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# Planning Accelerated Destructive Degradation Test with Competing Risks

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## **Abstract**

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## **Keywords**

compromise plan, competing risk, general equivalence theorem, Monte Carlo simulation, optimum plan, reliability

## **Disciplines**

Statistics and Probability

## **Comments**

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## Abstract

Accelerated destructive degradation tests (ADDTs) are widely used in manufacturing industries to obtain timely product reliability information, especially in applications where few or no failures are expected under use conditions in tests of practical length. An ADDT plan specifies the test conditions of accelerating variables, running time, and the corresponding allocation of test units to each condition. Usually, variables such as temperature, voltage, or pressure can be used as accelerating variables to accelerate degradation of a product. For some applications, however, tests at high-stress test conditions would result in more than one type of failure for test units, called competing risk problems. Careful test planning is important for efficient use of limited resources: test time, test units, and test facilities. This paper describes methods to find unconstrained and constrained optimum test plans for competing risk applications under a given test optimization criterion, such as minimizing the large-sample approximate variance of a failure-time distribution quantile at use conditions. A modified general equivalence theorem (GET) is used to verify the optimality of a given ADDT plan. Generally, an optimum test plan provides insight for constructing a good compromise test plan which tends to be more robust and practical. Monte Carlo simulations are used to provide visualization of the results that might be obtained from given test plans. The methods are illustrated with an application for an adhesive bond.

**KEY WORDS:** compromise plan; competing risk; general equivalence theorem; Monte Carlo simulation; optimum plan; reliability.

# 1 Introduction

## 1.1 Background

With the development of new technologies and strong global competition, manufactures are facing pressure to produce high reliability products and to do so quickly. This has motivated a strong interest in conducting up-front accelerated reliability tests on materials and components while products are being designed. Degradation tests are becoming more and more popular nowadays because degradation data can provide considerably more reliability information, compared to the traditional failure-time data (where time to failure is the response), especially in applications where few or no failures are expected within a long period of time. For most applications, to get reliability information quickly, degradation tests are accelerated by testing at higher than usual levels of accelerating variables such as temperature, voltage or humidity. Information obtained from tests at high levels of the accelerating variables is then extrapolated to the use conditions using a reasonable statistical model, often based on physics of failure knowledge. For some applications, degradation tests are destructive because the measurement process destroys or changes the physical/mechanical characteristics of test units. An accelerated degradation test with such degradation data is called an *accelerated destructive degradation test* (ADDT).

## 1.2 Motivation: Adhesive Bond C

The application motivating this work was an adhesive bond (adhesive bond C) which was to be used to attach two components of a product together. The experimental response was strength (in Newtons) of the adhesive bond and the strength deteriorated over time. In particular, the product engineers wanted to estimate the time at which 1% of the product would have an adhesive strength below 40 Newtons when operating at a room temperature of 25 °C (i.e., they wanted to estimate the 0.01 quantile of the failure-time distribution at use conditions). To obtain the degradation information quickly, temperature was to be used as an accelerating variable for the test. To measure the strength of the test units, engineers applied an increasing force until the two components attached by the adhesive bond broke apart. Thus the measurement process for this application was destructive. For the adhesive bond C, two types of failure were observed for the units tested at high-stress test conditions (i.e., high temperature for long running time). In particular, some units failed from *adhesive* failure and some failed from *cohesive* failure. Adhesive

failures occurred when one of the components detached from the adhesive bond. Cohesive failures occurred when the adhesive bonds to the material remained intact, but the components came apart due to a loss of cohesion within the layer of the adhesive material. Because of the destructiveness of measurement process, only one type of failure or the other could be observed on each test unit. Because of differences in the two effective activation energy values, the engineers were confident that only adhesive failure would occur on units operating at the normal operating temperatures, even after a long running time. Hence, the primary interest was to estimate the 0.01 quantile of the failure-time distribution for the adhesive failures.

Table 1 shows the accelerated destructive degradation test (ADDT) plan originally proposed for the adhesive bond C application in order to estimate the 0.01 quantile of the failure-time distribution at use conditions. The 15 units tested at time 0 would have no aging and were to serve as baseline units. A total of 59 additional test units were aged and measured after running according to the test conditions as shown in Table 1. Note that no test units were assigned at the highest stress test condition (16 Weeks, 70 °C), in an attempt to avoid cohesive failures.

Table 1: Original ADDT Plan for Adhesive Bond C.

Temperature °C	Weeks					Totals	
	0	2	4	6	12		16
70		4	4	4	4	0	16
60		5	0	5	5	5	20
50		6	0	6	6	5	23
—	15						15
Totals	15	15	4	15	15	10	74

### 1.3 Related Literature

There is a large amount of literature on planning accelerated tests, especially for accelerated life tests (ALTs) where failure time is the response. Nelson (2005a, 2005b) summarizes such test planning work. Some work that is particularly relevant to this paper is included in the following references. Nelson (1990, Chapter 6) describes methods for planning ALTs based on a simple model. Meeker and Escobar (1998, chapter 20) provide details and examples on how to plan a single-variable ALT. Escobar and Meeker (1995) describe methods for planning ALTs with two or more explanatory variables. There are some important differences between ALTs and accelerated degradation tests

because of their different responses. Boulanger and Escobar (1994) give methods for planning *repeated measures* accelerated degradation tests. Shi, Escobar, and Meeker (2009) show how to construct accelerated destructive degradation tests for applications with only one type of failure. In this paper, we describe methods for planning ADDTs for applications with more than one type of failure.

## 1.4 Overview

The remainder of this paper is organized as follows. Section 2 describes a class of degradation models for ADDT data with more than one type of failure mode and gives expressions for the degradation distribution and quantiles. Section 3 illustrates the failure-time distribution and quantiles induced by the degradation models. Section 4 talks about the criterion of accelerated destructive degradation test planning for applications with more than one type of failure. Section 5 gives a constrained optimum ADDT plan and applies a modification of the general equivalence theorem (GET) to verify the optimality of this test plan. Section 5 also presents a compromise ADDT plan that meets practical constraints. Section 6 uses Monte Carlo simulation to provide visualization of the results that might be obtained from given test plans. Section 7 contains some concluding remarks and extensions for future research work. Appendix A provides derivations and technical details about the large-sample approximations used in this work.

## 2 Degradation Models with Competing Risks

For applications resulting in data with more than one type of failure, the statistical competing risk model (e.g., David and Moeschberger 1978, and Crowder 2001) can be used. In this paper, we mainly focus on accelerated test planning in the applications with two types of failure where the marginal distribution of time to failure for one of the failure types is of primary interest. The generic responses corresponding to two types of failure are called the primary response and the competing response, respectively. In an ADDT with competing risks, for each experimental unit, we observe only the minimum of the primary response and the competing response.

## 2.1 Accelerated Degradation Model for the Primary Response

An important class of the degradation models for the primary response at time  $t$  and the accelerating variable  $\text{AccVar}$  is

$$Y = \beta_0 + \beta_1 \exp(\beta_2 x) \tau + \epsilon \quad (1)$$

where  $Y = h_d(\text{primary response})$  is transformed degradation,  $\tau = h_t(t)$  and  $x = h_a(\text{AccVar})$  are known monotone increasing transformations of time and the accelerating variable, respectively. These functions may be suggested by physics of failure or determined empirically.  $\epsilon$  is residual deviation that describes unit-to-unit variability. Here  $\epsilon/\sigma \sim \Phi(z)$ , where  $\Phi(z)$  is a completely specified cumulative distribution function (cdf), such as, the standardized normal cdf,  $\Phi_{\text{nor}}(z)$ , or the standardized smallest extreme value cdf,  $\Phi_{\text{sev}}(z)$ . The model parameters  $\boldsymbol{\theta} = (\beta_0, \beta_1, \beta_2, \sigma)$  are fixed but unknown, and will be estimated from ADDT data.

Model (1) is linear in the sense that for a specified  $x$ ,  $\mu(\tau, x) = \beta_0 + \beta_1 \exp(\beta_2 x) \tau$  is linear in  $\tau$ . The full model in (1) is, however, nonlinear in the parameters. The intercept  $\beta_0$  is the location parameter of the transformed degradation distribution when  $\tau = 0$ . The degradation rate of  $\mu(\tau, x)$  with respect to  $\tau$  at  $x$  is  $\omega(x) = \beta_1 \exp(\beta_2 x)$ . The sign of  $\beta_1$  determines whether the degradation is increasing or decreasing over time. For example if the degradation response is the size of a crack,  $\beta_1$  would be expected to be positive. On the other hand, if the degradation response is the strength of an adhesive bond,  $\beta_1$  would be expected to be negative.

## 2.2 Accelerated Degradation Model for the Competing Response

The degradation model for the competing response is parallel to the one for the primary response, and is given as

$$Y^{(c)} = \beta_0^{(c)} + \beta_1^{(c)} \exp(\beta_2^{(c)} x) \tau + \epsilon^{(c)}$$

where  $Y^{(c)} = h_d(\text{competing response})$  is the transformed competing degradation,  $\tau$  and  $x$  are the same as in Section 2.1, and again  $\epsilon^{(c)}/\sigma^{(c)} \sim \Phi(z)$ . The model parameters for this competing model are  $\boldsymbol{\theta}^{(c)} = (\beta_0^{(c)}, \beta_1^{(c)}, \beta_2^{(c)}, \sigma^{(c)})$  and they are also fixed but unknown.

### 2.3 Degradation Models for Adhesive Bond C

The degradation model for adhesive bond C is an extension of the model derived by Escobar et al. (2003) for a similar application without competing risks, adhesive bond B. Product engineers believe that these two types of bonds have the same characteristics for the adhesive strength, but for adhesive bond C, cohesive failures can, after a long running time, occur for units tested at high temperatures.

For the adhesive bond C application, the degradation models for the adhesive strength and cohesive strength are, respectively,

$$\begin{aligned} Y &= \beta_0 + \beta_1 \exp(\beta_2 x) \tau + \epsilon \quad \text{and} \\ Y^{(c)} &= \beta_0^{(c)} + \beta_1^{(c)} \exp(\beta_2^{(c)} x) \tau + \epsilon^{(c)} \end{aligned}$$

where

$$\begin{aligned} Y &= \log(\text{Adhesive Strength in Newtons}) \\ Y^{(c)} &= \log(\text{Cohesive Strength in Newtons}) \\ x &= -\frac{11605}{\text{Temperature in } ^\circ\text{C} + 273.15} \\ \tau &= \sqrt{\text{Time in Weeks}} \\ (\epsilon/\sigma) &\sim \Phi_{\text{nor}}(z) \quad \text{and} \quad (\epsilon^{(c)}/\sigma^{(c)}) \sim \Phi_{\text{nor}}(z). \end{aligned}$$

For this application, the accelerating variable is temperature and  $x$  is the Arrhenius transformed temperature. The denominator in  $x$  is temperature on the kelvin (K) scale and the numerator is the reciprocal of Boltzmann's constant in units of electronvolt per kelvin (eV/K). Under this parametrization,  $\beta_2$  and  $\beta_2^{(c)}$  have the interpretations of effective activation energy corresponding to the degradation mechanism for adhesive and cohesive strengths, respectively.

Figure 1 is an illustration of degradation distributions for the adhesive strength at 25 °C and different values of time for some specific values of the parameters  $\beta_0, \beta_1, \beta_2, \sigma$  corresponding to the maximum likelihood (ML) estimates given in Escobar et al. (2003). This figure gives a visual impression of the adhesive strength degradation distribution over time. For the cohesive strength, the degradation distribution should have a similar decreasing trend over time, but at a different

degradation rate from the adhesive strength because of differences in the parameters.

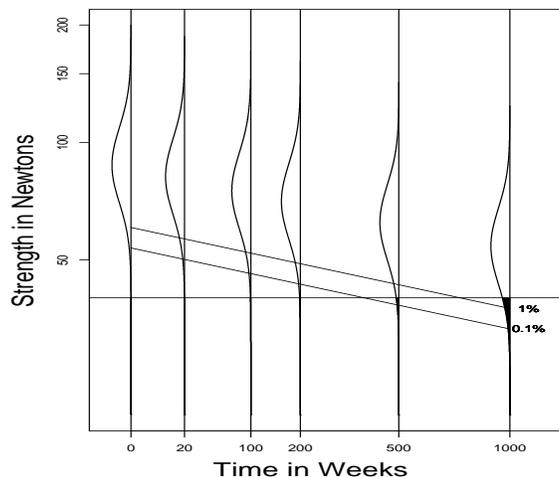


Figure 1: Adhesive bond C adhesive strength degradation distributions as a function of time at 25 °C. The strength axis is a logarithmic axis and the time axis is a square root axis. The horizontal line at  $\mathcal{D}_f = 40$  Newtons is the threshold for the failure-definition degradation level. At each time, the shaded area below the horizontal line is the failure probability at that specific time.

Due to the competing risk between two types of failure, we can only observe  $\min(Y, Y^{(c)})$ . If an adhesive failure occurs, then  $Y < Y^{(c)}$ ; if a cohesive failure occurs, then  $Y^{(c)} < Y$ . Because the adhesive strength is of primary interest, we consider an observation to be *exact* for an adhesive failure and *right-censored* for a cohesive failure. For right-censored observations, the adhesive strength is unobserved but known to be greater than the censored value. Such right-censored observations contain relatively little information about the adhesive strength. Thus, using the competing risk model for test planning should result in a test plan that limits the probability of having such censored observations in the accelerated test.

## 2.4 Degradation Distribution and Quantiles

For a given time and accelerating variable level, the cdf of the primary transformed degradation  $Y$  is

$$F_Y(y; \tau, x) = \Pr(Y \leq y; \tau, x) = \Phi \left[ \frac{y - \mu(\tau, x)}{\sigma} \right].$$

For the adhesive bond C example, the cdf for  $Y$ , the logarithm of the adhesive strength at a fixed test condition of time and temperature, can be obtained by replacing  $\Phi$  with  $\Phi_{\text{nor}}$ .

The  $p$  quantile function for the primary transformed degradation distribution at  $(\tau, x)$  is

$$y_p = \beta_0 + \beta_1 \exp(\beta_2 x) \tau + \sigma \Phi^{-1}(p)$$

where  $\Phi^{-1}(p)$  is the  $p$  quantile of a standard location-scale distribution. Substituting  $\Phi_{\text{nor}}^{-1}(p)$  for  $\Phi^{-1}(p)$ , one obtains the  $p$  quantile of the transformed degradation (log Newtons) for the adhesive strength in the adhesive bond C application, as shown in Figure 1 for  $p = 0.01$  and  $p = 0.001$ .

### 3 Failure-Time Distribution with Competing Risks

#### 3.1 Relationship Between Degradation and Failure

For some products, there is a gradual loss of performance as usage time increases (e.g., decreasing adhesive strength of an adhesive bond). A “soft failure” (see Section 13.4 of Meeker and Escobar 1998) occurs when performance reaches a critical level  $\mathcal{D}_f$  after a certain usage time. Because we are mainly interested in evaluating the failure-time distribution for the primary response, the failure-time  $T$ , for the competing risk applications, is defined as the time when the observed primary degradation crosses the critical level  $\mathcal{D}_f$ .

#### 3.2 Failure-Time Distribution and Quantiles

For decreasing degradation, when  $\beta_1$  is negative, the event of failure-time  $T$  being less than  $t$  is equivalent to an observed primary degradation response being less than the critical level  $\mathcal{D}_f$  at time  $t$  (i.e.,  $Y \leq y_f$ , where  $y_f = h_d(\mathcal{D}_f)$ ). Then the failure-time cdf is

$$\begin{aligned} F_T(t; x) &= \Pr(T \leq t) = \Pr(Y \leq y_f) = F_Y(y_f; \tau, x) \\ &= \Phi \left[ \frac{y_f - \mu(\tau, x)}{\sigma} \right] = \Phi \left( \frac{\tau - \nu}{\varsigma} \right), \quad \text{for } t \geq 0 \end{aligned} \quad (2)$$

where

$$\nu = \frac{(\beta_0 - y_f) \exp(-\beta_2 x)}{|\beta_1|} \quad \text{and} \quad \varsigma = \frac{\sigma \exp(-\beta_2 x)}{|\beta_1|}.$$

Note that in this model, there is a positive  $\Pr(T = 0) = \Phi(-\nu/\varsigma)$  at  $t = 0$ . This spike is sometimes called the dead-on-arrival (or DOA) probability.

From (2), the  $p$  quantile of the failure-time for decreasing degradation is

$$t_p = \begin{cases} h_t^{-1} [\nu + \varsigma \Phi^{-1}(p)] & \text{if } p \geq \Phi(-\nu/\varsigma) \\ 0 & \text{otherwise.} \end{cases}$$

For increasing degradation, when  $\beta_1$  is positive, the derivations of the failure-time cdf and quantiles are similar.

## 4 Test Planning with Competing Risks

### 4.1 ADDT Planning Information

For ADDT planning, there are usually practical constraints. For example, for the adhesive bond C application, there are three constraints for the ADDT planning, which are:

- 74 test units is the maximum sample size.
- 70 °C is the maximum temperature that can be used (The engineers involved in the testing felt that higher temperatures would cause the degradation models to break down).
- 16 weeks is the maximum time available for testing.

To do test planning, planning information for the parameters of the given degradation models is required. This is because the test plan evaluation criteria depend on the model parameters. Such planning information could come from previous experiments, past data, engineers' experience, etc. For the adhesive bond C application, planning values for parameters in the primary response degradation model are obtained from the data analysis in Escobar et al. (2003) because it is believed that the adhesive degradation properties of bond C are similar to those of bond B, as mentioned in Section 2.3. That is,  $\beta_0^\square = 4.471$ ,  $\beta_2^\square = 0.6364$ ,  $\sigma^\square = 0.158$ , and the degradation rate at 50 °C,  $\omega_{50}^\square = -0.1026$ . These parameters have clear practical interpretations, which makes them easier to elicit from experts when needed. With the given planning values, one gets  $\beta_1^\square = \omega_{50}^\square \exp(-\beta_2^\square x_{50}) = -0.1026 \exp(0.6364 \times 35.912) = -8.643 \times 10^8$ , where  $x_{50} = -11605/(50 + 273.15) = -35.912$  is the transformed temperature of 50 °C.

Planning values for parameters in the competing response degradation model are  $\beta_0^{(c)\square} = 5.7$ ,  $\beta_2^{(c)\square} = 0.6$ ,  $\omega_{50}^{(c)\square} = -0.23$ , and  $\sigma^{(c)\square} = 0.2$ . This information was obtained from a combination of limited

previous experience with cohesive failures in accelerated tests and engineering judgement. In some preliminary experiments, engineers did not find any cohesive failures at test conditions with low temperature and they would not expect to see any cohesive failures at use conditions, even after a long period of running time (i.e., with probability approaching one, adhesive failures would occur first at use conditions). Cohesive failures only happened at some high-stress test conditions. Under the specified planning values for the model parameters, Figure 2 shows a contour plot of the probability that the cohesive strength is less than adhesive strength (i.e., the right-censored probability) as a function of test conditions within the experimental region, and Figure 3 compares the mean degradation responses of adhesive and cohesive strength at various test conditions.

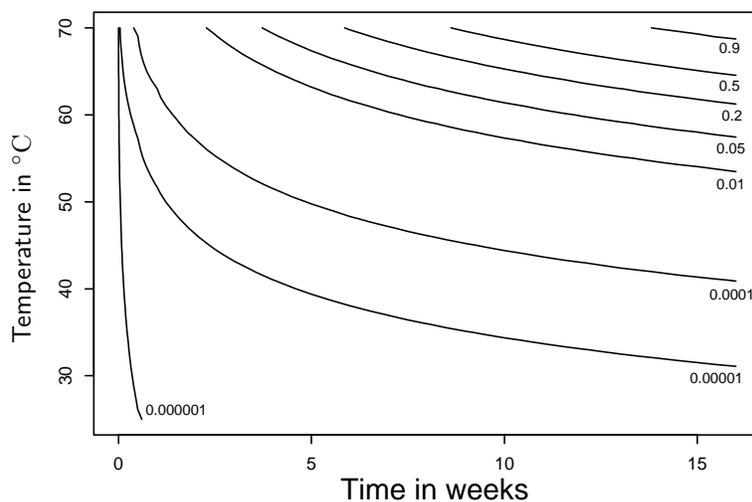


Figure 2: The probability of cohesive strength being less than adhesive strength (i.e., probability of a right-censored observation) under specified planning values for the model parameters.

## 4.2 Criterion for ADDT Planning with Competing Risks

For ADDT planning, the appropriate test planning criterion depends on the purpose of the experiment. For our application, the objective is to estimate  $t_p$ , a particular quantile of the failure-time distribution at use conditions. A commonly used criterion for test planning is to minimize  $\text{Avar}(\hat{t}_p)$ , the large-sample approximate variance of the maximum likelihood (ML) estimator of the specified failure-time quantile.

We use  $\mathbf{v} = (t, \text{AccVar})$  to denote a test condition specifying the time  $t$  and the accelerating

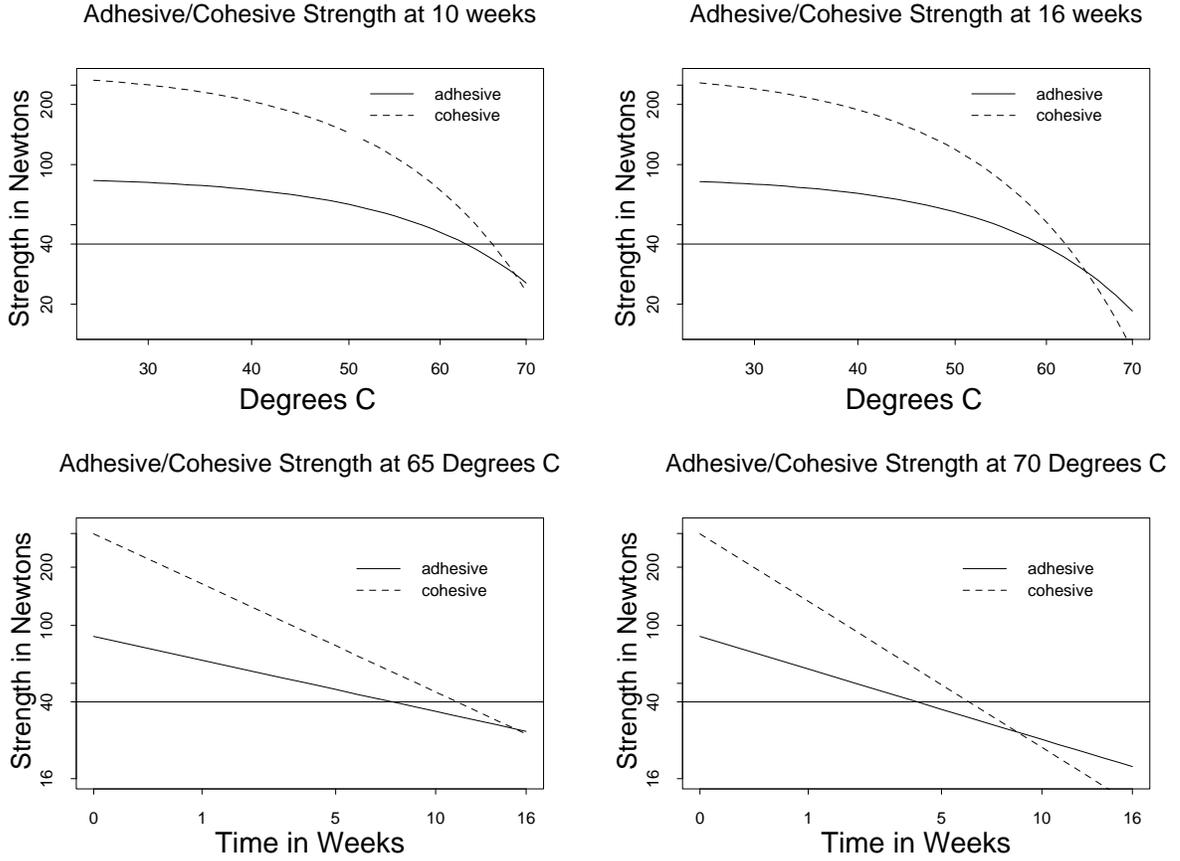


Figure 3: The mean degradation responses of adhesive and cohesive strength at various test conditions under specified planning values for the model parameters. The strength axis is a logarithmic axis, the time axis is a square root axis, and the temperature axis is an Arrhenius axis.

variable level AccVar. An ADDT plan, denoted by  $\xi$ , will specify a set of test conditions  $\mathbf{v}_i$ , and the corresponding proportional allocation  $\pi_i$  of test units at each  $\mathbf{v}_i$ . If a test plan has  $r$  test conditions, then the proportional allocations satisfy  $\pi_i > 0$ , for each  $i = 1, 2, \dots, r$ , and  $\sum_{i=1}^r \pi_i = 1$ .

Because  $h_t(t_p)$  is a monotone increasing function of  $t_p$ , minimizing  $\text{Avar}[h_t(\hat{t}_p)]$  is equivalent to minimizing  $\text{Avar}(\hat{t}_p)$ . Thus, from Appendix A.2, the optimization criterion is equivalent to finding a test plan  $\xi$  that maximizes the objective function

$$\Psi(\xi) = -\mathbf{c}'[\mathbf{I}_{\theta, \theta^{(c)}}(\xi)]^{-1}\mathbf{c}, \quad (3)$$

where  $\mathbf{c} = \partial h_t(t_p)/\partial \theta$  and  $\mathbf{I}_{\theta, \theta^{(c)}}(\xi)$  is the Fisher information matrix for test plan  $\xi$ . Details are defined in Appendix A. This criterion is known as *c optimality*.

## 5 ADDT Plan with Competing Risks

### 5.1 Initial Optimized ADDT Plan with Competing Risks

For applications with no competing risks and a single degradation model like that in (1), Shi, Escobar, and Meeker (2009) present an ADDT optimum plan structure, illustrated in Figure 4. Under the practical constraints of a maximum time  $t_M$  and a maximum accelerating variable level  $\text{AccVar}_M$ , the optimum plan with no competing risks is not unique. That is, there are many plans that provide the same smallest large-sample approximate variance for the ML estimator of a specified failure-time quantile. As shown in Figure 4, such optimum plans have some test units allocated at  $v_1$  (the baseline test condition) with  $t = 0$  (note from the model that the level of  $\text{AccVar}$  is not a factor at time  $t = 0$ ), some at the  $v_2$  test condition with  $\text{AccVar}_M$  and a time level larger than a lower boundary  $t_L$ , and some at the  $v_3$  test condition with  $t_M$  and an optimized  $\text{AccVar}$  value. The  $\text{AccVar}$  value for the  $v_3$  test condition and the proportional allocations  $\pi_1, \pi_2$  of test units are chosen to minimize the large-sample approximate variance for each value of time for  $v_2$  between  $t_L$  and  $t_M$ . For the no-competing-risk model, each such plan has the same optimum value for the objective function.

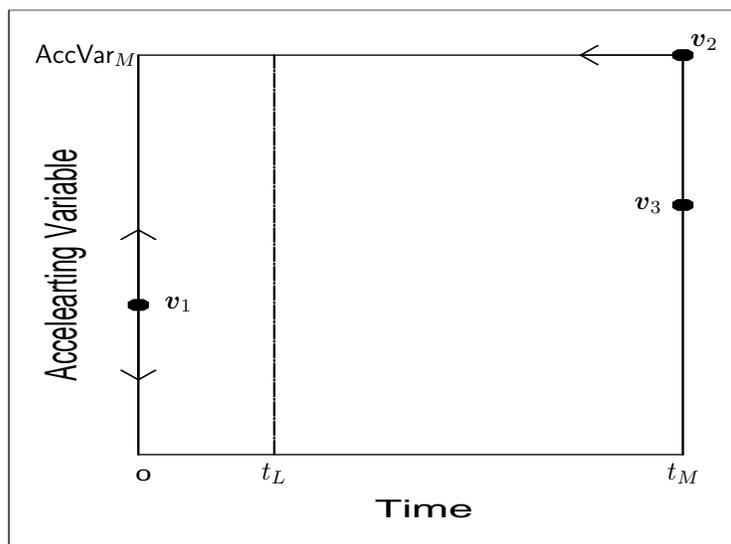


Figure 4: Optimum plan structure.

Generally, in a regression model, if all of the parameters are linear with respect to the explanatory

variables, the number of test conditions for the optimum plan will not be any larger than the number of parameters needed to define the response surface. Although no such definite conclusion exists for models that are nonlinear in their parameters, such as our degradation model, numerical experiments for the ADDT model have not found a counter example. Because there are three regression parameters in the degradation model described in (1), we expect that a non-degenerate optimum ADDT plan for an application with the competing risk degradation model could also be a three-point plan (i.e., the optimum test plan should have three test conditions). We will check this using the GET.

For the adhesive bond C application which has competing risks, our first attempt to find an optimum plan was to investigate test plans that have a structure similar to that for the no-competing-risk model, as shown in Figure 4. For most practical situations in which accelerated tests are used, all the test conditions for an optimum plan are on the boundaries of the experimental region and they spread out as much as possible, providing better estimates of the regression coefficients than closely-spaced test conditions. For the competing risk model, the probability of getting a cohesive failure (a right-censored observation, providing relatively little information about adhesive failures) is highest when  $\mathbf{v}_2$  is in the NE corner so we might expect the optimum  $\mathbf{v}_2$  to be at an interior point. Hence, we assumed similar  $\mathbf{v}_1$  and  $\mathbf{v}_3$  test conditions but explored the properties of test plans with the  $\mathbf{v}_2$  point moving within a subset of the experimental region. Then at each fixed test condition for  $\mathbf{v}_2$ , temperature for the  $\mathbf{v}_3$  and the proportional allocations  $\pi_1, \pi_2$  of test units were chosen to minimize  $\text{Avar}(\widehat{t}_{0.01})$ . Figure 5 shows the contour plot of the large-sample approximate standard error of  $\widehat{t}_{0.01}$ ,  $\text{Ase}(\widehat{t}_{0.01})$ , for each optimized plan as the test condition  $\mathbf{v}_2$  varies within the experimental region, under the usual constraint that proportional allocations for any test condition should be nonnegative and that they sum to 1.

Figure 5 indicates that having  $\mathbf{v}_2$  on the 70 °C boundary with time around 3 weeks results in a best test plan with minimum  $\text{Ase}(\widehat{t}_{0.01})$ . This optimization result for the competing risk model is different from the one with no-competing risk where many optimum plans are derived by moving the time for the  $\mathbf{v}_2$  point between a lower boundary  $t_L$  and  $t_M = 16$  weeks at 70 °C. The reason for different optimization conclusions between the two models is because that testing at temperatures approaching 70 °C will tend to cause cohesive failures.

As mentioned before, the boundary of the experimental region is of particular interest. Figure 6 shows  $\text{Ase}(\widehat{t}_{0.01})$  and the proportional allocations of test conditions obtained for different optimized

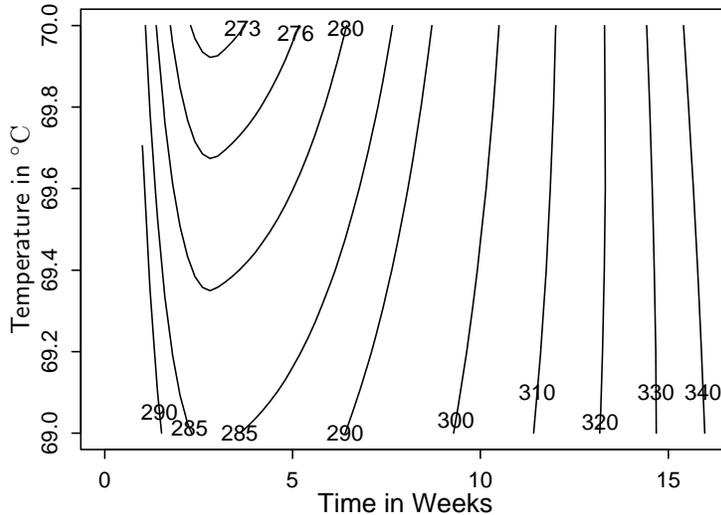


Figure 5: Contour plot of the large-sample approximate standard error of  $\hat{t}_{0.01}$  for optimized plans when the test condition  $\mathbf{v}_2$  moves within the experimental region.

test plans, when the point  $\mathbf{v}_2$  moves along the maximum 70 °C temperature line. Because all of the test plans are optimized under the condition that the proportional allocation for each test condition must be nonnegative and sum to one,  $\pi_1$  is close to zero for optimized plans when the time for  $\mathbf{v}_2$  is less than around 3 weeks. At the minimum  $\text{Ase}(\hat{t}_{0.01})$  value 272.06, the corresponding proportional allocation  $\pi_1$  is close to zero, which is nearly a degenerate optimum test plan with little practical appeal.

## 5.2 Constrained Optimum ADDT Plan with Competing Risks

In practical applications, engineers prefer to allocate a certain number of test units at the baseline condition at time 0. For the adhesive bond C application, instead of the initial optimized “degenerate” plan derived in Section 5.1, it is more meaningful to assign a fixed proportion of test units at the baseline condition and then find a constrained optimum ADDT plan. For purposes of illustration, we assign 10% of test units to the baseline test condition  $\mathbf{v}_1$ . This amount is typical of what engineers have used in previous ADDT experiments that we have seen. The time of condition  $\mathbf{v}_2$ , the temperature of condition  $\mathbf{v}_3$  and the proportional allocation  $\pi_2$  are then chosen to minimize  $\text{Avar}(\hat{t}_{0.01})$ . Table 2 shows the constrained optimum ADDT plan. The optimum  $\text{Ase}(\hat{t}_{0.01})$  for this

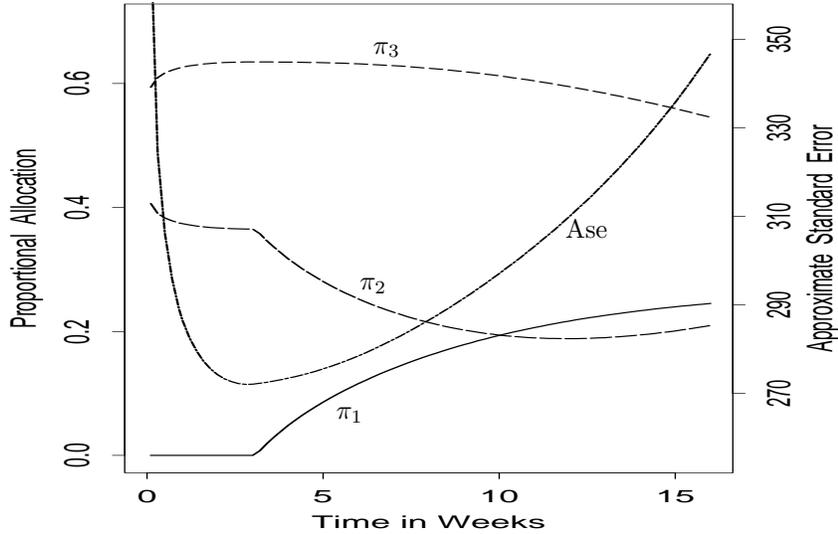


Figure 6: Proportional allocations and large-sample approximate standard error of  $\hat{t}_{0.01}$  as a function of time in weeks for  $\mathbf{v}_2$  showing different optimized plans arising from the constraint that proportional allocation for each test condition should be nonnegative and sum to one.

test plan is 274.62, only slightly larger than that for the initial optimum degenerate plan.

Table 2: Constrained optimum ADDT plan with competing risks. The — indicates that at time 0, the level of temperature has no effect on the model.

Test Condition	Time in weeks	Temperature in °C	Proportional Allocation
$\mathbf{v}_1$	0	—	0.1
$\mathbf{v}_2$	4.04	70	0.279
$\mathbf{v}_3$	16	53.519	0.621

### 5.3 General Equivalence Theorem

Shi, Escobar, and Meeker (2009) uses a variation of Whittle’s (1973) general equivalence theorem (GET) to check the optimality of a given ADDT plan without competing risks. For an ADDT with competing risks, the objective function shown in (3) has a form that is similar to the one in Shi, Escobar, and Meeker (2009), except that the expression of  $\mathbf{I}_{\boldsymbol{\theta}, \boldsymbol{\theta}^{(c)}}(\boldsymbol{\xi})$  is different. The derivation of  $\mathbf{I}_{\boldsymbol{\theta}, \boldsymbol{\theta}^{(c)}}(\boldsymbol{\xi})$  for the competing risk model is given in Appendix A. To use GET appropriately for checking the optimality of an ADDT plan with competing risks, some conditions need to be satisfied.

The proof for the satisfaction of GET conditions can be done in a manner that is similar to that presented in Appendix B.1 of Shi, Escobar, and Meeker (2009). To save space, however, the details of proof are not included here but just show the idea of GET for optimality check. For the GET, the directional derivative,  $\Lambda$ , of  $\Psi$  at  $\xi$  and in the direction of an alternative plan  $\eta$  is defined as

$$\begin{aligned}\Lambda(\xi, \eta) &= \lim_{\delta \rightarrow 0^+} \frac{\Psi[(1-\delta)\xi + \delta\eta] - \Psi(\xi)}{\delta} \\ &= \mathbf{c}' [\mathbf{I}_{\theta, \theta^{(c)}}(\xi)]^{-1} \mathbf{I}_{\theta, \theta^{(c)}}(\eta) [\mathbf{I}_{\theta, \theta^{(c)}}(\xi)]^{-1} \mathbf{c} - \mathbf{c}' [\mathbf{I}_{\theta, \theta^{(c)}}(\xi)]^{-1} \mathbf{c}\end{aligned}$$

where  $\mathbf{c}$ ,  $\mathbf{I}_{\theta, \theta^{(c)}}(\xi)$ , and  $\mathbf{I}_{\theta, \theta^{(c)}}(\eta)$  are evaluated using the planning values of the model parameters  $\theta^\square$  and  $\theta^{(c)\square}$ . Let  $\xi_v$  be a singular test plan that puts all units at the test condition  $v$ . Suppose a given test plan  $\xi$  has  $r$  test conditions,  $v_1, v_2, \dots, v_r$ . Then the plan  $\xi$  is optimal if it satisfies  $\Lambda(\xi, \xi_{v_1}) = \Lambda(\xi, \xi_{v_2}) = \dots = \Lambda(\xi, \xi_{v_r}) = 0$  and  $\Lambda(\xi, \xi_v) \leq 0$  for any other singular plan  $\xi_v$  in the experimental region.

For a constrained optimum ADDT plan with a fixed proportion of test units at the baseline condition, a modification to the GET is needed. Instead of allocating all units at a single test condition, the alternative test plan  $\eta$  must now put the same proportion of test units at the baseline condition  $v_1$  as in the constrained optimum plan, and the remaining proportion of units at a single test condition  $v$ . Let  $\xi_{v_1, v}$  denote a plan with two test conditions, one baseline condition  $v_1$  with a fixed proportion of test units, and the other condition  $v$  with all the remaining proportion of test units. For a given test plan  $\xi$  with a fixed test condition  $v_1$  and  $r - 1$  additional test conditions,  $v_2, v_3, \dots, v_r$ , the plan  $\xi$  is a constrained optimum plan if and only if it satisfies  $\Lambda(\xi, \xi_{v_1, v_2}) = \Lambda(\xi, \xi_{v_1, v_3}) = \dots = \Lambda(\xi, \xi_{v_1, v_r}) = 0$  and  $\Lambda(\xi, \xi_{v_1, v}) \leq 0$  for any other single condition  $v$  in the experimental region.

For the adhesive bond C application, this modified GET is used to check the optimality of the constrained test plan illustrated in Table 2. Figure 7 shows the directional derivatives  $\Lambda(\xi, \xi_{v_1, v})$  of this constrained optimum plan as a function of temperature and time for the test condition  $v$ , where  $\xi_{v_1, v}$  is a plan that puts 10% of the test units at the baseline condition  $v_1$  and the remaining 90% of the test units at the test condition  $v$ . Observe that, as expected, the directional derivatives are zero at those two unconstrained test conditions  $v_2, v_3$  of the constrained optimum plan and less than zero anywhere else. This behavior of the directional derivative curves also implies that the constrained optimum plan is unique.

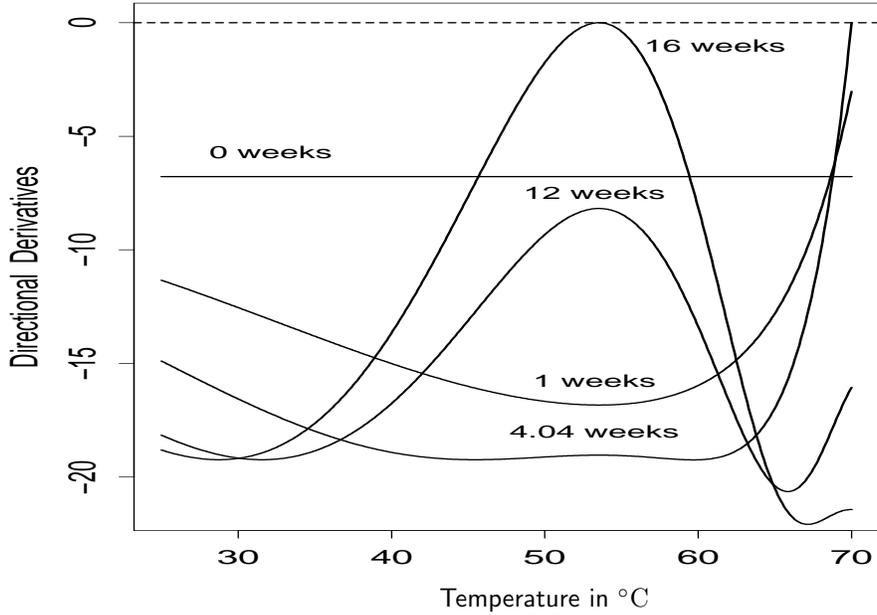


Figure 7: Directional derivatives  $\Lambda(\xi, \xi_{v_1, v})$  of the constrained optimum plan as a function of temperature and time for the test condition  $v$  in the adhesive bond C application.  $\xi_{v_1, v}$  is a plan that puts 10% test units at the baseline condition  $v_1$  and the remaining 90% test units at the test condition  $v$ .

#### 5.4 Compromise ADDT Plan with Competing Risks

An optimum plan results in the smallest large-sample approximate variance of the estimated failure-time quantile. An optimum plan, however, provides no information to check the adequacy of the model and tends to be highly sensitive to planning information specification errors (e.g., Meeker 1984). An optimum plan does, however, provide insight for obtaining good compromise test plans which tend to be more robust and useful in practical applications. A compromise test plan is generally suggested by the optimum plan but uses more test conditions to achieve robustness.

Shi, Escobar, and Meeker (2009) proposed an optimized compromise plan, whose structure is shown in Figure 8, for an adhesive bond application with no competing risks. This compromise plan has ten test conditions, numbered 1 to 10, including the baseline condition and nine additional test conditions consisting of equally spaced levels of time and temperature within the experimental region. For practical reasons (a limited number of temperature-controlled test chambers and the test groups of units at one time) the rectangular pattern is preferred by engineers. For our adhesive bond

C application with competing risks, we expect to obtain little useful information about adhesive strength by allocating test units at high-stress test conditions because of the high right-censored probabilities at those conditions, as shown in Figure 2. For this reason, some of the high-stress test conditions in the compromise plan structure of Figure 8 should be omitted to reduce the risk of cohesive failures and to get a better estimate of  $\hat{t}_{0.01}$  for competing risk applications. Table 3 compares  $\text{Ase}(\hat{t}_{0.01})$  for various compromise plans obtained by omitting different combinations of the high-stress test conditions. The proportional allocations for all the remaining test conditions in the plan are equal. For each compromise plan, beyond a baseline test condition, there are three equally spaced time levels and three equally spaced temperature levels. The highest time level is fixed at 16 weeks and the highest temperature level is fixed at 70 °C. The lowest time level and the lowest temperature level are chosen to minimize  $\text{Ase}(\hat{t}_{0.01})$ . The middle time level is the mean of the other two time levels and the middle temperature level is the mean of the other two temperature levels. As can be seen from Table 3, the plan after omitting three high-stress test conditions, numbered 7, 9, 10, appears to be the best optimized compromise plan with competing risks in the sense that it provides the smallest  $\text{Ase}(\hat{t}_{0.01})$ . Note that there is a trade off between the estimation precision and robustness of the test plan. We can obtain a test plan with a smaller  $\text{Ase}(\hat{t}_{0.01})$  by omitting more test conditions, but we have to pay for this by some loss of robustness. We would recommend the last compromise plan in Table 3 or a similar compromise plan.

Table 3: Comparison of different compromise ADDT plans with competing risks.

Omitted Conditions	Lowest Time in Weeks	Lowest Temperature in °C	Proportional Allocation	$\text{Ase}(\hat{t}_{0.01})$
None	8.19	50.437	1/10	389.83
10	7.56	50.182	1/9	372.47
9, 10	10.36	52.087	1/8	373.88
7, 10	7.89	48.673	1/8	368.26
7, 9, 10	9.39	51.552	1/7	364.44

## 6 Monte Carlo Simulation to Evaluate Test Plans

Monte Carlo simulation is a useful tool to provide visualization of the results obtained from a test plan. Simulation also provides a check on the adequacy of the large-sample approximations. We do simulations to evaluate the three ADDT plans illustrated before: the original plan in Table 1, the

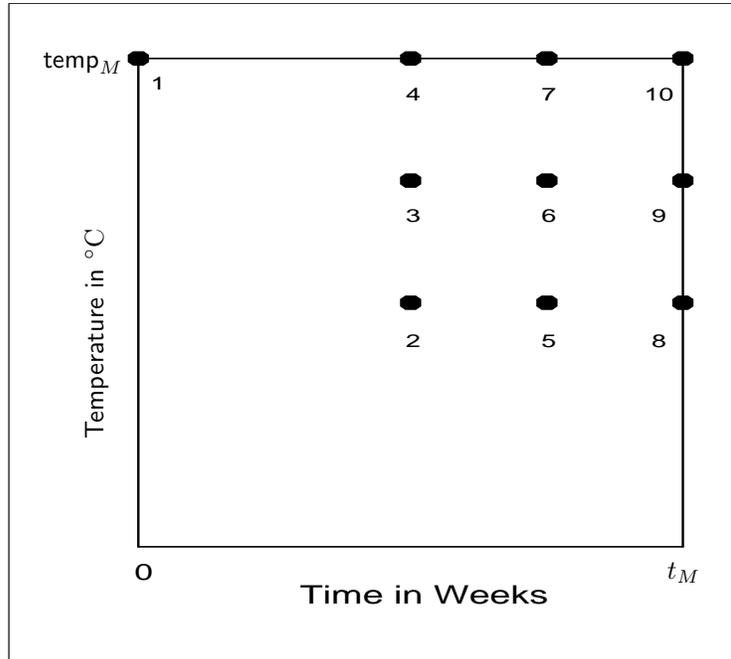


Figure 8: Compromise plan structure.

constrained optimum plan in Table 2, and the last compromise plan in Table 3. For the adhesive bond C application, the large-sample approximate standard error of the estimated 0.01 failure-time quantile,  $\text{Ase}(\hat{t}_{0.01})$ , for these test plans are 396.68, 274.62, and 364.44 respectively. For each of these test plans, a simulation trial consists of a set of 74 observations obtained according to the test plan, the given degradation models, and the planning information.

Figure 9 shows estimates of the 0.01 failure-time quantile versus temperature for 50 realizations of simulations for each test plan. The longer lines represent the values computed from the planning values, which we call the “true” values. The constrained optimum plan has the narrowest group of simulated lines. The spread of simulated lines for the compromise plan is wider than that for the constrained optimum plan but narrower than that for the original plan. These results are exactly consistent with the estimation precisions of different test plans, based on the large-sample approximate  $\text{Ase}(\hat{t}_{0.01})$ .

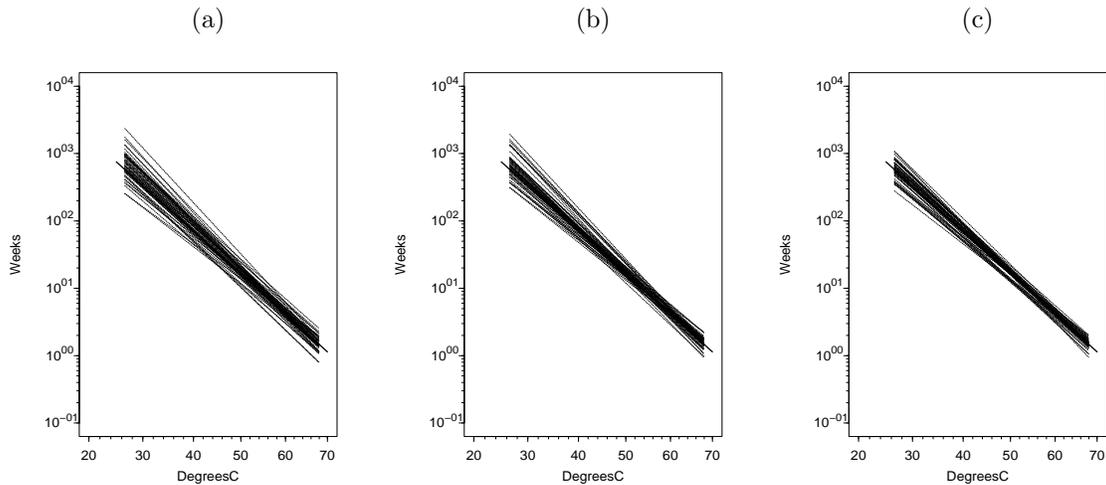


Figure 9: Simulation of 0.01 failure-time quantile estimates versus temperature for (a) the original plan, (b) the compromise plan, and (c) the constrained optimum plan.

## 7 Conclusions and Extensions

Accelerated destructive degradation testing is an important tool for making reliability inferences and predictions, especially when few or no failures are expected at use conditions within severe testing time constraints. Good ADDT plans can yield significant benefits to industry. However, accelerated tests might result in more than one type of failure for units tested at high-stress test conditions. Because of the competing risks problem, designing an appropriate ADDT plan could give more useful information about the primary degradation responses within the limited test time. The methodology presented in this paper can be extended in several important directions, suggesting areas for future research. These include the following:

1. Test plans for an accelerated degradation model with multiple accelerating variables (e.g., temperature and humidity) could be developed.
2. As explained in Section 2.1, the relationship between the location parameter and the transformed time is linear at a fixed accelerating variable level. The work in this paper could be extended to degradation models that have a nonlinear relationship (such a relationship might, for example, be suggested by a physics of failure model).
3. For some applications, prior knowledge about the failure mechanism might provide information about some of the model parameters. It is important that such prior information be used in

test planning as well as analysis. It would be useful to apply Bayesian methods like those described in Zhang and Meeker (2006) for ADDT planning.

## A Appendix: Technical Details

### A.1 The Fisher Information Matrix for ADDT with Competing Risks

Because of the competing risks, each obtained observation could be an exact or right random censoring value (for example, see Chapter 5 of Nelson 1982 or Chapter 9 of Lawless 2003). Escobar and Meeker (1998) show in detail how to compute the Fisher information matrix and large-sample approximate covariance matrix of maximum likelihood estimators for a wide class of parametric models, including models with right random censoring data. The specific approach used here follows the general approach outlined in Escobar and Meeker (1998).

Let  $\hat{\boldsymbol{\theta}}$  be the ML estimator of the parameters  $\boldsymbol{\theta} = (\beta_0, \beta_1, \beta_2, \sigma)'$  in the primary response model based on  $n$  observations. Under the usual regularity conditions (for example, see Appendix B.4 of Meeker and Escobar 1998),  $\hat{\boldsymbol{\theta}} \sim \text{MVN}(\boldsymbol{\theta}, \boldsymbol{\Sigma})$ , where  $\boldsymbol{\Sigma} = \mathbf{I}_{\boldsymbol{\theta}, \boldsymbol{\theta}^{(c)}}$ , and the Fisher information matrix  $\mathbf{I}_{\boldsymbol{\theta}, \boldsymbol{\theta}^{(c)}}$  with  $r$  test conditions is  $\mathbf{I}_{\boldsymbol{\theta}, \boldsymbol{\theta}^{(c)}} = n \sum_{i=1}^r \pi_i \mathbf{I}_i$ . Here  $\mathbf{I}_i$  is the contribution of one observation at a specific test condition  $\mathbf{v}_i = (\tau_i, x_i)$  to  $\mathbf{I}_{\boldsymbol{\theta}, \boldsymbol{\theta}^{(c)}}$ . For an ADDT with competing risks, following the approach given in Escobar and Meeker (1998),  $\mathbf{I}_i$  can be expressed in terms of parameters  $\boldsymbol{\theta}$  and  $\boldsymbol{\theta}^{(c)}$  as

$$\mathbf{I}_i = \frac{1}{\sigma^2} \begin{bmatrix} \left[ \int_{-\infty}^{\infty} f_{11}(-\infty, \frac{y-\mu_i}{\sigma}) h_i(y) dy \right] \mathbf{u}_i \mathbf{u}_i' & \left[ \int_{-\infty}^{\infty} f_{12}(-\infty, \frac{y-\mu_i}{\sigma}) h_i(y) dy \right] \mathbf{u}_i \\ \left[ \int_{-\infty}^{\infty} f_{12}(-\infty, \frac{y-\mu_i}{\sigma}) h_i(y) dy \right] \mathbf{u}_i' & \left[ \int_{-\infty}^{\infty} f_{22}(-\infty, \frac{y-\mu_i}{\sigma}) h_i(y) dy \right] \end{bmatrix}$$

where  $\mu_i$  is the location parameter for the primary response, defined in Section 2.1 as  $\mu_i = \beta_0 + \beta_1 \exp(\beta_2 x_i) \tau_i$  with  $\tau_i = h_t(t_i)$  and  $x_i = h_a(\text{AccVar}_i)$ .  $f_{jk}(-\infty, \frac{y-\mu_i}{\sigma})$ ,  $jk = 11, 12, 22$  are the elements of the Fisher information matrix, multiplied by  $\sigma^2$ , for right-censored observations (see Escobar and Meeker 1998 for details). These elements can be computed using the LSINF algorithm (see Escobar and Meeker 1994). Here  $h_i(y) = \frac{1}{\sigma^{(c)}} \phi\left(\frac{y-\mu_i^{(c)}}{\sigma^{(c)}}\right)$  is the pdf of right random censoring point  $y$  due to competing risks, where  $\mu_i^{(c)} = \beta_0^{(c)} + \beta_1^{(c)} \exp(\beta_2^{(c)} x_i) \tau_i$ , and  $\mathbf{u}_i$  is the vector of partial

derivatives of  $\mu_i$  with respect to the parameters  $(\beta_0, \beta_1, \beta_2)$ . That is

$$\mathbf{u}_i = \begin{bmatrix} \frac{\partial \mu_i}{\partial \beta_0} \\ \frac{\partial \mu_i}{\partial \beta_1} \\ \frac{\partial \mu_i}{\partial \beta_2} \end{bmatrix} = \begin{bmatrix} 1 \\ \exp(\beta_2 x_i) \tau_i \\ \beta_1 x_i \exp(\beta_2 x_i) \tau_i \end{bmatrix}$$

The Fisher information matrix is evaluated based on planning information for model parameters  $\boldsymbol{\theta}^\square$  and  $\boldsymbol{\theta}^{(c)\square}$ .

## A.2 Large-Sample Approximate Variance of $h_t(\hat{t}_p)$ and $\hat{t}_p$

When  $\hat{\boldsymbol{\theta}} \sim \text{MVN}(\boldsymbol{\theta}, \boldsymbol{\Sigma})$ , then for a scalar function  $\hat{g} = g(\hat{\boldsymbol{\theta}}) \sim \text{NOR}[g(\boldsymbol{\theta}), \text{Avar}(\hat{g})]$ , and the delta method gives

$$\text{Avar}(\hat{g}) = \left[ \frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right]' \boldsymbol{\Sigma} \left[ \frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right],$$

providing the large-sample approximate variance for the ML estimator of a desired function of  $\boldsymbol{\theta}$ .

$\text{Avar}[h_t(\hat{t}_p)]$  can be written as a function of  $\boldsymbol{\Sigma}$  using delta method and expressed as

$$\text{Avar}[h_t(\hat{t}_p)] = \mathbf{c}' \boldsymbol{\Sigma} \mathbf{c} = \mathbf{c}' \mathbf{I}_{\boldsymbol{\theta}, \boldsymbol{\theta}^{(c)}}^{-1} \mathbf{c},$$

where  $\mathbf{c} = \partial h_t(t_p) / \partial \boldsymbol{\theta}$ . For decreasing degradation,  $h_t(t_p) = \nu + \varsigma \Phi^{-1}(p)$  for  $p \geq \Phi(-\nu/\varsigma)$ . In this case the elements of  $\mathbf{c}$  are:

$$\begin{aligned} \frac{\partial h_t(t_p)}{\partial \beta_0} &= -\frac{1}{\beta_1 \exp(\beta_2 x)} & \frac{\partial h_t(t_p)}{\partial \beta_1} &= -\frac{h_t(t_p)}{\beta_1} \\ \frac{\partial h_t(t_p)}{\partial \beta_2} &= -x h_t(t_p) & \frac{\partial h_t(t_p)}{\partial \sigma} &= -\frac{\Phi^{-1}(p)}{\beta_1 \exp(\beta_2 x)}. \end{aligned}$$

Using the delta method again,

$$\text{Avar}(\hat{t}_p) = \left( \frac{\partial h_t^{-1}(z)}{\partial z} \Big|_{h_t(t_p)} \right)^2 \text{Avar}[h_t(\hat{t}_p)].$$

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