

8-2010

Zero forcing parameters and minimum rank problems

Francesco Barioli

University of Tennessee, Chattanooga

Wayne Barrett

Brigham Young University

Shaun M. Fallat

University of Regina

H. Tracy Hall

Brigham Young University

Leslie Hogben

Iowa State University, hogben@iastate.edu

See next page for additional authors

Follow this and additional works at: http://lib.dr.iastate.edu/math_pubs



Part of the [Algebra Commons](#)

The complete bibliographic information for this item can be found at http://lib.dr.iastate.edu/math_pubs/68. For information on how to cite this item, please visit <http://lib.dr.iastate.edu/howtocite.html>.

This Article is brought to you for free and open access by the Mathematics at Iowa State University Digital Repository. It has been accepted for inclusion in Mathematics Publications by an authorized administrator of Iowa State University Digital Repository. For more information, please contact digirep@iastate.edu.

Authors

Francesco Barioli, Wayne Barrett, Shaun M. Fallat, H. Tracy Hall, Leslie Hogben, Bryan Shader, P. van den Driessche, and Hein van der Holst

ZERO FORCING PARAMETERS AND MINIMUM RANK PROBLEMS*

FRANCESCO BARIOLI[†], WAYNE BARRETT[‡], SHAUN M. FALLAT[§],
H. TRACY HALL[¶], LESLIE HOGBEN^{||}, BRYAN SHADER^{**},
P. VAN DEN DRIESSCHE^{††} AND HEIN VAN DER HOLST^{‡‡}

March 2, 2010

1 **Abstract.** The zero forcing number $Z(G)$, which is the minimum number of vertices in a zero forcing set of a
2 graph G , is used to study the maximum nullity/minimum rank of the family of symmetric matrices described by
3 G . It is shown that for a connected graph of order at least two, no vertex is in every zero forcing set. The positive
4 semidefinite zero forcing number $Z_+(G)$ is introduced, and shown to be equal to $|G| - OS(G)$, where $OS(G)$ is the
5 recently defined ordered set number that is a lower bound for minimum positive semidefinite rank. The positive
6 semidefinite zero forcing number is applied to the computation of positive semidefinite minimum rank of certain
7 graphs. An example of a graph for which the real positive symmetric semidefinite minimum rank is greater than
8 the complex Hermitian positive semidefinite minimum rank is presented.

9 **Key words.** zero forcing number, maximum nullity, minimum rank, positive semidefinite zero forcing number,
10 positive semidefinite maximum nullity, positive semidefinite minimum rank, ordered set number

11 **AMS subject classifications.** (2010) 05C50, 15A03, 15A18, 15B57

12 **1. Introduction.** The *minimum rank problem* for a (simple) graph asks for the determination
13 of the minimum rank among all real symmetric matrices with the zero-nonzero pattern of off-
14 diagonal entries described by a given graph (the diagonal of the matrix is free); the maximum
15 nullity of the graph is the maximum nullity over the same set of matrices. This problem arose
16 from the study of possible eigenvalues of real symmetric matrices described by a graph and has
17 received considerable attention over the last ten years (see [7] and references therein). There
18 has also been considerable interest in the related *positive semidefinite minimum rank problem*,
19 where the minimum rank is taken over (real or complex Hermitian) positive semidefinite matrices
20 described by a graph (see, for example, [4, 6, 10, 12, 13, 15]).

21 Zero forcing sets and the zero forcing number were introduced in [1]. The zero forcing number
22 is a useful tool for determining the minimum rank of structured families of graphs and small graphs,

*Part of this research was done at the American Institute of Mathematics SQuaRE, “Minimum Rank of Symmetric Matrices described by a Graph,” and the authors thank AIM and NSF for their support.

[†]Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga TN, 37403 (francesco-barioli@utc.edu).

[‡]Department of Mathematics, Brigham Young University, Provo UT 84602 (wayne@math.byu.edu).

[§]Department of Mathematics and Statistics, University of Regina, Regina, SK, Canada (sfallat@math.uregina.ca).
Research supported in part by an NSERC Discovery grant.

[¶]Department of Mathematics, Brigham Young University, Provo UT 84602 (H.Tracy@gmail.com).

^{||}Department of Mathematics, Iowa State University, Ames, IA 50011, USA (lhogben@iastate.edu) and American Institute of Mathematics, 360 Portage Ave, Palo Alto, CA 94306 (hogben@aimath.org).

^{**}Department of Mathematics, University of Wyoming, Laramie, WY 82071, USA (bshader@uwyo.edu).

^{††}Department of Mathematics and Statistics, University of Victoria, Victoria, BC, V8W 3R4, Canada (pvdd@math.uvic.ca). Research supported in part by an NSERC Discovery grant.

^{‡‡}School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA (holst@math.gatech.edu). On leave from Eindhoven University of Technology.

23 and is motivated by simple observations about null vectors of matrices. The zero forcing process
 24 is the same as graph infection used by physicists to study control of quantum systems [5], and the
 25 zero forcing number is becoming a graph parameter of interest in its own right.

26 A *graph* $G = (V_G, E_G)$ means a simple undirected graph (no loops, no multiple edges) with a
 27 finite nonempty set of vertices V_G and edge set E_G (an edge is a two-element subset of vertices). All
 28 matrices discussed are Hermitian; the set of real symmetric $n \times n$ matrices is denoted by S_n and the
 29 set of (possibly complex) Hermitian $n \times n$ matrices is denoted by H_n . For $A \in H_n$, the *graph* of A ,
 30 denoted by $\mathcal{G}(A)$, is the graph with vertices $\{1, \dots, n\}$ and edges $\{\{i, j\} : a_{ij} \neq 0, 1 \leq i < j \leq n\}$.
 31 Note that the diagonal of A is ignored in determining $\mathcal{G}(A)$. The study of minimum rank has
 32 focused on real symmetric matrices (or in some cases, symmetric matrices over a field other than
 33 the real numbers), whereas much of the work on positive semidefinite minimum rank involves
 34 (possibly complex) Hermitian matrices. Whereas it is well known that using complex Hermitian
 35 matrices can result in a lower minimum rank than using real symmetric matrices, one of the issues
 36 in the study of minimum positive semidefinite rank has been whether or not using only real matrices
 37 or allowing complex matrices matters to minimum positive semidefinite rank. Example 4.1 below
 38 shows that complex Hermitian positive semidefinite minimum rank can be strictly lower than real
 39 symmetric positive semidefinite minimum rank.

40 Let G be a graph. The *set of real symmetric matrices described by G* is

$$41 \quad \mathcal{S}(G) = \{A \in S_n : \mathcal{G}(A) = G\}.$$

42 The *minimum rank* of G is

$$43 \quad \text{mr}(G) = \min\{\text{rank } A : A \in \mathcal{S}(G)\}$$

44 and the *maximum nullity* of G is

$$45 \quad \text{M}(G) = \max\{\text{null } A : A \in \mathcal{S}(G)\}.$$

46 Clearly $\text{mr}(G) + \text{M}(G) = |G|$, where the *order* $|G|$ is the number of vertices of G . The *set of*
 47 *real positive semidefinite matrices described by G* and the *set of Hermitian positive semidefinite*
 48 *matrices described by G* are, respectively,

$$49 \quad \mathcal{S}_+(G) = \{A \in S_n : \mathcal{G}(A) = G \text{ and } A \text{ is positive semidefinite}\}$$

$$50 \quad \mathcal{H}_+(G) = \{A \in H_n : \mathcal{G}(A) = G \text{ and } A \text{ is positive semidefinite}\}.$$

51 The *minimum positive semidefinite rank* of G and *minimum Hermitian positive semidefinite rank*
 52 of G are, respectively,

$$53 \quad \text{mr}_+^{\mathbb{R}}(G) = \min\{\text{rank } A : A \in \mathcal{S}_+(G)\} \text{ and } \text{mr}_+^{\mathbb{C}}(G) = \min\{\text{rank } A : A \in \mathcal{H}_+(G)\}.$$

54 The *maximum positive semidefinite nullity* of G and the *maximum Hermitian positive semidefinite*
 55 *nullity* of G are, respectively,

$$56 \quad \text{M}_+^{\mathbb{R}}(G) = \max\{\text{null } A : A \in \mathcal{S}_+(G)\} \text{ and } \text{M}_+^{\mathbb{C}}(G) = \max\{\text{null } A : A \in \mathcal{H}_+(G)\}.$$

57 Clearly $\text{mr}_+^{\mathbb{R}}(G) + \text{M}_+^{\mathbb{R}}(G) = |G|$ and $\text{mr}_+^{\mathbb{C}}(G) + \text{M}_+^{\mathbb{C}}(G) = |G|$. There are a variety of symbols in
 58 the literature (see, for example, [4, 15]) for these parameters, including $\text{msr}(G)$ and $\text{hmr}_+(G)$ for

59 what we denote by $\text{mr}_+^{\mathbb{C}}(G)$. Clearly $\text{M}_+^{\mathbb{R}}(G) \leq \text{M}(G)$ and $\text{mr}(G) \leq \text{mr}_+^{\mathbb{R}}(G)$ for every graph G ,
60 and it is well known that these inequalities can be strict (for example, any tree T that is not a
61 path has $\text{mr}(T) < \text{mr}_+^{\mathbb{R}}(T)$).

62 We need some additional graph terminology. The *complement* of a graph $G = (V, E)$ is the
63 graph $\overline{G} = (V, \overline{E})$, where \overline{E} consists of all two element sets from V that are not in E . We denote
64 the complete graph on n vertices by K_n ; a complete graph is also called a clique. The *degree* of
65 vertex v in graph G is the number of edges incident with v , and the minimum degree of the vertices
66 of G is denoted by $\delta(G)$. A set of subgraphs of G , each of which is a clique and such that every
67 edge of G is contained in at least one of these cliques, is called a *clique covering* of G . The *clique*
68 *covering number* of G , denoted by $\text{cc}(G)$, is the smallest number of cliques in a clique covering of
69 G .

70 OBSERVATION 1.1. [7] For every graph G , $\text{mr}_+^{\mathbb{R}}(G) \leq \text{cc}(G)$, so $|G| - \text{cc}(G) \leq \text{M}_+^{\mathbb{R}}(G)$.

71 For an $n \times n$ matrix A and $W \subseteq \{1, \dots, n\}$, the principal submatrix $A[W]$ is the submatrix of
72 A lying in the rows and columns that have indices in W . For a graph $G = (V_G, E_G)$ and $W \subseteq V_G$,
73 the *induced subgraph* $G[W]$ is the graph with vertex set W and edge set $\{\{v, w\} \in E_G : v, w \in W\}$.
74 The induced subgraph $\mathcal{G}(A)[W]$ of the graph of A is naturally associated with the graph of the
75 the principal submatrix for W , i.e., $\mathcal{G}(A[W])$. The subgraph induced by $\overline{W} = V_G \setminus W$ is usually
76 denoted by $G - W$, or in the case W is a singleton $\{v\}$, by $G - v$.

77 The *path cover number* $\text{P}(G)$ of G is the smallest positive integer m such that there are m
78 vertex-disjoint induced paths P_1, \dots, P_m in G that cover all the vertices of G (i.e., $V_G = \dot{\cup}_{i=1}^m V_{P_i}$).
79 A graph is *planar* if it can be drawn in the plane without crossing edges. A graph is *outerplanar*
80 if it has such a drawing with a face that contains all vertices. Given two graphs G and H , the
81 *Cartesian product* of G and H , denoted $G \square H$, is the graph whose vertex set is the Cartesian
82 product of V_G and V_H , with an edge between two vertices exactly when they are identical in one
83 coordinate and adjacent in the other.

84 Let $G = (V_G, E_G)$ be a graph. A subset $Z \subseteq V_G$ defines an initial set of black vertices (with all
85 the vertices not in Z white), called a *coloring*. There are no constraints on permissible colorings;
86 instead there are constraints on how new colorings can be derived. The *color change rule* (for the
87 zero forcing number) is to change the color of a white vertex w to black if w is the unique white
88 neighbor of a black vertex u ; in this case we say u *forces* w and write $u \rightarrow w$. Given a coloring of
89 G , the *derived set* is the set of black vertices obtained by applying the color change rule until no
90 more changes are possible. A *zero forcing set* for G is a subset of vertices Z such that if initially
91 the vertices in Z are colored black and the remaining vertices are colored white, the derived set is
92 V_G . The *zero forcing number* $\text{Z}(G)$ is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V_G$.

93 THEOREM 1.2. [1, Proposition 2.4] For any graph G , $\text{M}(G) \leq \text{Z}(G)$.

94 Suppose $S = (v_1, v_2, \dots, v_m)$ is an ordered subset of vertices from a given graph G . For each
95 k with $1 \leq k \leq m$, let G_k be the subgraph of G induced by $\{v_1, v_2, \dots, v_k\}$, and let H_k be the
96 connected component of G_k that contains v_k . If for each k , there exists a vertex w_k that satisfies:
97 $w_k \neq v_l$ for $l \leq k$, $\{w_k, v_k\} \in E$, and $\{w_k, v_s\} \notin E$, for all v_s in H_k with $s \neq k$, then S is called an
98 *ordered set of vertices* in G , or an OS-set. As defined in [10], the *OS number* of a graph G , denoted

99 by $\text{OS}(G)$, is the maximum of $|S|$ over all OS-sets S of G .

100 THEOREM 1.3. [10, Proposition 3.3] *For any graph G , $\text{OS}(G) \leq \text{mr}_+^{\text{C}}(G)$.*

101 In Section 2 we establish several properties of the zero forcing number, including the nonunique-
102 ness of zero forcing sets. In Section 3 we introduce the positive semidefinite zero forcing number
103 as an upper bound for maximum positive semidefinite nullity, show that the sum of the positive
104 semidefinite zero forcing number and the OS number is the order of the graph, and apply the
105 positive semidefinite zero forcing number to the computation of positive semidefinite minimum
106 rank. Section 4 provides the first example showing that $\text{mr}_+^{\text{R}}(G)$ and $\text{mr}_+^{\text{C}}(G)$ need not be the
107 same (described as unknown in [7]).

108 **2. Properties of the zero forcing number.** In this section, we establish several properties
109 of the zero forcing number, including the non-uniqueness of zero forcing sets and its relationship
110 to path cover number. We need some additional definitions related to the zero forcing number.

111 DEFINITION 2.1. A *minimum zero forcing set* is a zero forcing set Z such that $|Z| = \text{Z}(G)$.

112 Zero forcing chains of digraphs were defined in [2]. We give an analogous definition for graphs.
113

114 DEFINITION 2.2. Let Z be a zero forcing set of a graph G .

- 115 • Construct the derived set, recording the forces in the order in which they are performed.
116 This is the *chronological list of forces*.
- 117 • A *forcing chain* (for this particular chronological list of forces) is a sequence of vertices
118 (v_1, v_2, \dots, v_k) such that for $i = 1, \dots, k - 1$, $v_i \rightarrow v_{i+1}$.
- 119 • A *maximal forcing chain* is a forcing chain that is not a proper subsequence of another
120 zero forcing chain.

121 Note that a zero forcing chain can consist of a single vertex (v_1) , and such a chain is maximal if
122 $v_1 \in Z$ and v_1 does not perform a force.

123 As noted in [1], the derived set of a given set of black vertices is unique; however, a chronological
124 list of forces (of one particular zero forcing set) usually is not. At Rocky Mountain Discrete
125 Mathematics Days held Sept. 12 – 13, 2008 at the University of Wyoming, the following questions
126 were raised.

127 QUESTION 2.3. *Is there a graph that has a unique minimum zero forcing set?*

128 QUESTION 2.4. *Is there a graph G and a vertex $v \in V_G$ such that v is in every minimum zero
129 forcing set?*

130 We show the answers to both these questions are negative for nontrivial connected graphs.

131 DEFINITION 2.5. Let Z be a zero forcing set of a graph G . A *reversal* of Z is the set of last
132 vertices of the maximal zero forcing chains of a chronological list of forces.

133 Each vertex can force at most one other vertex and can be forced by at most one other vertex,
134 so the maximal forcing chains are disjoint, and the elements of Z are the initial vertices of the

135 maximal forcing chains. Thus the cardinality of a reversal of Z is the same as the cardinality of Z .

136 THEOREM 2.6. *If Z is a zero forcing set of G then so is any reversal of Z .*

137 *Proof.* Write the chronological list of forces in reverse order, reversing each force (call this the
138 reverse chronological list of forces) and let the reversal of Z for this list be denoted W . We show
139 the reverse chronological list of forces is a valid list of forces for W . Consider the first “force”
140 $u \rightarrow v$ on the reverse chronological list. All neighbors of u except v must be in W , since when the
141 last force $v \rightarrow u$ of Z was done, each of them had the white neighbor u and thus did not force any
142 vertex previously (in the original chronological list of forces). Thus $u \rightarrow v$ is a valid force for W .
143 Continue in this manner or use induction on $|G|$. \square

144 COROLLARY 2.7. *No connected graph of order greater than one has a unique minimum zero*
145 *forcing set.*

146 LEMMA 2.8. *Let G be a connected graph of order greater than one and let Z be a minimum*
147 *zero forcing set. Every $z \in Z$ has a neighbor $w \notin Z$.*

148 *Proof.* Suppose not. Then there is a vertex $z \in Z$ such that every neighbor of z is in Z (and
149 z does have at least one neighbor v). Since z cannot perform a force, z is in the reversal W of Z .
150 Using the reversed maximal forcing chains, no neighbor of z performs a force. So $W \setminus \{z\}$ is a zero
151 forcing set of smaller cardinality, because after every vertex except z is black, v can force z . \square

152 THEOREM 2.9. *If G is a connected graph of order greater than one, then*

$$153 \quad \bigcap_{Z \in ZFS(G)} Z = \emptyset,$$

154 *where $ZFS(G)$ is the set of all minimum zero forcing sets of G .*

155 *Proof.* Suppose not. Then there exists $v \in \bigcap_{Z \in ZFS(G)} Z$. In particular, for each Z and each
156 reversal W of Z , v is in both Z and W . This means that there is a maximal forcing chain consisting
157 of only v , or in other words v does not force any other vertex.

158 Let Z be a zero forcing set. If there is no chronological list of forces in which a neighbor of
159 v performs a force, then replace Z by its reversal (since, by Lemma 2.8, v originally had a white
160 neighbor u , in the reversal u performs a force). Let $u \rightarrow w$ be the first force in which the forcing
161 vertex u is a neighbor of v . We claim that $Z \setminus \{v\} \cup \{w\}$ is a zero forcing set for G . The forces
162 can proceed until u is encountered as a forcing vertex. At that time, replace $u \rightarrow w$ by $u \rightarrow v$, and
163 then continue as in the original chronological list of forces. \square

164 Next we show that for any graph the zero forcing number is an upper bound for the path cover
165 number.

166 PROPOSITION 2.10. *For any graph G , $P(G) \leq Z(G)$.*

167 *Proof.* Let Z be a zero forcing set. The vertices in a forcing chain induce a path in G because
168 the forces in a forcing chain occur chronologically in the order of the chain (since only a black
169 vertex can force). The maximal forcing chains are disjoint, contain all the vertices of G , and the
170 elements of the set Z are the initial vertices of the maximal forcing chains. Thus $P(G) \leq |Z|$. By
171 choosing a minimum zero forcing set Z , $P(G) \leq Z(G)$. \square

172 In [14] it was shown that for a tree T , $P(T) = M(T)$, and in [1] it was shown that for a tree,
 173 $P(T) = Z(T)$ (and thus $M(T) = Z(T)$). In [3] it was shown that for graphs in general, $P(G)$
 174 and $M(G)$ are not comparable. However, Sinkovic has established the following relationship for
 175 outerplanar graphs: If G is an outerplanar graph, then $M(G) \leq P(G)$ [16]. The next example
 176 shows that neither outerplanar graphs nor 2-trees require $M(G) = Z(G)$ or $P(G) = Z(G)$ (a 2-tree
 177 is constructed inductively by starting with a K_3 and connecting each new vertex to 2 adjacent
 178 existing vertices).

179 **EXAMPLE 2.11.** Let G_{12} be the graph shown in Figure 2.1, called the pinwheel on 12 vertices.
 Note that G_{12} is an outerplanar 2-tree. The set $\{1, 2, 6, 10\}$ is a zero forcing set for G_{12} , so

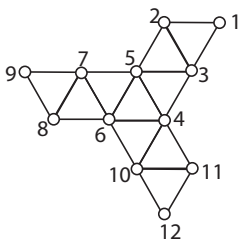


FIG. 2.1. The graph G_{12} for Example 2.11, the pinwheel on 12 vertices

180 $Z(G_{12}) \leq 4$. We show that $Z(G_{12}) \geq 4$, which implies $Z(G_{12}) = 4$. Suppose to the contrary that
 181 Z is a zero forcing set for G_{12} and $|Z| = 3$. To start the forcing, at least two of the vertices must
 182 be in one of the sets $\{1, 2, 3\}, \{7, 8, 9\}, \{10, 11, 12\}$; without loss of generality, assume that two or
 183 three black vertices are in $\{1, 2, 3\}$. Then after several forces the vertices $\{1, 2, 3, 4, 5\}$ are black,
 184 and at most one additional vertex $v \notin \{1, 2, 3, 4, 5\}$ is in Z . To perform another force with only one
 185 more black vertex v , either 6 or 7 must be black, and 5 can force the other, but then no additional
 186 forces can be performed, so Z was not a zero forcing set for G_{12} . Clearly G_{12} can be covered by 9
 187 triangles, so $cc(G_{12}) \leq 9$ and $M_+^{\mathbb{R}}(G_{12}) \geq 3$, by Observation 1.1. It is easy to find a path covering
 188 of 3 paths, so $M_+^{\mathbb{R}}(G_{12}) = M(G_{12}) = P(G_{12}) = 3$ and $mr_+^{\mathbb{R}}(G_{12}) = mr(G_{12}) = cc(G_{12}) = 9$. Since
 189 G_{12} is chordal, $mr_+^{\mathbb{C}}(G_{12}) = cc(G_{12})$ [4], and thus $M_+^{\mathbb{C}}(G_{12}) = 3$.

191 **3. The positive semidefinite zero forcing number.** In this section, we introduce the
 192 positive definite zero forcing number, relate it to maximum positive semidefinite nullity and to the
 193 OS number, and apply it to compute maximum positive semidefinite nullity of several families of
 194 graphs. The definitions and terminology for zero forcing (coloring, derived set, etc.) are the same as
 195 as for the zero forcing number $Z(G)$, but the color change rule is different.

196 **DEFINITION 3.1.**

- 197 • The *positive semidefinite color change rule* is:
 198 Let B be the set consisting of all the black vertices. Let W_1, \dots, W_k be the sets of vertices
 199 of the k components of $G - B$ (note that it is possible that $k = 1$). Let $w \in W_i$. If $u \in B$
 200 and w is the only white neighbor of u in $G[W_i \cup B]$, then change the color of w to black.
- 201 • The *positive semidefinite zero forcing number of a graph G* , denoted by $Z_+(G)$, is the
 202 minimum of $|X|$ over all positive semidefinite zero forcing sets $X \subseteq V_G$ (using the positive

203 semidefinite color change rule).

204 Forcing using the positive semidefinite color change rule can be thought of as decomposing the
 205 graph into a union of certain induced subgraphs and using ordinary zero forcing on each of these
 206 induced subgraphs. The application of the positive semidefinite color change rule is illustrated in
 207 the next example.

208 **EXAMPLE 3.2.** Let T be a tree. Then $Z_+(T) = 1$, because any one vertex v is a positive
 209 semidefinite zero forcing set. Formally, this can be established by induction on $|T|$: If v is a
 210 leaf, it forces its neighbor; if not a decomposition takes place. In either case smaller tree(s) are
 211 obtained. It has been known for a long time (see, for example, [7]) that $M_+^C(T) = 1$, but the use
 212 of Z_+ provides an easy proof of this result, because $M_+^C(T) = 1$ is an immediate consequence of
 213 $Z_+(T) = 1$ by Theorem 3.5 below.

214 **OBSERVATION 3.3.** *Since any zero forcing set is a positive definite zero forcing set,*

215
$$Z_+(G) \leq Z(G).$$

216 **EXAMPLE 3.4.** The pinwheel G_{12} shown in Figure 2.1 has $Z_+(G_{12}) = 3 = M_+^R(G_{12})$ because
 217 $X = \{4, 5, 6\}$ is a positive semidefinite zero forcing set ($G_{12} - X$ is disconnected, and X is a zero
 218 forcing set for $G[\{1, 2, 3, 4, 5, 6\}]$, etc.).

219 For any graph G that is the disjoint union of connected components $G_i, i = 1, 2, \dots, k, Z_+(G) =$
 220 $\sum_{i=1}^k Z_+(G_i)$ (the analogous results for M, M_+^R, M_+^C and Z are all well known).

221 **THEOREM 3.5.** *For any graph $G, M_+^C(G) \leq Z_+(G).$*

222 *Proof.* Let $A \in \mathcal{H}_+(G)$ with $\text{null } A = M_+^C(G)$. Let $\mathbf{x} = [x_i]$ be a nonzero vector in $\ker A$.
 223 Define B to be the set of indices u such that $x_u = 0$ and let W_1, \dots, W_k be the sets of vertices of
 224 the k components of $G - B$. We claim that in $G[B \cup W_i], w \in W_i$ cannot be the unique neighbor
 225 of any vertex $u \in B$. Once the claim is established, if X is a positive semidefinite zero forcing set
 226 for G , then the only vector in $\ker A$ with zeros in positions indexed by X is the zero vector, and
 227 thus $M_+^C(G) \leq Z_+(G)$.

228 To establish the claim, renumber the vertices so that the vertices of B are last, the vertices of
 229 W_1 are first, followed by the vertices of W_2 , etc. Then A has the block form

230
$$A = \begin{bmatrix} A_1 & 0 & \dots & 0 & C_1^* \\ 0 & A_2 & \dots & 0 & C_2^* \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & A_k & C_k^* \\ C_1 & C_2 & \dots & C_k & D \end{bmatrix}.$$

231 Partition \mathbf{x} conformally as $\mathbf{x} = [\mathbf{x}_1^T, \dots, \mathbf{x}_k^T, 0]^T$, and note that all entries of \mathbf{x}_i are nonzero,
 232 $i = 1, \dots, k$. Then $A\mathbf{x} = 0$ implies $A_i\mathbf{x}_i = 0, i = 1, \dots, k$. Since A is positive semidefinite, each
 233 column in C_i^* is in the span of the columns of A_i by the column inclusion property of Hermitian
 234 positive semidefinite matrices [8]. That is, for $i = 1, \dots, k$, there exists Y_i such that $C_i^* = A_i Y_i$.
 235 Thus $C_i\mathbf{x}_i = Y_i^* A_i \mathbf{x}_i = 0$, and $w \in W_i$ cannot be the unique neighbor in W_i of any vertex $u \in B$.
 236 \square

237 Theorem 3.5 is also a consequence of Theorem 3.6 below and Theorem 1.3 above, but using
 238 that as a justification obscures the motivation for the definition and the connection between zero
 239 forcing and null vectors that is given in the short direct proof.

240 In [15, Theorem 2.10] it is shown that $|G| - Z(G) \leq OS(G)$. A similar method can be used to
 241 show an a more precise relationship between Z_+ and the OS number.

242 **THEOREM 3.6.** *For any graph $G = (V, E)$ and any ordered set S , $V \setminus S$ is a positive semidefinite
 243 forcing set for G , and for any positive semidefinite forcing set X for G , there is an order that makes
 244 $V \setminus X$ an ordered set for G . Thus $Z_+(G) + OS(G) = |G|$.*

245 *Proof.* Let X be a positive semidefinite zero forcing set for G such that $|X| = Z_+(G)$. Let v_i
 246 be the vertex colored black by the i th application of the positive semidefinite color change rule.
 247 We show that $S = (v_t, v_{t-1}, \dots, v_1)$ is an OS set for G , where $t = |G| - Z_+(G)$. Further define
 248 $X_0 = X$, and $X_{i+1} = X_i \cup \{v_{i+1}\}$, for $i = 0, 1, \dots, t-1$. For each v_i , since it was initially white and
 249 then colored black on the i th application of the positive semidefinite color change rule, there exists
 250 a vertex $w_i \in X_i$ (the current black vertices) such that v_i is the only neighbor in the subgraph of G
 251 induced by $X_i \cup H_1$, where the subgraph $G \setminus X_i$ has components H_1, H_2, \dots, H_p with $v_i \in H_1$. Since
 252 X is a positive semidefinite zero forcing set, no other vertex from the set $\{v_{i+1}, v_{i+2}, \dots, v_t\}$ (the
 253 remaining white vertices) can be in H_1 and be a neighbor of w_i . Hence the set $(v_t, v_{t-1}, \dots, v_1)$ is
 254 an OS-set. Therefore $t \leq OS(G)$. Thus

$$255 \quad |G| - Z_+(G) \leq OS(G). \quad (3.1)$$

256 For the converse, we use the fact that if $S = (v_1, v_2, \dots, v_m)$ is an OS set, then the set $S \setminus \{v_m\}$
 257 is also an OS set. Suppose $S = (v_1, v_2, \dots, v_m)$ is an OS set with $|S| = OS(G)$. Then we claim that
 258 $V \setminus S$ is a positive semidefinite zero forcing set. So color the vertices $V \setminus S$ black, and suppose the
 259 subgraph G_m induced by the vertices of $\{v_1, \dots, v_m\}$ has components induced by U_1, U_2, \dots, U_ℓ .
 260 Let $v_m \in U_1$. Since S is an OS-set there exists a vertex $w_m \in V \setminus S$ such that $w_m v_m \in E$ and
 261 $w_m v_s \notin E$ for all other $v_s \in U_1$. This implies that v_m can be colored black under the positive
 262 semidefinite color change rule. Since $S \setminus \{v_m\}$ is also an OS-set for G , we may continue this
 263 argument and deduce that $V \setminus S$ is a positive semidefinite zero forcing set. Hence

$$264 \quad |G| - OS(G) = |V \setminus S| \geq Z_+(G), \quad (3.2)$$

265 as the positive semidefinite zero forcing number is defined as a minimum over all such zero forcing
 266 sets. From (3.1) and (3.2), $Z_+(G) + OS(G) = |G|$. \square

267 **COROLLARY 3.7.** *For every graph G ,*

$$268 \quad \delta(G) \leq Z_+(G).$$

269 *Proof.* By [15, Corollary 2.19], $OS(G) \leq |G| - \delta(G)$. Combining this with Theorem 3.6 gives
 270 the result. \square

271 Another consequence of Theorem 3.6 is that there are examples of graphs for which Z_+ may
 272 not be equal to M_+^C . For example, in [15] it was shown that the Möbius Ladder on 8 vertices,

273 sometimes denoted by ML_8 or V_8 , satisfies $OS(ML_8) = 4$ and $\text{mr}_+^{\mathbb{C}}(ML_8) = 5$. In this case, by
 274 Theorem 3.6, it follows that $Z_+(ML_8) = 4$, and hence $Z_+(ML_8) > 3 = M_+^{\mathbb{C}}(ML_8)$.

275 In [1], the zero forcing number was used to establish the minimum rank/maximum nullity of
 276 numerous families of graphs. The positive semidefinite zero forcing number is equally effective.
 277 Here we apply it to two families of graphs. The set of vertices associated with (the same) positive
 278 semidefinite zero forcing set in each copy of G is a positive semidefinite zero forcing set for $G \square H$.

279 PROPOSITION 3.8. *For all graphs G and H , $Z_+(G \square H) \leq \min\{Z_+(G)|H|, Z_+(H)|G|\}$.*

280 COROLLARY 3.9. *If T is a tree and G is a graph, then $Z_+(T \square G) \leq |G|$.*

281 THEOREM 3.10. *If T is a tree of order at least two, then $M_+^{\mathbb{R}}(T \square K_r) = M_+^{\mathbb{C}}(T \square K_r) =$
 282 $Z_+(T \square K_r) = r$.*

283 *Proof.* Let T be a tree of order $n \geq 2$. By Corollary 3.9, $Z_+(T \square K_r) \leq r$. We show
 284 $r \leq M_+^{\mathbb{R}}(T \square K_r)$ by constructing a matrix $A \in \mathcal{S}_+(T \square K_r)$ of rank at most $(n-1)r$, and the
 285 result then follows from Theorem 3.5. The construction is by induction on n . Let P_2 denote the
 286 path on 2 vertices. To show that $\text{mr}_+^{\mathbb{R}}(P_2 \square K_r) = r$, choose a nonsingular matrix $M \in \mathcal{S}_+(K_r)$
 287 such that $M^{-1} \in \mathcal{S}_+(K_r)$ (for example, $M = I + J$, where I is the identity matrix and J is the
 288 all 1s matrix). Then $B = \begin{bmatrix} M & I \\ I & M^{-1} \end{bmatrix} \in \mathcal{S}_+(P_2 \square K_r)$ and $\text{rank } B = \text{rank } M = r$. Without loss of
 289 generality, in T vertex n is adjacent only to vertex $n-1$. We order the vertices (i, j) of $T \square K_r$
 290 lexicographically. By the induction hypothesis, there is a matrix $C \in \mathcal{S}_+((T-n) \square K_r)$ such that
 291 $\text{rank } C = (n-2)r$; let $C' = C \oplus 0_{r \times r}$. Using $B \in \mathcal{S}_+(P_2 \square K_r)$ already constructed with rank r , let
 292 $B' = 0_{(n-2)r \times (n-2)r} \oplus B$. Then for $\alpha \in \mathbb{R}$ chosen to avoid cancellation, $A = C' + \alpha B' \in \mathcal{S}_+(T \square K_r)$
 293 and $\text{rank } A \leq (n-2)r + r = (n-1)r$. \square

294 A book with $m \geq 2$ pages, denoted B_m [9, p. 14], is m copies of a 4-cycle with one edge in
 295 common, or equivalently, $B_m = K_{1,m} \square P_2$, where $K_{1,m}$ is the complete bipartite graph with partite
 296 sets of 1 and m vertices. For $m \geq 2$, $t \geq 3$, we call m copies of a t -cycle with one edge in common
 297 a *generalized book*, denoted by B_m^t (obviously, $B_m = B_m^4$).

298 PROPOSITION 3.11. *If B_m^t is a generalized book, then $M_+^{\mathbb{R}}(B_m^t) = M_+^{\mathbb{C}}(B_m^t) = Z_+(B_m^t) = 2$.*

299 *Proof.* The two vertices in the common edge are a positive semidefinite zero forcing set, so
 300 $Z_+(B_m^t) \leq 2$. Thus by Theorem 3.5, $M_+^{\mathbb{C}}(B_m^t) \leq 2$. Since B_m^t is not a tree, $M_+^{\mathbb{R}}(B_m^t) \geq 2$ [12]. \square

301 **4. Real versus complex minimum positive semidefinite rank.** Clearly $\text{mr}_+^{\mathbb{C}}(G) \leq$
 302 $\text{mr}_+^{\mathbb{R}}(G)$ for every graph G . Previously it was not known whether $\text{mr}_+^{\mathbb{C}}(G)$ could differ from $\text{mr}_+^{\mathbb{R}}(G)$
 303 [7, p. 578]. In this final section we provide an example of a graph for which these parameters are
 304 not identical.

305 EXAMPLE 4.1. The “ k -wheel with 4 hubs” (for k at least 3) is the graph on $4k+4$ vertices such
 306 that the outer cycle has $4k$ vertices, and each of the 4 hubs is attached to every 4th vertex of the
 307 cycle, and no others; this graph is denoted $H_4(k)$, and $H_4(3)$ is shown in Figure 4.1. This family
 308 arose in Hall’s investigation of graphs having minimum rank 3 [11]. We show $\text{mr}_+^{\mathbb{C}}(\overline{H_4(3)}) = 3$ and
 309 $\text{mr}_+^{\mathbb{R}}(\overline{H_4(3)}) = 4$. As numbered in Figure 4.1, $H_4(3)$ is bipartite with partite sets consisting of the
 310 odd vertices and the even vertices. By [2, Theorem 3.1], $\text{mr}_+^{\mathbb{R}}(\overline{H_4(3)}) = \text{mr}^{\mathbb{R}}(Y_{\overline{H_4(3)}})$ where $Y_{\overline{H_4(3)}}$

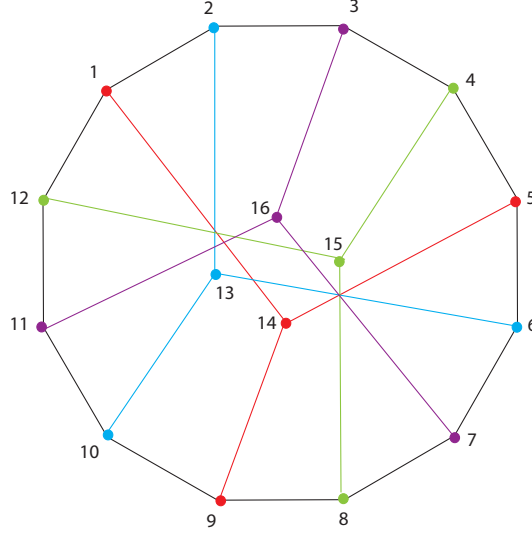


FIG. 4.1. The the 3-wheel on 4 hubs, $H_4(3)$, for Example 4.1

311 is the biadjacency zero-nonzero pattern of $\overline{H_4(3)}$ and $\text{mr}^{\mathbb{R}}(Y_{\overline{H_4(3)}})$ is the asymmetric minimum
 312 rank over the real numbers (Theorem 3.1 applies to $\overline{H_4(3)}$ because $H_4(3)$ is a bipartite graph).
 313 The same method used to prove Theorem 3.1 also shows that $\text{mr}_+^{\mathbb{C}}(\overline{H_4(3)}) = \text{mr}^{\mathbb{C}}(Y_{\overline{H_4(3)}})$ where
 314 $\text{mr}^{\mathbb{C}}(Y_{\overline{H_4(3)}})$ is the asymmetric minimum rank over the complex numbers (in [2, Remark 3.2] it is
 315 noted that the method in Theorem 3.1 is valid for constructing a symmetric matrix over an infinite
 316 field, and the same reasoning applies to constructing a Hermitian matrix over \mathbb{C} by using Hermitian
 317 adjoints in place of transposes). After scaling rows and columns, a minimum rank matrix having
 318 zero-nonzero pattern $Y_{\overline{H_4(3)}}$ has the form

$$319 \quad A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & a_{3,8} & a_{3,10} & a_{3,12} & a_{3,14} & 0 \\ 1 & 0 & 0 & a_{5,8} & a_{5,10} & a_{5,12} & 0 & a_{5,16} \\ 1 & a_{7,4} & 0 & 0 & a_{7,10} & a_{7,12} & a_{7,14} & 0 \\ 1 & a_{9,4} & a_{9,6} & 0 & 0 & a_{9,12} & 0 & a_{9,16} \\ 1 & a_{11,4} & a_{11,6} & a_{11,8} & 0 & 0 & a_{11,14} & 0 \\ 0 & 1 & 0 & a_{13,8} & 0 & a_{13,12} & a_{13,14} & a_{13,16} \\ 1 & 0 & a_{15,6} & 0 & a_{15,10} & 0 & a_{15,14} & a_{15,16} \end{bmatrix}.$$

320 where the displayed entries a_{ij} are nonzero (real or complex) numbers. Since the principal subma-
 321 trix in the first three rows and columns is nonsingular, $\text{rank } A = 3$ implies that rows 4 through 8
 322 are linear combinations of rows 1 through 3. Computations show that the following assignments

323 of variables are necessary:

$$\begin{aligned}
324 \quad & a_{5,8} = (a_{3,8} - 1)a_{7,4}, \quad a_{5,10} = (a_{3,10} - 1)a_{7,4} + a_{7,10}, \quad a_{5,12} = a_{3,12}a_{7,4} + a_{7,12}, \quad a_{5,16} = -a_{7,4}, \\
325 \quad & a_{7,14} = -a_{3,14}a_{7,4}, \quad a_{9,16} = a_{9,4} - a_{7,4}, \quad a_{9,6} = a_{9,4}, \quad a_{9,12} = a_{3,12}a_{7,4} + a_{7,12} - a_{3,12}a_{9,4} + a_{3,12}a_{9,6}, \\
326 \quad & a_{7,10} = a_{7,4} - a_{3,10}a_{7,4} - a_{9,4}, \quad a_{9,4} = (1 - a_{3,8})a_{7,4}, \quad a_{11,4} = a_{7,4}, \quad a_{11,14} = a_{3,14}(a_{11,6} - a_{11,4}), \\
327 \quad & a_{7,12} = -a_{3,12}a_{11,6}, \quad a_{11,8} = a_{3,8}a_{11,6}, \quad a_{3,8} = a_{3,10}(a_{7,4} - a_{11,6})/a_{7,4}, \quad a_{13,16} = 1, \quad a_{13,14} = -a_{3,14}, \\
328 \quad & a_{13,12} = -a_{3,12}, \quad a_{3,10} = 1, \quad a_{13,8} = a_{11,6}/a_{7,4}, \quad a_{15,16} = -a_{7,4}, \quad a_{15,14} = a_{3,14}a_{15,6}, \\
329 \quad & a_{15,10} = -a_{11,6} + a_{15,6}, \quad a_{11,6} = a_{7,4} + a_{15,6}.
\end{aligned}$$

330 After making these assignments, rows 4 - 7 are linear combinations of rows 1, 2, and 3, and in
331 order for row 8 to be a linear combinations of rows 1, 2, and 3, it is necessary and sufficient that

$$332 \quad 1 + \frac{a_{7,4}}{a_{15,6}} + \left(\frac{a_{7,4}}{a_{15,6}} \right)^2 = 0. \tag{4.1}$$

333 Clearly (4.1) has a solution if and only if the field contains a primitive third root of unity. Thus
334 $\text{mr}^{\mathbb{C}}(Y_{H_4(3)}) = 3$ whereas $\text{mr}^{\mathbb{R}}(Y_{H_4(3)}) = 4$, giving

$$335 \quad \text{mr}_+^{\mathbb{C}}(\overline{H_4(3)}) = 3 < 4 = \text{mr}_+^{\mathbb{R}}(\overline{H_4(3)}).$$

336

337

REFERENCES

- 338 [1] AIM Minimum Rank – Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioabă, D.
339 Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelsen, S. Narayan, O. Pryporova, I.
340 Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, and A. Wangsness). Zero forcing sets
341 and the minimum rank of graphs. *Linear Algebra and its Applications*, 428/7: 1628–1648, 2008.
- 342 [2] F. Barioli, S. M. Fallat, D. Hershkowitz, H. T. Hall, L. Hogben, H. van der Holst, and B. Shader. On the
343 minimum rank of not necessarily symmetric matrices: a preliminary study. *Electronic Journal of Linear
344 Algebra*, 18: 126–145, 2009.
- 345 [3] F. Barioli, S. M. Fallat, and L. Hogben. Computation of minimal rank and path cover number for graphs.
346 *Linear Algebra and its Applications*, 392: 289–303, 2004.
- 347 [4] M. Booth, P. Hackney, B. Harris, C. R. Johnson, M. Lay, L. H. Mitchell, S. K. Narayan, A. Pascoe, K.
348 Steinmetz, B. D. Sutton, W. Wang, On the minimum rank among positive semidefinite matrices with a
349 given graph. *SIAM Journal of Matrix Analysis and Applications*, 30: 731–740, 2008.
- 350 [5] D. Burgarth and V. Giovannetti. Full control by locally induced relaxation. *Physical Review Letters* PRL 99,
351 100501 (2007).
- 352 [6] Y. Colin de Verdière. Multiplicities of eigenvalues and tree-width of graphs. *J. Combin. Theory Ser. B* 74:
353 121–146, 1998.
- 354 [7] S. Fallat and L. Hogben. The minimum rank of symmetric matrices described by a graph: a survey. *Linear
355 Algebra and its Applications*, 426: 558–582, 2007.
- 356 [8] S. Fallat and C. R. Johnson. Olga, matrix theory, and the Taussky unification problem. *Linear Algebra and
357 Its Applications*, 280: 39–49, 1998.
- 358 [9] J. A. Gallian, A Dynamic Survey of Graph Labeling. *Electronic Journal of Combinatorics*, DS6 (219 pp.),
359 <http://www.combinatorics.org/Surveys/ds6.pdf>, January 31, 2009.
- 360 [10] P. Hackney, B. Harris, M. Lay, L. H. Mitchell, S. K. Narayan, A. Pascoe, Linearly independent vertices and
361 minimum semidefinite rank, *Linear Algebra and Its Applications*, 431: 1105–1115, 2009.
- 362 [11] H. T. Hall. Minimum rank 3 is difficult to determine. Preprint.
- 363 [12] H. van der Holst. Graphs whose positive semi-definite matrices have nullity at most two. *Linear Algebra and
364 Its Applications*, 375: 1–11, 2003.

- 365 [13] H. van der Holst. On the maximum positive semi-definite nullity and the cycle matroid of graphs. *Electronic*
366 *Journal of Linear Algebra*, 18: 192–201, 2009.
- 367 [14] C. R. Johnson and A. Leal Duarte. The maximum multiplicity of an eigenvalue in a matrix whose graph is a
368 tree. *Linear and Multilinear Algebra* 46: 139–144, 1999.
- 369 [15] L. Mitchell, S. Narayan, and A. Zimmer, Lower bounds in minimum rank problems, *Linear Algebra and its*
370 *Applications*, 432: 430-440, 2010.
- 371 [16] J. Sinkovic. Maximum nullity of outerplanar graphs and the path cover number. To appear in *Linear Algebra*
372 *and its Applications*.

This is a manuscript of an article from *Linear Algebra and its Applications* 433 (2010): 401, doi:10.1016/j.laa.2010.03.008. Posted with permission. © 2010. This manuscript version is made available under the CC-BY-NC-ND 4.0 license <http://creativecommons.org/licenses/by-nc-nd/4.0/>