation coefficient between the two factors. Although for the soybean market k_x is estimated to have a posterior probability of being positive greater than 97.5%, its magnitude is small when compared to the lean hog market (for which $k_x = 0.65$ at the median), and it has little impact on the model's ability to fit the historical data. There is large overlap on the credible interval of σ_e for Models 1 and 2. The estimates of σ_e describe the inferred standard deviation on the noise terms that allow for some deviation between theoretical and observed log-futures prices.

Seasonality is important and significant in the soybean market. There is only one seasonality parameter ($\lambda_{y,1,cos}$) whose 95% posterior density region includes zero. Furthermore, the model with seasonality (Model 3) yields a credible interval of σ_e with an upper bound smaller than the lower bound of the corresponding credible interval generated by its counterpart without seasonality (Model 2). The non-seasonal part of risk premia associated with the net convenience yield process is estimated to have more than 97.5% posterior probability of being negative in all of the models.

Figures 3 and 4 show the term structure of median soybean futures prices implied by the three models on January 15, 2010 and November 15, 2000, respectively. From figure 3, we can see that Model 3 precisely captures the seasonality feature of the CME data for maturities shorter than 20 months. For completeness, figure 3 also shows the settlement price for contracts with positive open interest but zero trading volume. These prices were set by the CME to calculate the margins that need to be posted, but are not prices at which trading actually occurred on January 15, 2010. The estimated futures curve suggests that such zero-volume settlement prices significantly understated the seasonality that characterizes the soybean market.

On January 15, 2010, the soybean price was relatively high. The futures curve implied by Model 3 shows a market expectation of a reduction in price levels to the market's estimate of production costs. For contracts with a short time to maturity, the curvature of the futures curve implied by Schwartz's model depends on the relative value of the net convenience yield. If we evaluate the drift at the risk-neutral long-term mean of the net convenience yield, it is negative. Consequently, the futures curve has a constant negative slope in the long run. With a short time to maturity (e.g., less than 24 months), Models 1 and 2 predict similar futures prices. However, as time to maturity increases, the difference becomes noticeable, with Model 2 predicting a lower value of long-term futures prices. The futures curve implied by Model 3 follows the trend of Model 2 but with seasonality. We observe a local maximum when time to maturity is equal to 6 months, which corresponds to a maturity date of July 15, 2010. July is right before the U.S. harvest season. At that time, the supply is at the lowest point of the year. So it is not surprising to expect the spot price to be highest on that month.

Figure 4 shows the term structure of the futures curve predicted by Models 1 through 3 on November 15, 2000. Compared to January 2010, the soybean spot price was much lower on November 2000. The net convenience yield is also well below the long-term mean implied by Model 1. Since the stochastic process of the net convenience yield in Schwartz's model is assumed to be mean-reverting, the net convenience yield is expected to increase in the following months. And since the net convenience yield's speed of mean reversion is much lower in the soybean market than in the lean hog market, it takes a longer time for the net convenience yield to reach its long-term mean. As the net convenience yield recovers, the futures price is expected to increase at a decreasing rate. Finally, when the net convenience yield reaches its long-term mean, the risk-neutral process of the spot price has a negative risk-neutral drift. Consequently, the term structure of the long-term futures curve is expected to have a negative slope in Schwartz's model. On the other hand, price mean reversion is assumed in Model 2. On that date, a low value of the spot price and the net convenience yield is implied by Model 2, so both x_t and y_t are expected to increase. As a result, Model 2 predicts that futures prices will increase at a decreasing rate with time to maturity.

Comparison Among Models

Figures 1 through 4 suggest that Model 3 dominates Models 1 and 2 in terms of fitting historical data, at least for the dates selected. To provide a more rigorous comparison of the model specifications, we computed the Bayesian deviance information criterion (*DIC*) advocated by Spiegelhalter *et al.* (2002),

$$DIC \equiv D(\overline{\theta}) + 2p_D, \tag{28}$$

where $D(\theta) \equiv -2log(Pr(data \mid \theta))$, $p_D \equiv \overline{D(\theta)} - D(\overline{\theta})$ measures the complexity of the model, $\overline{\theta}$ represents the posterior means of the parameters, $\overline{D(\theta)}$ is the mean deviance, and $D(\overline{\theta})$ is the deviance of the means. *DIC* may be interpreted as a classical estimation of fit, $D(\overline{\theta})$, plus twice the effective number of parameters, p_D. Spiegelhalter *et al.* proposed that *DIC* inferences could follow similar guidelines to AIC tests, where differences of less than 2 show similar support among models, whereas differences greater than 3 indicate stronger support for one model over another.

The *DIC* comparison results are reported in table 3. Both the lean hog and soybean results are shown and the results are the same across the commodities. The differences between the *DIC*s of Models 1 and 2 exceed 20, which indicates strong statistical evidence for the inclusion of mean reversion in the spot price. The addition of seasonality parameters to the model is also strongly favored as can be seen by comparing the *DIC*s of Model 3 to those of Models 1 and 2. The *DIC* results parallel the inferences that can be drawn from figures 1 through 4.

95 Percent Credible Band of Futures Prices

Given the substantial uncertainty associated with long-term commodity prices, it is useful to look at the 95 percent credible band of the futures prices predicted by our model (Model 3). Figure 5 shows such a band for soybean futures on January 15, 2010. By construction, the predicted futures curve goes through the futures prices with 2 and 12 months until maturity, as those two futures prices were taken to be the ones perfectly correlated with the latent factors.

The band corresponding to parameter variability shows the 95 percent credible band of the futures curve induced by the uncertainty in model parameters only. It is observed that the 95 percent credible band is very tight for futures prices with a short time to maturity. As time to maturity increases, the variability in the Bayesian estimates leads to a wider 95 percent credible band. When time to maturity is near 100 months, the width of the band exceeds $1/bushel.

The "total variability" band shows the 95 percent credible band of futures prices when we consider both the uncertainty in the model parameters and the observation errors. Allowing for observation errors has a negligible effect on the median value of the estimated futures prices, because errors are assumed to have zero mean. However, observation errors have a dramatic effect on the 95 percent credible band of futures prices. The total variability band is much wider than the parameter variability band for short maturities. As time to maturity increases, however, parameter uncertainty accounts for a larger share of the futures uncertainty relative to the observation errors.

Figure 6 for the lean hog market tells a similar story to its counterpart for the soybean market, figure 5. The width of the 95 percent "parameter variability" credible band increases with time to maturity. One striking difference with figure 5 is that the 95 percent "total variability" credible band is slightly wider for short times to maturity. This result is somewhat counterintuitive at first glance. Note, however, that in equation (27) the error term is added to the logarithm of futures prices instead of futures prices themselves. Thus, when taking the exponential of the logarithm of futures prices including errors to compute futures prices, a wider band is obtained for larger values of the logarithm of futures prices. On this particular date (January 15, 2010), futures prices are decreasing with time to maturity (ignoring seasonal effects). As a result, in this instance the observation error volatility dominates the volatility in model parameters, which leads to a wider 95 percent credible band with short times to maturity.

Conclusions and Future Research

With the purpose of developing a method to estimate the long-term futures curve for agricultural futures, we generalize Schwartz's two-factor model by allowing for both mean reversion in spot prices and seasonality. These are key features of agricultural commodity markets. Closed-form futures pricing formulas are derived. We show that Schwartz's model is a special case of our model. Soybean and lean hog futures price data from the CME are employed to estimate the models by means of a Bayesian MCMC algorithm. Estimates for Schwartz's model are obtained by imposing the corresponding restrictions to our model.

We show results for the markets during two historical pricing periods as examples. The first example represents a period of relatively high prices, whereas the second corresponds to a period of low prices. In both instances the results suggest an intuitive relationship between the short-term futures we observe and the long-run expected production cost. The addition of mean reversion and seasonality is supported by the model estimates and futures price projections are improved with their incorporation. The evolution of this model and the projections of long-term futures prices from it could provide support for the continued development of agricultural swaps. The price projections could also support development of long-term price risk management tools and insurance products.

As any model, the one advocated here relies on simplifying assumptions. An important one is that commodity prices revert to a seasonal, deterministic long-term mean.[14] In addition, it is known that when commodity prices display mean reversion, option prices are substantially lower than they would be under geometric Brownian motion (Casassus and Collin-Dufresne (2005)). Furthermore, adding seasonality into the model also affects the estimated instantaneous volatility, which in turn changes the fair value of options written on futures. Generalizing the present model to incorporate a stochastic long-term mean, or analyzing the model implications just mentioned, are beyond the scope of the present paper. However, they seem important topics to address in future research.

Notes

[1]It could be argued that the method we propose here is in fact an econometric model, and therefore subject to the failings of these models. While it is true that the method we propose depends on econometric estimation, the purpose of this estimation is to use the term structure of existing short-term futures to estimate the long-term futures curve. This philosophy is very different from the long-run supply and demand parameters that are typically used to drive results in structural econometric models of the type evaluated by Just and Rausser.

[2]As pointed out by an anonymous reviewer, even though multi-year rollover hedges might seem appealing in the absence of long-term futures, rollover strategies do not allow one to lock in current futures prices for crops to be harvested one or more years later. Lence and Hayenga (2001) provide a theoretical model explaining the failure of multiyear rollover hedging strategies, and empirical evidence supporting their model.

[3]Under the risk-neutral measure, futures prices for a fixed maturity are martingales. The risk-neutral measure and the physical measure differ to the extent that there are risk premia. Thus, if the spot price exhibits mean reversion in the physical measure and there are no risk premia, the futures curve must show a tendency for long-term futures to revert back to the spot price's long-term mean.

[4]Futures prices are risk-neutral expectations of future spot prices, and are martingales in the risk-neutral measure.

[5]In fact, it is this restriction which allows us to estimate long-term futures prices from the prices of short-term futures contracts.

[6]This restriction on ϕ_0 and ϕ follows from (20) and the fact that the spot price is the same as the futures price with instantaneous maturity (i.e., $S(t) = F(t,t)$). To see this, note that application of (20) yields $x_t \equiv \ln(S(t)) = \ln(F(t,t)) = \ln(E_t^Q\{\exp[\phi_0 + \phi\mu(t)]\}) = \ln(\exp[\phi_0 + \phi\mu_t]) = \phi_0 + \phi\ [x_t, y_t]^{'}$, which can only be satisfied if $\phi_0 = 0$ and $\phi^{'} = [1,0]$.

[7]In 1997, the hogs futures contract switched from live hogs to lean hogs. Live hog prices were converted to lean hog prices using the standardized conversion rate $LeanHogPrice = LiveHogPrice/0.74$. This conversion rate was the accepted rate at the time and has remained the accepted conversion rate between live and lean hog prices.

[8]Credible intervals are the Bayesian analogs of confidence intervals in frequentist statistics.

[9]For the period under analysis, the average annual interest rate corresponding to three-month treasury bills was 5.60 percent. It must be noted, however, that the interest rate does not change the analysis in any substantive way. As implied by (12) and (14), the main impact of adopting a different value for the interest rate (r) is to induce an equal change in the estimated y_t component of the net convenience yield, and a change of the same absolute value but opposite sign in the risk premium (λ_x). The model could be extended by explicitly modeling stochastic interest rates. However, Schwartz (1997) and Trolle and Schwartz (2009) show that for commodity futures the pricing error arising from ignoring the stochastic nature of interest rates is negligible.

[10]The perfectly correlated futures prices are selected to be among the observed data.

[11]Calculating $Prob(V^{(prop)} \mid \Phi^{j}_{-V}, f_t^{(j)})$ and $Prob(V^{(j)} \mid \Phi^{j}_{-V}, f_t^{(j)})$ will again resort to the empirical equations (i.1) and (i.2).

[12]See Appendix D for graphs of the posterior distributions of key parameters.

[13]The half-life expresses the expected time it takes the impact from a given shock to the process to level off by half the size of the shock. The half-life in the Ornstein-Uhlenbeck process is calculated as $\ln(2)/k$. In our case $\ln(2)/1.07 = 0.65$ years, which is about 7.7 months.

[14]See Tang (2010) for a model with stochastic long-term mean for energy commodities.

Appendix A

According to equation (22), $\frac{\partial \beta_1(t,T)}{\partial t} = k_x \beta_1(t,T)$. Together with the boundary condition $\beta_1(T,T)$ $= 1$, this implies that $\beta_1(t,T) = \exp(k_x(t-T))$. Also from equation (22),

$$\begin{aligned}
\frac{\partial \beta_2(t,T)}{\partial t} &= \beta_1(t,T) + k_y \beta_2(t,T) \\
&= \exp(k_x(t-T)) + k_y \beta_2(t,T).
\end{aligned}$$

Therefore, $\beta_2(t,T) = \frac{\exp(k_x(t-T)) - \exp(k_y(t-T))}{k_x - k_y}$.

Using the above expressions for $\beta_1(t,T)$ and $\beta_2(t,T)$, equation (23) can be written as

$$\frac{\partial \alpha(t,T)}{\partial t} = (r - \sigma_x^2/2)\beta_1(t,T) + \psi(t)\beta_2(t,T) - \frac{1}{2}\beta'(t,T)V\beta(t,T).$$

Hence,

$$\begin{aligned}
\alpha(t,T) \;=\; & \frac{r-\sigma_x^2/2}{k_x}(\exp(k_x(t-T))-1) + \frac{\psi_0}{k_x-k_y}[(\frac{1}{k_x}(\exp(k_x(t-T))-1) - (\frac{1}{k_y}(\exp(k_y(t-T))-1)] \\
& + \sum_{h=1}^{2} \frac{\psi_{h,\cos}}{k_x-k_y}(\frac{1}{k_x^2+4\pi^2h^2} - \frac{1}{k_y^2+4\pi^2h^2}) \times \\
& \{k_x[\exp(k_x(t-T))\cos(2\pi ht) - \cos(2\pi hT)] + 2\pi h[\exp(k_x(t-T))\sin(2\pi ht) - \sin(2\pi hT)]\} \\
& + \sum_{h=1}^{2} \frac{\psi_{h,\sin}}{k_x-k_y}(\frac{1}{k_x^2+4\pi^2h^2} - \frac{1}{k_y^2+4\pi^2h^2}) \times \\
& \{k_x[\exp(k_x(t-T))\sin(2\pi ht) - \sin(2\pi hT)] + 2\pi h[\exp(k_x(t-T))\cos(2\pi ht) - \cos(2\pi hT)]\} \\
& - \frac{1}{2}\{[\frac{\sigma_x^2}{2k_x} + \frac{\rho_{xy}\sigma_x\sigma_y}{k_x(k_x-k_y)} + \frac{\sigma_y^2}{2k_x(k_x-k_y)^2}] \times [\exp(2k_x(t-T))-1] \\
& - [\frac{2\rho_{xy}\sigma_x\sigma_y}{k_x^2-k_y^2} + \frac{2\sigma_y^2}{(k_x+k_y)(k_x-k_y)^2}] \times [\exp((k_x+k_y)(t-T))-1] \\
& + \frac{\sigma_y^2}{2k_y(k_x-k_y)^2}[\exp(2k_y(t-T))-1]\}.
\end{aligned}$$

Appendix B

Equation (25) can be written as

$$
\begin{aligned}
x_{t+\Delta} - x_t &= (r - \sigma_x^2/2 - k_x x_t - y_t + \lambda_x(t))\Delta + \sqrt{\Delta}\varepsilon_{1,t}, \\
y_{t+\Delta} - y_t &= (\psi(t) - k_y y_t + \lambda_y(t))\Delta + \sqrt{\Delta}\varepsilon_{2,t},
\end{aligned}
$$

where $t = \frac{1}{12}, \frac{2}{12}, ..., \frac{T-1}{12}$, T is the total number of observation dates, and $cov(\varepsilon_{1,i}, \varepsilon_{2,j}) = V$ if $i = j$, and is zero otherwise. The above two equations can be rearranged to yield

$$
\begin{aligned}
z_{1,t} &= \lambda_{x,0} + \sum_{h=1}^{2} (\lambda_{x,h,\cos} Cos(2\pi ht) + \lambda_{x,h,\sin} Sin(2\pi ht)) + \frac{1}{\sqrt{\Delta}}\varepsilon_{1,t}, \\
z_{2,t} &= \lambda_{y,0} + \sum_{h=1}^{2} (\lambda_{y,h,\cos} Cos(2\pi ht) + \lambda_{y,h,\sin} Sin(2\pi ht)) + \frac{1}{\sqrt{\Delta}}\varepsilon_{2,t},
\end{aligned}
$$

where $z_{1,t} \equiv \frac{x_{t+\Delta} - x_t}{\Delta} - (r - \sigma_x^2/2 - k_x x_t - y_t)$ and $z_{2,t} \equiv \frac{y_{t+\Delta} - y_t}{\Delta} - (\psi(t) - k_y y_t)$. The latter equations can be expressed as the matrix equality

$$
\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} + \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix},
$$

where $Z_i \equiv [z_{i,1/12},\ z_{i,2/12},\ \cdots,\ z_{i,(T-1)/12}]'$ for $i = 1$ and 2, $\gamma_1 \equiv [\lambda_{x,0},\ \lambda_{x,1,\cos},\ \lambda_{x,1,\sin},\ \lambda_{x,2,\cos},\ \lambda_{x,2,\sin}]'$, $\gamma_2 \equiv [\lambda_{y,0},\ \lambda_{y,1,\cos},\ \lambda_{y,1,\sin},\ \lambda_{y,2,\cos},\ \lambda_{y,2,\sin}]'$, $\xi_i \equiv [\frac{1}{\sqrt{\Delta}}\varepsilon_{i,1/12},\ \frac{1}{\sqrt{\Delta}}\varepsilon_{i,2/12},\ \cdots,\ \frac{1}{\sqrt{\Delta}}\varepsilon_{i,(T-1)/12}]'$ for $i = 1$ and 2, and

$$
X_1 = X_2 \equiv \begin{bmatrix}
1, & \cos(2\pi \times \frac{1}{12}), & \sin(2\pi \times \frac{1}{12}), & \cos(4\pi \times \frac{1}{12}), & \sin(4\pi \times \frac{1}{12}) \\
1, & \cos(2\pi \times \frac{2}{12}), & \sin(2\pi \times \frac{2}{12}), & \cos(4\pi \times \frac{2}{12}), & \sin(4\pi \times \frac{2}{12}) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1, & \cos(2\pi \times \frac{T-1}{12}), & \sin(2\pi \times \frac{T-1}{12}), & \cos(4\pi \times \frac{T-1}{12}), & \sin(4\pi \times \frac{T-1}{12})
\end{bmatrix}.
$$

By employing obvious notation, the above matrix equation can be written more compactly as $Z = X\ \Gamma + \Xi$, where $cov(\Xi) = \Sigma \otimes I_{T-1}$ and $\Sigma \equiv \frac{1}{\Delta}V$. This is a standard problem of Bayesian inference on the SUR model. Giles (2001) points out that Γ satisfies a multivariate normal distribution with mean equal to $[X^{'}\ (\Sigma^{-1} \otimes I_{T-1})\ X]^{-1}\ X^{'}\ (\Sigma^{-1} \otimes I_{T-1})\ Z$ and variance equal to $[X^{'}$ $(\Sigma^{-1} \otimes I_{T-1})\ X]^{-1}$.

Appendix C

[Insert table 4 here.]

Appendix D

[Insert figure 7 here.]

[Insert figure 8 here.]

Table 1: Parameter estimates for the lean hog futures market.

Parameters	Model 1			Model 2			Model 3		
	2.5%	50%	97.5%	2.5%	50%	97.5%	2.5%	50%	97.5%
k_x				0.414	0.657	1.015	0.447	0.647	0.902
k_c/k_y	2.330	2.608	2.887	1.461	1.935	2.355	1.027	1.312	1.648
$\lambda_c/\lambda_{y,0}$	-0.449	-0.287	-0.126	-0.369	-0.225	-0.099	-0.208	-0.151	0.100
$\lambda_{y,1,sin}$							0.637	0.699	0.768
$\lambda_{y,1,cos}$							-0.485	-0.391	-0.301
$\lambda_{y,2,sin}$							-1.898	-1.758	-1.612
$\lambda_{y,2,cos}$							1.461	1.603	1.744
$u_{x,0}$	-0.061	-0.020	0.021	-0.049	-0.006	0.036	-0.004	0.018	0.038
$u_{x,1,sin}$							0.020	0.051	0.083
$u_{x,1,cos}$							0.068	0.093	0.117
$u_{x,2,sin}$							-0.128	-0.108	-0.089
$u_{x,2,cos}$							-0.267	-0.247	-0.226
$u_{c,0}/u_{y,0}$	-0.198	-0.085	0.027	-6.597	-5.254	-3.817	-4.230	-3.527	-2.887
$u_{y,1,sin}$							-0.457	-0.307	-0.155
$u_{y,1,cos}$							-2.277	-2.114	-1.945
$u_{y,2,sin}$							2.375	2.716	3.057
$u_{y,2,cos}$							2.908	3.260	3.613
σ_x	0.425	0.449	0.473	0.409	0.430	0.454	0.327	0.342	0.356
σ_y	1.282	1.447	1.612	0.931	1.107	1.304	0.532	0.617	0.702
$\rho_{x,y}$	0.868	0.884	0.898	0.608	0.728	0.803	0.508	0.628	0.726
ρ	0.156	0.190	0.225	0.161	0.196	0.231	0.061	0.099	0.137
σ_e	0.077	0.081	0.084	0.076	0.080	0.083	0.044	0.046	0.047

Note: The three quantities denote respectively the 2.5, 50 and 97.5 percentiles of the posterior probability band.

Table 2: Parameter estimates for the soybean futures market.

Parameters	Model 1			Model 2			Model 3		
	2.5%	50%	97.5%	2.5%	50%	97.5%	2.5%	50%	97.5%
k_x				0.005	0.022	0.038	0.038	0.052	0.063
k_c/k_y	0.950	1.061	1.170	0.978	1.075	1.172	1.031	1.124	1.230
$\lambda_c/\lambda_{y,0}$	-0.061	-0.046	-0.033	-0.054	-0.040	-0.025	-0.038	-0.025	-0.013
$\lambda_{y,1,sin}$							0.183	0.200	0.218
$\lambda_{y,1,cos}$							-0.026	-0.008	0.010
$\lambda_{y,2,sin}$							-0.339	-0.245	-0.159
$\lambda_{y,2,cos}$							0.231	0.315	0.395
$u_{x,0}$	0.020	0.030	0.038	0.024	0.032	0.041	0.029	0.037	0.046
$u_{x,1,sin}$							0.079	0.091	0.103
$u_{x,1,cos}$							0.052	0.064	0.076
$u_{x,2,sin}$							-0.108	-0.096	-0.083
$u_{x,2,cos}$							-0.113	-0.102	-0.091
$u_{c,0}/u_{y,0}$	0.005	0.014	0.023	-0.249	-0.138	-0.018	-0.438	-0.361	-0.262
$u_{y,1,sin}$							0.203	0.249	0.289
$u_{y,1,cos}$							-0.651	-0.604	-0.559
$u_{y,2,sin}$							0.508	0.636	0.768
$u_{y,2,cos}$							0.223	0.356	0.501
σ_x	0.247	0.257	0.268	0.247	0.258	0.269	0.241	0.251	0.262
σ_y	0.245	0.263	0.281	0.246	0.262	0.279	0.241	0.256	0.275
$\rho_{x,y}$	0.669	0.700	0.729	0.647	0.682	0.716	0.625	0.662	0.696
ρ	0.076	0.107	0.138	0.080	0.109	0.140	0.084	0.115	0.148
σ_e	0.0305	0.0315	0.0332	0.0305	0.0313	0.0316	0.0247	0.0255	0.0263

Note: The three quantities denote respectively the 2.5, 50 and 97.5 percentiles of the posterior probability band.

Table 3: Deviance results for the lean hog and soybean futures prices.

	Lean Hogs			Soybean		
Parameters	Model 1	Model 2	Model 3	Model 1	Model 2	Model 3
$\bar{D}$	-6225.24	-6297.56	-9553.98	-14253.26	-14277.38	-15400.50
$D(\overline{\theta})$	-6233.04	-6307.04	-9576.64	-14261.18	-14288.16	-15425.34
p_D	7.80	9.48	22.66	7.92	10.78	24.84
DIC	-6217.44	-6288.08	-9531.32	-14245.34	-14266.60	-15375.66

Table 4: Gelman-Rubin test statistics for the lean hog and soybean futures markets.

	Lean Hogs			Soybean		
Parameters	Model 1	Model 2	Model 3	Model 1	Model 2	Model 3
k_x		1.047	1.007		1.038	1.001
k_c/k_y	1.017	1.014	1.050	1.044	1.042	1.047
$\lambda_c/\lambda_{y,0}$	1.010	1.002	1.006	1.013	1.001	1.079
$\lambda_{y,1,sin}$			1.014			1.001
$\lambda_{y,1,cos}$			1.031			1.006
$\lambda_{y,2,sin}$			1.006			1.017
$\lambda_{y,2,cos}$			1.001			1.032
$u_{x,0}$	1.010	1.001	1.001	1.003	1.002	1.029
$u_{x,1,sin}$			1.027			1.006
$u_{x,1,cos}$			1.003			1.001
$u_{x,2,sin}$			1.001			1.015
$u_{x,2,cos}$			1.001			1.001
$u_{c,0}/u_{y,0}$	1.012	1.061	1.084	1.027	1.033	1.032
$u_{y,1,sin}$			1.007			1.006
$u_{y,1,cos}$			1.001			1.007
$u_{y,2,sin}$			1.001			1.040
$u_{y,2,cos}$			1.008			1.003
σ_x	1.004	1.001	1.051	1.002	1.004	1.001
σ_y	1.009	1.007	1.011	1.068	1.064	1.079
$\rho_{x,y}$	1.003	1.045	1.011	1.013	1.004	1.013
ρ	1.007	1.004	1.001	1.026	1.039	1.007
σ_e	1.001	1.001	1.002	1.072	1.006	1.001

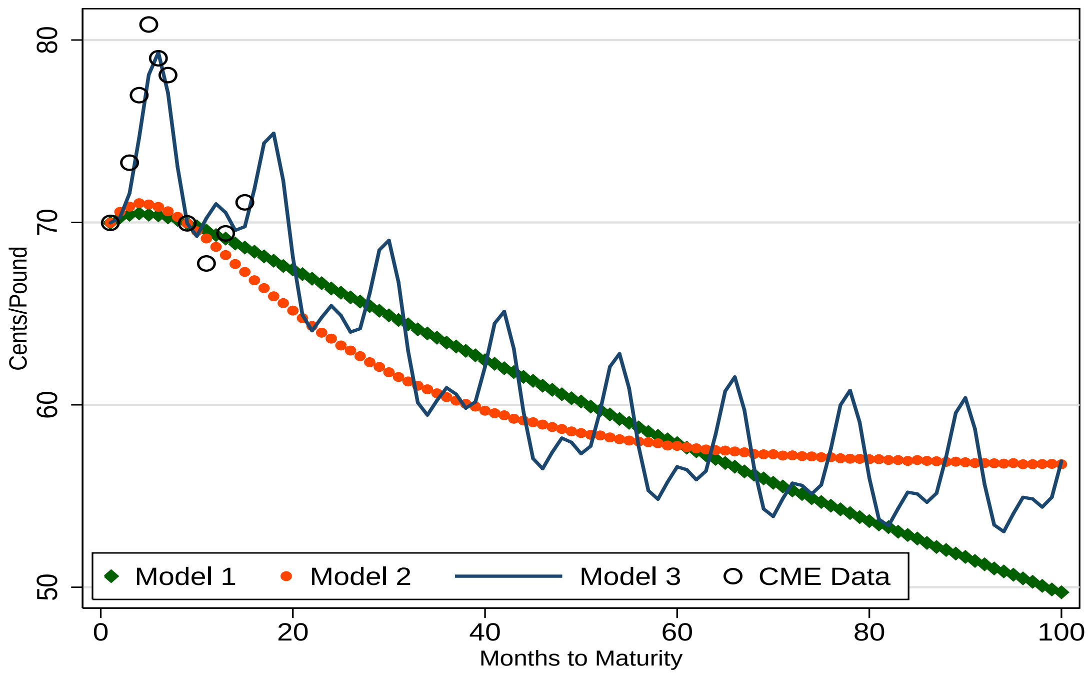

Figure 1: Projection of lean hog futures prices on January 15, 2010.

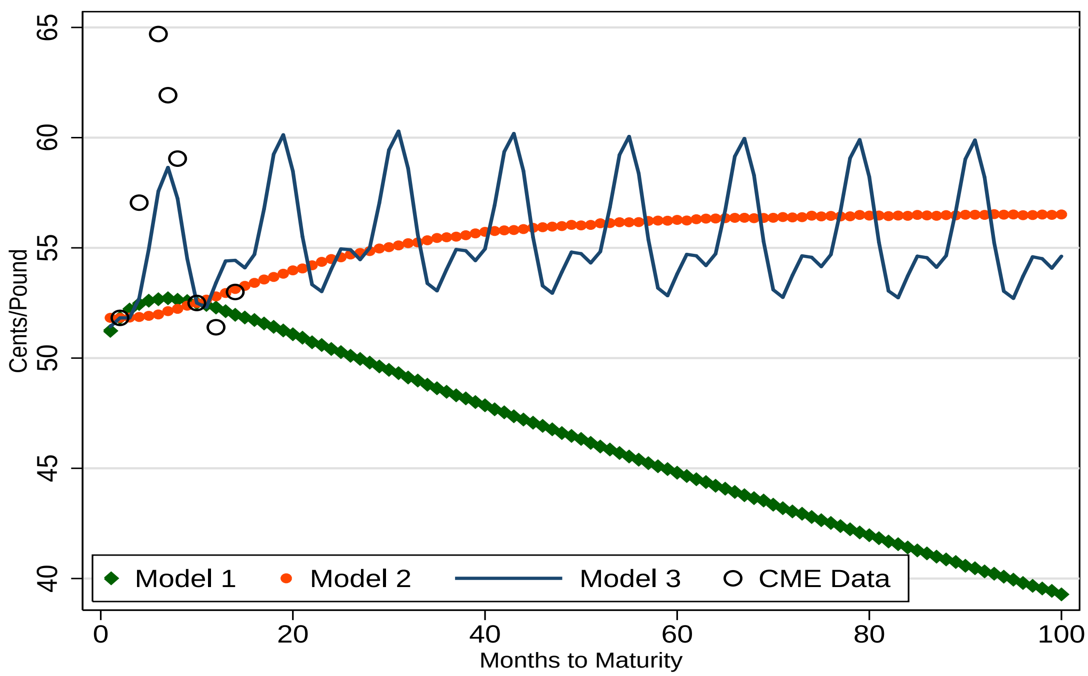

Figure 2: Projection of lean hog futures prices on December 16, 2002.

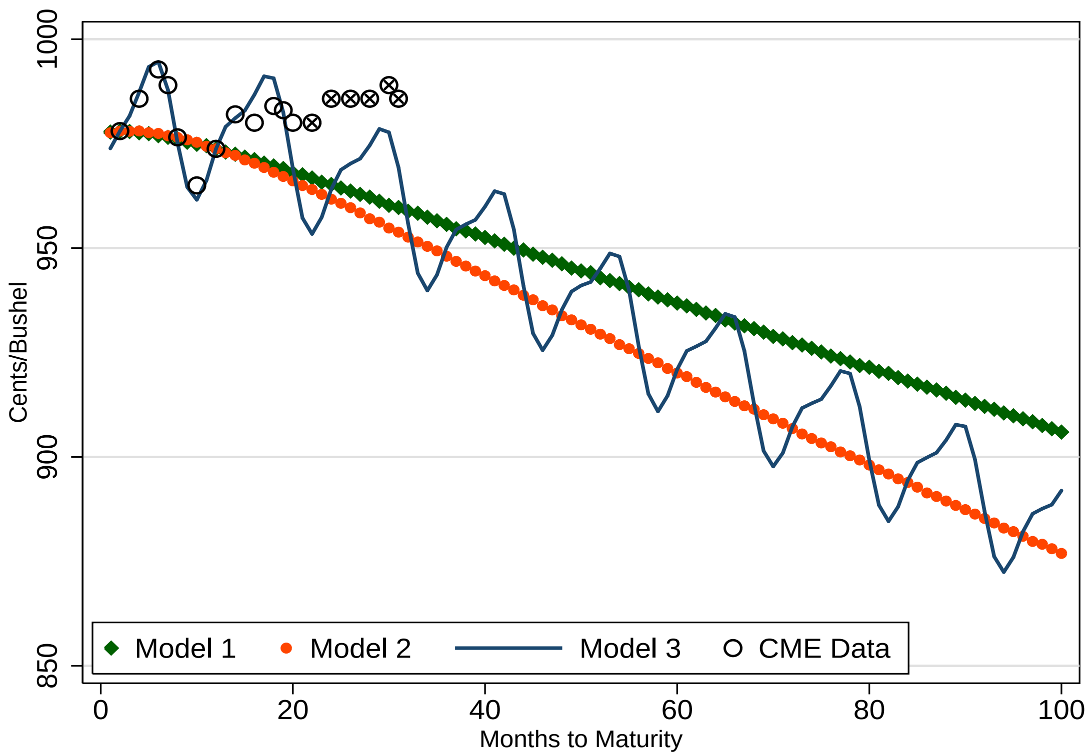

Figure 3: Projection of soybean futures prices on January 15, 2010.

Note: $\otimes$ represents CME data with zero volume.

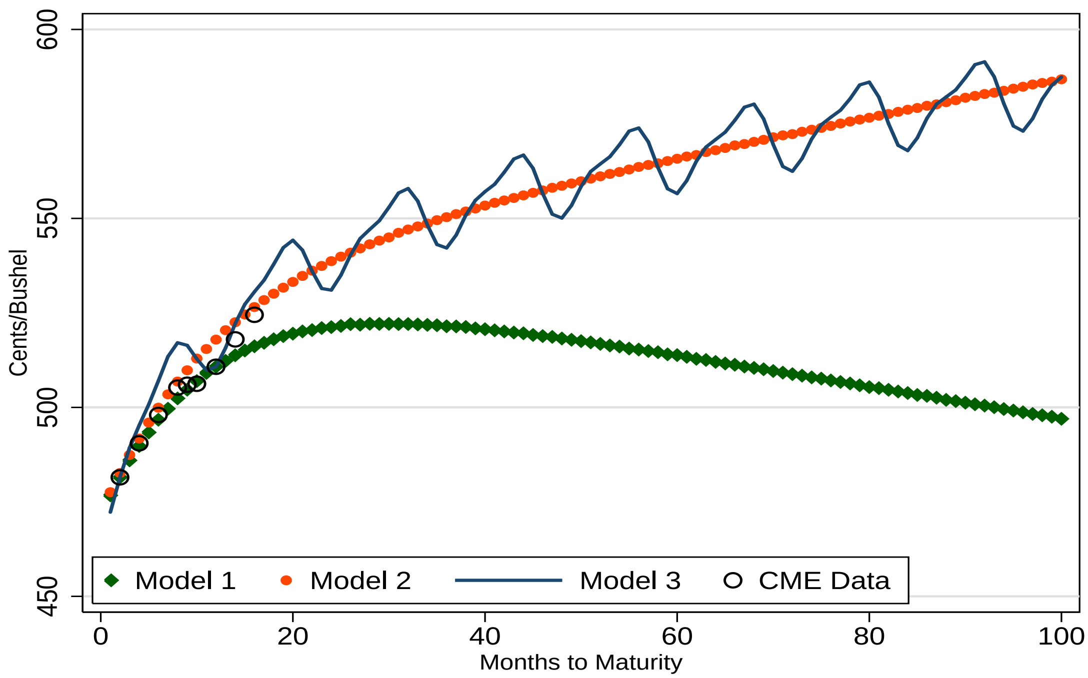

Figure 4: Projection of soybean futures prices on November 15, 2000.

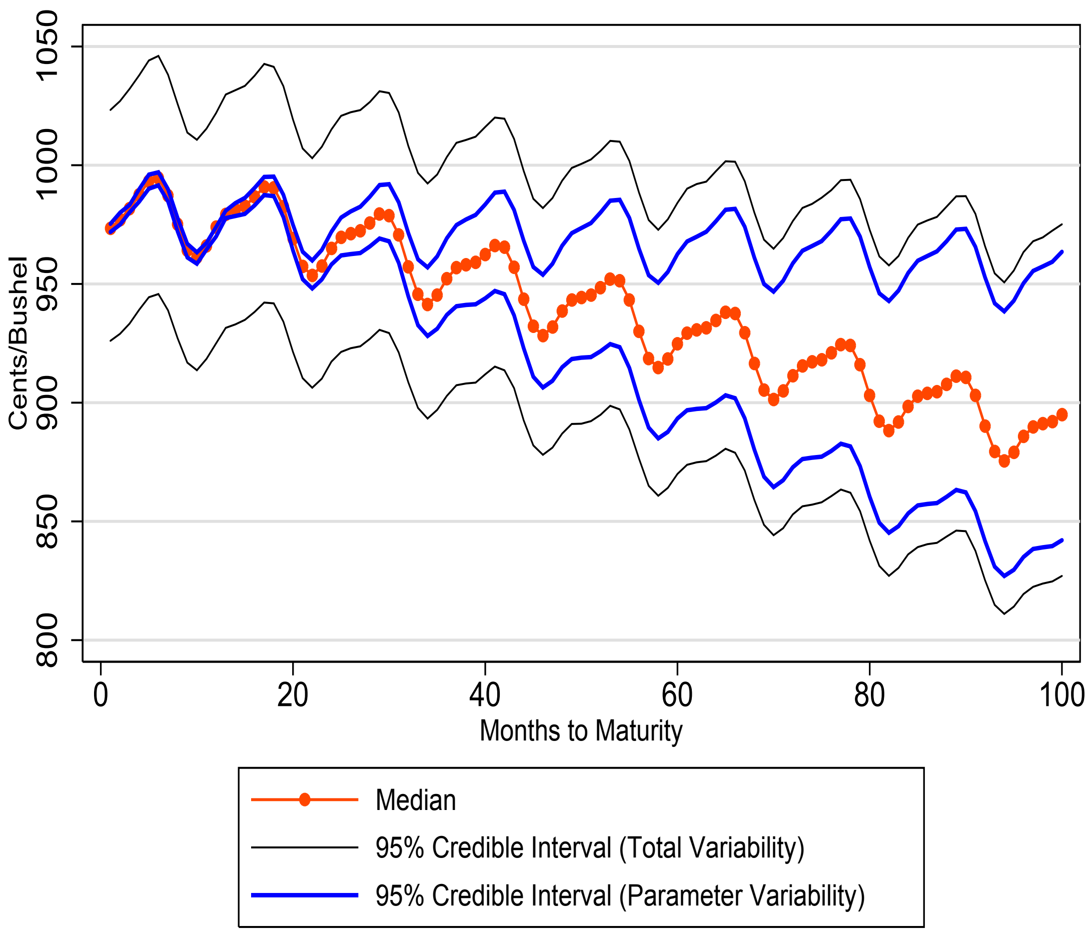

Figure 5: 95 percent credible band of futures prices predicted by Model 3 for soybean market on January 15, 2010.

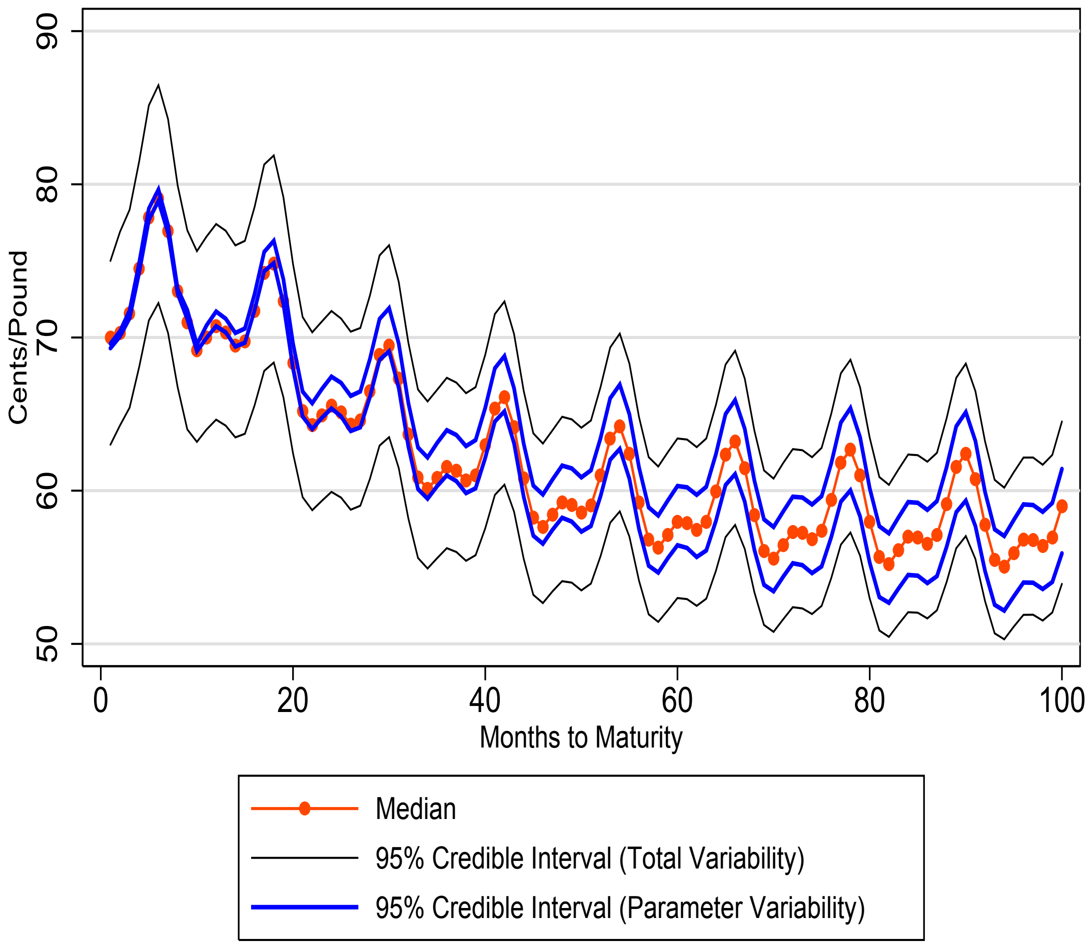

Figure 6: 95 percent credible band of futures prices predicted by Model 3 for lean hog market on January 15, 2010.

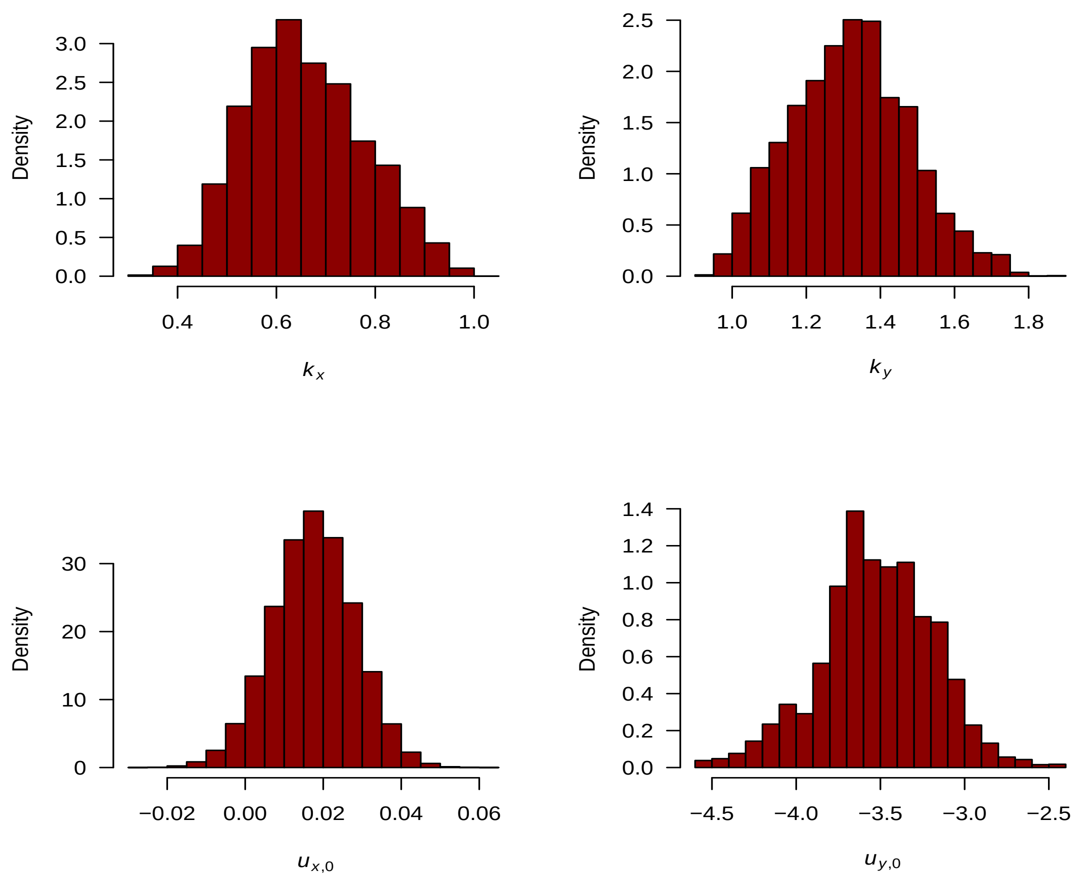

Figure 7: Posterior distributions of selected parameters for Model 3, corresponding to lean hog futures prices.

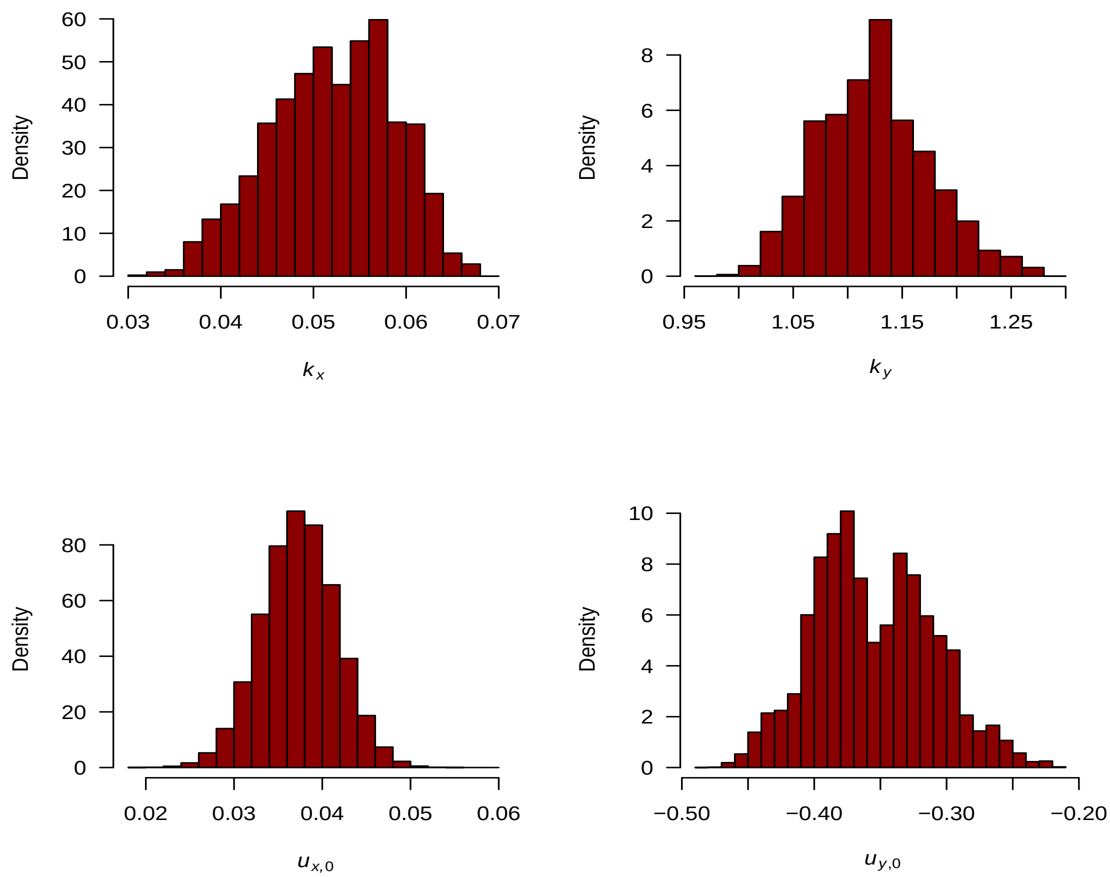

Figure 8: Posterior distributions of selected parameters for Model 3, corresponding to soybean futures prices.

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