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Coverage Probabilities of Simultaneous Confidence Bands and Regions for Log-Location-Scale Distributions

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Keywords

confidence region, cumulative distribution function, failure probability, quantile, statistical uncertainty

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Coverage Probabilities of Simultaneous Confidence Bands and Regions for Log-Location-Scale Distributions

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Abstract

A simultaneous confidence band (*SCB*) for the cumulative distribution function (cdf) of a random variable can be used to assess the statistical uncertainty of the estimated cdf. Cheng and Iles (1983) presented a general approach of constructing an *SCB* for the cdf of a continuous random variable derived from a $100(1-\alpha)\%$ simultaneous confidence region (*SCR*) for the parameters of the distribution. The Cheng and Iles *SCB* procedure includes the true cdf with probability at least $(1-\alpha)$. This paper identifies the conditions under which the coverage probability for the *SCB* procedure is exactly $(1-\alpha)$. A small simulation illustrates the important theoretical results in the paper.

Key Words: Confidence region; Cumulative distribution function; Failure probability; Quantile; Statistical uncertainty.

1 Introduction

1.1 The problem

When estimating an unknown cumulative distribution function (cdf), it is important to assess the precision of the cdf estimate. Some applications require a simultaneous confidence band (*SCB*) that contains the entire unknown cdf with a certain confidence level. Cheng and Iles (1983) described a general method of constructing an *SCB* for the cdf of a continuous random variable. For the location-scale and log-location-scale models, which include the most popular distribution families used in lifetime modeling, there are explicit forms for the upper and lower boundaries of the *SCB*.

A random variable Y belongs to the location-scale family of distributions if its cdf has the form $F_Y(y;\boldsymbol{\theta}) = \Phi[(y-\mu)/\sigma]$, $-\infty < y < \infty$. Here, $\boldsymbol{\theta} = (\mu, \sigma)'$, μ is the location parameter, σ is the scale parameter, $\Phi(z)$ is the cdf of $(Y-\mu)/\sigma$, and $\Phi(z)$ does not depend on any unknown parameters. The normal and the smallest extreme value are location-scale distributions. A positive random variable T belongs to the log-location-scale family of distribution if $Y = \log(T)$ belongs to the location-scale family of distributions. Thus the cdf of T has the form $F_T(t;\boldsymbol{\theta}) = \Phi\{[\log(t)-\mu]/\sigma\}, t > 0$. The lognormal and the Weibull are among the important distributions of this family. The results in this paper apply to complete and censored data from the continuous location-scale and log-location-scale families.

A summary of the approach of Cheng and Iles (1983) to obtain an *SCB* for a cdf $F(y;\boldsymbol{\theta})$ is as follows. First, a $100(1-\alpha)\%$ simultaneous confidence region (*SCR*), denoted by $SCR(\boldsymbol{\theta})$, is identified for the unknown parameters $\boldsymbol{\theta} = (\mu, \sigma)'$ of the cdf. The *SCR* for $\boldsymbol{\theta}$ can be obtained from Wald statistics with expected information, estimated expected information, or local information (e.g., Escobar, Hong, and Meeker 2009). It can also be obtained through inversion of a likelihood ratio or a score statistic or a parametric bootstrap procedure (e.g., Jeng and Meeker 2001). Then one obtains the graph of all the cdfs $F(y;\boldsymbol{\theta})$ when $\boldsymbol{\theta}$ is in the $SCR(\boldsymbol{\theta})$. The *SCB* is the S-shaped region in the plane swept by the graph. Figure 1 gives an illustration of the S-shaped *SCB*.

The probability that the *SCB* procedure includes the true cdf is *at least* $(1-\alpha)$ if the $SCR(\boldsymbol{\theta})$ includes the true value of $\boldsymbol{\theta}_0 = (\mu_0, \sigma_0)'$ with probability (w.p.) $(1-\alpha)$. The approach of Cheng and Iles (1983) cannot, generally, ensure that the coverage probability (CP) of the *SCB* procedure is exactly $(1-\alpha)$ when the corresponding *SCR* procedure for the parameter

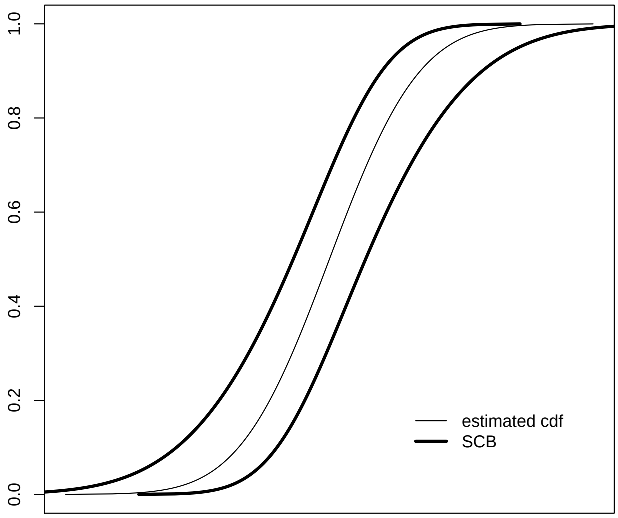

Figure 1: Illustration of an SCB.

$\boldsymbol{\theta}$ has a CP of $(1-\alpha)$. For the location-scale and log-location-scale families considered here, however, the CP for the SCB procedure is exactly $(1-\alpha)$ when the SCR procedure satisfies the mild conditions given in **Theorem 1** or **Corollary 1** of Section 2.

1.2 General approaches for constructing $\boldsymbol{SCB}$s

Two important functions of the model parameters μ and σ are the failure probability $F_T(t_e;\boldsymbol{\theta})$ at t_e and the p quantile t_p of the distribution. Let $y_e = \log(t_e)$ and $y_p = \log(t_p)$. Then $F_T(t_e;\boldsymbol{\theta}) = \Phi\left[(y_e-\mu)/\sigma\right]$ and $y_p = \mu + z_p\,\sigma$ where $z_p = \Phi^{-1}(p)$ is the p quantile of $\Phi(z)$.

Two alternative approaches are available for obtaining SCBs for the log-location-scale family using the general method of Cheng and Iles (1983). The first one obtains SCBs directly for quantiles. In particular, for each $0<p<1$, the SCB for the p quantile is the solution to the optimization problems

$$\max_{\boldsymbol{\theta}\in SCR(\boldsymbol{\theta})} (\mu + z_p\sigma) \quad \text{and} \quad \min_{\boldsymbol{\theta}\in SCR(\boldsymbol{\theta})} (\mu + z_p\sigma)\,. \tag{1}$$

The $\max_{\boldsymbol{\theta}\in SCR(\boldsymbol{\theta})}$ and $\min_{\boldsymbol{\theta}\in SCR(\boldsymbol{\theta})}$ in (1) are well defined because all SCRs considered in this paper are *closed* sets in the topological sense. The probability distributions in the log-location-scale and location-scale family that are considered in this paper are continuous distributions.

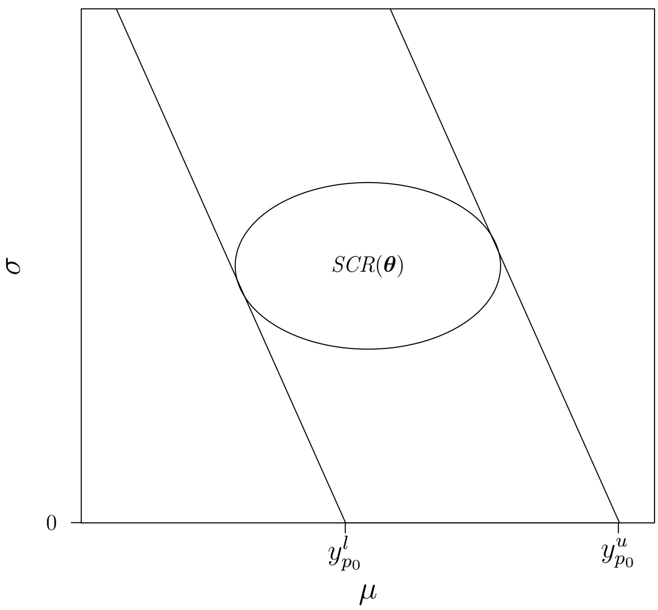

Figure 2: Illustrates of finding the solutions of (1) graphically for a particular p_0. $y^u_{p_0} = \max_{\boldsymbol{\theta} \in SCR(\boldsymbol{\theta})}(\mu + z_{p_0}\sigma)$ and $y^l_{p_0} = \min_{\boldsymbol{\theta} \in SCR(\boldsymbol{\theta})}(\mu + z_{p_0}\sigma)$.

Thus, the probability that the true parameter is on the boundary of the SCR is zero. That is, the closeness requirement for the SCR has no effect on the coverage probability of the procedure. Figure 2 illustrates how to find the solutions of (1) graphically for a particular p_0.

The second approach obtains the SCB directly for cumulative probabilities. In particular, the SCB is defined by finding for each $-\infty < y_e < \infty$, the solution for the optimization problems

$$\max_{\boldsymbol{\theta} \in SCR(\boldsymbol{\theta})} \Phi\left(\frac{y_e - \mu}{\sigma}\right) \quad \text{and} \quad \min_{\boldsymbol{\theta} \in SCR(\boldsymbol{\theta})} \Phi\left(\frac{y_e - \mu}{\sigma}\right). \tag{2}$$

For these approaches, Cheng and Iles (1983) gave closed-form expressions for the upper and lower boundaries of the SCB derived from an expected information ellipsoidal $SCR(\boldsymbol{\theta})$. Escobar, Hong, and Meeker (2009) extended the work of Cheng and Iles (1983) and studied the SCBs derived from local information, expected information, and estimated expected information for approaches (1) and (2), and considered special features of non-ellipsoidal SCRs that can occur when using the expected information.

2 Coverage probabilities for the SCB

Let $\boldsymbol{\theta}_0 = (\mu_0, \sigma_0)'$ be the true parameter of a log-location-scale distribution.

Definition 1. *The CP for an SCR procedure is defined as* $\mathbf{Pr}[\boldsymbol{\theta}_0 \in SCR(\boldsymbol{\theta})]$.

Definition 2. *The CP for an SCB procedure using approach (1) is* $\mathbf{Pr}(\mu_0 + z_p\sigma_0 \in SCB \ \textit{ for all } p)$ *and for an SCB procedure using approach (2), the CP is* $\mathbf{Pr}\{\Phi[(y_e - \mu_0)/\sigma_0] \in SCB \ \textit{ for all } y_e\}$.

For log-location-scale distributions, Cheng and Iles (1983, Section 3.1) showed that the SCB procedure derived from an expected information ellipsoidal SCR has exactly $(1-\alpha)$ CP when the SCR procedure has exactly $(1-\alpha)$ CP. In this paper, we present a more general result for the log-location-scale family, giving the necessary and sufficient conditions on an SCR such that if the SCR procedure has a CP of $(1-\alpha)$, then the corresponding SCB procedure has exactly $(1-\alpha)$ CP.

We first give several definitions needed in the development that follows.

Definition 3. *An SCB procedure is regular if* ***Condition 1 (C1)*** *below holds.*

C1: *There exists a* $p_0 \in (0,1)$ *such that w.p. 1 at least one of the SCB boundaries* $\min_{\boldsymbol{\theta} \in SCR(\boldsymbol{\theta})}(\mu + z_{p_0}\sigma)$ *or* $\max_{\boldsymbol{\theta} \in SCR(\boldsymbol{\theta})}(\mu + z_{p_0}\sigma)$ *is finite.*

C1 is not an stringent condition because not satisfying ***C1*** would imply an SCB equal to the entire plane, which makes little sense. Now we define a regular SCR procedure.

Definition 4. *An SCR procedure for* $\boldsymbol{\theta}$ *is regular if the corresponding SCB procedure is regular.*

The CP property of an SCB procedure is closely related the convexity of the corresponding SCR.

Definition 5. *The convex hull of an SCR, denoted by coSCR, is defined as the smallest convex set containing the SCR.*

Remark 1. $\min_{\boldsymbol{\theta} \in coSCR(\boldsymbol{\theta})}(\mu + z_p\,\sigma) = \min_{\boldsymbol{\theta} \in SCR(\boldsymbol{\theta})}(\mu + z_p\,\sigma)$ *and* $\max_{\boldsymbol{\theta} \in coSCR(\boldsymbol{\theta})}(\mu + z_p\,\sigma) = \max_{\boldsymbol{\theta} \in SCR(\boldsymbol{\theta})}(\mu + z_p\,\sigma)$ *for a given* p.

To prove this, it is straightforward to show that $\min_{\boldsymbol{\theta} \in coSCR(\boldsymbol{\theta})}(\mu + z_p\,\sigma) \leq \min_{\boldsymbol{\theta} \in SCR(\boldsymbol{\theta})}(\mu + z_p\,\sigma)$. If $\min_{\boldsymbol{\theta} \in coSCR(\boldsymbol{\theta})}(\mu + z_p\,\sigma) < \min_{\boldsymbol{\theta} \in SCR(\boldsymbol{\theta})}(\mu + z_p\,\sigma)$ holds, then one can always construct a convex set containing $SCR(\boldsymbol{\theta})$ that is smaller than the $coSCR(\boldsymbol{\theta})$. But this is in contradiction with the definition of $coSCR(\boldsymbol{\theta})$. Hence $\min_{\boldsymbol{\theta} \in coSCR(\boldsymbol{\theta})}(\mu + z_p\,\sigma) = \min_{\boldsymbol{\theta} \in SCR(\boldsymbol{\theta})}(\mu + z_p\,\sigma)$ for a given p. A similar argument shows that $\max_{\boldsymbol{\theta}\in coSCR(\boldsymbol{\theta})}(\mu + z_p\,\sigma) = \max_{\boldsymbol{\theta}\in SCR(\boldsymbol{\theta})}(\mu + z_p\,\sigma)$. Thus, **Remark 1** follows.

The following theorem gives the general result.

Theorem 1. *Consider the **Statement 1 (S1)** and **Condition 2 (C2)** as given below.*

***S1:** The SCB procedure using approach (1) or (2) includes the true cdf w.p.* $(1-\alpha)$ *if the SCR procedure includes the true parameter* $\boldsymbol{\theta}_0$ *w.p.* $(1-\alpha)$.

***C2:** For the SCR procedure,* $\boldsymbol{\theta}_0 \in SCR(\boldsymbol{\theta})$ *if and only if* $\boldsymbol{\theta}_0 \in coSCR(\boldsymbol{\theta})$.

*Then for a regular SCR procedure, that **C2** holds w.p. 1 is a necessary and sufficient condition for **S1** to hold.*

The proof of this theorem is given in the appendix. The rational for ***C2*** is that the *SCR* should behave exactly like (except for a probability 0 set) its convex hull in including the true parameter $\boldsymbol{\theta}_0$. When the *SCR* is not convex, ***C2*** will not hold if there is a positive probability that the true parameter $\boldsymbol{\theta}_0 \in coSCR(\boldsymbol{\theta}) \setminus SCR(\boldsymbol{\theta})$. The *SCR* in Figure 3 illustrates this situation. This *SCR* was computed using a subset of the left truncated and right censored high-voltage power transformer lifetime data described in Hong, Meeker, and McCalley (2009) consisting of those units from manufacturer MB. Figure 3 shows the 95% likelihood *SCR* for $\boldsymbol{\theta}_0$ for the Weibull distribution. All the observations from MB are truncated which causes the non-convexity of the *SCR*. When the shaded area shown in Figure 3 has a positive probability of capturing $\boldsymbol{\theta}_0$, ***C2*** will not hold w.p. 1.

In practice, however, ***C2*** is difficult to verify. The following general corollary gives an easier result to use in practice.

Corollary 1. *For an SCR procedure, if the SCR is convex w.p. 1, then **C1** is a sufficient condition for **S1** to hold.*

To prove this corollary notice that ***C1*** implies that the *SCR* procedure is regular, and the convexity of the *SCR* implies ***C2*** holds w.p. 1. Thus the corollary follows from **Theorem 1**.

3 Simulation Study

In this section, we describe a simulation conducted to illustrate some of the theoretical results in Section 2. The simulation study mainly provides insights on the relationship between the CPs of *SCR*s and the corresponding *SCB*s.

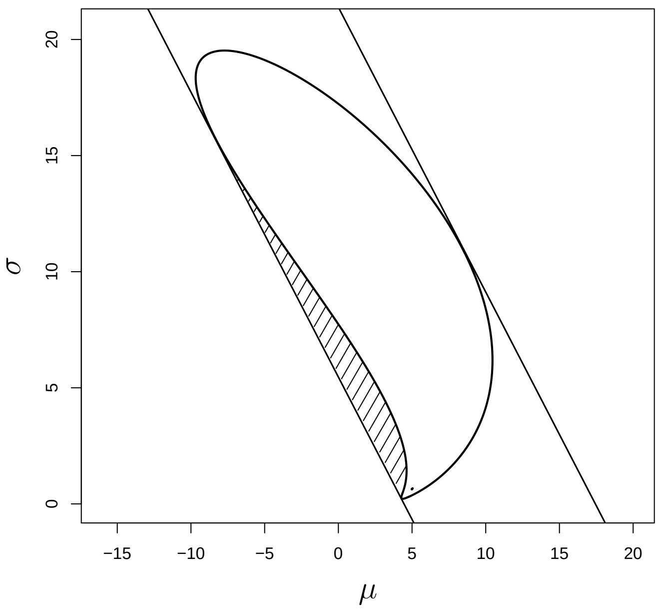

Figure 3: 95% likelihood *SCR* for $\boldsymbol{\theta}_0 = (\mu_0, \sigma_0)'$ for the Weibull distribution, computed form the lifetimes of high-voltage power transformer manufactured by MB.

We generated complete data from the lognormal distribution with $\boldsymbol{\theta}_0 = (0, 1)'$ for different sample sizes n. The estimate of the parameter, $\widehat{\boldsymbol{\theta}}$ was obtained by maximum likelihood estimation. We consider convex and non-convex *SCR*s for the parameter as follows.

$$
\begin{aligned}
\text{Case I}\ : \quad & SCR(\boldsymbol{\theta}) = \left\{(\mu, \sigma)' : (\boldsymbol{\theta} - \widehat{\boldsymbol{\theta}})'\widehat{\mathrm{Var}}(\widehat{\boldsymbol{\theta}})^{-1}(\boldsymbol{\theta} - \widehat{\boldsymbol{\theta}}) \le \gamma_1\right\} \\
\text{Case II}: \quad & SCR(\boldsymbol{\theta}) = \{(\mu, \sigma)' : \widehat{\mu} - \gamma_2\widehat{\mathrm{se}}_{\widehat{\mu}} \le \mu \le \widehat{\mu} + \gamma_2\widehat{\mathrm{se}}_{\widehat{\mu}}, \widehat{\sigma} - \gamma_2\widehat{\mathrm{se}}_{\widehat{\sigma}} \le \sigma \le \widehat{\sigma} + \gamma_2\widehat{\mathrm{se}}_{\widehat{\sigma}}\} \\
\text{Case III}: \quad & SCR(\boldsymbol{\theta}) = \left\{(\mu, \sigma)' : \widehat{\mu} - \gamma_3\widehat{\mathrm{se}}_{\widehat{\mu}} \le \mu \le \widehat{\mu} + \gamma_3\widehat{\mathrm{se}}_{\widehat{\mu}}, \widehat{\sigma} - \frac{\gamma_3}{2}\widehat{\mathrm{se}}_{\widehat{\sigma}} \le \sigma \le \widehat{\sigma} + \gamma_3\widehat{\mathrm{se}}_{\widehat{\sigma}}\right\} \\
& \qquad \cup \left\{(\mu, \sigma)' : \widehat{\mu} - \frac{\gamma_3}{2}\widehat{\mathrm{se}}_{\widehat{\mu}} \le \mu \le \widehat{\mu} + \gamma_3\widehat{\mathrm{se}}_{\widehat{\mu}}, \widehat{\sigma} - \gamma_3\widehat{\mathrm{se}}_{\widehat{\sigma}} \le \sigma < \widehat{\sigma} - \frac{\gamma_3}{2}\widehat{\mathrm{se}}_{\widehat{\sigma}}\right\}
\end{aligned}
$$

where $\widehat{\mathrm{Var}}(\widehat{\boldsymbol{\theta}}) = \widehat{\sigma}^2\mathrm{diag}(1, 0.5)/n$, $\widehat{\mathrm{se}}_{\widehat{\mu}} = \widehat{\sigma}\sqrt{1/n}$, $\widehat{\mathrm{se}}_{\widehat{\sigma}} = \widehat{\sigma}\sqrt{0.5/n}$, and n is the sample size. γ_1, γ_2, and γ_3 are critical values that can be chosen to provide different CPs for the procedures. Figure 4 illustrates the shapes of the *SCR*s for these three cases. The first two *SCR*s are convex and the third one is not. The third *SCR* is an artificial example for purposes of illustration. Non-convex *SCR*s, however, exists in real applications, especially when the likelihood approach is used to construct an *SCR*, as shown in Figure 3. We obtained the corresponding *SCB*s for each of cases that are illustrated in Figure 5.

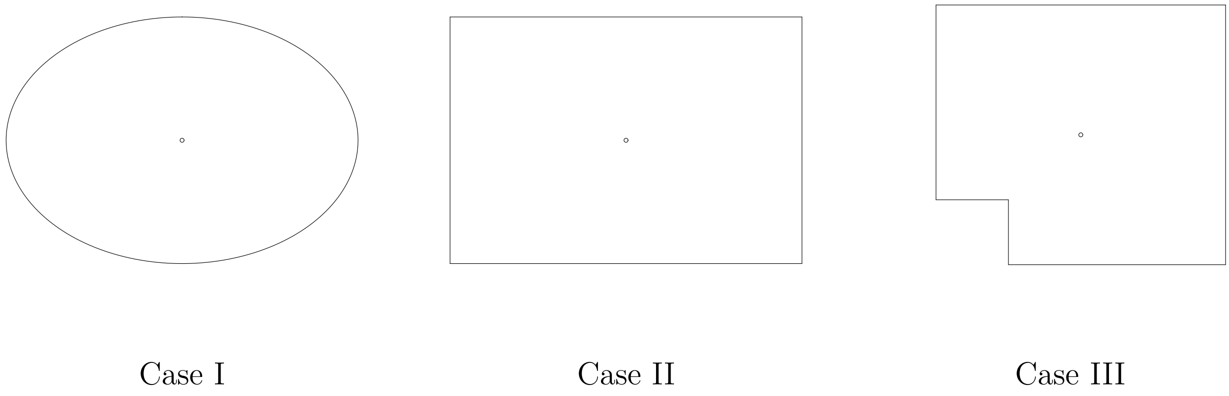

Figure 4: Illustration of the *SCR*s for the three cases.

The CPs of *SCR*s and *SCB*s were estimated using simulation. For each sample size, 10,000 samples were generated and the CPs of the *SCR*s (*SCB*s) were estimated as the proportion of samples that capture the true parameter (true cdf). For each case, the CP estimates were obtained as a function of the corresponding critical value $\gamma_i, i = 1, 2, 3$. Figure 6 shows the CP estimates for the *SCB*s versus the CP estimates for the *SCR*s for different sample sizes; $n = 20, 100$, and $1,000$. To obtain the results shown in Figure 6, a wide range of γ_i values were chosen to ensure that the range of the CP estimate of the *SCR* contains the interval $(0.9, 1)$. The CP estimates for the *SCR*s and *SCB*s are always equal for Cases I and II. For Case III, where the *SCR* is not convex, the CP estimate for the *SCB* is larger than the CP estimate of the *SCR*. Interestingly, for case III, the differences between the CP estimates for the *SCB*s and the corresponding *SCR*s are larger for larger sample sizes. This is caused by the fact that the distribution of $\widehat{\sigma}$ is left skewed when the sample size is small and it becomes more symmetric when the sample size is larger. Thus, $\mathbf{Pr}[\boldsymbol{\theta}_0 \in coSCR(\boldsymbol{\theta}) \setminus SCR(\boldsymbol{\theta})]$ increases with sample size for Case III, causing the differences as shown in Figure 6.

4 Concluding remarks

Using **Corollary 1**, it can be verified that the *SCB* procedures in **Theorems 2** through **7** in Escobar, Hong, and Meeker (2009) have a CP exactly equal to $(1 - \alpha)$ if they are based on *SCR* procedures that have a CP exactly equal to $(1 - \alpha)$. **Corollary 1** is useful in the calibration of an *SCB* procedure. If an *SCR* procedure satisfies condition ***C1*** and the *SCR* is convex w.p. 1, then the *SCB* procedure has a CP exactly equal to the CP of the corresponding

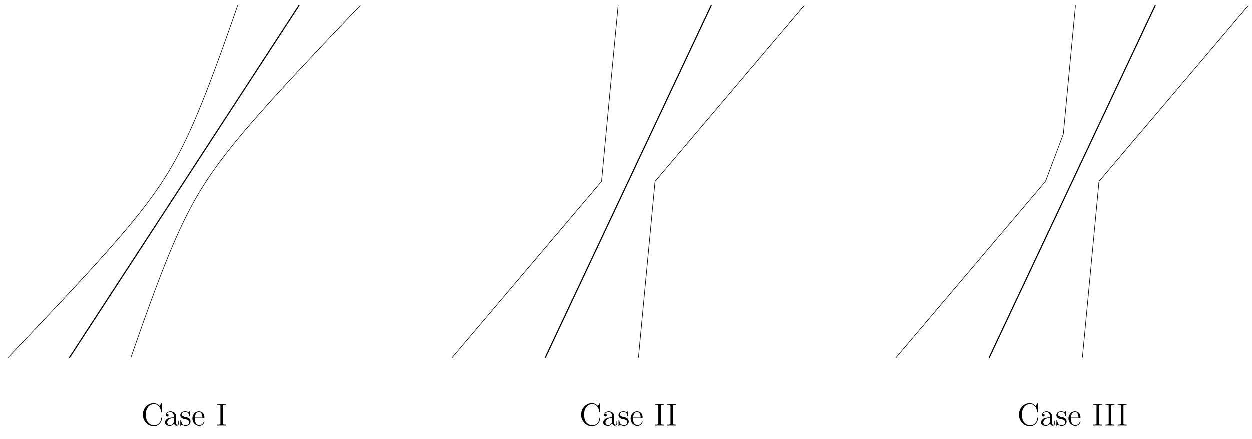

Figure 5: Illustration of the SCBs for the three cases.

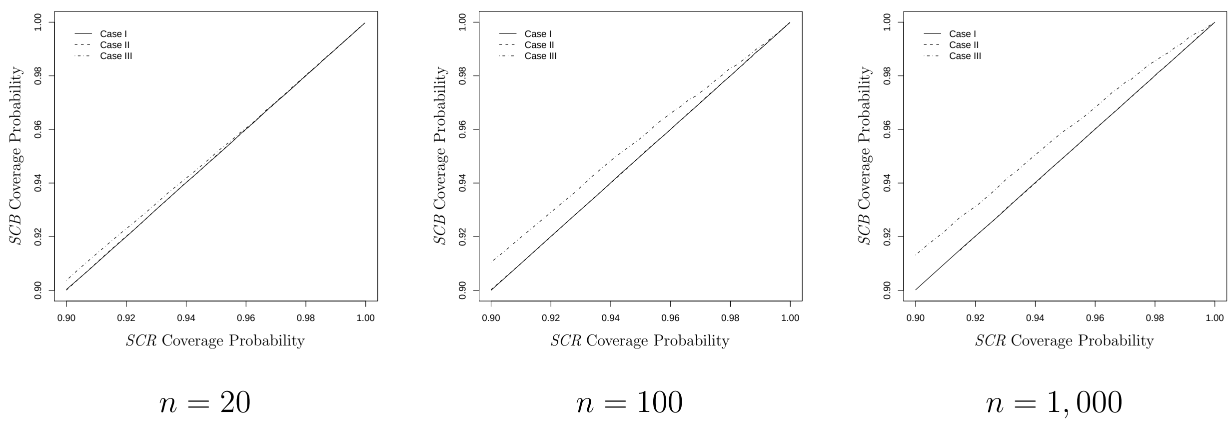

Figure 6: The relationship between the coverage probabilities of the SCRs and the SCBs.

SCR procedure. Thus, it suffices to calibrate the *SCR* procedure instead of calibrating the *SCB* procedure. See Escobar, Hong, and Meeker (2009) for more details on calibrating *SCR* procedures.

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Appendix: Proof of Theorem 1

The following Lemma is used in proving **Theorem 1**.

Lemma 1. *For a fixed* $SCR(\boldsymbol{\theta})$*, if* ***C1*** *and* ***C2*** *hold, the condition*

$$\min_{\boldsymbol{\theta}\in SCR(\boldsymbol{\theta})} (\mu + z_p\sigma) \leq \mu_0 + z_p\sigma_0 \leq \max_{\boldsymbol{\theta}\in SCR(\boldsymbol{\theta})} (\mu + z_p\sigma) \quad \textit{for all } 0 < p < 1 \tag{3}$$

implies $\boldsymbol{\theta}_0 \in SCR(\boldsymbol{\theta})$.

Proof. The proof here is by contradiction. Suppose that $\boldsymbol{\theta}_0 \notin SCR(\boldsymbol{\theta})$. Then ***C2*** implies that $\boldsymbol{\theta}_0 \notin coSCR(\boldsymbol{\theta})$. Because $SCR(\boldsymbol{\theta})$ is closed, $\boldsymbol{\theta}_0 \notin coSCR(\boldsymbol{\theta})$ and the fact that two non-intersecting convex sets can be separated by a line, there exits a line $\sigma = a+b\mu$ (or a line $\mu = c$) that *strictly* separates $\boldsymbol{\theta}_0$ and $coSCR(\boldsymbol{\theta})$. Without loss of generality, we can assume $\sigma_0 > a+b\mu_0$ and $\sigma < a + b\mu$ for all $\boldsymbol{\theta} = (\mu, \sigma)' \in coSCR(\boldsymbol{\theta})$. First consider the case of $b \neq 0$. When the separating line is of the type $\sigma = a + b\mu$ with $b \neq 0$ define $z_p = -1/b$. If the separating line is just $\mu = c$, then define $z_p = 0$. In the cases considered above, $\mu_0 + z_p\sigma_0 < \min_{\boldsymbol{\theta}\in SCR(\boldsymbol{\theta})} (\mu + z_p\sigma)$ or $\mu_0 + z_p\sigma_0 > \max_{\boldsymbol{\theta}\in SCR(\boldsymbol{\theta})} (\mu + z_p\sigma)$ which contradicts (3). Hence the only possibility is that $\boldsymbol{\theta}_0 \in coSCR(\boldsymbol{\theta})$. Then using ***C2***, $\boldsymbol{\theta}_0 \in SCR(\boldsymbol{\theta})$.

If $b = 0$ (i.e., the separating line is $\sigma = a$), we use ***C1*** to show that there is another separating line $\sigma = a_1 + b_1\mu$ with slope $b_1 \neq 0$. Therefore, the proof given above for the case of a non-null slope can be used with this new separating line to show that $\boldsymbol{\theta}_0 \in SCR(\boldsymbol{\theta})$. To obtain the separating line $\sigma = a_1 + b_1\mu$, without loss of generality, assume that there is a $0 < p_0 < 1$ such that $c_0 = \min_{\boldsymbol{\theta}\in SCR(\boldsymbol{\theta})}(\mu + z_{p_0}\sigma)$ is finite. It can be verified that there exists a new separating line with slope $b_1 \neq 0$ and through the point $(c_0 - z_{p_0}a, a)$. □

We now prove **Theorem 1**.

Proof. We only prove approach (1) as the proof for approach (2) is similar. By the construction of approach (1), any $\boldsymbol{\theta} \in SCR(\boldsymbol{\theta})$ satisfies (3). Thus, $\boldsymbol{\theta}_0 \in SCR(\boldsymbol{\theta})$ implies $\boldsymbol{\theta}_0$ satisfies (3) (i.e., $\mu_0 + z_p\sigma_0 \in SCB$ for all p). Hence

$$\mathbf{Pr}[\boldsymbol{\theta}_0 \in SCR(\boldsymbol{\theta})] \leq \mathbf{Pr}(\mu_0 + z_p\sigma_0 \in SCB \text{ for all } p). \tag{4}$$

Because the SCR is regular, ***C1*** holds w.p. 1.

For the "if" part, we have that ***C2*** holds w.p. 1. For any fixed $SCR(\boldsymbol{\theta})$, ***C1*** and ***C2*** hold. If $\mu_0 + z_p\sigma_0 \in SCB$ for all p, i.e., (3) holds, by **Lemma 1**, $\boldsymbol{\theta}_0 \in SCR(\boldsymbol{\theta})$. Thus,

$$\mathbf{Pr}[\boldsymbol{\theta}_0 \in SCR(\boldsymbol{\theta})] \geq \mathbf{Pr}(\mu_0 + z_p\sigma_0 \in SCB \text{ for all } p). \tag{5}$$

By (4) and (5) $\mathbf{Pr}[\boldsymbol{\theta}_0 \in SCR(\boldsymbol{\theta})] = \mathbf{Pr}(\mu_0 + z_p\sigma_0 \in SCB \text{ for all } p)$, which means that the SCB procedure is exact.

To prove the "only if" part, for any fixed $SCR(\boldsymbol{\theta})$, $\boldsymbol{\theta}_0 \in SCR(\boldsymbol{\theta})$ implies $\boldsymbol{\theta}_0 \in coSCR(\boldsymbol{\theta})$. Because of (4) and that the SCB procedure is exact,

$$1 - \alpha = \mathbf{Pr}[\boldsymbol{\theta}_0 \in SCR(\boldsymbol{\theta})] \leq \mathbf{Pr}[\boldsymbol{\theta}_0 \in coSCR(\boldsymbol{\theta})] \leq \mathbf{Pr}(\mu_0 + z_p\sigma_0 \in SCB \text{ for all } p) = 1 - \alpha.$$

Thus, $\mathbf{Pr}[\boldsymbol{\theta}_0 \in SCR(\boldsymbol{\theta})] = \mathbf{Pr}[\boldsymbol{\theta}_0 \in coSCR(\boldsymbol{\theta})]$. Hence, $\mathbf{Pr}[\boldsymbol{\theta}_0 \in coSCR(\boldsymbol{\theta}) \text{ and } \boldsymbol{\theta}_0 \notin SCR(\boldsymbol{\theta})] = 0$. This means that $\boldsymbol{\theta}_0 \in coSCR(\boldsymbol{\theta})$ implies $\boldsymbol{\theta}_0 \in SCR(\boldsymbol{\theta})$ holds w.p. 1. Thus, ***C2*** holds w.p. 1, proving the theorem. □

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