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Matching with Generalized Lexicographic Choice Rules

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Keywords

Matching with contracts, lexicographic choice, dynamic capacity, stability, strategy-proofness, respecting improvements

Disciplines

Behavioral Economics | Economic Theory

Matching with Generalized Lexicographic Choice Rules *

Orhan Aygün[†] and Bertan Turhan[‡]

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1 Introduction

Many real-life problems involve matching agents to institutions that are composed of divisions such as firms, hospitals, or schools. These divisions are endowed with their own choice rules. That is, each division has a well-defined choice rule that selects from any given set of alternatives and a capacity. In some applications, choice rules are induced[1] from a strict ranking of alternatives. In many others, the choices are more complicated and cannot be generated so easily. To make it precise, consider a choice rule that selects applicants on the basis of their merit scores but also requires that the division select a minimum number of disabled applicants whenever possible. Such a choice rule is obviously not a responsive choice rule.

Moreover, in many real-world institutions, there are cross-division constraints, in the sense that the number of available positions in a division might depend on the number of applicants hired by other divisions. For example, consider a business school whose departments—economics, finance, marketing, etc.— are in the job market to hire new faculty. Suppose that the finance group hires first, followed by the economics department, which is then followed by the marketing department. The number of available positions for the marketing group might be increased if either the finance or the economics department hire fewer new faculty than what they initially planned, provided that the total budget of the business school is not exceeded.

In this paper, we study a many-to-one matching with contracts that incorporates a theory of choice of institutions that are composed of divisions. We formulate a new and practical family of choice rules, *Generalized Lexicographic Choice Rules* (GLCR), for institutions. Each institution has a total capacity, i.e., a number of available positions, and a pre-specified linear order at which it fills its divisions. Each division is endowed with a choice rule, i.e., a *sub-choice rule*. Each sub-choice rule has two inputs: the set of available options and its (dynamic) capacity. For each division, both of the inputs depend on the choices made by the divisions that precede it. The set of available options can be thought of as the set of remaining options from the choices of divisions that precede it. The dynamic capacity of each division is a function of the number of unfilled seats of the divisions that precede it given by an exogenously specified capacity transfer function. The overall choice rule of an institution is then defined as the union its divisions' sub-choice rules. The collection of these sub-choice rules and the capacity transfer scheme identify an overall choice rule.

We impose three conditions on sub-choice rules: *Substitutability, size monotonicity* and

[1] These are choice rules for which there is a rational strict preference relation that always selects the $q - best$ elements whenever available. Such choice rules are referred to as "*q-responsive*."

quota monotonicity. A choice rule is *substitutable* if no two alternatives x and y are complementary, in the sense that gaining access to x makes y desirable. A choice rule is *size monotonic* if it chooses weakly more alternatives whenever the set of available alternatives expands. A choice rule satisfies *quota monotonicity* if the following conditions hold: (1) If there is an increase in capacity, then the choice rule selects every alternative it chose beforehand, given any set of alternatives, and (2) If the capacity of a division increases by κ, then the difference between the number of alternatives chosen under increased capacity and the initial capacity cannot exceed κ. There are many non-responsive choice rules that are crucial for real-world applications and satisfy these three conditions[2]. One such family of choice rules in GLCR is introduced in our companion paper, *Aygün and Turhan* (2019b), in the context of matching problems in India with complex reservation constraints.

We impose a mild condition, *monotonicity,* on capacity transfer functions, à la *Westkamp* (2013). The monotonicity condition requires that (1) whenever weakly more seats are left unfilled in *every* division preceding the j^{th} division, weakly more slots should be available for the j^{th} division, and (2) an institution cannot decrease its total capacity in response to increased demand in some divisions. Our companion paper, *Aygün and Turhan* (2019b), introduces several such monotonic capacity transfer functions in the context of comprehensive affirmative action constraints in college admissions and public sector job hiring in India.

We show how markets with generalized lexicographic choice rules can be cleared by the cumulative offer mechanism (COM). To illustrate this, we borrowed the novel observability theory of *Hatfield et al.* (2019). We prove that when sub-choice rules satisfy substitutability, size monotonicity, and quota monotonicity, and when the capacity transfer functions are monotonic, then the overall choice rules of institutions satisfies: the irrelevance of rejected contracts condition (Proposition 1), the observable substitutability (Proposition 2), the observable size monotonicity (Proposition 3), and the non-manipulation via contractual terms (Proposition 4).

Our first main result, Theorem 1, follows by the characterization result of *Hatfield et al.* (2019). The authors show that when choice rules of institutions are observably substitutable, observably size monotonic and non-manipulable via contractual terms, the COM is the unique mechanism which is stable and strategy-proof (for agents). Therefore, in marketplaces in which institutions' choice rules can be modeled, as in the GLCR family, the COM is the uniquely stable and strategy-proof mechanism.

We define a choice-based notion of improvement. We say that a *choice rule of a division* is an *improvement* over another choice rule for an individual if (1) whenever the latter choice

[2]Note that the responsiveness of a choice rule implies substitutability, size monotonicity, and quota monotonicity.

rule selects an individual's contract, the former selects a contract (not necessarily the same contract) from her as well, and (2) when no contract of the individual is selected by either of the choice rules, then both choice rules select the same set. We extend this improvement notion to institutions' choice rules by requiring that conditions (1) and (2) are satisfied for every division. Our second main result states that the COM respects improvements. This result has important implications for real-world applications.

Finally, we note that the generality of our framework enables novel market design applications. The GLCR family via the COM offers a satisfactory solution to many practical real-world assignment problems. We present one such application in our companion paper *Aygün and Turhan* (2019b) in the context of comprehensive affirmative action in India for admission to publicly funded edicational institutions and government sponsored jobs. However, we believe that the theory we develop in this paper might help to design centralized marketplaces beyond the Indian case.

Practical Applications

1. College Admission and Government Job Recruitment in India India has been using one of the most comprehensive affirmative action policies in the world for decades. This policy is embedded in its constitution. There are two types of reservations in India: *vertical* (also called *social*) and *horizontal* (also called *special*) *reservations*[3]. Vertical reservations have been provided as a level playing field for historically disadvantaged castes and tribes. At each institution, certain fractions of available seats are reserved for people from *Scheduled Castes* (SC), *Scheduled Tribes* (ST) and *Other Backward Classes* (OBC). The remaining members of society are collectively categorized under the *General Category* (GC)[4]. Within each vertical category, horizontal reservations are implemented for specific groups, such as *disabled people*, *women*, *people from hill areas*, etc. For each horizontal reservation category, a certain minimum number of such individuals must be admitted within each vertical category. Each vertical category in an institution can be modeled as a division in our framework.

Two recents papers, *Sönmez and Yenmez* (2019a&b), study affirmative action in India with comprehensive affirmative action. The authors formulate the complex Indian affirmative action constraints and introduce *"vertical"* and *"horizontal"* reservations terminology. *Sönmez and Yenmez* (2019a) formulate shortcomings of the choice procedure given in the Supreme Court judgement in *Anil Kumar Gupta vs. State of U. P.* (1995), and document that these

[3] *Aygün and Turhan* (2019a) consider caste-based reservations only and assume away special reservations for simplicity. See also *Aygün and Turhan* (2017).

[4] In 2019, the Union Government of India approved 10 percent reservation in government jobs and publicly funded educational institutions for the *Economically Weaker Section* (EWS) in the GC. The EWS is a subcategory of people belonging to the GC having an annual family income less than a certain amount.

shortcomings are the main cause of numerous lawsuits in India. The authors provide an alternative choice rule to resolve these shortcomings.

Aygün and Turhan (2019b) formulates a sub-choice function for divisions that respects merit scores while also satisfying horizontal reservations. We show that these *non-responsive* sub-choice rules are *substitutable*, *size monotonic*, and *quota monotonic*. We design different monotonic capacity transfer functions depending on the application in India. Therefore, following the theory developed in this paper, we show that the COM can satisfactorily clear several different two-sided many-to-one matching markets in India with comprehensive affirmative action constraints. Our approach in *Aygün and Turhan* (2019b) is sharply different than *Sönmez and Yenmez*'s (2019a&b) approach. In particular, we consider

- applicants' preferences not only over institution but also over vertical category they are admitted under[5], and

- reverting unfilled OBC seats to GC, which is also called OBC de-reservation.

Therefore, the theory we developed in this paper allows us to formulate in the Indian affirmative action problem in its full generality.

2. Regional Quotas in Residency Matching in Japan This real-world market design problem is first introduced by *Kamada and Kojima* (2015) and later studied by *Kamada and Kojima* (2017 and 2018), *Kojima et al.* (2018), and *Goto et al.* (2017).

In 2008, the Japanese government introduced a regional cap that restricts the total number of medical resident matches within each of the 47 prefectures, in order to regulate the geographical distributions of doctors. Each prefecture consists of multiple hospitals. Prefectures can be modeled as institutions, and hospitals in a prefecture can be modeled as divisions of the institution. In Japan, hospitals' (divisions') sub-choice rules are responsive. Hence, the axioms we imposed on sub-choice rules are trivially satisfied. Depending on the specific government goals, monotonic capacity transfer schemes among hospitals in the same prefecture can be formulated so that the COM performs satisfactorily.

Related Literature

The matching problem with generalized lexicographic choice rules is a special case of the matching with contracts model of *Fleiner* (2003) and *Hatfield and Milgrom* (2005). *Fleiner*

[5]It is *optional* for SC, ST and OBC applicants to report the vertical category they belong. Many candidates from these groups do not reveal their caste and tribe membership and utilize its benefits. *Aygün and Turhan* (2019b) provide further evidence on why candidates have preferences also over the ctegory through which they are admitted.

(2003)[6] introduced a matching model with contracts in which institutions are allowed to select multiple contracts with a given agent. *Hatfield and Milgrom* (2005) assume a substitutes condition, which was later weakened by *Hatfield and Kojima* (2010), who maintained the stability and strategy-proofness of the COM.

Hatfield et al. (2019) characterize when stable and strategy-proof matching is possible in many-to-one matching setting with contracts. The authors introduce three novel conditions—observable substitutability, observable size monotonicity, and non-manipulability via contractual terms—and show that when these conditions are satisfied, the COM is the unique mechanism that is stable and strategy-proof (for agents). Moreover, they show that their three conditions are necessary in the following sense: When the choice rule of some institution fails any of their three conditions, they can construct unit-demand choice rules for the other institutions, such that no stable and strategy-proof mechanism exists. We utilize their observability theory in the following sense: Each choice rule in the GLCR class satisfies observable substitutability, observable size monotonicity, and non-manipulation via contractual terms conditions of *Hatfield et al.* (2019), as well as the irrelevance of rejected contracts in *Aygün and Sönmez* (2013). Hence, following their characterization result, the COM is the unique stable and strategy-proof mechanism in our environment.

The closest paper to ours is *Hatfield et al.* (2017). The authors show how to model institutions' choices with cross-division constraints using the framework of matching with contracts. Cross-division constraints introduce new complexities that render prior approaches to proving stability and strategy-proofness inapplicable, including the approaches found in *Hatfield and Milgrom* (2005), *Hatfield and Kojima* (2010), *Kominers and Sönmez* (2016), and *Hatfield and Kominers* (2019). Building upon the observable substitutability theory of *Hatfield et al.* (2019), the authors are able to show that stable and strategy-proof matching is possible in the presence of cross-division constraints. They introduce a model of institutional choice in which each institution has a set of divisions, as in our setting, and flexible allotment capacities that vary as a function of the set of contracts available. Each institution is modeled as having an allotment function that determines how many positions are allocated to each division, given the set of available contracts. Our capacity transfer functions are different than their allotment functions with respect to their very definitions and the assumptions imposed.

Aygün and Turhan (2019a) study a model of dynamic reserves that is similar to the lexicographic choice rules considered in this paper. However, the sub-choice rules are *q-responsive* in their setting, whereas in the GLCR family the sub-choices might not be q-responsive.

[6]Fleiner's results cover *Hatfield and Milgrom*'s (2005) result regarding stability. However, *Fleiner* (2003) does not analyze incentives.

Hence, the model of *Aygün and Turhan* (2019a) cannot accommodate the matching problems in India with both vertical and horizontal reservations, even though it can accommodate the matching problems with vertical reservations only.

Our paper is related to *Afacan* (2017). The author is the first to define a choice-based improvement notion. We also define a choice-based improvement notion in the absence of priorities. We adapt his notion for overall choice rules to our sub-choice rules. The difference between his notion and ours is critical, because he assumes that overall choice rules satisfy unilateral substitutability, size monotonicity, and the irrelevance of rejected contracts conditions. Both the unilateral substitutability and size monotonicity of overall choice rules might be violated in our setting. His analysis—to show that the COM respects improvements—relies on the existence of the agent-optimal stable matching while ours—our proof for Theorem 2—does not.

Our paper is also related to the research agenda on matching with constraints studied in a series of papers: *Kamada and Kojima* (2015, 2017 and 2018), *Kojima et al.* (2018), and *Goto et al.* (2017). In these papers, constraints are imposed on subsets of institutions as a joint restriction, as opposed to being imposed at each institution. A leading example in these papers is the medical matching problem in Japan, in which the government imposes *regional caps*, that serve as upper bounds on the number of doctors that can be placed in each region of the country.

Our paper also contributes to the matching-theoretical school choice literature initiated by *Abdulkadiroğlu and Sönmez* (2003), in which the authors introduced a simple affirmative action policy with *type-specific quotas*. *Kojima* (2012) investigates the consequences of affirmative action policy with type-specific quotas on students' welfare. The author provides examples in which all minority students are made worse off under this type of affirmative action. To circumvent inefficiencies caused by majority quotas, *Hafalır et al.* (2013) offer *minority reserves*. *Echenique and Yenmez* (2015) characterize choice rules for schools that regard students as substitutes while expressing preferences for a diverse student body.

Lexicographic choice rules are also studied in *Alva* (2016), *Kominers and Sönmez* (2016), *Chambers and Yenmez* (2018), and *Doğan et al.* (2018) among others. *Alva* (2016) formulates institutional choice functions that are lexicographic. The author analyzes the relationship between the properties imposed on divisions' choice rules and that of institutions' overall choice rule. *Chambers and Yenmez* (2018) consider lexicographic choice rules from an axiomatic perspective. They show that lexicographic choice rules satisfy acceptance and path independence, and that there are path independent choice rules that are not lexicographic. *Doğan et al.* (2018) provide a characterization of lexicographic choice rules and a characterization of deferred acceptance mechanism that operate based on a lexicographic choice structure. Our focus on cross-division constraints while designing the GLCR family and analyzing choice-based improvement notion in our setting differentiate our work from these papers.

2 Matching with Contracts Setting

There is a finite set of *agents* $\mathcal{I} = \{i_1, ..., i_n\}$ and a finite set of *institutions* $\mathcal{S} = \{s_1, ..., s_m\}$. There is a finite set of *contracts* $\mathcal{X}$. Each contract $x \in \mathcal{X}$ is associated with an agent $\mathbf{i}(x)$ and an institution $\mathbf{s}(x)$. There may be many contracts for each agent-institution pair. We call a set of contracts $X \subseteq \mathcal{X}$ an *outcome*, with $\mathbf{i}(X) \equiv \underset{x\in X}{\cup}\{\mathbf{i}(x)\}$ and $\mathbf{s}(X) \equiv \underset{x\in X}{\cup}\{\mathbf{s}(x)\}$. For any $i \in \mathcal{I} \cup \mathcal{S}$, we let $X_i \equiv \{x \in X \mid i \in \{\mathbf{i}(x), \mathbf{s}(x)\}\}$. An outcome $X \subseteq \mathcal{X}$ is *feasible* if $\mid X_i \mid \leq 1$ for all $i \in \mathcal{I}$.

Each agent $i \in \mathcal{I}$ has unit demand over contracts in $\mathcal{X}_i$ and an outside option $\emptyset_i$. The strict preference of agent i over $\mathcal{X}_i \cup \{\emptyset_i\}$ is denoted by P_i. A contract $x \in \mathcal{X}_i$ is *acceptable for* i (with respect to P_i) if $xP_i\emptyset_i$. Agent preferences over contracts are extended preferences over outcomes in the natural way.

Each institution $s \in \mathcal{S}$ has multi-unit demand and is endowed with a choice rule C^s that describes how s would choose from any offered set of contracts. We let $\overline{q}^s$ denote the physical capacity of institution s. We assume throughout that for all $X \subseteq \mathcal{X}$ and for all $s \in \mathcal{S}$, the choice rule C^s (1) only selects contracts to which s is a party, i.e., $C^s(X) \subseteq X_s$, and (2) selects at most one contract with any given agent and selects at most $\overline{q}^s$ contracts, i.e., $C^s(X)$ is feasible.

For any $X \subseteq \mathcal{X}$ and $s \in \mathcal{S}$, we denote by $R^s(X) \equiv X \setminus C^s(X)$ the set of contracts that s *rejects* from X.

Stability

A feasible outcome $Y \subseteq \mathcal{X}$ is *stable* if it is

1. *Individually rational:* $C^s(Y) = Y_s$ for all $s \in \mathcal{S}$, and $Y_iR_i\emptyset_i$ for all $i \in \mathcal{I}$.

2. *Unblocked:* There does not exist a nonempty $Z \subseteq (\mathcal{X} \setminus Y)$, such that $Z_s \subseteq C^s(Y \cup Z)$ for all $s \in \mathbf{s}(Z)$ and ZP_iY for all $i \in \mathbf{i}(Z)$.

Stability requires that neither agents nor institutions wish to unilaterally walk away from their assignments, and that agents and institutions cannot benefit by recontracting outside of the match.

Mechanisms

A *mechanism* $\mathcal{M}(\cdot; C)$ maps preference profiles $P = (P_i)_{i \in \mathcal{I}}$ to outcomes, given a profile of institutional choice rules $C = (C^s)_{s \in \mathcal{S}}$. Unless otherwise stated, we assume that the choice rules of the institutions are fixed and write $\mathcal{M}(P)$ in place of $\mathcal{M}(P; C)$.

A mechanism $\mathcal{M}$ is *stable* if $\mathcal{M}(P)$ is a stable outcome for every preference profile P. A mechanism $\mathcal{M}$ is *strategy-proof* if for every preference profile P and for each agent $i \in \mathcal{I}$, there is no $\widetilde{P}_i$, such that $\mathcal{M}(\widetilde{P}_i, P_{-i}) P_i \mathcal{M}(P)$.

Cumulative offer mechanisms constitute a particularly important class of mechanisms. In a cumulative offer mechanism, $\mathcal{C}^{\vdash}$, agents propose contracts according to a strict ordering $\vdash$ of the elements of $\mathcal{X}$. In every step, some agent who does not currently have a contract held by any institution proposes his most preferred contract that has not yet been proposed. Then, each institution chooses its most preferred set of contracts according to its choice rule and holds this set until the next step. When multiple agents are able to propose in the same step, the agent who actually proposes is determined by the ordering $\vdash$. The mechanism terminates when no agent is able to propose; at that point, each institution is assigned the set of contracts it is holding. (We describe cumulative offer mechanisms formally in Appendix B.)

3 Institutions' Choice Rules for Stable and Strategy-Proof Matching

In their novel analysis, *Hatfield et al.* (2019) characterized the conditions for institutional choice rules to guarantee the existence of stable and strategy-proof mechanisms. They also showed that when stable and strategy-proof matching is possible, the outcome of any such mechanism coincides with that of a cumulative offer mechanism. Moreover, the outcomes of all cumulative offer mechanisms coincide.

In their seminal work, *Hatfield and Milgrom* (2005) showed that the *substitutes* and *size monotonicity* of institutions' choice rules are sufficient[7] for stable and strategy-proof matching in many-to-one matching settings with contracts. A choice rule is *substitutable* if no two contracts x and y are "complementary" in the sense that gaining access to x makes y desirable. Formally, a choice rule C^s satisfies **substitutability** if for all $x, y \in \mathcal{X}$ and $X \subseteq \mathcal{X}$, $y \notin C^s(X \cup \{y\})$ implies $y \notin C^s(X \cup \{x, y\})$. Substitutability is the *monotonicity*

[7] The analysis of Hatfield and Milgrom (2005) implicitly assumes the *irrelevance of rejected contracts* (IRC) condition. Please see Aygün and Sönmez (2013) for details. Alkan (2002) refers to the same condition as *consistency*.

of the rejection function: C^s is substitutable if and only if we have $R^s(X) \subseteq R^s(Y)$ for all sets of contracts X and Y such that $X \subseteq Y$.

The choice rule of institution $s \in \mathcal{S}$ is *size monotonic*[8] if s chooses weakly more contracts whenever the set of available contracts expands. That is, C^s is size monotonic if for all contracts $x \in \mathcal{X}$ and sets of contracts $X \subseteq \mathcal{X}$, we have $\mid C^s(X) \mid \leq \mid C^s(X \cup \{x\} \mid$. Substitutable and size monotonic choice rules satisfy the IRC condition. A choice rule C^s satisfies the IRC if for all $X \subset \mathcal{X}$ and $x \in \mathcal{X} \setminus X$, $x \notin C^s(X \cup \{x\})$ implies $C^s(X) = C^s(X \cup \{x\})$.

Hatfield et al. (2019) showed that for a cumulative offer mechanism to be stable and strategy-proof, substitutability and size monotonicity need *only* hold during the running of the mechanism itself. However, in that case, we also need to rule out intra-institutional manipulation. Following their terminology, we will give definitions for three properties that, if satisfied, guarantee the existence of a stable and strategy-proof mechanism. We first introduce necessary concepts. An **offer process** for a given institution $s \in \mathcal{S}$, with choice rule C^s, is a finite sequence of distinct contracts $(x^1, x^2, ..., x^M)$, such that for all $m = 1, ..., M$, $x^m \in \mathcal{X}_s$. We say that an offer process $(x^1, x^2, ..., x^M)$ for s is **observable** if, for all $m = 1, ..., M$, $\mathbf{i}(x^m) \notin \mathbf{i}(C^s(\{x^1, ..., x^{m-1}\}))$, i.e, an observable offer process for institution s is a sequence of contract offers proposed by agents, such that an agent can propose x^m only if that agent is rejected by s when this institution has access to $\{x^1, x^2, ..., x^{m-1}\}$.

Definition 1. (*Hatfield et al., 2019*) A choice rule C^s of an institution $s \in \mathcal{S}$ is *observable substitutable* if there does not exist an observable offer process $(x^1, ...x^M)$ for s such that

$$x^t \notin C^s(\{x^1, ..., x^t, ...x^{M-1}\} \text{ but } x^t \in C^s(\{x^1,...,x^t,...,x^M\}).$$

In other words, if $(x^1, ..., x^m)$ is an observable offer process, choice rule C^s satisfies observable substitutability if in an economy where s is the only institution no contract that is rejected at step $m-1$ of the cumulative offer process is accepted at step m. This condition weakens the usual substitutability condition by requiring the set of rejected contracts to expand only at sets of contracts that can be observed in the cumulative offer process.

Proposition 3 of *Hatfield et al.* (2019) indicates that if the choice function of every institution is observably substitutable, then for every preference profile P and any two orderings $\vdash$ and $\vdash'$, $\mathcal{C}^{\vdash}(P) = \mathcal{C}^{\vdash'}(P)$. This implies that all cumulative offer mechanisms are equivalent, i.e., the cumulative offer process is equivalent to the deferred acceptance mechanism described by *Gale and Shapley* (1962).

Another important observation of *Hatfield et al.* (2019) is that if choice rule of every institution is observably substitutable and mechanism $\mathcal{M}$ is stable and strategy-proof, then

[8] Hatfield and Milgrom (2005) call size monotonicity the *law of aggregate demand*.

for any preference profile P, $\mathcal{M}$ is equivalent to the cumulative offer mechanism. This implies that if the choice rules of all institutions satisfy observable substitutability, and we want to prove that a strategy-proof mechanism exists, it is enough to focus on the cumulative offer process, as it is the only candidate. The authors also find that if the choice function of each institution is observably substitutable, then for any preference profile P, the cumulative offer process is stable.

Another desirable property of the cumulative offer mechanism is strategy-proofness for agents. Without it, it is a complicated mechanism for participating agents. Even if the mechanism is stable with respect to the reported preferences, it may no longer be stable for the true preferences. However, observable substitutability is not enough to guarantee strategy-proofness—for this, two other properties, which are defined below, are required.

Definition 2. (*Hatfield et al.*, 2019) A choice rule C^s of an institution $s \in \mathcal{S}$ satisfies *observable size monotonicity* if there does not exist an offer process $(x^1, ..., x^M)$ for s such that

$$| C^b(\{x^1, ..., x^M\}) | < | C^b(\{x^1, ..., x^{M-1}\}) | .$$

In other words, observable size monotonicity requires that the size of the accepted set of contracts weakly increases along observable offer processes. This condition weakens the usual *size monotonicity* condition in that it only has to be satisfied by observable offer processes.

Definition 3. (*Hatfield et al.*, 2019) A choice rule C^s of an institution $s \in \mathcal{S}$ is *non-manipulable via contractual terms* if there does not exist an ordering $\vdash$ and preference profile P for agents $\mathcal{I}$, under which only contracts with s are acceptable, with some agent $i \in \mathcal{I}$, with a preference relation $\widetilde{P}_i$, for which only contracts with s are acceptable, such that

$$C^{\vdash}(\widetilde{P}_i, P_{-i}) P_i C^{\vdash}(P_i, P_{-i}).$$

In other words, a choice rule C^s satisfies the non-manipulability via contractual terms condition if no agent $i \in I$ can profit from reporting a non-truthful preference relation that only finds contracts with s acceptable.

Cumulative Offer Algorithm

The cumulative offer algorithm, which is the generalization of the agent-proposing deferred acceptance algorithm of *Gale and Shapley* (1962), is the central allocation mechanism used in the matching with contracts framework. We now introduce the cumulative offer process for matching with contracts. Here, we provide an intuitive description of this algorithm; we give a more technical statement in Appendix B.

In the **cumulative offer process**, agents propose contracts to institutions in a sequence of steps $l = 1, 2, ...$:

Step 1: Some agent $i^1 \in I$ proposes his most-preferred contract, $x^1 \in X_{i^1}$. Institution $s(x^1)$ holds x^1 if $x^1 \in C^{s(x^1)}(\{x^1\})$ and rejects x^1 otherwise. Set $A^2_{s(x^1)} = \{x^1\}$, and set $A^2_{s'} = \emptyset$ for each $s' \neq s(x^1)$; these are the sets of contracts available to institutions at the beginning of Step 2.

Step 2: Some agent $i^2 \in I$, for whom no contract is currently held by any institution, proposes his most-preferred contract that has not yet been rejected, $x^2 \in X_{i^2}$. Institution $s(x^2)$ holds the contract in $C^{s(x^2)}(A^2_{s(x^2)} \cup \{x^2\})$ and rejects all other contracts in $A^2_{s(x^2)} \cup \{x^2\}$; institutions $s' \neq s(x^2)$ continue to hold all contracts they held at the end of Step 1. Set $A^3_{s(x^2)} = A^2_{s(x^2)} \cup \{x^2\}$ and set $A^3_{s'} = A^2_{s'}$ for each $s' \neq s(x^2)$.

Step l: Some agent $i^l \in I$, for whom no contract is currently held by any institution, proposes his most-preferred contract that has not yet been rejected, $x^l \in X_{i^l}$. Institution $s(x^l)$ holds the contract in $C^{s(x^l)}(A^l_{s(x^l)} \cup \{x^l\})$ and rejects all other contracts in $A^l_{s(x^l)} \cup \{x^l\}$; institutions $s' \neq s(x^l)$ continue to hold all contracts they held at the end of Step $l-1$. Set $A^{l+1}_{s(x^l)} = A^l_{s(x^l)} \cup \{x^l\}$ and set $A^{l+1}_{s'} = A^l_{s'}$ for each $s' \neq s(x^l)$.

If at any time no agent is able to propose a new contract—that is, if all agents for whom no contracts are on hold have proposed all contracts they find acceptable—then the algorithm terminates. The outcome of the cumulative offer process is the set of contracts held by institutions at the end of the last step before termination.

In the cumulative offer process, agents propose contracts sequentially. Institutions accumulate offers, choosing a set of contracts at each step (according to C^s) to hold from the set of all previous offers. The process terminates when no agent wishes to propose a contract.

4 Matching with Generalized Lexicographic Choice Rules

We now introduce a model of institutional choice in which each institution has a set of *divisions* and *dynamic capacities* for those divisions that vary as a function of the number of unused slots in the preceding divisions. We model an institution s as having a set of divisions $\mathcal{K}_s = \{1, ..., K_s\}$. Each division $k \in \mathcal{K}_s$ has an associated *sub-choice* rule $C^s_k : 2^{\mathcal{X}} \times \mathbb{Z}_{\geq 0} \longrightarrow 2^{\mathcal{X}_s}$ that specifies the contracts division k chooses given a set of offers and a dynamic capacity to fill them. We require that each division k never chooses more contracts than its dynamic capacity, i.e., for a set of contracts $Y \subseteq \mathcal{X}_s$ and a dynamic capacity of κ, we must have $\mid C_k^s(Y;\kappa) \mid \leq \kappa$.

The choice procedure is lexicographic. Institution s starts filling positions in division 1. It then fills positions in division 2, and so on and so forth. We let $\overline{q}_1^s$ be the given capacity of division 1. Given a set of contracts $Y \equiv Y^1 \subseteq \mathcal{X}$ and its given capacity $\overline{q}_1^s$, $C_1^s(Y^1;\overline{q}_1^s)$ denotes the set of chosen contracts by division 1. We let $r_1 = \overline{q}_1^s - \mid C_1^s(Y;\overline{q}_1^s) \mid$ be the number of remaining slots in division 1. The dynamic capacity of division 2 is then defined as $q_2^s = q_2^s(r_1)$. We remove every agent's contract that was chosen by division 1 for the rest of the procedure. Given the set of remaining contracts Y^2 and its dynamic capacity q_2^s, division 2 chooses $C_2^s(Y^1;q_2^s(r_1))$. We let $r_2 = q_2^s(r_1) - \mid C_2^s(Y^2;q_2^s) \mid$ be the number of vacant slots from division 2. In general, given the number of vacant slots $r_1, r_2, ..., r_{k-1}$, the dynamic capacity of division k is given by $q_k^s(r_1, ..., r_{k-1})$. Given the set remaining contracts Y^k and its dynamic capacity $q_k^s(r_1, ..., r_{k-1})$, division k chooses $C_k^s(Y^k; q_k^s(r_1, ..., r_{k-1}))$. We let $r_k = q_k^s(r_1, ..., r_{k-1}) - \mid C_k^s(Y^k; q_k^s(r_1, ..., r_{k-1})) \mid$ be the number of vacant slots from division k. All the remaining contracts of agents whose contracts chosen by division k is removed from Y^k for the rest of the procedure.[9]

Given an initial capacity of the first division $\overline{q}_s^1$, a **capacity transfer scheme** of institution s is a sequence of capacity functions $q^s = (\overline{q}_1^s, (q_k^s)_{k=2}^{K_s})$, where $q_k^s : \mathbb{Z}_+^{k-1} \longrightarrow \mathbb{Z}_+$ for all $k \in \mathcal{K}_s$ and such that $\overline{q}_1^s + q_2^s(0) + q_3^s(0,0) + \cdots + q_K^s(0, ..., 0) = \overline{q}^s$.

We also impose a mild condition on capacity transfer functions, à la *Westkamp* (2013). A capacity transfer scheme q^s is **monotonic** if, for all $j \in \{2, ..., K_s\}$ and all pairs of sequences $(r_l, \widetilde{r}_l)$, such that $\widetilde{r}_l \geq r_l$ for all $l \leq j-1$,

$$q_s^j(\widetilde{r}_1, ..., \widetilde{r}_{j-1}) \geq q_s^j(r_1, ..., r_{j-1}), \ \textit{and}$$

$$\sum_{m=2}^{j} [q_s^m(\tilde{r}_1, ..., \widetilde{r}_{m-1}) - q_s^m(r_1, ..., r_{m-1})] \leq \sum_{m=1}^{j-1} [\widetilde{r}_m - r_m].$$

Monotonicity of capacity transfer schemes requires that (1) whenever weakly more seats are left unfilled in *every* division preceding the j^{th} division, weakly more slots should be available for the j^{th} division, and (2) an institution cannot decrease total capacity in response to increased demand in some divisions.

The tuple $\left(\mathcal{I}, \mathcal{S}, P, (C_k^s(\cdot,\cdot), q^s)_{s\in\mathcal{S}, k\in\mathcal{K}_s}\right)$ denotes a problem. Note that the collection of sub-choice rules together with a capacity transfer function fully identify the overall choice rule and are hence regarded as the primitives of the model.

[9]The formal description of the generalized lexicographic choice rules is given in Appendix A.

Conditions on Divisions' Sub-Choice Rules

We impose three conditions on divisions' sub-choice rules: *Substitutability* (S), *size monotonicity* (SM), and *quota monotonicity* (QM).

We already defined substitutability and size monotonicity in Section 3. We now introduce quota monotonicity.

Definition 4. A sub-choice function $C_k^s(\cdot;\cdot)$ satisfies *quota monotonicity* if for any $q, q^{'} \in \mathbb{Z}_+$ such that $q < q^{'}$, for all $Y \subseteq \mathcal{X}$

$$C_k^s(Y, q) \subseteq C_k^s(Y, q^{'}), \ \textit{and}$$

$$\mid C_k^s(Y, q^{'}) \mid - \mid C_k^s(Y, q) \mid \leq q^{'} - q.$$

QM requires choice rules to satisfy two conditions. First, given any set of contracts, if there is an increase in the capacity, we require the choice rule to select every contract it was choosing before increasing its capacity. It might choose some additional contracts. Second, if the capacity of a division increases by κ, then the difference between the number of contracts chosen with the increased capacity and the initial capacity cannot exceed κ.

Our first result relates the three conditions imposed on the sub-choice rules to the IRC that is satisfied by the overall choice rule in the GLCR family.

Proposition 1. *Suppose that every division's sub-choice rule satisfies S, SM, and QM. Then, the institution's overall choice rule satisfies the IRC condition.*

Note that Proposition 1 does not refer to the monotonicity of the capacity transfer functions. That is, Proposition 1 holds even when the capacity transfer functions fail the monotonicity property. In the proof of Proposition 1, it can be seen that sub-choice rules that satisfy the IRC condition are sufficient for the overall choice rule to satisfy the IRC under any given capacity transfer function. Notice that the substitutability and size monotonicity of a choice rule imply the IRC condition (*Aygün and Sönmez*, 2013).

Our next result, Proposition 2, relates three conditions imposed on the sub-choice rules together with monotonicity of the capacity transfer functions to the observable substitutability of the overall choice rule.

Proposition 2. *Suppose that every division's sub-choice rule satisfies S, SM, and QM. If the capacity transfer scheme is monotonic, then the institution's overall choice rule is observably substitutable.*

From the results of *Hatfield et al.* (2019) we know that the COM is stable when the choice rule of every institution satisfies the observable substitutability. Another desirable property of allocation mechanisms is strategy-proofness for agents. However, observable substitutability is not sufficient to guarantee strategy-proofness of the cumulative offer mechanism. For this, we need to show that choice rules in the GLCR family satisfy observable size monotonicity and non-manipulation via contractual terms.

Proposition 3. *Suppose that every division's sub-choice rule satisfies S, SM, and QM. If the capacity transfer scheme is monotonic, then the institution's overall choice rule is observably size monotonic.*

Our last result in this section relates conditions imposed on sub-choice rules together with the monotonicity of the capacity transfer functions to the non-manipulation via contractual terms condition of overall choice rules.

Proposition 4. *Suppose that every division's sub-choice rule satisfies S, SM, and QM. If the capacity transfer scheme is monotonic, then the institution's overall choice rule is not manipulable via contractual terms.*

We are now ready to present our first main result.

Theorem 1. *Suppose that every division's sub-choice rule satisfies S, SM, and QM at each institution. If the capacity transfer scheme of each institution is monotonic, then the COM is stable and strategy-proof.*

Theorem 1 is useful for practical real-life applications, as it states that a stable and strategy-proof matching mechanism is possible under the GLCR family of institutional choice. The design of different mechanisms for matching problems in India in *Aygün and Turhan* (2019b), both for college admissions and job matching in the government sector via vertical and horizontal reservations, utilizes Theorem 1.

5 Respect for Improvements

Respect for improvement is an attractive property of matching mechanisms. In some settings, especially in meritocratic systems, it is rather crucial. The property is first defined by *Balinski and Sönmez* (1999) in a priority-based setting. The authors showed that deferred acceptance respects improvements in the sense that making one student more highly ranked in schools' priority rankings improves their deferred acceptance outcome. In the matching with contracts setting, *Sönmez and Switzer* (2013), *Sönmez* (2013), and *Kominers and*

Sönmez (2016) introduce choice rule specific improvement notions in the presence of ranking lists. In our setting, we may not have ranking lists. As opposed to these papers, we define a notion of improvement over choice rules regardless of the presence of a ranking list. This choice-based improvement notion was first introduced by *Afacan* (2017) for overall choice rules.

Definition 5. A sub-choice rule of division k at institution s $\widetilde{C}^s_k(\cdot;\cdot)$ is an **improvement** over $C^s_k(\cdot;\cdot)$ for agent i if, for any set of contracts $X \subseteq \mathcal{X}$ and for any integer $\kappa \in \mathbb{Z}_+$, the following hold:

1. if $x \in C^s_k(X;\kappa)$ such that $\mathbf{i}(x) = i$, then $y \in \widetilde{C}^s_k(X;\kappa)$ for some $y \in X$ such that $\mathbf{i}(y) = i$;
2. if $i \notin \mathbf{i}[C^s_k(X;\kappa) \cup \widetilde{C}^s_k(X;\kappa)]$, then $C^s_k(X;\kappa) = \widetilde{C}^s_k(X;\kappa)$.

The first condition states that if a contract of agent i is chosen from a given set under the sub-choice rule C^s_k, then a contract of the same agent (not necessarily the same one) must be chosen under $\widetilde{C}^s_k$ given that division k has the same capacity under both sub-choice rules. It is important to note here that it is not a problem if agent i prefers x over y. As the cumulative offer algorithm is run, agents make offers in decreasing order of their preferences. If agent i offers y at some point, it means that x was rejected in earlier steps. *Hatfield et al.* (2019) show that renegotiation does not take place during a cumulative offer process if institutions' choice rules satisfy observable substitutability. We assume that sub-choice rules satisfy S, SM, and QM. We also assume that capacity transfer functions are monotonic. Thus, by our Proposition 2, institutional overall choice rules are observably substitutable. The second condition states that if no contract of agent i is chosen from a given set under both C^s_k and $\widetilde{C}^s_k$, then the chosen sets are the same, given that division k has the same capacity under choice rules C^s_k and $\widetilde{C}^s_k$.

This improvement notion in conjunction with QM imply the following: If $x \in C^s_k(X;\kappa)$ such that $\mathbf{i}(x) = i$, then $y \in \widetilde{C}^s_k(X;\kappa')$ for some $y \in X$, such that $\mathbf{i}(y) = i$ for any $\kappa' \geq \kappa$. Note that QM requires that when the capacity increases, the choice rule selects a superset of the set it was selecting beforehand.

Consider two overall choice rules $\widetilde{C}^s$ and C^s for institution s. Each rule takes the same monotonic capacity transfer function q^s as input and both of their sub-choice rules satisfy properties S, SM, and QM. We say that an overall choice rule $\widetilde{C}^s$ is an **improvement** over C^s for agent i if $\widetilde{C}^s_k(\cdot;\cdot)$ is an improvement over $C^s_k(\cdot;\cdot)$ for agent i at each division $k = 1, ..., K_s$. Finally, we say that $\widetilde{C} \equiv (\widetilde{C}^s)_{s\in S}$ is an **improvement** over $C \equiv (C^s)_{s\in S}$ for agent i if $\widetilde{C}^s$ is an improvement over C^s for agent i at each institution $s \in S$.

Definition 6. Mechanism φ **respects improvements** if for any problem (P, C) and $\widetilde{C}$ such that $\widetilde{C} = (\widetilde{C}^s_k(\cdot;\cdot), q^s)_{s\in S, k\in\mathcal{K}_s}$ is an improvement over $C = (C^s_k(\cdot;\cdot), q^s)_{s\in S, k\in\mathcal{K}_s}$ for agent i where sub-choice rules $(\widetilde{C}^s_k(\cdot;\cdot))_{k=1}^{K_s}$ and $(C^s_k(\cdot;\cdot))_{k=1}^{K_s}$ satisfy S, SM, and QM at each institution $s \in S$ and capacity transfer schemes $(q^s)_{s\in S}$ are all monotonic,

$$\varphi(P, \widetilde{C}) R_i \varphi(P, C).$$

We are now ready to present our second main result.

Theorem 2. *Suppose that at each institution divisions' sub-choice rules satisfy S, SM, and QM and capacity transfer function is monotonic. Then, the COM respects improvements.*

Theorem 2 has significant implications in real-world applications where a meritocratic system is integrated with affirmative action constraints, such as the matching problems in India with vertical and horizontal reservations. *Aygün and Turhan* (2019b) design choice rules for divisions for matching problems in India with these constraints by taking meritocratic component into account. The meritocratic sub-choice rule that takes horizontal reservations into account we propose is not q-responsive. Yet, we show that it satisfies S, SM, and QM. Moreover, the capacity transfer functions in India—for transferring otherwise vacant OBC slots to others—is shown to be monotonic in our setting. Hence, the COM respects improvements in regards to our design.

An important implication of respecting improvements in the case of Indian college admissions and job matching problems is that it incentivizes applicants to declare all horizontal reservation types they have.

6 Conclusion

This paper introduces a new and practical family of choice rules motivated by real-life institutional allocation and choice problems. Institutions are divided into divisions where each division is endowed with a choice rule that satisfies S,SM, and QM. Interaction between divisions, in the sense of capacity transfers, are allowed. The capacity transfer functions are assumed to be monotonic. The overall choice rule of an institution is then defined as the union of its divisions' sub-choices. The paper proves that each such choice rule satisfies the three novel conditions introduced by *Hatfield et al.* (2019), namely, observable substitutability, observable size monotonicity and non-manipulation via contractual terms, together with the irrelevance of rejected contracts condition of *Aygün and Sönmez* (2013). As a result, in many-to-one matching frameworks, the COM, with respect to such overall choice rules, becomes stable and strategy-proof. We define a choice-based notion of improvement and show that the COM respects improvements. This result has important implications for real-world applications.

Our results can be used to design practical marketplaces. One such example is shown in our companion paper, *Aygün and Turhan* (2019b), in the context of matching problems in India that implement comprehensive affirmative action constraints, i.e., vertical and horizontal affirmative action constraints. In our companion paper, we define a sub-choice rule that satisfies S, SM, and QM. The sub-choice rule we designed considers additional policy goals that are specific to India. The design also includes monotonic capacity transfer functions, so that the COM appears as the unique stable and strategy-proof matching mechanism. Our construction can be used not only for college admissions under constraints in India but also in job matching processes for government-sponsored job recruitments. The COM also respects improvements which has significant implications for the Indian case. We believe the theory we developed in this paper will find other attractive real-life applications beyond India.

7 Appendix

A. Formal Description of the Generalized Lexicographic Choice Procedure

Given a set contracts $Y \equiv Y^1 \subseteq \mathcal{X}$, a capacity $\overline{q}^s$ for institution s, and a capacity $\overline{q}_1^s$ for division 1, we compute the chosen set $C^s(Y;\overline{q}^s)$ in K steps where division 1 chooses in step 1, division 2 chooses in step 2, and so on and so forth.

Step 1 Given Y^1 and $\overline{q}_1^s$, division 1 chooses $C_1^s(Y^1;\overline{q}_1^s)$. Let $r_1 = \overline{q}_1^s - \mid C_1^s(Y^1;\overline{q}_1^s) \mid$. Let $Y^2 \equiv Y^1 \setminus \{x \in Y^1 \mid \mathbf{i}(x) \in \mathbf{i}[C_1^s(Y^1;\overline{q}_1^s)]\}$.

Step k ($2 \leq k \leq K$): Given the set of remaining contracts Y^k and its dynamic capacity $q_k^s(r_1,...,r_{k-1})$, division k chooses $C_k^s(Y^k;q_k^s(r_1,...,r_{k-1}))$. Let $r_k = q_k^s(r_1,...,r_{k-1}) - \mid C_k^s(Y^k;q_k^s(r_1,...,r_{k-1})) \mid$. Let $Y^{k+1} \equiv Y^k \setminus \{x \in Y^k \mid \mathbf{i}(x) \in \mathbf{i}[C_k^s(Y^k;q_k^s(r_1,...,r_{k-1}))]\}$.

The union of divisions' choices is the institution's chosen set, i.e.,

$$C^s(Y;\overline{q}^s) \equiv C_1^s(Y^1;\overline{q}_1^s) \cup \bigcup_{k=2}^{K} C_k^s(Y^k;q_k^s(r_1,...,r_{k-1})).$$

B. Formal Description of the Cumulative Offer Process

The **cumulative offer process associated with proposal order** Γ is the following algorithm

1. Let $l = 0$. For each $s \in S$, let $D_s^0 \equiv \emptyset$, and $A_s^1 \equiv \emptyset$.

2. For each $l = 1, 2, ...$
   Let i be the $\Gamma_l$$-$*maximal* agent $i \in I$, such that $i \notin i(\underset{s \in S}{\cup} D_s^{l-1})$ and $\underset{P^i}{max}(X \backslash (\underset{s \in S}{\cup} A_s^l))_i \neq \emptyset_i$— that is, the first agent in the proposal order who wants to propose a new contract— if such an agent exists. (If no such agent exists, then proceed to Step 3, below.)
   (**a**) Let $x = \underset{P^i}{max}(X \setminus (\underset{s \in S}{\cup} A_s^l))_i$ be i's most preferred contract that has not been proposed.
   (**b**) Let $s = s(x)$. Set $D_s^l = C^s(A_s^l \cup \{x\})$ and set $A_s^{l+1} = A_s^l \cup \{x\}$. For each $s^{'} \neq s$, set $D_{s^{'}}^l = D_{s^{'}}^{l-1}$ and $A_{s^{'}}^{l+1} = A_{s^{'}}^l$.

3. Return the outcome

$$Y \equiv (\underset{s \in S}{\cup} D_s^{l-1}) = (\underset{s \in S}{\cup} C^s(A_s^l)),$$

   which consists of contracts held by institutions at the point when no agents want to propose additional contract.

Here, the sets D_s^{l-1} and A_s^l denote the set of contracts **held by** and **available to** institution s at the beginning of the cumulative offer process step l. We say that a contract z is **rejected** during the cumulative offer process if $z \in A_{s(z)}^l$ but $z \notin D_{s(z)}^{l-1}$ for some l.

C. Proofs

Before we prove the results, we first introduce some notation:

- If $X^M = \{x^1, ..., x^M\}$ is an observable offer process, we say $X^m = \{x^1, .., x^m\}$, i.e., X^m are the contracts proposed up to step m of the observable offer process X^M.

- $H_k(X^m)$ denotes the set of contracts available to division k in the computation of $C^s(X^m)$.

- $F_k(X^m) = \underset{n \leq m}{\cup} H_k(X^n)$, i.e., $F_k(X^m)$ is the set of all contracts that were available to division k at some point of offer process $X^m = \{x^1, ..., x^m\}$.

- $q_k^{m-1}(r_1, ..., r_{k-1})$ is the dynamic capacity of division k at step $m-1$ of the observable offer process and $q_k^m(\widetilde{r}_1, ..., \widetilde{r}_{k-1})$ is the dynamic capacity of division k at step m of the observable offer process.

- $R_k(X^m; q_k^m(\widetilde{r}_1, ..., \widetilde{r}_{k-1}))$ is the set of contracts rejected by division k at step m of the observable offer process X^m.

We first prove the following lemma which will be key for proving our results.

Lemma 1. *For all divisions $k \in \{1, ..., K_s\}$ and for all $m \in \{1, ..., M\}$ where M is the last step of observable offer process $X^M = \{x^1, ..., x^M\}$:*

1. $C_k(H_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1})) \subseteq H_k(X^m)$.
2. $C_k(F_k(X^m); q_k^m(\widetilde{r}_1, ..., \widetilde{r}_{k-1})) \subseteq C_k(H_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1})) \cup [H(X^m) \backslash H_k(X^{m-1})]$.
3. $C_k(H_k(X^m); q_k^m(\widetilde{r}_1, ..., \widetilde{r}_{k-1})) = C_k(F_k(X^m); q_k^m(\widetilde{r}_1, ..., \widetilde{r}_{k-1}))$.
4. $q_k^{m-1}(r_1, ..., r_{k-1}) \geq q_k^m(\widetilde{r}_1, ..., \widetilde{r}_{k-1})$.
5. $R_k(F_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1})) \subseteq R_k(F_k(X^m); q_k^m(\widetilde{r}_1, ..., \widetilde{r}_{k-1}))$.

Proof of Lemma 1 We use mathematical induction on pairs (m, k) ordered in the following way:

$$(1,1), (1,2), ..., (1,K), (2,1), (2,2), ..., (2,K), ..., (M,1), (M,2), ..., (M,K).$$

Initial Step: Consider $m = 1$ and any division $k = 1, ..., K$. Note that $X^{m-1} = X^0 = \emptyset$ and $X^1 = \{x^1\}$. Since $H_k(X^0) = H_k(\emptyset) = \emptyset$, condition (1) holds trivially because $\emptyset \subseteq H_k(X^1)$ for all $k = 1, ..., K$. Condition (2) also holds because it reduces to $C_k(F_k(X^1) \subseteq H_k(X^1) = F_k(X^1)$. Condition (3) also holds trivially since $H_k(X^1) = F_k(X^1)$. Condition (4) holds at the pair $(1,1)$ as for the first division the initial capacity is given exogenously, i.e., $\overline{q}_1^s$. Condition (5) reduces to $R_k(\emptyset) = \emptyset \subseteq R_k(F_k(X^1))$ and it trivially holds.

Inductive assumption: Assume that conditions (1)-(5) hold for

- every (m', k) with $m' < m$ and $k = 1, ..., K$,

- every (m, k') with $k' < k$.

We need to show that conditions (1)-(5) hold for the pair (m, k). We start with condition (1).

(1) Take $z \in C_k(H_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1}))$. If z is chosen by division k, then it must have been rejected by all divisions that precede it. Hence, we have

$$(\{x^1, ..., x^{m-1}\})_{\mathbf{i}(z)} \subseteq \cap_{k' < k} R_{k'}(H_{k'}(X^{m-1})).$$

By inductive assumptions (2) and (3), for all $k' < k$, we have

$$C_{k'}(H_{k'}(X^m); q_{k'}^m(\widetilde{r}_1, ..., \widetilde{r}_{k'-1})) \subseteq C_{k'}(H_{k'}(X^{m-1}); q_{k'}^{m-1}(r_1, ..., r_k)) \cup [H_{k'}(X^m) \setminus H_{k'}(X^{m-1})].$$

Note that all the contracts of agent $\mathbf{i}(z)$ are in $H_{k'}(X^{m-1})$ for all $k' < k$. Hence, $\mathbf{i}(z) \notin \mathbf{i}[H_{k'}(X^m) \setminus H_{k'}(X^{m-1})]$. We also know that $\mathbf{i}(z) \notin \mathbf{i}[C_{k'}(H_{k'}(X^{m-1}); q_{k'}^{m-1}(r_1, ..., r_k))]$ because z is chosen by division k in the offer process X^{m-1}. Then we have

$$z \notin C_{k'}(H_{k'}(X^m); q_{k'}^m(\widetilde{r}_1, ..., \widetilde{r}_{k'-1})), \ \forall k' < k.$$

This means z is not chosen by any division that precedes division k, i.e., $(\{x^1, ..., x^{m-1}\})_{\mathbf{i}(z)} \subseteq \cap_{k' < k} R_{k'}(H_{k'}(X^m))$. Therefore, we have $z \in H_k(X^m)$.

(4) By the inductive assumption (4) holds for (i) every (m', k) with $m' < m$ and $k = 1, ..., K$, and (ii) every (m, k') with $k' < k$. To show that $q_k^{m-1}(r_1, ..., r_{k-1}) \geq q_k^m(\widetilde{r}_1, ..., \widetilde{r}_{k-1})$, we will first compare $r_{k'}^{m-1}$ and $\widetilde{r}_{k'}^m$. By definition,

$$r_{k'}^{m-1} = q_{k'}^{m-1}(r_1, ..., r_{k'-1}) - \mid C_{k'}(H_{k'}(X^{m-1}); q_{k'}^{m-1}(r_1, ..., r_{k'-1})) \mid .$$

By inductive assumption, we have $q_{k'}^{m-1}(r_1, ..., r_{k'-1}) \geq q_{k'}^m(\widetilde{r}_1, ..., \widetilde{r}_{k'-1})$. We combine it with the inequality above and get

$$r_{k'}^{m-1} \geq q_{k'}^m(\widetilde{r}_1, ..., \widetilde{r}_{k'-1}) - \mid C_{k'}(H_{k'}(X^{m-1}); q_{k'}^{m-1}(r_1, ..., r_{k'-1})) \mid .$$

By the inductive assumption (3), for all $k' < k$, we have

$$C_{k'}(H_{k'}(X^{m-1}); q_{k'}^{m-1}(\widetilde{r}_1, ..., \widetilde{r}_{k'-1})) = C_{k'}(F_{k'}(X^{m-1}); q_{k'}^{m-1}(\widetilde{r}_1, ..., \widetilde{r}_{k'-1})).$$

By the size monotonicity of the sub-choice functions and the fact that $F_{k'}(X^{m-1}) \subseteq F_{k'}(X^m)$, we have

$$\mid C_{k'}(F_{k'}(X^{m-1}); q_{k'}^{m-1}(r_1, ..., r_{k'-1})) \mid \leq \mid C_{k'}(F_{k'}(X^m); q_{k'}^{m-1}(r_1, ..., r_{k'-1})) \mid .$$

Hence,

$$r_{k'}^{m-1} \geq q_{k'}^{m}(\widetilde{r}_1, ..., \widetilde{r}_{k'-1}) - \mid C_{k'}(F_{k'}(X^{m-1}); q_{k'}^{m-1}(r_1, ..., r_{k'-1})) \mid .$$

By QM, we have

$$\mid C_{k'}(F_{k'}(X^{m}); q_{k'}^{m-1}(r_1, ..., r_{k'-1})) \mid - \mid C_{k'}(F_{k'}(X^{m}); q_{k'}^{m}(\widetilde{r}_1, ..., \widetilde{r}_{k'-1})) \mid \leq$$

$$q_{k'}^{m-1}(r_1, ..., r_{k'-1}) - q_{k'}^{m}(\widetilde{r}_1, ..., \widetilde{r}_{k'-1}).$$

Rearranging the terms gives us

$$q_{k'}^{m-1}(r_1, ..., r_{k'-1}) - \mid C_{k'}(F_{k'}(X^{m}); q_{k'}^{m-1}(r_1, ..., r_{k'-1})) \mid \geq$$

$$q_{k'}^{m}(\widetilde{r}_1, ..., \widetilde{r}_{k'-1}) - \mid C_{k'}(F_{k'}(X^{m}); q_{k'}^{m}(\widetilde{r}_1, ..., \widetilde{r}_{k'-1})) \mid = \widetilde{r}_{k'}^{m}.$$

Combining the last inequality with $r_{k'}^{m-1} \geq q_{k'}^{m}(\widetilde{r}_1, ..., \widetilde{r}_{k'-1}) - \mid C_{k'}(F_{k'}(X^{m-1}); q_{k'}^{m-1}(r_1, ..., r_{k'-1})) \mid$ gives us that $r_{k'}^{m-1} \geq \widetilde{r}_{k'}^{m}$ for all $k' = 1, ..., k-1$. Hence, by the monotonicity of the capacity transfers we conclude that $q_k^{m-1}(r_1, ..., r_{k-1}) \geq q_k^{m}(\widetilde{r}_1, ..., \widetilde{r}_{k-1})$.

(5) By (4), we already know that $q_k^{m-1}(r_1, ..., r_{k-1}) \geq q_k^{m}(\widetilde{r}_1, ..., \widetilde{r}_{k-1})$. By QM, we have

$$C_k(F_k(X^{m-1}); q_k^{m}(\widetilde{r}_1, ..., \widetilde{r}_{k-1})) \subseteq C_k(F_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1})).$$

Hence, we have that

$$F_k(X^{m-1}) \setminus C_k(F_k(X^{m-1}); q_k^{m}(\widetilde{r}_1, ..., \widetilde{r}_{k-1})) \supseteq$$

$$F_k(X^{m-1}) \setminus C_k(F_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1})) = R_k(F_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1})).$$

Then, by substitutability of the sub-choice rules and the fact that $F_k(X^{m-1}) \subseteq F_k(X^{m})$,

$$R_k(F_k(X^{m}); q_k^{m}(\widetilde{r}_1, ..., \widetilde{r}_{k-1})) = F_k(X^{m}) \setminus C_k(F_k(X^{m}); q_k^{m}(\widetilde{r}_1, ..., \widetilde{r}_{k-1})) \supseteq$$

$$F_k(X^{m-1}) \setminus C_k(F_k(X^{m-1}); q_k^{m}(\widetilde{r}_1, ..., \widetilde{r}_{k-1})).$$

Hence, we conclude that

$$R_k(F_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1})) \subseteq R_k(F_k(X^m); q_k^m(\widetilde{r}_1, ..., \widetilde{r}_{k-1})).$$

(2) By the definition of choice rules, we have $C_k(F_k(X^m); q_k^m(\widetilde{r}_1, ..., \widetilde{r}_{k-1})) \subseteq F_k(X^m)$. We can decompose $F_k(X^m)$ as follows:

$$F_k(X^m) = (F_k(X^m)) \setminus F_k(X^{m-1})) \cup [R_k(F_k(X^{m-1})) \cup C_k(F_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1}))],$$

since $F_k(X^{m-1}) = R_k(F_k(X^{m-1})) \cup C_k(F_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1}))$ by definition. If we replace $R_k(F_k(X^{m-1}))$ by $R_k(F_k(X^m))$ and use the fact that $R_k(F_k(X^{m-1})) \subseteq R_k(F_k(X^m))$ by (5), then we have

$$C_k(F_k(X^m); q_k^m(\widetilde{r}_1, ..., \widetilde{r}_{k-1})) \subseteq (F_k(X^m)) \setminus F_k(X^{m-1})) \cup [R_k(F_k(X^m)) \cup C_k(F_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1}))].$$

But, by definition, $C_k(F_k(X^m); q_k^m(\widetilde{r}_1, ..., \widetilde{r}_{k-1}) \cap R_k(F_k(X^m)) = \emptyset$. Hence, the above inclusion relation can be written as

$$C_k(F_k(X^m); q_k^m(\widetilde{r}_1, ..., \widetilde{r}_{k-1})) \subseteq (F_k(X^m)) \setminus F_k(X^{m-1})) \cup C_k(F_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1})).$$

By the definition of $F_k(X^m)$ and $F_k(X^{m-1})$, we know that

$$F_k(X^m) \setminus F_k(X^{m-1}) \equiv H_k(X^m) \setminus \cup_{n<m} H_k(X^n) \subseteq H_k(X^m) \setminus H_k(X^{m-1}).$$

Then, we can conclude that

$$C_k(F_k(X^m); q_k^m(\widetilde{r}_1, ..., \widetilde{r}_{k-1})) \subseteq (H_k(X^m)) \setminus H_k(X^{m-1})) \cup C_k(F_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1})).$$

Finally, by the inductive assumption (3), we have

$$C_k(H_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1})) = C_k(F_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1})).$$

Hence, we can conclude that

$$C_k(F_k(X^m); q_k^m(\widetilde{r}_1, ..., \widetilde{r}_{k-1})) \subseteq (H_k(X^m)) \setminus H_k(X^{m-1})) \cup C_k(H_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1})).$$

(3) By (2) we have that

$$C_k(F_k(X^m); q_k^m(\widetilde{r}_1, ..., \widetilde{r}_{k-1})) \subseteq (H_k(X^m)) \setminus H_k(X^{m-1})) \cup C_k(H_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1})).$$

By (1) we have $C_k(H_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1})) \subseteq H_k(X^m)$. Then, combining (1) and (2) gives us

$$C_k(F_k(X^m); q_k^m(\widetilde{r}_1, ..., \widetilde{r}_{k-1})) \subseteq H_k(X^m).$$

Hence, since the sub-choice rules satisfy the IRC, we can conclude that

$$C_k(H_k(X^m); q_k^m(\widetilde{r}_1, ..., \widetilde{r}_{k-1})) = C_k(F_k(X^m); q_k^m(\widetilde{r}_1, ..., \widetilde{r}_{k-1})).$$

Proof of Proposition 1 Take a set of contracts $Y \subseteq \mathcal{X}$ and a contract $z \in \mathcal{X} \setminus Y$, such that $z \notin C^s(Y \cup \{z\}; \overline{q}^s)$. We need to prove that $C^s(Y; \overline{q}^s) = C^s(Y \cup \{z\}; \overline{q}^s)$. Before starting our proof, note that the substitutability and size monotonicity of divisions' sub-choice rules imply they satisfy the IRC.

Consider two different choice processes for institution s: one starts with Y and one starts with $Y \cup \{z\}$. Let Y^j and $\widetilde{Y}^j$ denote the set of contracts division j receives under choice processes starting with Y and $Y \cup \{z\}$, respectively. Note that $Y^1 \equiv Y$ and $\widetilde{Y}^1 \equiv Y \cup \{z\}$. Let r_j and $\widetilde{r}_j$ denote the number of vacant seats at division k in the choice processes starting with Y and $Y \cup \{z\}$, respectively.

Consider division 1. Under both choice processes, the given capacity of division 1 is the same, i.e., $\overline{q}_1^s$. Since $z \notin C^s(Y \cup \{z\}; \overline{q}^s)$ we know that $z \notin C_1^s(Y \cup \{z\}; \overline{q}_1^s)$. Since the sub-choice rule of division 1 satisfies the IRC, we have that $C_1^s(Y \cup \{z\}; \overline{q}_1^s) = C_1^s(Y; \overline{q}_1^s)$. We also have that $r_1 = \widetilde{r}_1$ since

$$\mid C_1^s(Y \cup \{z\}; \overline{q}_1^s) \mid = \mid C_1^s(Y; \overline{q}_1^s) \mid .$$

Inductive assumption: Suppose that for all divisions $j = 1, ..., k-1$ we have that $C_j^s(Y^j; q_j^s(r_1, ..., r_{j-1})) = C_j^s(\widetilde{Y}^j; q_j^s(\widetilde{r}_1, ..., \widetilde{r}_{j-1}))$.

We will now prove that for division k we have that

$$C_k^s(Y^k; q_k^s(r_1, ..., r_{k-1})) = C_k^s(\widetilde{Y}^k; q_k^s(\widetilde{r}_1, ..., \widetilde{r}_{k-1})).$$

The inductive assumption implies that for all $j = 1, ..., k-1$ we have $r_j = \widetilde{r}_j$. Hence, we have that

$$q_k^s(r_1, ..., r_{k-1}) = q_k^s(\widetilde{r}_1, ..., \widetilde{r}_{k-1}).$$

Since $z \notin C^s(Y \cup \{z\}; \overline{q}^s)$, we have that

$$z \notin C_k^s(\widetilde{Y}^k; \overline{q}_k^s(\widetilde{r}_1, ..., \widetilde{r}_{k-1})).$$

Note that the inductive assumption also implies that $\widetilde{Y}^k \equiv Y^k \cup \{z\}$. By the IRC property of the sub-choice rules, we have that

$$C_k^s(Y^k; q_k^s(r_1, ..., r_{k-1})) = C_k^s(\widetilde{Y}^k; q_k^s(\widetilde{r}_1, ..., \widetilde{r}_{k-1})).$$

Hence, we have $C^s(Y; \overline{q}^s) = C^s(Y \cup \{z\}; \overline{q}^s)$.

Proof of Proposition 2 Consider an institution $s \in \mathcal{S}$ and observable offer process $X \equiv \{x^1, ..., x^M\}$ for s. Let X^{M-1} be the offer process $\{x^1, ..., x^{M-1}\}$. Suppose that $y \in R^s(X^{M-1})$. Since y is rejected by institution s when s faces the offer process X^{M-1}, it must be rejected by all divisions $k = 1, ..., K$. Let r_k and $\widetilde{r}_k$ denote the number of unfilled seats of division k for the choice processes starting with X^{M-1} and X^M, respectively. Hence, for all $k = 1, ..., K$, we have that

$$y \in R_k^s(H_k(X^{M-1}); q_k^{M-1}(r_1, ..., r_{k-1})).$$

Since the sub-choice rules of divisions satisfy substitutability, we have that

$$y \in R_k^s(F_k(X^{M-1}); q_k^{M-1}(r_1, ..., r_{k-1})).$$

for all $k = 1, ..., K$. By (5) of Lemma 1, we have that

$$R_k(F_k(X^{M-1}); q_k^{M-1}(r_1, ..., r_{k-1})) \subseteq R_k(F_k(X^M); q_k^M(\widetilde{r}_1, ..., \widetilde{r}_{k-1}))$$

for each division $k = 1, ..., K$. This implies that

$$y \in R_k^s(F_k(X^M); q_k^M(\widetilde{r}_1, ..., \widetilde{r}_{k-1}))$$

for each division $k = 1, ..., K$. Therefore,

$$y \notin C_k^s(F_k(X^M); q_k^M(\widetilde{r}_1, ..., \widetilde{r}_{k-1})).$$

By (3) of Lemma 1, we have that

$$C_k^s(H_k(X^M); q_k^M(\widetilde{r}_1, ..., \widetilde{r}_{k-1})) = C_k^s(F_k(X^M); q_k^m(\widetilde{r}_1, ..., \widetilde{r}_{k-1})).$$

Hence, we have that

$$y \notin C_k^s(H_k(X^M); q_k^M(\widetilde{r}_1, ..., \widetilde{r}_{k-1}))$$

for all $k = 1, ..., K$. Thus, under the choice procedure that defines C^s, we have that $y \notin R^s(X^M)$, as desired.

Proof of Proposition 3 Consider an institution $s \in \mathcal{S}$ and observable offer processes $X^{M-1} \equiv \{x^1, ..., x^{M-1}\}$ and $X^M \equiv \{x^1, ..., x^M\}$ for s. Let r_k and $\widetilde{r}_k$ be the number of unfilled slots of division k under choice procedures starting with the observable offer processes X^{M-1} and X^M, respectively. Let $H_k(X^{M-1})$ and $H_k(X^M)$ denote the sets of contracts division k faces under the choice procedures starting with X^{M-1} and X^M, respectively. Let $F_k(X^{M-1}) = \underset{n \le M-1}{\cup} H_k(X^n)$ and $F_k(X^M) = \underset{n \le M}{\cup} H_k(X^n)$, i.e., $F_k(X^{M-1})$ and $F_k(X^M)$ are the set of all contracts available to division k at some point of offer process $X^{M-1} = \{x^1, ..., x^{M-1}\}$ and $X^M = \{x^1, ..., x^M\}$, respectively. For the ease of notation, we let $\mathbf{r}_j = (r_1, ..., r_{j-1})$ and $\widetilde{\mathbf{r}}_j = (\widetilde{r}_1, ..., r_{j-1})$. Similarly, let $\mathbf{r}_k = (r_1, ..., r_{k-1})$ and $\widetilde{\mathbf{r}}_k = (\widetilde{r}_1, ..., r_{k-1})$.

Note that by (3) of Lemma 1 we can replace H sets with F sets as follows:

$$C_j^s(H_j(X^{M-1}); q_j^{M-1}(\mathbf{r}_j) = C_j^s(F_1(X^{M-1}); q_j^{M-1}(\mathbf{r}_j))$$

and

$$C_j^s(H_j(X^M); q_j^M(\widetilde{\mathbf{r}}_j)) = C_j^s(F_j(X^M); q_j^M(\widetilde{\mathbf{r}}_j)).$$

By (4) of Lemma 1, we know that

$$q_k^{M-1}(\mathbf{r}_k) \ge q_k^M(\widetilde{\mathbf{r}}_k).$$

By the second condition of monotonic capacity transfer functions, we have

$$\sum_{j=1}^{k} (q_j^{M-1}(\mathbf{r}_j) - q_j^M(\widetilde{\mathbf{r}}_j)) \le \sum_{j=1}^{k-1} [r_j - \widetilde{r}_j].$$

Replacing

$$r_j = q_j^{M-1}(\mathbf{r}_j) - \mid C_j^s(F_j(X^{M-1}); q_j^{M-1}(\mathbf{r}_j)) \mid$$

and

$$\widetilde{r}_j = q_j^M(\widetilde{\mathbf{r}}_j) - \mid C_j^s(F_j(X^M); q_j^M(\widetilde{\mathbf{r}}_j)) \mid$$

gives us the following:

$$\sum_{j=1}^{k} [q_j^{M-1}(\mathbf{r}_j) - q_j^M(\widetilde{\mathbf{r}}_j)] \le$$

$$\sum_{j=1}^{k-1}[q_j^{M-1}(\mathbf{r}_j) - \mid C_j^s(F_j(X^{M-1}); q_j^{M-1}(\mathbf{r}_j)) \mid - q_j^M(\widetilde{\mathbf{r}}_j) + \mid C_j^s(F_j(X^M); q_j^M(\widetilde{\mathbf{r}}_j)) \mid].$$

Readjusting the terms on the right and left sides gives us

$$0 \leq q_k^{M-1}(\mathbf{r}_j) - q_k^M(\widetilde{\mathbf{r}}_j) \leq \sum_{j=1}^{k-1}[C_j^s(F_k(X^M); q_k^M(\widetilde{\mathbf{r}}_j)) - C_j^s(F_j(X^{M-1}); q_j^{M-1}(\mathbf{r}_j))].$$

By QM of sub-choice rules, we have the following:

$$\mid C_k^s(F_k(X^{M-1}); q_k^{M-1}(\mathbf{r}_k)) \mid - \mid C_k^s(F_k(X^{M-1}); q_k^M(\widetilde{\mathbf{r}}_k)) \mid \leq q_k^{M-1}(\mathbf{r}_j) - q_k^M(\widetilde{\mathbf{r}}_k)$$

$$\leq \sum_{j=1}^{k-1}[C_j^s(F_k(X^M); q_k^M(\widetilde{\mathbf{r}}_j) - C_j^s(F_j(X^{M-1}); q_j^{M-1}(\mathbf{r}_j))].$$

By SM of the sub-choice rules, we also have the following:

$$\mid C_k^s(F_k(X^{M-1}); q_k^{M-1}(\mathbf{r}_j)) \mid \leq \mid C_k^s(F_k(X^M); q_k^{M-1}(\mathbf{r}_j)) \mid$$

since, by definition, $F_k(X^{M-1}) \subseteq F_k(X^M)$. Combining the inequalities above, we obtain

$$\mid C_k^s(F_k(X^{M-1}); q_k^{M-1}(\mathbf{r}_k)) \mid - \mid C_k^s(F_k(X^M); q_k^M(\widetilde{\mathbf{r}}_k) \mid \leq q_k^{M-1}(\mathbf{r}_j) - q_k^M(\widetilde{\mathbf{r}}_j)$$

$$\leq \sum_{j=1}^{k-1}[C_j^s(F_j(X^M); q_j^M(\widetilde{\mathbf{r}}_j)) - C_j^s(F_j(X^{M-1}); q_j^{M-1}(\mathbf{r}_j))].$$

Hence, we have

$$\sum_{j=1}^{k}\mid C_j^s(F_j(X^{M-1}); q_j^{M-1}(\mathbf{r}_j)) \mid \leq \sum_{j=1}^{k}\mid C_j^s(F_j(X^M); q_j^M(\widetilde{\mathbf{r}}_j)) \mid .$$

Applying (3) of Lemma 1 again gives us

$$\sum_{j=1}^{k}\mid C_j^s(H_j(X^{M-1}); q_j^{M-1}(\mathbf{r}_j)) \mid \leq \sum_{j=1}^{k}\mid C_j^s(H_j(X^M); q_j^M(\widetilde{\mathbf{r}}_j)) \mid .$$

Since the inequality above holds for all $k = 1, ..., K$, we have

$$\sum_{j=1}^{K} | C_j^s(H_j(X^{M-1}); q_j^{M-1}(\mathbf{r}_j) | \leq \sum_{j=1}^{K} | C_j^s(H_j(X^M); q_j^M(\widetilde{\mathbf{r}}_j)) |,$$

which is the desired conclusion.

Proposition 2 of Hatfield et al. (2019)

Suppose that C^s is a choice rule for institution $s \in \mathcal{S}$ that is observably substitutable and manipulable by agent $i \in \mathcal{I}$ via contractual terms. In this case, there exists a preference profile P and preferences $\widetilde{P}_i$ under which only contracts with s are acceptable, with P_i of the form

$$P_i : \ z^1 P_i \cdots P_i z^M,$$

and $\widetilde{P}_i$ of the form

$$\widetilde{P}_i : \ z^0 \widetilde{P}_i z^1 \widetilde{P}_i \cdots \widetilde{P}_i z^M,$$

such that either

1. $\mathcal{C}_i(P_i, P_{-i}) = \emptyset$ while $\mathcal{C}_i(\widetilde{P}_i, P_{-i}) P_i \emptyset$, or
2. $\mathcal{C}_i(\widetilde{P}_i, P_{-i}) = \emptyset$ while $\mathcal{C}_i(P_i, P_{-i}) \widetilde{P}_i \emptyset$.

Before starting to prove Proposition 4 we will first prove a lemma and then introduce some extra notation to ease our proof.

Lemma 2. *Let $X^m = \{x^1, ..., x^m\}$ and $Y^n = \{y^1, ..., y^n\}$ be two observable offer processes, such that $X^m \subseteq Y^n$. Then, for all divisions $k = 1, ..., K$,*

1. $F_k(X^m) \subseteq F_k(Y^n)$, and
2. $R_k(F_k(X^m); q_k^m(r_1, ..., r_{k-1})) \subseteq R_k(F_k(Y^n); q_k^n(\widetilde{r}_1, ..., \widetilde{r}_{k-1}))$, and
3. $q_k^m(r_1, ..., r_{k-1}) \geq q_k^n(\widetilde{r}_1, ..., \widetilde{r}_{k-1})$, where $(r_1, ..., r_k)$ and $(\widetilde{r}_1, ..., \widetilde{r}_{k-1})$ are the vector of the number of vacant seats in choice procedures starting with offer sets X^m and Y^n, respectively.

Proof of Lemma 2 We proceed by mathematical induction on divisions $k = 1, ..., K$.

For the first division, i.e., $k = 1$, $F_1(X^m) \equiv X^m$ and $F_1(Y^n) \equiv Y^n$ by definition. Hence, we have by our assumption $F_1(X^m) \subseteq F_1(Y^n)$. Thus, (1) is satisfied. Since the capacity of the first division is given, regardless of the offer set, statement (3) is trivially satisfied.

The substitutability of the sub-choice rules implies that $R_1(F_1(X^m); \overline{q}_1^s) \subseteq R_1(F_1(Y^n); \overline{q}_1^s)$. Therefore, (2) is also satisfied for $k = 1$.

Inductive assumption: Suppose that (1)-(3) are satisfied for all divisions $j < k$.

We now need to show that (1)-(3) hold for division k.

We start by showing that (3) holds for k. By our inductive assumption (3) and QM, for all $j < k$,

$$| C_j^s(F_j(X^m); q_j^m(r_1, ..., r_{j-1})) | - | C_j^s(F_j(X^m); q_j^n(\widetilde{r}_1, ..., \widetilde{r}_{j-1})) | \leq q_j^m(r_1, ..., r_{j-1}) - q_j^n(\widetilde{r}_1, ..., \widetilde{r}_{j-1}).$$

Rearranging the terms gives us

$$q_j^m(r_1, ..., r_{j-1}) - | C_j^s(F_j(X^m); q_j^m(r_1, ..., r_{j-1})) | \geq q_j^n(\widetilde{r}_1, ..., \widetilde{r}_{j-1}) - | C_j^s(F_j(X^m); q_j^n(\widetilde{r}_1, ..., \widetilde{r}_{j-1})) | .$$

By our inductive assumption (1) and the size monotonicity, we have

$$q_j^m(r_1, ..., r_{j-1}) - | C_j^s(F_j(X^m); q_j^m(r_1, ..., r_{j-1})) | \geq q_j^n(\widetilde{r}_1, ..., \widetilde{r}_{j-1}) - | C_j^s(F_j(Y^n); q_j^n(\widetilde{r}_1, ..., \widetilde{r}_{j-1})) |,$$

which implies that $r_j \geq \widetilde{r}_j$ for all $j < k$. Then, by monotonicity of capacity transfer functions,

$$q_k^m(r_1, ..., r_{k-1}) \geq q_k^n(\widetilde{r}_1, ..., \widetilde{r}_{k-1}).$$

Hence, (3) holds for division k.

To show (1), consider $z \in F_k(X^m)$. There are two cases to consider:

1. $\mathbf{i}(z) \notin \mathbf{i}[C_j(F_j(X^m); q_j^m(r_1, ..., r_{j-1}))]$, for all $j < k$.

In this case, we know that all contracts of agent $\mathbf{i}(z)$ are rejected by all divisions that precede division k, i.e.,

$$(X^m)_{\mathbf{i}(z)} \subseteq \cap_{j<k} R_j(F_j(X^m); q_j^m(r_1, ..., r_{j-1})).$$

By our inductive assumption (2), we have

$$(X^m)_{\mathbf{i}(z)} \subseteq R_j(F_j(Y^n); q_j^n(\widetilde{r}_1, ..., \widetilde{r}_{j-1})),$$

for some $j < k$. Then, all contracts of agent $\mathbf{i}(z)$ in X^m are considered by some division at some $n^{'} \leq n$ in the choice procedure starting with Y^n. Hence, we have that $(X^m)_{\mathbf{i}(z)} \subseteq F_k(Y^n)$. Since $z \in F_k(X^m)$ was chosen arbitrarily, we conclude that $F_k(X^m) \subseteq F_k(Y^n)$.

2. $\mathbf{i}(z) \in \mathbf{i}[C_j(F_j(X^m); q_j^m(r_1, ..., r_{j-1}))]$ for some $j < k$.

Let y be the contract of agent $\mathbf{i}(z)$ that is chosen by some division j that precedes division k in the choice procedure starting with X^m, i.e.,

$$y = [C_j(F_j(X^m); q_j^m(r_1, ..., r_{j-1}))]_{\mathbf{i}(z)}.$$

Since X^m is an observable offer process and the overall choice rule of institution s, $C^s(\cdot; \overline{q}^s)$, is observably substitutable, we have

$$(X^m)_{\mathbf{i}(z)} \setminus \{y\} \subseteq F_k(X^m).$$

However, $y \notin F_k(X^m)$. Otherwise, we would contradict with the fact that y is chosen by some division $j < k$ in the choice procedure starting with X^m. Recall that as n increases, the sets $F_j(X^n)$ expand and the overall choice rules are observably substitutable.

Now, if $\mathbf{i}(z) \notin \mathbf{i}[C_j(F_j(Y^n); q_j^n(\widetilde{r}_1, ..., \widetilde{r}_{j-1}))]$ for all $j < k$, then it means $(X^m)_{\mathbf{i}(z)} \subseteq F_k(Y^n)$ and the proof concludes. Otherwise, suppose that there exists some $w \in C_j(F_j(Y^n); q_j^n(\widetilde{r}_1, ..., \widetilde{r}_{j-1}))$ for some $j < k$ and $\mathbf{i}(w) = \mathbf{i}(z)$. Then, since Y^n is an observable offer process and C^s satisfies observable substitutability, we have that $(Y^n)_{\mathbf{i}(z)} \setminus \{w\} \subseteq F_k(Y^n)$.

- If $w \notin (X^m)_{\mathbf{i}(z)}$, then $(X^m)_{\mathbf{i}(z)} \subseteq (Y^n)_{\mathbf{i}(z)} \setminus \{w\} \subseteq F_k(Y^n)$ and the proof concludes.

- If $w = y$, then $(X^m)_{\mathbf{i}(z)} \setminus \{y\} \subseteq (Y^n)_{\mathbf{i}(z)} \setminus \{w\} \subseteq F_k(Y^n)$ and the proof concludes.

- If $w \in (X^m)_{\mathbf{i}(z)} \setminus \{y\}$, then, by the fact that X^m is an observable offer process, there must exist $m^{'} < m$ such that $w \in R_j(F_j(X^{m^{'}}); q_j^{m^{'}}(r_1, ..., r_{j-1})$. By the construction of the F sets, the sub-choice rules that satisfy QM and (4) of Lemma 1, we have $w \in R_j(F_j(X^m); q_j^m(r_1, ..., r_{j-1}))$. By (1) and (3) of Lemma 1 and the fact that sub-choice rules satisfy substitutability, we have $w \in R_j(F_j(Y^n); q_j^n(\widetilde{r}_1, ..., \widetilde{r}_{j-1}))$. It contradicts $w \in C_j(F_j(Y^n); q_j^n(\widetilde{r}_1, ..., \widetilde{r}_{j-1}))$. Hence, $w \in (X^m)_{\mathbf{i}(z)} \setminus \{y\}$ cannot be the case.

Some Extra Notation for the proof of Proposition 4

Consider an arbitrary agent i, and let $z^0, z^1, ..., z^L$ be an arbitrary sequence of contracts in $\mathcal{X}_i$. Fix a profile of preferences P_{-i} for all other agents, and let P_i and $\widetilde{P}_i$ be given by

$$P_i : \; z^1 P_i \cdots P_i z^L,$$

$$\widetilde{P}_i : \; z^0 \widetilde{P}_i z^1 \widetilde{P}_i \cdots \widetilde{P}_i z^L.$$

We first fix an ordering $\vdash$ over the set of contracts $\mathcal{X}$. We let $X^N = \{x^1, ..., x^N\}$ be the observable offer process induced by the COM with ordering $\vdash$ under the preferences (P_i, P_{-i}) when institution s is the only institution available. Suppose also that $\widehat{X} = \{\widehat{x}^1, ..., \widehat{x}^{\widehat{N}}\}$ is the observable offer process induced by the COM with ordering $\vdash$ under the preferences $(\widetilde{P}_i, P_{-i})$ when institution s is the only institution available.

Lemma 3. *If $z^0 \notin C^s(\widehat{X}^{\widehat{N}}; \overline{q}^s)$, then $R_s(X^N; \overline{q}^s) \subseteq R_s(\widehat{X}^{\widehat{N}}; \overline{q}^s)$ and for all divisions $k = 1, ..., K$ we have that $F_k(X^N) \subseteq F_k(\widehat{X}^{\widehat{N}})$.*

Proof of Lemma 3 We proceed by mathematical induction on pairs (m, k) in the following order:

$$(1,1), (1,2), ..., (1,K), (2,1), (2,2), ..., (2,K), ..., (N,1), (N,2), ..., (N,K),$$

and at each step we show the following:

$$F_k(X^m) \subseteq F_k(\widehat{X}^{\widehat{N}})$$

$$R_k(F_k(X^m); q_k^m(r_1, ..., r_{k-1})) \subseteq R_k(F_k(\widehat{X}^{\widehat{N}}); q_k^{\widehat{N}}(\hat{r}_1, ..., \hat{r}_{k-1})),$$

where $q_k^m(r_1, ..., r_{k-1})$ and $q_k^{\widehat{N}}(\hat{r}_1, ..., \hat{r}_{k-1})$ are the dynamic capacity of division k in the choice processes starting with X^m and $\widehat{X}^{\widehat{N}}$, respectively.

For the base case $(1,1)$, it must be that x^1 is either the highest-ranked contract of some agent $\mathbf{i}(x^1) \neq i$ or $x^1 = z^1$. In the former case, x^1 must be offered at some step of the offer process $\widehat{X}^{\widehat{N}}$, as it is the best contract agent $\mathbf{i}(x^1)$ want to offer. In the latter case, since z^0 is rejected by our assumption, i must offer her second-best contract under $\widetilde{P}_i$, $z^1 = x^1$, at some step in the offer process $\widehat{X}^{\widehat{N}}$. Hence, in both cases, $x^1 \in \widehat{X}^{\widehat{N}}$. Since $F_1(\widehat{X}^{\widehat{N}}) = \widehat{X}^{\widehat{N}}$, we have that $F_1(\{x^1\}) \subseteq F_1(\widehat{X}^{\widehat{N}})$. Then, by substitutability of the sub-choice rules, we have that

$$R_1(F_1(X^1); \overline{q}_1^s) \subseteq R_1(F_1(\widehat{X}^{\widehat{N}}); \overline{q}_1^s).$$

We now show that both inclusion relations hold for (m, k) if they both hold for

- every pair (m', k), such that $m' < m$ and $k = 1, ..., K$, and
- every pair (m, k') with $k' < k$.

We first show that they hold for $(m, 1)$, given that they are satisfied for every pair (m', k), such that $m' < m$ and $k = 1, ..., K$. By the inductive assumption for pairs $(m-1, k)$, we have

$$R_k(F_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1})) \subseteq R_k(F_k(\widehat{X}^{\widehat{N}}); q_k^{\widehat{N}}(\hat{r}_1, ..., \hat{r}_{k-1}))$$

for all $k = 1, ..., K$. By the observability of X^m we have

$$(\{x^1, ..., x^{m-1}\})_{\mathbf{i}(x^m)} \subseteq R_k(H_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1}))$$

for all $k = 1, ..., K$. Moreover, by the substitutability of the sub-choice rules, we have

$$(\{x^1, ..., x^{m-1}\})_{\mathbf{i}(x^m)} \subseteq R_k(H_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1})) \subseteq R_k(F_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1})).$$

Hence, we have

$$R_k(H_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1})) \subseteq R_k(F_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1})).$$

Therefore, given that $R_k(F_k(X^{m-1}); q_k^{m-1}(r_1, ..., r_{k-1})) \subseteq R_k(F_k(\widehat{X}^{\widehat{N}}); q_k^{\widehat{N}}(\hat{r}_1, ..., \hat{r}_{k-1}))$ for all $k = 1, ..., K$, we find

$$(\{x^1, ..., x^{m-1}\})_{\mathbf{i}(x^m)} \subseteq R_k(F_k(\widehat{X}^{\widehat{N}}); q_k^{\widehat{N}}(\hat{r}_1, ..., \hat{r}_{k-1})),$$

for all $k = 1, ..., K$.

By (3) of Lemma 1, there exists an $N' \leq \widehat{N}$ such that

$$(\{x^1, ..., x^{m-1}\})_{\mathbf{i}(x^m)} \subseteq R_k(F_k(\widehat{X}^{N'}); q_k^{N'}(\hat{r}_1, ..., \hat{r}_{k-1})),$$

for all $k = 1, ..., K$. By the observability of the offer process $\widehat{X}^{\widehat{N}}$ and the fact that it represents all the offers made under the cumulative offer process for $(\widetilde{P}_i, P_{-i})$, there must exist some step $\widetilde{n}$ at which x^m is proposed in $\widehat{X}^{\widehat{N}}$. Recall that $\widehat{X}^{\widehat{N}} \equiv F_1(\widehat{X}^{\widehat{N}})$. Therefore, we have that $x^m \in \widehat{X}^{\widehat{N}} \equiv F_1(\widehat{X}^{\widehat{N}})$. Also, by the inductive assumption for the pair $(m-1, 1)$, we have that $X^{m-1} \equiv F_1(X^{m-1}) \subseteq F_1(\widehat{X}^{\widehat{N}})$. Moreover, since we know that $F_1(X^m) \equiv X^m = \{x^m\} \cup X^{m-1}$, we have

$$F_1(X^m) \subseteq F_1(\widehat{X}^{\widehat{N}}),$$

which is the first condition we want to show for the pair $(m, 1)$. Then, by substitutability of the sub-choice rules, we can conclude that

$$R_1(F_1(X^m); \overline{q}_1^s) \subseteq R_1(F_1(\widehat{X}^{\widehat{N}}; \overline{q}_1^s),$$

which ends our proof for the pair $(m, 1)$.

We now show that they hold for the pair (m, k) when $k > 1$, given that they hold for every pair $(m^{'}, k)$ such that $m^{'} < m$ and $k = 1, ..., K$, and every pair $(m, k^{'})$ with $k^{'} < k$. We will first show that $F_k(X^m) \subseteq F_k(\widehat{X}^{\widehat{N}})$. By our inductive assumption on the pair $(m-1, k)$, it is sufficient to show that $H_k(X^m) \subseteq F_k(\widehat{X}^{\widehat{N}})$, since $F_k(X^m) \equiv F_k(X^{m-1}) \cup H_k(X^m)$. Take $z \in H_k(X^m)$. Since $z \in H_k(X^m)$, all contracts of agent $\mathbf{i}(z)$ must have been rejected by all divisions that preceded division k, i.e.,

$$z \in \cap_{k'<k} R_{k'}(H_{k'}(X^m); q_{k'}^m(r_1, ..., r_{k'-1})).$$

By the substitutability of the sub-choice functions, we can replace $H_{k'}(X^m)$ sets by $F_{k'}(X^m)$ sets for all $k^{'} < k$, i.e.,

$$z \in \cap_{k'<k} R_{k'}(F_{k'}(X^m); q_{k'}^m(r_1, ..., r_{k'-1})).$$

By our inductive assumption on pairs $(m, k^{'})$ with $k^{'} < k$, we have

$$R_{k'}(F_{k'}(X^m); q_{k'}^m(r_1, ..., r_{k'-1})) \subseteq R_{k'}(F_{k'}(\widehat{X}^{\widehat{N}}); q_{k'}^{\widehat{N}}(\hat{r}_1, ..., \hat{r}_{k'-1})).$$

Thus, we have that $z \in \cap_{k'<k} R_{k'}(F_{k'}(\widehat{X}^{\widehat{N}}); q_{k'}^{\widehat{N}}(\hat{r}_1, ..., \hat{r}_{k'-1}))$.

By (3) of Lemma 1, we have that

$$C_{k'}^s(H_{k'}(\widehat{X}^{\widehat{N}}); q_{k'}^{\widehat{N}}(\hat{r}_1, ..., \hat{r}_{k'-1})) = C_{k'}^s(F_{k'}(\widehat{X}^{\widehat{N}}); q_{k'}^{\widehat{N}}(\hat{r}_1, ..., \hat{r}_{k'-1})),$$

for all $k^{'} < k$. Therefore, if there were a $k^{'} < k$, such that $z \in C_{k'}^s(H_{k'}(\widehat{X}^{\widehat{N}}); q_{k'}^{\widehat{N}}(\hat{r}_1, ..., \hat{r}_{k'-1}))$, that would imply $z \in C_{k'}^s(F_{k'}(\widehat{X}^{\widehat{N}}); q_{k'}^{\widehat{N}}(\hat{r}_1, ..., \hat{r}_{k'-1}))$, which contradicts

$$z \in \cap_{k'<k} R_{k'}(F_{k'}(\widehat{X}^{\widehat{N}}); q_{k'}^{\widehat{N}}(\hat{r}_1, ..., \hat{r}_{k'-1})).$$

Then, the following must hold:

$$z \in \cap_{k'<k} R_{k'}(H_{k'}(\widehat{X}^{\widehat{N}}); q_{k'}^{\widehat{N}}(\hat{r}_1, ..., \hat{r}_{k'-1})).$$

Therefore, there must exist some step $\bar{n}$ of the offer process $\widehat{X}^{\widehat{N}}$ such that $z \in H_k(\widehat{X}^{\bar{n}})$, and hence $z \in F_k(\widehat{X}^{\bar{n}}) \subseteq F_k(\widehat{X}^{\widehat{N}})$. So, we can conclude that $F_k(X^m) \subseteq F_k(\widehat{X}^{\widehat{N}})$, which is the first condition we wanted to show for the pair (m, k). To show the second condition, we apply Lemma 2 which then gives us

$$R_k(F_k(X^m); q_k^m(r_1, ..., r_{k-1})) \subseteq R_k(F_k(\widehat{X}^{\widehat{N}}); q_k^{\widehat{N}}(\hat{r}_1, ..., \hat{r}_{k-1})).$$

This completes our induction and ends the proof.

Proof of Proposition 4 By Proposition 2 of *Hatfield et al.* (2019), it is sufficient to show that when s is the only institution, the following two conditions hold:

1. If $\mathbf{i}[\mathcal{C}(P_i, P_{-i})] = \emptyset$, then either $\mathbf{i}[\mathcal{C}(\widetilde{P}_i, P_{-i})] = \emptyset$ or $\mathbf{i}[\mathcal{C}(\widetilde{P}_i, P_{-i})] = \{z^0\}$, and

2. If $\mathbf{i}[\mathcal{C}(\widetilde{P}_i, P_{-i})] = \emptyset$, then $\mathbf{i}[\mathcal{C}(P_i, P_{-i})] = \emptyset$.

To show (1), note that Lemma 3 implies that if $\mathbf{i}[\mathcal{C}(P_i, P_{-i})] = \emptyset$ and $z^0 \notin \mathcal{C}(\widetilde{P}_i, P_{-i})$, then $R_s(X^N) \subseteq R_s(\widehat{X}^{\widehat{N}})$. Moreover, if $\mathbf{i}[\mathcal{C}(P_i, P_{-i})] = \emptyset$, then $\{z^1, ..., z^L\} \subseteq R_s(X^N)$ and hence $\{z^1, ..., z^L\} \subseteq R_s(\widehat{X}^{\widehat{N}})$. Thus, $\mathbf{i}[\mathcal{C}(\widetilde{P}_i, P_{-i})] = \emptyset$.

To show (2), first notice that by Proposition 1 of *Hatfield et al.* (2019) the COM is order independent, since our overall choice rules are observably substitutable and observably size monotonic. Hence, we can consider X^N and $\widehat{X}^{\widehat{N}}$ to be generated by the cumulative offer process with respect to the same proposal ordering $\vdash$ in which all of the agents' contracts other than i precede all of the contracts associated with i. Under this choice of $\vdash$, there must exist an λ such that

1. $x^m = \hat{x}^m$ for all $m < \lambda$,

2. $x^\lambda = z^1$, and

3. $\hat{x}^\lambda = z^0$.

That is, λ is the first step of each cumulative offer process with respect to the order $\vdash$ at which agent i proposes. The offer process $\widehat{X}^{\widehat{N}}$ ends with the rejection of the contract z^L, since z^L follows all contracts with agents other than i according to our specific ordering $\vdash$ and the fact that $\mathbf{i}[\mathcal{C}(\widetilde{P}_i, P_{-i})] = \emptyset$. At each step after λ, exactly one contract is newly rejected, since the overall choice rule of the institution is observable substitutable and size monotonic. Formally, the following holds:

1. $\mid R^s(\widehat{X}^{\tilde{m}}) \setminus R^s(\widehat{X}^{\tilde{m}-1}) \mid = 1$ for all $\tilde{m} = \lambda, \lambda+1, ..., \widehat{N}$, and

2. $z^L \in R^s(\widehat{X}^{\widehat{N}}) \setminus R^s(\widehat{X}^{\widehat{N}-1})$.

For the offer process X^N, we must have $\mid R^s(X^m) \setminus R^s(X^{m-1}) \mid = 1$ for all $m = \lambda, \lambda + 1, ..., N-1$.

Notice that $X^{\lambda-1} = \widehat{X}^{\lambda-1}$. Hence, we have $\mid C^s(X^{\lambda-1}; \overline{q}^s) \mid = \mid C^s(\widehat{X}^{\lambda-1}; \overline{q}^s) \mid$. Moreover, since $\mid R^s(\widehat{X}^{\tilde{m}}) \setminus R^s(\widehat{X}^{\tilde{m}-1}) \mid = 1$ for all $\tilde{m} = \lambda, \lambda+1, ..., \widehat{N}$ we must have that

$$\mid C^s(X^{\lambda-1}; \overline{q}^s) \mid = \mid C^s(\widehat{X}^{\widehat{N}}; \overline{q}^s) \mid .$$

Similarly, since $\mid R^s(X^m) \setminus R^s(X^{m-1}) \mid = 1$ for all $m = \lambda, \lambda+1, ..., N-1$, we have

$$\mid C^s(X^{N-1}; \overline{q}^s) \mid = \mid C^s(\widehat{X}^{\widehat{N}}; \overline{q}^s) \mid .$$

But, since $X^N \subseteq \widehat{X}^{\widehat{N}}$, the observable size monotonicity of C^s implies that

$$\mid C^s(X^N; \overline{q}^s) \mid \leq \mid C^s(\widehat{X}^{\widehat{N}}; \overline{q}^s) \mid .$$

Therefore, we must have

$$R^s(X^N; \overline{q}^s) \setminus R^s(X^{N-1}; \overline{q}^s) \neq \emptyset.$$

Toward a contradiction, suppose that $y \in R^s(X^N; \overline{q}^s) \setminus R^s(X^{N-1}; \overline{q}^s) \neq \emptyset$ and $y \neq z^L$. Note that y is the least-preferred acceptable contract of agent $\mathbf{i}(y)$ with respect to $P_{\mathbf{i}(y)}$ where $\mathbf{i}(y) \neq i$. Then, Lemma 3 implies that there is some step $m^* \geq \lambda$, such that $y \in R^s(\widehat{X}^{m^*}) \setminus R^s(\widehat{X}^{m^*-1})$. But, since $\mid R^s(\widehat{X}^{m^*}) \setminus R^s(\widehat{X}^{m^*-1}) \mid = 1$ and y is the least preferred acceptable contract for $\mathbf{i}(y)$, the cumulative offer process for $(\widetilde{P}_i, P_{-i})$ would end at step m^* with the rejection of y. This contradicts the fact that the cumulative offer process for $(\widetilde{P}_i, P_{-i})$ ends with the rejection of z^L.

Proof of Theorem 1. By Propositions (1)-(4), the overall choice rule of each institution satisfies the IRC condition, observable substitutability, observable size monotonicity, and non-manipulation via contractual terms. Then, by Theorem 4 (*Hatfield et al.*, 2019), the COM is the unique stable and strategy-proof mechanism.

Proof of Theorem 2. Let Φ denote the COM. The contract agent i receives for the problem (P, C) is denoted by $\Phi_i(P, C)$. Consider a problem (P, C) and $\widetilde{C}$, which is an improvement over C for agent i such that each sub-choice rule satisfies S, SM, and QM under both C and $\widetilde{C}$. Moreover, each institution's capacity transfer function is monotonic under both $\widetilde{C}$ and C. Let x and y be the contracts agent i receives under the cumulative offer processes with regards to C and $\widetilde{C}$, respectively. That is,

$$\Phi_i(P, C) = x \quad and \quad \Phi_i(P, \widetilde{C}) = y.$$

Toward a contradiction, assume that xP_iy. Note that $yR_i\emptyset$ (with the possibility that $y = \emptyset$) because the cumulative offer algorithm returns an individually rational match for agents. Consider the false preference, $\widetilde{P}_i$, for agent i such that x is the only contract agent i. That is,

$$\widetilde{P}_i : \ x - \emptyset.$$

We will first show that $\Phi_i(\widetilde{P}_i, P_{-i}, \widetilde{C}) = x$. We prove this claim in two steps. In the first step, we will show that $\Phi_i(\widetilde{P}_i, P_{-i}, C) = x$. Toward a contradiction, suppose that $\Phi_i(\widetilde{P}_i, P_{-i}, C) = \emptyset$. Recall that $\Phi_i(P, C) = x$. Hence, agent i can report P_i instead of $\widetilde{P}_i$ and and obtain x. That means the COM is manipulable at the preference profile $(\widetilde{P}_i, P_{-i})$. However, we established in Theorem 1 that the COM is strategy-proof. This is a contradiction. Therefore, $\Phi_i(\widetilde{P}_i, P_{-i}, C) = x$ must hold.

In the second step, we will show that $\Phi_i(\widetilde{P}_i, P_{-i}, \widetilde{C}) = x$. In the first step, we showed that $\Phi_i(\widetilde{P}_i, P_{-i}, C) = x$. Recall that $\widetilde{C}$ is an improvement over C for agent i. Let $\mathbf{s}(x) = s$. We know that the cumulative offer process is order independent. Let agent i be the last agent to propose contracts. Then, by the definition of improvements, the cumulative offer processes under choice profiles $\widetilde{C}$ and C are identical without agent i. Since $\widetilde{C}^s$ is an improvement over C^s for agent i, the division in institution $\mathbf{s}(x)$ that selects x at $(\widetilde{P}_i, P_{-i}, C)$ selects x at $(\widetilde{P}_i, P_{-i}, \widetilde{C})$, as well. Hence, we have $\Phi_i(\widetilde{P}_i, P_{-i}, \widetilde{C}) = x$.

Therefore, agent i has an incentive to report $\widetilde{P}_i$ at problem $(P, \widetilde{C})$. This contradicts the fact that the COM is strategy-proof under our assumptions. Thus, we must have yR_ix.

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