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Keywords
growth, structural change, urbanization, choice of techniques, productivity

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Supply Side Structural Change*

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Abstract

The interest rate and the rate of economic growth are often regarded as roughly constant as economies grow. Moreover, the agricultural sector and rural population typically shrink. We show that an otherwise standard growth model that includes a backward and an advanced sector can account for these regularities. The mechanism works as follows: as the economy accumulates capital, labor flows from the backward sector to the advanced one. This migration prevents the usual diminishing marginal returns of capital. As a result, the interest rate and the growth rate of the economy remain constant during the transition to the steady state.

The model predicts that developed countries must experience a *sudden* slowdown in their growth rates once the backward sector fully disappears. Productivity, as measured by the Solow residuals, also must slow down. Cross-country evidence supports these predictions of the model.

JEL classifications: O14, O15, O18, O41, O47
Keywords: Growth, Structural Change, Urbanization, Choice of Techniques, Productivity Slowdown.

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1 Introduction

In an influential paper, Kaldor (1963) observed that the growth rate of output, the real interest rate, the labor income share, and the capital-output ratio remain roughly constant as economies grow. This characterization of growing economies, known as the *Kaldor facts*, is widely accepted and used as a criterion of admissibility for models of economic growth (e.g., Barro and Sala-I-Martin (1995)).

Around the same time, another group of researchers including Clark (1940), Lewis (1954), Kuznets (1957,73), and Chenery (1965) stressed another more conspicuous regularity of economic growth. They observed that growing economies also experience a structural transformation characterized by a significant reallocation of labor from agriculture into manufacturing and services, and a simultaneous urbanization of population. For an illustration of the magnitude of this transformation, consider the evolution of the share of urban population in currently developed countries. It has increased from roughly 11% in 1800, to 32% in 1900, to 65% in 1980 (Bairoch, 1988, Table 13.4).

In the light of this evidence, it is natural to ask whether or not current models of economic growth can account simultaneously for the *Kaldor facts* and the *structural change facts*. To date, results in the literature are somewhat negative. It appears that the forces required to produce reallocation of factors across sectors also affect the path of the interest interest and/or the growth rate of the economy. Kongsamut, Rebelo and Xie (2001) (henceforth, KRX), for example, study the path of the interest rate in a model that includes two traditional mechanisms used to explain structural change: an income elasticity of the demand for agricultural products that is less than one and sector-specific technological change. They conclude that the interest rate is constant only if a knife-edge condition is satisfied.

We propose an alternative explanation for the simultaneous ocurrence of Kaldor type of facts and structural change facts. It is based on the observation that rural and urban activities are fundamentally different. In particular, at early stages of development rural activities are land and labor intensive, while urban activities are capital and labor intensive. From this perspective, we regard the rural-urban migration as a process of gradual adoption of more capital-intensive technologies.

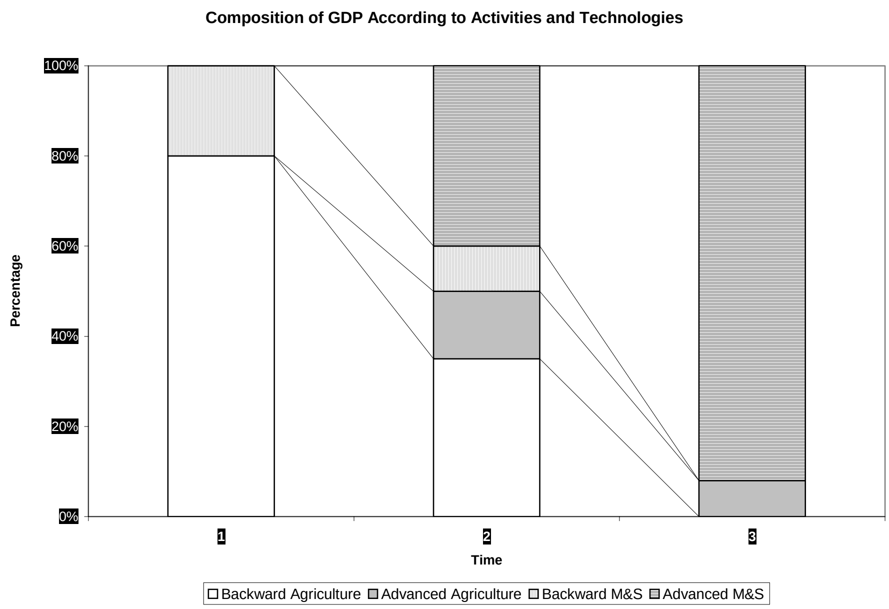

Figure 1: Structural Change

Figure 1 describes in more detail our view about structural change. At any point in time GDP is composed of agricultural goods, manufacturing goods and services. Moreover, these goods and services are produced using different production technologies, called *backward* and *advanced* technologies. As the economy grows, the composition of GDP shifts to manufacturing and services and away from agriculture. This transformation is emphasized by traditional theories of structural change and explained mainly as a consequence of Engel's law.

But another transformation is taking place simultaneously: the underlying rural-urban migration implies that farms are being substituted by factories. We think of this process as one in which backward technologies are abandoned and new advanced technologies are adopted. We study this side of the structural change in the paper, and explain it as resulting from changes in production technologies.

The mechanism works as follows. Consider an economy characterized by the use of a labor-intensive technique[1]. Then a capital-biased technological revolution occurs: a capital-

[1] From now on we use the terms technique and technology as in Atkinson and Stigliz (1969). A technique is a *blue print* describing how inputs can be combined to produce a certain amount of output. Technology

intensive technique suddenly becomes available. The lack of capital in the economy limits the extent to which the new technique is adopted. It is only through a gradual process of capital accumulation that the new advanced technique is fully adopted and the old backward technique is completely abandoned.

The 'industrial revolution' just described triggers a gradual process of capital accumulation, technological adoption, and labor reallocation from the *backward* technique to the *advanced* one. The interest rate of the economy is determined by the marginal product of capital in the advanced sector. It remains constant during the transition path because additional capital accumulated is exactly matched by the labor that migrates from the backward sector. In fact, for an interval of aggregate capital-labor ratios, the aggregate production function takes an $AK + BL$ form in spite of the fact that individual techniques are strictly concave in each input. As a result, the transition path of this economy resembles the one of an AK model (Rebelo, 1990) characterized by a constant interest rate, a constant growth rate, and labor reallocation.[2]

The model in this paper also offers a novel explanation for the well-known productivity slowdown. Growth in our economy suddenly slows down once the structural change is completed. This occurs because additional capital in the advanced sector is no longer matched with labor coming from the backward sector, and as a result decreasing marginal returns on capital set in. We show that growth accounting in our model would incorrectly conclude that productivy also slows down. We present empirical evidence that supports these predictions of the model.

Finally, we analyze other predictions of the model regarding the role of transitional dynamics in economic growth and income differences created by slight variations in fundamentals.

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is the set of available techniques.

[2] Glachant (2000) has also shown that an extension of this model can also produce a kind of Kuznetz curve: inequality jumps when the capital-intensive technique is introduced, monotonically rises during the structural change, and falls when the adoption of the new technique ends.

2 Related Literature

This paper shares the spirit of Kongsamut, Rebelo and Xie (2001). They noted that balanced growth models are consistent with *Kaldor facts* but inconsistent with the observed reallocation of labor between activities. They seek to characterize "generalized balanced paths", paths along which different sectors grow at different rates but for which the interest rate remains constant. They proposed a demand side explanation that relies on a non-homothetic preferences (so that Engel's law holds), and balanced growth paths. In contrast, we propose a supply side explanation that relies on biased-technological change and transitional paths.

Our explanation of the Kaldor facts and the structural change dynamics has at least two advantages when compared with the alternative proposed by KRX. First, the initial distribution of capital among sectors is not crucial in our model, and preferences and technologies can adopt standard functional forms. We thus avoid the knife-edge condition required by KRX. Second, the structural change occurs regardless of whether the economy is open or closed. In our model, labor reallocation occurs naturally as capital is accumulated. In constrast, KRX require a closed economy. Only then changes in the composition of the demand, brought about by Engel's law, affect the supply. This is an important limitation because evidence indicates that most economies have experienced structural transformation regardless of their degree of openness.

A second related paper is Zeira (1998)[3]. He studies problems associated with the adoption of capital-biased innovations in cases for which countries differ in their productivity levels. We leave adoption problems aside, and focus on the adjustment path of an economy that has already decided to adopt the new technology. Such adjustment is instantaneous in Zeira because his economy is small and has full access to international capital markets. We assume an economy with no access to international capital markets, and as result the transition is slow and the interest rate is endogenously determined.

Other related papers are Hansen and Prescott (1998) and Goodfriend and McDermott (1995). They use models similar to ours but focus on very different issues than the ones

[3]The first version of this paper was written before Zeira's paper was published. Thus, some similar results were independently derived.

in this paper. The production side of our economy follows the ideas of R. Jones (1974) regarding the endogenous choice of commodities by a small open economy. He studies the efficient choice of commodities by a country with fixed factor endowments, where output prices and technologies are determined by the world economy. We study a related question but allow for endogenous prices and factor accumulation[4]. Finally, Echevarria (1997) and Coleman and Caselli (2001) calibrate economies similar to the one in KRX and therefore share similar limitations.

This paper is divided into seven sections. Section 3 reviews some evidence about structural change. Section 4 sets up the model and describes the properties of the technology. Section 5 characterizes the equilibrium and derives the predictions of the model. We study four aspects of the equilibrium: (i) its consistency with Kaldor facts and structural change facts; (ii) its ability to account for large income differences when parameters are slightly different; (iii) the contribution of transitional growth to economic growth; and (iv) the growth and productivity slowdown predicted by the model. Section 6 elaborates on some extensions, including the introduction of land and multiple goods into the model. Section 7 concludes.

3 Evidence on Structural Change

The rapid structural transformations of growing economies are among the most robust regularities of economic growth. These transformations include massive reallocation of labor from agriculture into manufacturing and services, and rapid urbanization. Kongsamut *et al.* (2001) recently reviewed extensive evidence regarding the sectoral reallocation of labor and output for a wide set of countries. Their sample includes U.S. data for one hundred years, data for 22 countries for at least 50 years, and data for 123 countries for the period from 1973 to1989. They document that economic growth is accompanied by a significant decline of the agricultural sector, both as a share of the GDP and as a share of labor employment.

Regarding the association between economic growth and urbanization, Bairoch (1988) presents vast cross-country evidence from ancient times until 1980. He shows that the so-called 'Industrial Revolution' brought about a new era of economic growth characterized by

[4] Another related study is Burmeister and Dobell (1970).

a fundamental new role for cities, and an urban explosion. This process began in England, was then followed by some European countries and the U.S., and later by most other countries in the world.

Figures 2 and 3 display recent evidence about urbanization for 14 less-developed countries and for 15 developed countries for the period 1960-1995. According to Figure 2, the urban share of population in less-developed countries has steadily increased and shows no signs of reaching a steady level. The urban share increased on average around 20 percentage points during the period. In some cases, like Korea, it doubled in less than 30 years.

Figure 3 displays a different pattern for the urban share of population in developed countries. In those cases, the urban share steadily increased until the mid seventies at a pace similar to the one in the less-developed countries, but then the share virtually stopped growing. During the period 1960-75, the urban share rose 7.9 points in developed economies and 8.6 points in less developed countries. For the period 1975-95, the share rose around 12.1 points in less developed countries but only 2.2 points in currently developed countries.

In summary, we observe that growing economies experience an increase in their urban share of population, and that the urban share stops increasing once the economy reaches a certain level of development.

4 The Model

Consider a closed economy that produces output using a labor-intensive technique. At time zero the economy suddenly faces a drastic, unexpected and biased technological change - a new capital-intensive technique is discovered. Although the new technique is more productive than the old one, it is not adopted immediately because capital is scarce or nonexistent at the moment of the invention.

We start by describing in detail the production possibilities, and then preferences. For convenience, we present only the planner's problem, although the results hold for a competitive equilibrium.

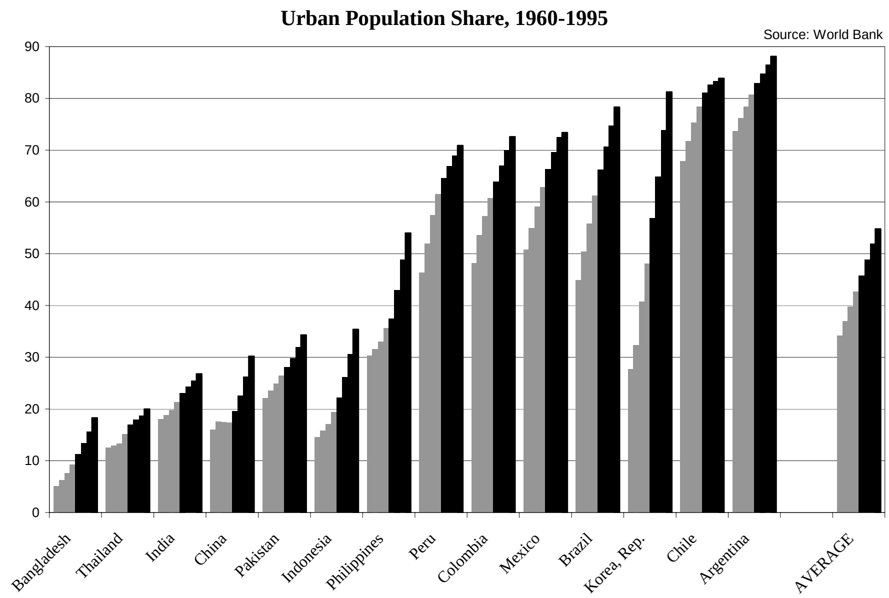

Figure 2: Share of Urban Population, Selected Less Developed Countries

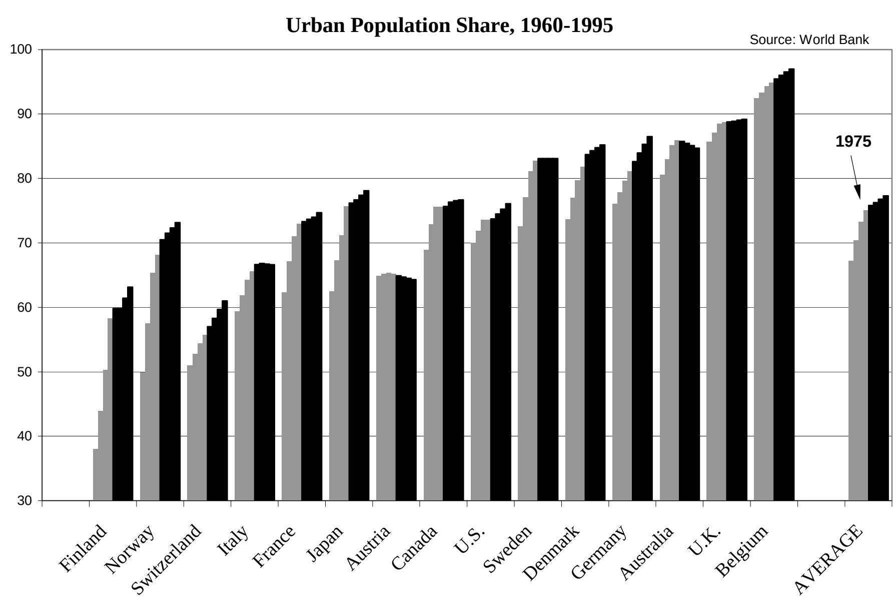

Figure 3: Share of Urban Population, Selected Developed Countries

4.1 Technology

The economy produces only one good which may be consumed or accumulated as productive capital. There are two factors, capital and labor, and two techniques of production (after the discovery), a *backward* one and an *advanced* one. In this section we simply assume that first is a rural technique while the second is an urban technique. Section 6.1 below provides a complete model, that includes land in which this interpretation is meaninful. The results there are identical as in the model without land.

Let K denote capital, L labor, and A productivity. Assumption 1 states that the technology satisfies standard requirements.

Assumption 1. *The good in the economy can be produced using any combination of two C^2 linear homogenous techniques, $F^i(K_i, A_iL_i)$, $i = 1, 2$. Techniques display strictly positive first derivatives and are strictly concave in each argument. In particular,* $\frac{\partial}{\partial x}\left(\frac{\partial F^i(x,y)}{\partial x}\right) < 0$ *and* $\frac{\partial}{\partial y}\left(\frac{\partial F^i(x,y)}{\partial y}\right) < 0$ *for $i = 1, 2$. In addition, $F^2(K_i, A_iL_i)$ satisfies the Inada conditions.*

The following assumption states the existence of a capital-labor ratio threshold, $\widetilde{k}$, such that F^1 dominates F^2 for ratios below $\widetilde{k}$ while the opposite occurs for ratios above $\widetilde{k}$.

Assumption 2. (Unit isoquant crosses only once in R^{++}) $\exists$ a unique $\widetilde{k} > 0$ such that $F^1(\widetilde{k}, 1) = F^2(\widetilde{k}, 1)$. In addition, $F^1(K, L) > F^2(K, L)$ $\forall$ $K/L < \widetilde{k}$ and $F^1(K, L) < F^2(K, L)$ $\forall$ $K/L > \widetilde{k}$.

Assumption 2 guarantees two properties of the technology. First, no technique completely dominates the other. Second, reswitching is avoided, i.e. the case in which a technique that has been abandoned could be re-adopted as the capital-ratio increases. In the light of the second assumption we call F^1 the backward or labor-intensive technique, and F^2 the advanced or capital-intensive technique. The single crossing property is a strong requirement. For example, it is not satisfied by arbitrary combinations of Cobb-Douglas or CES production functions. This property, however, is not crucial for our results. If isoquants cross more than once, reswitching can be avoided if isoquants cross only once for capital-labor ratios below the steady state level.

Finally, we allow for labor-augmenting technological progress and population growth.

Assumption 3. Technology and population evolve exogenously according to the rules

$$A_{it} = A_i e^{xt}, L_t = e^{nt}. \quad (1)$$

Figure 4 displays a pair of unit isoquants, one for each technique, satisfying Assumptions 1 and 2. The isoquants cross exactly once so that the backward technique is more productive – requires less inputs per unit of output – for low capital-labor ratios. This relation reverses for high capital-labor ratios. For the case in which techniques cannot be combined, the aggregate isoquant for the economy is just the envelope of the individual isoquants. In that case, it would be efficient to use the backward technique if the aggregate capital-labor ratio is below $\widetilde{k}$, and switch to the advanced technique if the aggregate capital-labor ratio surpasses that level.

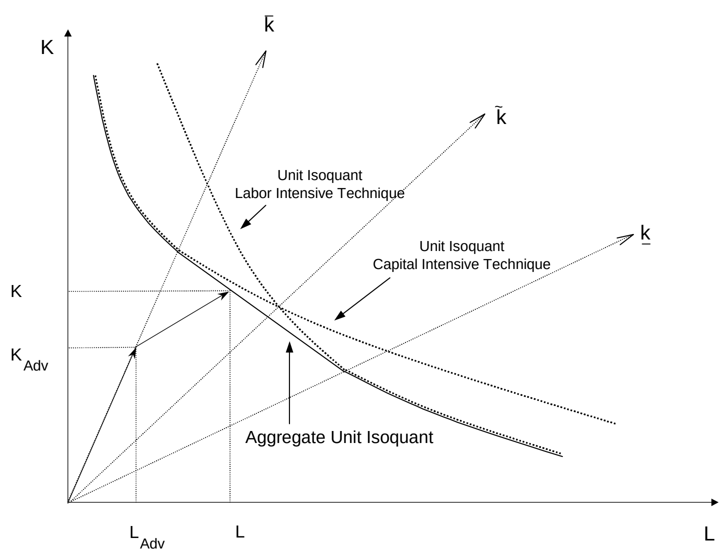

Figure 4: Individual and Aggregate Isoquants

An alternative and relevant case for our purpose arises if techniques can be freely combined. In that case, factors are efficiently allocated among techniques, and aggregate production is determined by the aggregate production function defined as[5]

[5] The maximum is well-defined by Weierstrass Theorem.

$$F_t(K_t, L_t) \equiv \max_{\substack{0 \le K_1 \le K \\ 0 \le L_1 \le L}} F^1(K_{1t}, A_{1t}L_{1t}) + F^2\left(K_t - K_{1t}, A_{2t}\left(L_t - L_{1t}\right)\right). \tag{*}$$

Let $k_{it} \equiv \frac{K_{it}}{e^{(x+n)t}}$ be the efficient capital-labor ratio allocated to technique i, measured in efficiency units of labor, and $k_t \equiv \frac{K_t}{e^{(x+n)t}}$ be the aggregate capital-labor ratio. Marginal products must be equal in any interior solution, i.e.,

$$F_L^1(k_{1t}, A_1) = F_L^2(k_{2t}, A_2) \text{ and } F_k^1(k_{1t}, A_1) = F_k^2(k_{2t}, A_2). \tag{2}$$

The solution also must satisfy the aggregate constraint $K_t = K_{1t} + K_{2t}$, or

$$k_t = k_{1t}l_{1t} + k_{2t}(1 - l_{1t}), \tag{3}$$

where $0 \le l_{1t} \equiv L_{1t}/L_t \le 1$. Define $f^i(k_t) \equiv F^i(K_t, A_tL_t)/e^{(x+n)t} = F^i(k_t, A_i)$ for $i = 1, 2$. Using the equilibrium relations $K_1 + K_2 = K$, $L_1 + L_2 = L$, and the assumption that F^i is linear homogenous, and we can recast (*) as:

$$f(k_t) \equiv \frac{F_t(K_t, L_t)}{e^{(x+n)t}} = \max_{\substack{k_1, l_1 \\ 0 \le k_1 l_1 \le k_t}} f^1(k_{1t})l_{1t} + f^2\left(\frac{k_t - k_1 l_1}{1 - l_1}\right)(1 - l_{1t}) \tag{**}$$

The following proposition collects the main properties of the aggregate technology.

Proposition 1 *Let assumptions 1, 2 and 3 hold, and consider the solution to problem (**). There exist* $0 < \underline{k} < \overline{k}$ *such that (i) for* $k_t \le \underline{k}$*, only technique 1 is used; (ii) for* $k_t \ge \overline{k}$*, only technique 2 is used; (ii) for* $\overline{k} > k_t > \underline{k}$*, both techniques are used in such way that the* $k_1 = \underline{k}$ *and* $k_2 = \overline{k}$*, the fraction of total labor allocated to the second technique is*

$$l_{2t} = \left\{ \begin{array}{cc} 0 & \text{for } k_t < \underline{k} \\ = \frac{1}{\overline{k} - \underline{k}}\left(k_t - \underline{k}\right) & \text{for } \overline{k} \ge k_t > \underline{k} \\ 1 & \text{for } k_t > \overline{k} \end{array} \right\}, \tag{4}$$

and the aggregate production function satisfies

$$f(k_t) = \left\{ \begin{array}{cc} f^1(k_t) & \text{for } k_t < \underline{k} \\ \theta_1 + \theta_2 \cdot k_t & \text{for } \overline{k} \ge k_t > \underline{k} \\ f^2(k_t) & \text{for } k_t > \overline{k} \end{array} \right\}, \tag{5}$$

where $\theta_1 = A_1 F_l^1(\underline{k}, A_1) = A_2 F_l^2(\overline{k}, A_2)$ *and* $\theta_2 = F_k^2(\overline{k}, A_2)$.

Proof. See Appendix 1.1. ■

Corollary 2 *The aggregate production, F, is a differentiable linear homogenous function in (K, L).*

Figure 4 provides a graphical description of the results in the previous proposition. Since both techniques are linear homogenous, the aggregate unit isoquant is just the envelope of the convex combinations of the unit isoquants. As a result, the aggregate isoquant displays a linear segment in the region where techniques are combined. This region is known as the *cone of diversification* (Jones, 1974), a cone delimited by $\underline{k}$ and $\bar{k}$. Figure 4 also illustrates the efficient allocation of factors across techniques. Factors are allocated so that the capital-labor ratios in the backward and advanced techniques are $\underline{k}$ and $\bar{k}$ respectively.

Alternatively, note that (2) is a time homogenous system of two equations and two unknowns, k_1 and k_2, that can be solved *independently* of (3). Assumptions 1 and 2 assure that this system has a unique solution, and that $k_1 < k_2$. Let $\underline{k} \equiv k_1$ and $\overline{k} \equiv k_2$ be the solutions to (2). This pair is not necessarily the solution to (**) because (3) needs also to be satisfied. A brief inspection of the last equation, however, reveals that $(\underline{k}, \bar{k})$ is in fact the solution to (**) as long as $\underline{k} \leq k_t \leq \overline{k}$. It also follows that the marginal products of capital and labor remain constant as k_t moves along this interval.

It is apparent from Figure 4 that the aggregate production function is linear in the cone of diversification. It has the form $AK + BL$ in the cone, a result that resembles the AK model of Rebelo (1990). As in the AK model, this economy can also sustain unlimited endogenous growth if $\bar{k} \to \infty$. In that case, the marginal product of capital ultimately becomes constant. In the case of finite $\overline{k}$ the economy eventually faces decreasing marginal returns, and endogenous growth is typically bounded. The important observation is that growth along the cone of diversification inherits similar properties as the AK model.

Consider the evolution of this economy as k_t increases. As long as the ratio is below $\underline{k}$, only the backward technique is employed and the marginal product of capital decreases. For $k_t \in [\underline{k}, \overline{k}]$, both techniques are employed. As k_t increases, a larger fraction of factors are allocated into the advanced technique, the marginal products of labor and capital are constant, and the elasticity of substitution is infinity. Finally, once the aggregate capital-output ratio exceeds $\bar{k}$, only the advanced technique is operated, and a decreasing marginal product of capital reappears.

A useful example arises when the backward technique is linear in labor,

$$F^1(K_t, A_{1t}L_t) = A_{1t}L_t. \tag{6}$$

In this case $\underline{k} = 0$. Although this formulation does not satisfy Assumption 1, Proposition 1 still applies.

Lemma 3 *Let F^1 satisfy (6). Then $\underline{k} = 0$ and $\bar{k}$ is defined implicitly by the equation $F_l^2(\overline{k}, A_2) = \frac{A_1}{A_2}$. In addition $l_{2t} = \frac{k_t}{\bar{k}}$ for $\overline{k} \geq k_t$, $l_{2t} = 1$ for $k_t > \overline{k}$ and the aggregate production function satisfies:*

$$f(k_t) = \left\{ \begin{array}{cc} A_1 & \text{for } k_t = 0 \\ A_1 + f_k^2(\overline{k}) \cdot k_t & \text{for } \overline{k} \geq k_t > \underline{k} \\ f^2(k_t) & \text{for } k_t > \overline{k} \end{array} \right\}. \tag{7}$$

Proof. The marginal product of labor in the backward technique is A_{1t} and $A_{2t}F_2(K_t, A_{2t}L_{2t})$ in the advanced one. Since $\lim_{L \to 0} F_2(K, A_2L_2) = \infty$ for $K_t > 0$, it is efficient to use the advanced technique whenever $K_t > 0$. Therefore $\underline{k} = 0$. The efficiency condition for labor allocation could be express in terms of the aggregate capital labor ratio and l_{2t},

$$A_1 \leq A_2 \cdot F_2(k_t, A_2 l_{2t}^*) \qquad \text{with equality if } l_{2t} < 1. \tag{8}$$

$\overline{k}$ is the aggregate capital-labor ratio that makes $l_{2t}^* = 1$. Other results follow from Proposition 1. ■

4.2 Preferences

The planner seeks to maximize the utility of a representative, infinite-lived household

$$\int_0^\infty e^{(n-\rho+(1-\theta)x)t} \frac{c_t^{1-\theta}}{1-\theta} dt, \tag{9}$$

where $c_t \equiv \frac{C_t}{L_t e^{xt}}$, C is the household consumption, L is labor (and population), n is the growth rate of labor, $\rho (> n + (1-\theta)x)$ is the rate of time preference, and $\theta (> 0)$ is the (negative of the) elasticity of marginal utility. The planner faces the aggregate resource constraint

$$\frac{dk_t}{dt} \equiv \dot{k}_t = f(k_t) - c_t - (\delta + x + n)k_t. \tag{10}$$

We have followed the standard procedure of writing the utility function and the resource constraint in efficiency units of labor[6], i.e., dividing all level variables by $e^{(n+x)t}$.

5 Efficient Allocation

An optimal allocation for this problem must satisfy the following Euler equation,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left(r_t - \rho - \theta x + \mu_t\right), \tag{11}$$

where r_t is the (net) marginal product of capital, which is also interest rate in a decentralized equilibrium, μ_t is a Lagrange multiplier equal to zero if $k_t > 0$ and positive if $k_t = 0$. The Lagrange multiplier is required if capital is not essential for production, a case that occurs, for example, when F^1 is linear in labor.

The interest rate can be derived from (5) as

$$r_t = r(k_t) = \left\{ \begin{array}{lr} f_k^1(k_t) - \delta & \text{for } k_t < \underline{k} \\ \theta_2 - \delta & \text{for } \overline{k} \geq k_t > \underline{k} \\ f_k^2(k_t) - \delta & \text{for } k_t > \overline{k} \end{array} \right\}. \tag{12}$$

where $f_k(k)$ is the first derivative. Notice that the interest rate is constant over the interval $\overline{k} \geq k_t > \underline{k}$. This is a major departure from the standard neoclassical model (e.g., King and Rebelo (1993)) which exhibits an interest rate that is sensitive to the level of capital. In contrast, the marginal product of capital can be constant or remain in a sensible interval even for low levels of capital in our model. For example, if $\underline{k} = 0$, the interest rate has an upper bound given by $r(0) = r(\overline{k}) = \theta_2 - \delta$.

5.1 Steady State

The steady state level of capital, k^*, can be determined from (11) and (12). $k^* = 0$ if $f_k(0) < \delta + \rho + \theta x$, and otherwise it is implicitly determined by the equation

[6] Details can be found, for example, in Barro and Sala-I-Martin (1995).

$$f_k(k^*) = (\delta + \rho + \theta x)\,. \tag{13}$$

If $f_k(0) \geq \delta+\rho+\theta x$ then a solution to (13) exists because F^2 satisfies the Inada conditions (Assumption 1). There are three cases to consider, depending on which inequality holds true: $f_k(\overline{k}) \gtreqless f_k(k^*)$.

- *Case 1.* $f_k(\overline{k}) > \delta + \rho + \theta x = f_k(k^*)$. In this case $k^* > \overline{k}$ and $f_k(k^*) = f_k^2(k^*)$. This is a case in which the economy only employs the advanced technique in the steady state.

- *Case 2.* $f_k(\overline{k}) = \delta + \rho + \theta x = f_k(k^*)$. In this case there is a continuum of unstable steady states characterized by $\underline{k} \leq k^* \leq \overline{k}$. The scale at which both techniques are operated is undetermined. If the economy starts below $\underline{k}$, it converges monotonically to $\underline{k}$, and if the economy starts above $\overline{k}$, it converges monotonically to $\overline{k}$. If the initial level of capital lies between $\underline{k}$ and $\overline{k}$, the economy remains there forever. This multiplicity of the steady states could explain why economies with apparently identical preferences and technologies may perform very differently. In this case, initial conditions completely determine the steady state consumption, per-capita output, and capital levels.

- *Case 3.* $f_k(\overline{k}) < \delta + \rho + \theta x = f_k(k^*)$. This case represents an economy that only operates the backward technique in the steady state. In this case $k^* \leq \underline{k}$ and $f_k(k^*) = f_k^1(k^*)$.

Note that for given preferences and technologies, the economy generally posseses a unique steady state, unless the parameters satisfy Case 2. It is revealing to consider the case in which the economy is 'trapped' in the backward technique forever (Case 3). Suppose $F^1 \equiv A_1 L_{1t}$ and $F^2(K, A_2 L_2) \equiv K^\alpha (A_2 L_2)^{1-\alpha}$. In that case, the backward technique dominates the advanced technique as long as[7]

$$\frac{A_1^{1-\alpha}}{A_2} > (1-\alpha)^{(1-\alpha)} \left(\frac{\alpha}{\rho + \theta x + \delta}\right)^{\alpha}. \tag{14}$$

[7]We already show the advanced technique is eventually abandoned if $k^* < \overline{k}$. We can rewrite this inequality using the definitions of k^* (from 13) and $\overline{k}$ (from 8), which provide the equation in the text.

Thus, an economy that is particularly productive with the backward technique (large A_1), or particularly inefficient with the advanced technique (low A_2), or that faces a high rate of depreciation, or that displays a high rate of labor augmenting technological progress may find it more efficient to stay with the backward technique. Preferences also play a crucial role. An economy with a high discount rate, or high coefficient of risk aversion, may prefer to keep the backward technique in order to avoid capital accumulation.

Taxes could also prevent an economy from adopting advanced techniques. Consider a tax on capital and labor income earned in the advanced sector. This seems a plausible case since countries tend to tax primarily incomes earned in cities and formal sectors, and subsidize agricultural activities. Such income taxes have the same effect as a lower A_2 and higher A_1 and therefore can deter the adoption of advanced techniques.

5.2 Transitional Dynamics

In the remainder of the paper we focus on economies that eventually fully adopt the advanced technique.

Assumption 4: $f_k(\overline{k}) > \delta + \rho + \theta x$.

We study the equilibrium dynamics of an economy with an initial level of capital below its steady state. We focus on four aspects of the equilibrium path: (i) its consistency with Kaldor facts and structural change facts; (ii) its ability to explain large income differences induced by small differences in parameters; (iii) the contribution of transitional growth to total economic growth; and (iv) the growth and productivity slowdown predicted by the model.

5.2.1 Kaldor Facts and Structural Change

Using equations (11) and (12), we can rewrite the growth rate of consumption as

$$\frac{\dot{c}_t}{c_t} = \left\{ \begin{array}{cc} \frac{1}{\theta}\left(f_k^1(k_t) - f_k(k^*)\right) & \text{for } k_t < \overline{k} \\ \frac{1}{\theta}\left(f_k(\overline{k}) - f_k(k^*)\right) & \text{for } \underline{k} \leq k_t \leq \overline{k} \\ \frac{1}{\theta}\left(f_k^2(k_t) - f_k(k^*)\right) & \text{for } k_t \geq \overline{k} \end{array} \right\}. \tag{15}$$

Since $k^* > \bar{k}$, consumption grows as long as $k_t < k^*$, and it does so at a constant rate as long as k_t remains in the interval $(\underline{k}, \overline{k})$. Above the cone of diversification, the economy behaves exactly as in the Ramsey model. In particular, consumption grows at a decreasing rate. Figure 5 illustrates a phase diagram for this case, constructed using equations (10) and (15), and the usual transversality condition. The phase diagram reveals the following important lemma.

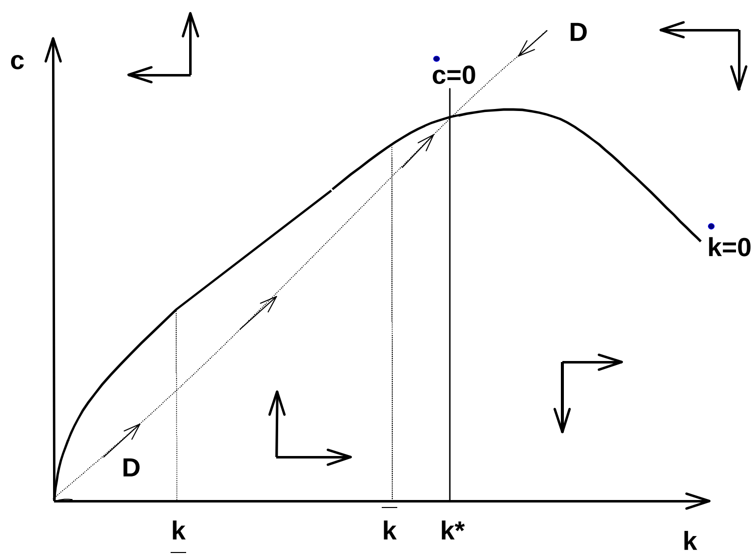

Figure 5: Phase Diagram

Lemma 4 *The optimal allocations of capital and consumption converges monotonically to their steady state levels. In particular, if* $k_0 < k^*$*, the model exhibits increasing sequences of capital, output, and consumption over time.*

Figure 6 displays the equilibrium values of three key variables in the model as a function of k: the interest rate, described by equation (12), the growth rate of the economy, given by equation (15), and the share of labor in the advanced sector, described by equation (4). As indicated, we interpret this share as the urban share of population. The following Proposition summarizes the "growth" properties of the model.

Proposition 5 *(Kaldor and structural change facts). Let technology and preferences satisfy*

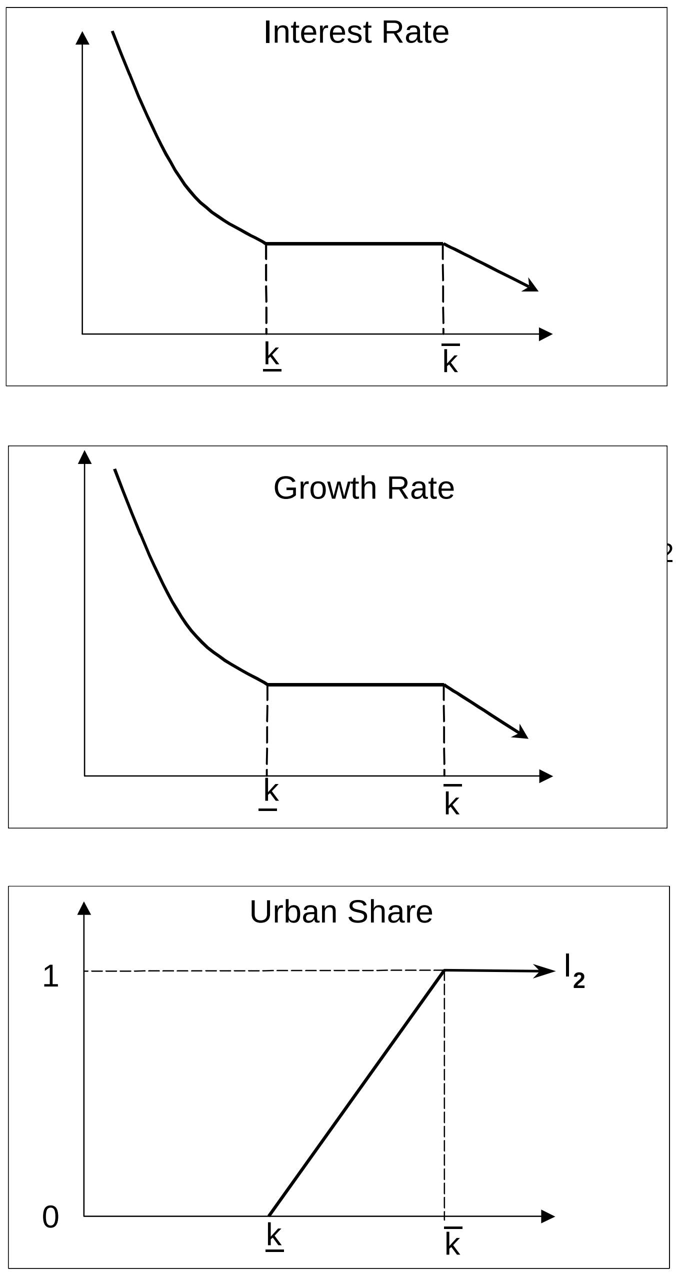

Figure 6: Key Variables

Assumptions 1 through 4, and let $k_0 < \bar{k}$. Then, the equilibrium path of the economy goes through a transient period in which the growth rate of consumption and the interest rate are constant, the capital-output ratio and the capital-labor ratio increase, and the urban share of population gradually increases. Finally, the labor share of income may be constant.

Proof. k_t increases monotonically toward k^* by Lemma 4 and the fact that $k_0 < \overline{k}$. As long as $k_t \in [\underline{k}, \overline{k}]$ the growth rate of consumption and the interest rate remain unchanged, and the urban share of population increases according to equations (15), (12), and (4). Since the production function $f(k^*)$ is concave, the average product, $f(k)/k$, is decreasing. As a result, the capital-output ratio increases along the transition. In addition, simple inspection of (5) reveals that $f(k)/k$ is strictly decreasing for $k \in [\underline{k}, \overline{k}]$. For $k_t > \bar{k}$ the ratio is also strictly decreasing since $f(k)$ is strictly concave. The labor share is constant if the larger capital intensity of the advanced technique reflects human capital. ■

Proposition 5 states that the transition path along the cone of diversification is consistent with most of the Kaldor facts, and with labor reallocation. The only major discrepancy concerns the capital-output ratio. According to Kaldor, growing economies exhibit a constant ratio but our model unequivocally predicts an increasing ratio. Our contetion is that the evidence strongly indicates that the capital-output ratio has significantly increased in growing economies. A strong case on this respect is made by Zeira (1998, Section VIII) who presents convincing evidence supporting this claim. Additional evidence can be found. Figure 7, for example, displays the non-residential capital-output ratios for the U.S. and for the set of currently developed countries considered by Maddison (1991, Table 3.10)) for the period 1890-1987[8]. According to the Figure, the ratio is hardly constant for any country. For example, the ratio increased by a factor of 1.88 for the set of "other countries" during the 100 year span.

The upward trend becomes sharper if human capital is included. In this respect, Judson (1995) has constructed different measures of total capital, including human and physical capital, for a cross section of countries and different periods, as shown in Figure 8. The upward trend is clear, a fact also stressed by Judson.

[8] The sample includes France, Germany, Japan, Netherlands and the U.K

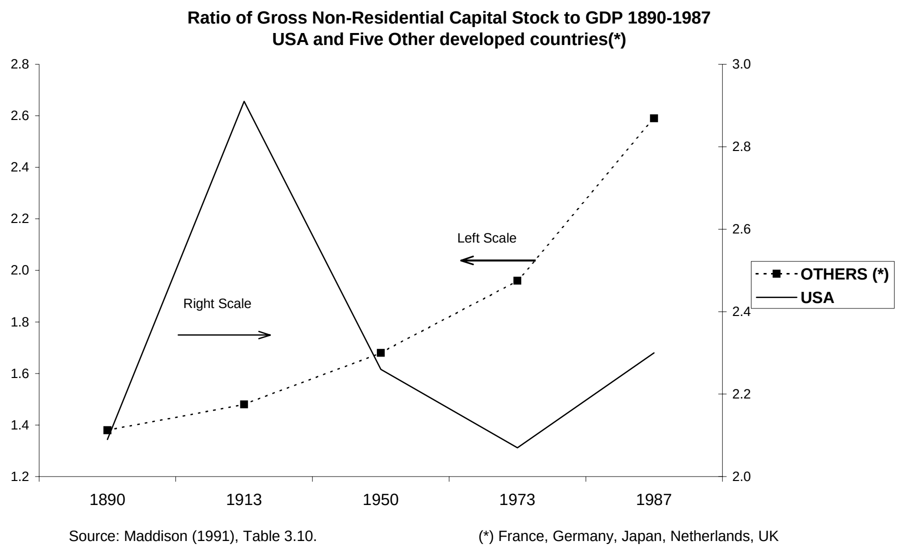

Figure 7: Capital-Output Ratio Developed Countries

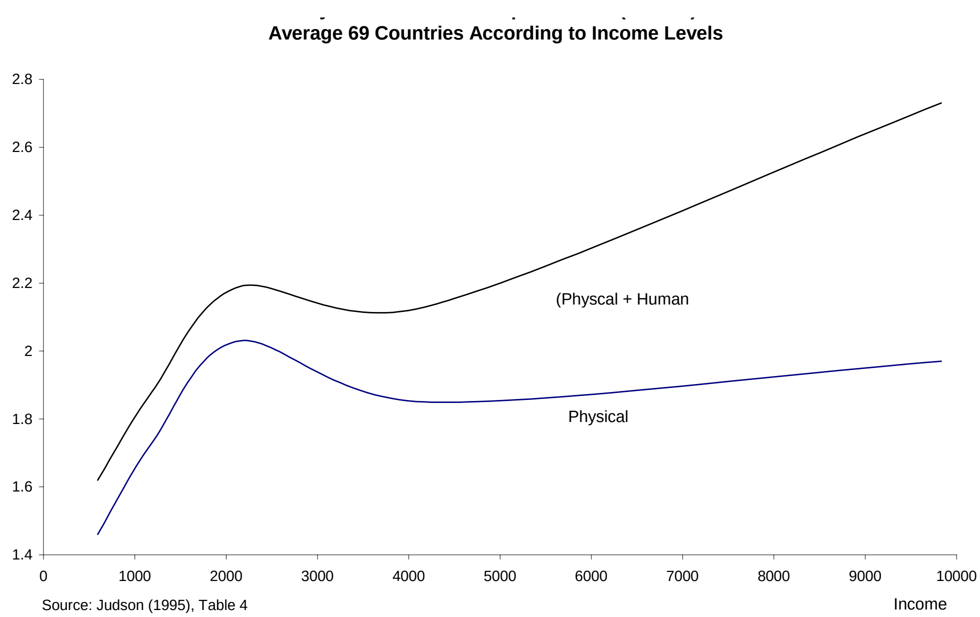

Figure 8: Capita-Output Ratio Vs Income Levels

In conclusion, the simple model constructed here can explain why one usually observes Kaldor facts strongly associated with structural change regularities. In order to grow faster, countries must adopt new capital-intensive technologies, i.e., labor needs to be reallocated. As we see below, this reallocation favors cities because they provide advantages for the adoption of capital intensive technologies relative to rural places. Thus, growth requires urbanization. Before introducing cities, we analyze three other predictions of the model regarding amplification, transitional growth, and productivity slowdown.

5.2.2 Amplification

The model also has a significant amplification effect in the sense that small changes in the parameters of the model can induce large changes in output. The reason is that changes in parameters can affect the adoption decision, and thus drive the economy from one side of the cone of diversification to the other side. Consequently, countries with slightly different preferences, productivities, and/or taxes can display large differences in per-capita income[9]. In this case, government policies could be particularly effective in pushing a country to a higher income level. These results contrast with the predictions of a standard model in which steady state variables are continuous functions of the parameters, and government policies can only induce marginal changes in per-capita income.

How strong is this amplification effect? The model produces additional growth, compared with the standard model, due to capital deepening along the cone of diversification. Output growth along the cone is given by

$$g^C \equiv \frac{f^2(\overline{k})}{f^1(\underline{k})} = \frac{F^2(\overline{k}, A_2)}{F^1(\underline{k}, A_1)}.$$

The following lemma establishes an important relationship between g^C and the capital share of income.

Lemma 6 *Let $\alpha(k)$ be the capital share of income when the aggregate capital-labor ratio is k. Then,*

$$g^C = \frac{1-\alpha(\underline{k})}{1-\alpha(\overline{k})}.$$

[9] Zeira (1998) studies how this model amplifies differences in productivities. See footnote (3).

Proof. Using the Euler rule for linear homogenous functions, g^C can be written as

$$g^C = \frac{F_L^2(\overline{k}, A_2)A_2 + F_k^2(\overline{k}, A_2)\overline{k}}{F_L^1(\underline{k}, A_1)A_1 + F_k^1(\underline{k}, A_1)\underline{k}}.$$

Along the cone, the marginal product of labor is constant so that $F_L^2(\overline{k}, A_2)A_2 = F_L^1(\underline{k}, A_1)A_1$. Therefore,

$$\begin{aligned} g^C &= 1 - \alpha(\underline{k}) + \frac{F_k^2(\overline{k}, A_2)\overline{k}}{F_L^1(\underline{k}, A_1)A_1 + F_k^1(\underline{k}, A_1)\underline{k}} \\ &= 1 - \alpha(\underline{k}) + \frac{F_k^2(\overline{k}, A_2)\overline{k}}{F_L^2(\overline{k}, A_2)A_2 + F_k^2(\overline{k}, A_2)\overline{k}} g^C \\ &= 1 - \alpha(\underline{k}) + \alpha(\overline{k}) g^C. \end{aligned}$$

Solving for g^C, the required result follows. ■

Thus, growth along the cone is fully reflected in the capital share of income, a natural result since the advanced technique is more capital-intensive. The larger the difference in capital intensities, as measured by α, the larger the multiplier. In fact, the multiplier can be very large if $\alpha(\overline{k})$ is close to 1, i.e., as the advanced technique becomes an AK technique. The popularity of AK models in the literature suggests that an $\alpha(\overline{k})$ close to 1 is an important case to consider. The amplification effect in that case may be considerable.

It is important to stress that the increasing capital share in our model is not inconsistent with the Kaldor observation that the labor share is constant. Capital may include physical and human capital, and as a result, part of the capital payments can in fact be accrued by workers.

5.2.3 Transitional Growth

How much growth could be due to transitional dynamics? Early studies (e.g. King and Rebelo, (1993)) found that transitional growth can only play a minor role in explaining growth regularities because of the implausible changes required in the interest rate. To illustrate the point, consider a case in which the only available technique is $F^2(K_t, A_t L_t) = K_t^{\alpha}(A_t L_t)^{1-\alpha}$ with $A_0 = 1$. Per-capita output is given by $y_t^p = A_t k_t^{\alpha}$ and the equilibrium interest rate satisfies $r_t + \delta = k_t^{\alpha-1}$. Let g be the growth rate of per-capita output between t_0 and t. Then, it follows that

$$g = \frac{A_t}{A_{t_0}} \left(\frac{r_t + \delta}{r_{t_0} + \delta} \right)^{\frac{\alpha}{\alpha - 1}} \tag{16}$$

King and Rebelo seek to explain the sevenfold increase in U.S. per-capita output ($g = 7$). They allow technological progress to explain half of the observed growth ($\frac{A_t}{A_{t_0}} = \sqrt{7}$) and ask if the remaining growth can be explained by transitional dynamics, i.e., $\left(\frac{r_t+\delta}{r_{t_0}+\delta}\right)^{\frac{\alpha}{\alpha-1}} = \sqrt{7}$. For a standard value of $\alpha = 1/3$, the gross interest rate requires a sevenfold drop during the transition; for $\alpha = 1/2$, $\sqrt{7}$ times; and for $\alpha = 2/3$, $\sqrt[4]{7}$ times. Due to the lack of evidence supporting long term significant changes in the interest rates, King and Rebelo conclude that transitional dynamics could only play a minor role in the observed growth, particularly because α seems to be below 1/2.

We ask now the same question in our model. As we pointed out, transient growth can occur in our model without any change in the interest rate as the economy moves along the cone of diversification. According with Lemma 6, growth in the cone is equal to $g^C = \frac{1-\alpha(\underline{k})}{1-\alpha(\overline{k})}$. Suppose F^1 requires no capital and $F^2 = K_t^\alpha (A_t L_t)^{1-\alpha}$, as before. In that case, $\alpha(\underline{k}) = 0$, $\alpha(\overline{k}) = \alpha$, and $g^C = \frac{1}{1-\alpha}$. Then we require $g^C \left(\frac{r_t+\delta}{r_{t_0}+\delta}\right)^{\frac{\alpha}{\alpha-1}} = \sqrt{7}$ or $\left(\frac{r_t+\delta}{r_{t_0}+\delta}\right)^{\frac{\alpha}{\alpha-1}} = \sqrt{7}(1-\alpha)$. Thus, for $\alpha = 1/3$, the gross interest rate $(r+\delta)$ requires a $\frac{7}{2.25}$ drop instead of the sevenfold required before; for $\alpha = 1/2$, it must fall $\frac{\sqrt{7}}{2}$times instead of $\sqrt{7}$, and for $\alpha = 2/3$ it must fall $\frac{\sqrt[4]{7}}{1.73}$ times instead of $\sqrt[4]{7}$.

In conclusion, the required fall in the interest rate is less drastic in our model. The results also suggest that a large capital share is still needed if transitional dynamics is to be important for growth. Sensible results are obtained with $\alpha = 1/2$. This high value for α is plausible if human capital is included as part of capital. In any case, the role of human capital in our model needs not be as important as some studies suggest (for example, Barro *et al.* (1995) pick α around 0.8).

5.2.4 Productivity Slowdown

A major puzzle in economic growth is the significant and persistent slowdown in growth experienced by most developed countries since the early 70s. Although several explanations have been advanced, there is still no consensus about what caused the slowdown. Our model gives a natural explanation for the puzzle. It predicts that a *permanent* growth slowdown occurs when the economy surpasses the capital level $\bar{k}$. At that point the structural change is completed, capital accumulation in the advanced sector cannot be matched with labor arriving from the backward sector, and the economy starts facing decreasing marginal returns of capital. As a result, a growth slowdown occurs at the time when the economy reaches a steady level of urban population. According to the model, the growth slowdown is equal to the transitional growth as given by equation (11) evaluated at $k = \overline{k}$,

$$\text{Growth Slowdown} = \frac{1}{\theta} \left(r(\overline{k}) - r(k^*) \right).$$

This prediction of the model is supported by empirical evidence. According to Figure 3, the share of urban population increased steadily until the early 70s, when it suddenly almost stabilized. On average, it rose 0.32 points per year during the period from 1960 to 1975 but only 0.11 points per year during the period 1975-95. During this last period, most developed economies also experienced a major growth slowdown (See Maddison, 1991).

Inverse causality can be argued: the slowdown in income growth could explain the urbanization slowdown due to Engel's law. This is certainly a plausible argument, but it cannot be the whole story for at least three reasons. First, the urbanization slowdown was expected, regardless of the income trend, because the urban ratio was growing at an unsustainable rate given its upper bound of 1 (See Figure 3). Second, the evidence in Figure 2 shows no slowdown of the urban ratio in developing countries in spite of their dramatic income 'meltdown' during the 80s (Ben-David and Papell (1997)). This suggests that Engel's law is not the main determinant of urbanization. Finally, the urbanization slowdown seems too strong to be explained by Engel's law. Consider the case of the U.S. presented in Figure 9. Income continued to increase at a significant rate even after the slowdown while the urban ratio almost stopped growing.

Apparent drawbacks of this explanation are the findings in several studies indicating that slowdown is associated with a slowdown in productivity growth. There is no productivity slowdown in our model since productivity always grows at the constant rate x. However, the predictions of the model can be reconciled with the evidence of productivity. To show this, note that standard exercises compute series of productivity as Solow residuals. Those exercises typically assume that the aggregate production function is of Cobb-Douglas form

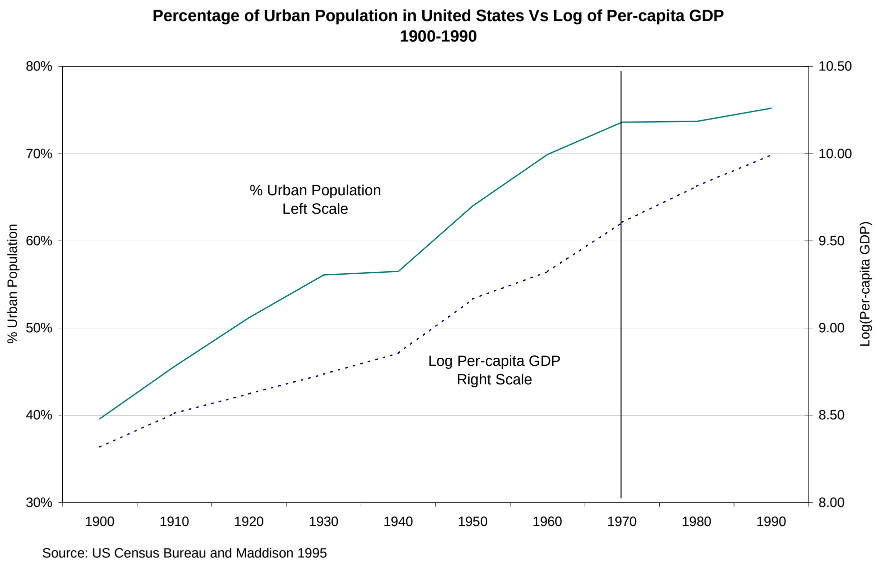

Figure 9: Urban Share of Population Vs GDP Percapita

with a capital share of around 1/3. If the true production function is given by (5), then those exercises produce spurious series of productivity.

Let $\widehat{\alpha}$ be the estimated capital elasticity of output (usually 1/3), y^p per-capita income, and k^p per-capita capital stock. The Solow residuals (in levels) can be computed as

$$R_t = \frac{y_t^p}{(k_t^p)^{\widehat{\alpha}}} = \frac{y_t e^{xt}}{(k_t e^{xt})^{\widehat{\alpha}}} = \frac{y_t}{(k_t)^{\widehat{\alpha}}} e^{(1-\widehat{\alpha})xt}.$$

If the true production function is Cobb-Douglas with capital share $\widehat{\alpha}$, then Solow residuals are a correct measure of productivity: $R_t = e^{(1-\widehat{\alpha})xt}$. If, instead, the true production function is given by equation (5) and $\underline{k} = 0$ (to simplify), then the Solow residuals become

$$R(k_t, \widehat{\alpha}) = \left\{ \begin{array}{ll} (k_t)^{-\widehat{\alpha}} (A_1 + f_k^2(\overline{k}) \cdot k_t) e^{(1-\widehat{\alpha})xt} & \text{for } 0 \leq k_t \leq \overline{k} \\ (k_t)^{\alpha-\widehat{\alpha}} e^{(1-\widehat{\alpha})xt} & \text{for } k_t \geq \overline{k} \end{array} \right\},$$

and the growth rates of the residual are given by

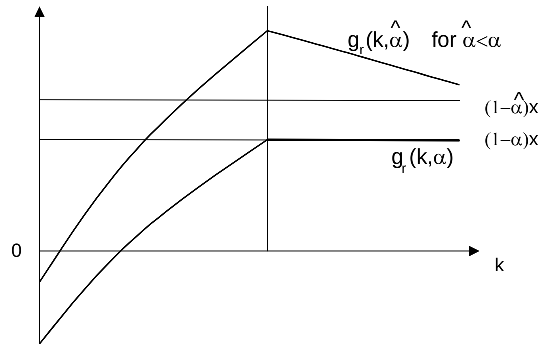

Figure 10: Productivity Slowdown

$$\left(\frac{\partial}{\partial t}\log R\right)(k_t,\widehat{\alpha}) = \left\{ \begin{array}{ll} -\left(\widehat{\alpha} - f_k^2(\overline{k})\frac{k_t}{y_t}\right)\dot{k}_t + (1-\widehat{\alpha})x & \text{for } 0 \leq k_t \leq \overline{k} \\ (\alpha - \widehat{\alpha})\dot{k}_t + (1-\widehat{\alpha})x & \text{for } k_t \geq \overline{k} \end{array} \right\}. \tag{17}$$

Figure 10 illustrates this equation and the following lemma summarizes its properties. Note that $(1-\alpha)x$ is the rate of "Hicks neutral" technological progress.

- **Lemma 7** *(Growth of the Solow Residual) Let* $g^r(k_t,\widehat{\alpha}) \equiv \frac{\partial}{\partial t}\log R_t$. *Then, (i)* $g_t^r(k,\widehat{\alpha})$ *is continuous in* k*; (ii)* $g_t^r(k,\widehat{\alpha})$ *is increasing in* k *for* $0 \leq k_t \leq \overline{k}$*; (iii) given* $\widehat{\alpha}$ *and* $k_t \geq \bar{k}$, $g_t^r(k_t,\widehat{\alpha})$ *converges monotonically toward* $(1-\widehat{\alpha})x$*; (iv)* $g_t^r(\overline{k},\widehat{\alpha}) > (1-\widehat{\alpha})x$ *for* $\widehat{\alpha} < \alpha$. *Also,* $g_t^r(\bar{k},\widehat{\alpha})$ *is decreasing in* $\widehat{\alpha}$*; (v) if* $\widehat{\alpha} = \alpha$, $g_t^r(\overline{k},\widehat{\alpha}) = (1-\widehat{\alpha})x$, *for* $k_t \geq \overline{k}$*;*

  Proof. Continuity in part (i) follows from the fact that $f_k^2(\overline{k})\frac{\bar{k}}{y} = \alpha$; $g_t^r(k_t)$ increasing for $0 \leq k_t \leq \overline{k}$ follows from the facts that k/y is strictly increasing (Proposition 2) and $\dot{k}_t$ is decreasing for that interval so that $\left(\widehat{\alpha} - f_k^2(\overline{k})\frac{k_t}{y_t}\right)\dot{k}_t$ is decreasing. The other properties follow from direct inspection of equation (17) and the fact that $\dot{k}_t$ monotonically converges toward zero. ■

From the lemma, it is clear that a "productivity slowdown" can be observed if the capital share on income, $\widehat{\alpha}$, is below the true capital elasticity of output, α. This is precisely a point made by the recent growth literature. Several papers have suggested that α is larger than 0.5 if the role of human capital is considered (e.g. King and Rebelo (1990), Mankiw *et al.* (1992), Barro and Sala-I-Martin (1990), among others). If that is the case, the model can easily explain a growth slowdown accompanied by a wrongly measured 'productivity slowdown'. The following proposition summarizes the discussion and follows from the last lemma.

Proposition 8 *("Productivity Slowdown"). Let* $\widehat{\alpha} \leq \alpha$. *Then* $g^r(k_t, \widehat{\alpha})$ *is strictly increasing for* $0 \leq k_t \leq \overline{k}$ *and decreasing for* $k_t > \overline{k}$.

6 Extensions

In this section we extend the model in two dimensions. First, we introduce land into the model and provide a justification for interpreting labor in the advanced sector as urban labor. According to this explanation, workers in the advanced sector agglomerates around the most productive piece of land. The second extension is designed to account for another feature of the structural change: the change in the composition of output. For this purpose we introduce agricultural and non-agricultural goods into the model. We show that the extended model can account for changes in the composition of output and for the decay of the agricultural labor share, even under homothetic preferences.

6.1 Land and Agglomeration

Given the structure of the model, we would like to interpret l_1 as the share of rural population, and l_2 as the share of urban population. For that purpose, we introduce space into the model and induce agglomeration in a way that allows the results of the previous section to remain unaffected.

Suppose land is homogeneously spread along a straight line of length 1, although any other spatial configuration is equally useful. Locations are denoted by j, for $j \in [0,1]$ and each location has one unit of land. Land is another factor of production, homogenous from the perspective of the backward sector (for example, all land is equally fertile), but heterogeneous from the perspective of the advanced one. In particular, land productivity is highest at location j^* (due, for example, to the existence of a port or natural resource) and it smoothly decreases with the distance from j^*. Thus, efficiency dictates that the production of the advanced sector must agglomerate around j^*.

More precisely, suppose the backward technique requires one unit of land per worker, i.e., $F^1(K_1, A_1(L_1, T_1)) = F^1(K_1, A_1 \min\{L_1, T_1\})$, where T_1 stands for land. Assume also that there is enough land so that labor is the limiting factor. In this case, $F^1(K_1, A_1L_1, T_1)$ $= F^1(K_1, A_1L_1)$ as before, and labor in the backward sector can be considered rural since it is spread along the line.

The advanced technique, on the other hand, requires $\epsilon < 1$ units of land per worker. Production at location j with the advanced technique is given by $F^2(K_2^j, A_2^j(L_2^j, T(j))) =$ $F^2(K_2^j, A_2^j \min\left\{L_2^j, 1/\epsilon\right\})$. For ϵ sufficiently close to 0, all labor in the advanced sector agglomerates at j^*. Total advanced production is $F^2(K_2^{j^*}, A_2^{j^*} L_2^{j^*})$, as in the previous section. The advantage of this formulation is that we can now interpret labor in the advanced sector as urban labor because it agglomerates at a single location.

This approach to agglomeration retains the results of the previous section and preserves the efficiency of the competitive equilibrium. It is supported by the evidence that most cities are located in places with particular geographic advantages, such as coasts, rivers, minerals, or fertile soil (e.g. Fujita *et al* (1999)). It is also supported by the evidence about city-size distribution (See Córdoba, 2002).

An alternative to assumming a best location is to allow for externalities in the advanced sector, as in Henderson (1974). According to this approach, factors are more productive working together than apart. The competitive equilibrium in this case is not efficient in general; but the qualitative results obtained above can still apply. To see this, suppose F^2 is given by $F^2(K_2^j, K_2^{A,j}, L_2^j)$, where K_2^j is individual capital, $K_2^{A,j}$ is aggregate capital in the advanced sector at location j, and F^2 is linear homogenous in its three arguments. A positive externality is introduced by making F^2 increasing in K_2^A. In a competitive equilibrium, marginal products must be equal as long as techniques are combined, i.e.

$$F_l^1(k_{1t},1) = F_l^2(k_{2t},k_{2t}^A,1)$$
$$F_k^1(k_{1t},1) = F_k^2(k_{2t},k_{2t}^A,1),$$

where k_1, k_2 and k_2^A are capital-labor ratios in each technique, and F_k^2 is the partial derivative of F^2 with respect to private capital. Aggregate constraints also impose $k_{2t} = k_{2t}^A$ as all firms using F^2 locate together because of the externality[10]. The two previous equalities thus form a system of two equations and two unknowns, k_1 and k_2, that can be solved independently of the aggregate capital-labor ratio. This is the mechanism that gives rise to a production function with linear segments, as shown in the previous section.

6.2 Multiple goods

We can extend the model to capture changes in the composition of output, and to allow for non-zero rural population in the steady state. In particular, we aim to capture an increasing share of manufactured output and a decreasing share of agriculture output. For notational ease, suppose there is no exogenous technological progress, no population growth, and total labor equals 1. Suppose there are two essential goods in the economy: an agricultural good and a manufactured good. The manufacturing sector in this model supplies consumption and investment goods, while the agricultural sector only supplies consumer goods. A convenient but not essential assumption is to suppose that both goods can be produced with exactly the same technique. Let $F^1(K,L) = A_1 L$ and $F^2(K,L) = A_2 K^\alpha L^{1-\alpha}$. Since both goods can be produced with the same techniques then the relative price of the goods is 1[11]. Let c_A and c_M be the consumption of agricultural and manufactured good respectively and let the instantaneous utility function be given by

$$u(c_A,c_M) = \frac{1}{1-\sigma}\left((c_A - \underline{c})^{1-\gamma}\, c_M^\gamma\right)^{1-\sigma},$$

where $\underline{c}$ is a minimum consumption requirement and it satisfies $A_1 > \underline{c} \geqq 0$. Equating

[10] This is not the only possible equilibrium agglomeration, but it is the only one robust to sensible refinements.

[11] This fact can be seen by equating marginal products of capital and labor across sectors.

the relative price to the marginal rate of substitution we have

$$1 = \frac{u_{c_A}(c_A, c_M)}{u_{c_M}(c_A, c_M)} = \frac{1-\gamma}{\gamma}\frac{c_M}{c_A - \underline{c}}. \tag{18}$$

Let l_a be the share of total labor in agriculture. We want to see what happens to l_a as the economy grows. Before the industrial revolution, $c_A = A_1 l_a$ and $c_M = A_1(1 - l_a)$. Substituting these two equalities into (20) and solving from l_a we obtain

$$l_a = 1 - \gamma + \frac{\gamma \underline{c}}{A_1}.$$

Now consider a steady state of this economy in which only F^2 is used. Such a steady state exists if and only if $k^* > \overline{k}$ where $\overline{k} = \left(\frac{A_1}{(1-\alpha)A_2}\right)^{\frac{1}{\alpha}}$. Efficiency implies that the same capital-labor ratio, k^*, is used in both sectors. Note also that k^* is the aggregate level of capital. Therefore, $c_A = A_2 k^{*\alpha} l_a^*$ and $c_M = A_2 k^{*\alpha}(1 - l_a^*) - \delta k^*$. Substituting the two last expressions into (18) and solving for l_a^* :

$$\begin{aligned} l_a^* &= (1-\gamma)\left(1 - \frac{\delta k^*}{A_2 k^{*\alpha}}\right) + \gamma \frac{\underline{c}}{A_2 k^{*\alpha}} \\ &< (1-\gamma) + \gamma \frac{\underline{c}}{A_2 k^{*\alpha}} \\ &< (1-\gamma) + \gamma \frac{\underline{c}}{A_1} = l_a \end{aligned}$$

The last inequality follow from requiring $k^* > \overline{k}$ so that the advanced technique is adopted. In that case $A_2 k^{*\alpha} > A_2 \overline{k}^{\alpha} = \frac{A_1}{(1-\alpha)} > A_1$. Therefore, economic growth in this model result in a lower the labor share in agriculture and a larger labor share in the manufacturing sector. This is true even if $\underline{c} = 0$, i.e., even with homothetic preferences. The shift in the composition of output in this case is driven by the fact that capital is a manufactured good. In this case. $l_a - l_a^* = (1-\gamma)\frac{\delta k^*}{A_2 k^{*\alpha}} = (1-\gamma)\delta \frac{k^*}{y^*}$. Suppose large values for $\frac{k^*}{y^*}$, δ and γ so that we can find an upper bound for $l_a - l_a^*$. Let $\frac{k^*}{y^*} = 4$, $\delta = 10\%$ and $\gamma = 0.5$. Then, $l_a - l_a^* = 0.2$. These computations suggest that a significant amount of structural change can be explained with homothetic preferences. However it also suggests that some non-homotheticity is required.

7 Conclusions

We show that capital-biased technological inventions can induce adjustment dynamics that are surprisingly consistent with very diverse and essential regularities of economic growth. First, it is consistent with most of the Kaldor facts: the interest rate, growth, and labor share are constant, and the capital-labor ratio is increasing along the adjustment path. Second, the transition displays a gradual reallocation of labor from the backward sector to the advanced one, which is consistent with cross-country evidence on urbanization and structural change. Third, the model predicts that a growth slowdown takes place once the structural change is completed, a prediction consistent with important evidence about growth, productivity slowdown, and urban shares. Fourth, the model can also explain the Kuznets curve (Glachant (2000)). Fifth, the model predicts that the capital-output ratio increases, a prediction that conflicts with one of the Kaldor facts but that is strongly supported by the empirical evidence. Finally, the model provides an important amplification mechanism: small changes in preferences or technologies can induce large differences in per-capita income (Zeira (1998)).

We mantain that these features of the model support the use of capital-biased inventions in growth models of the type considered in this paper. These types of inventions induce more realistic transitional dynamics that complement the standard labor-augmenting technological progress. We also regard the explanation proposed in this paper for the structural change as complementary to the more traditional explanation based on Engel's law. We find that in order to account for the main features of the data, preferences may not be homothetic, and technologies may not be strictly concave in each input. Lastly, our approach provides support for the use of $AK + BL$ technologies in growth models. They naturally arise when technological progress is dramatic and biased.

8 References

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9 Appendix 1.1

Proposition 1 is proved through a series of lemmata. To reduce notation, assume no technological progress nor population growth and $A_1 = A_2 = 1$. Define $G : [0, K] \times [0, L] \to \Re^+$ by

$$G(K_1, L_1; K, L) \equiv F^1(K_1, L_1) + F^2(K - K_1, L - L_1)$$

Let $D(K, L)$ be the domain of $G : D(K, L) = [0, K] \times [0, L]$. Define H as the arg max of G:

$$H(K, L) = \left\{ \begin{array}{c} (K_1^*, L_1^*) : G(K_1^*, L_1^*) \geq G(K_1, L_1) \\ \text{for } \forall (K_1, L_1) \in D(K, L) \end{array} \right\}$$

G and D satisfy the assumptions of Weierstrass Theorem so that $H(K, L)$ is not empty. Note that $F(K, L) = \max\limits_{0 \leq K_1 \leq K, 0 \leq L_1 \leq L} G(K_1, L_1; K, L)$. Let $MRTS^i(k) = \frac{F_L^i(k,1)}{F_K^i(k,1)}$ be the marginal rate of technical substitution for technique $i = 1, 2$.

Lemma 9 *Let $K/L = \widetilde{k}$. Then, (i) $F_L^1(K, L) > F_L^2(K, L)$, $F_K^2(K, L) > F_K^1(K, L)$ and $MRTS^1(K/L) > MRTS^2(K/L)$; (ii) $F(K, L) > F^1(K, L)$ and $F(K, L) > F^2(K, L)$ (it is optimal to use both techniques).*

Proof. (i) From Assumption 1.2, $F^2(K + \Delta K, L) > F^1(K + \Delta K, L)$ for $\Delta K > 0$. Since the $F^i()$'s are linear homogenous and differentiable, it follows that $F_K^2(\widetilde{k}, 1) = F_K^2(K, L) > F_K^1(K, L) = F_K^1(\widetilde{k}, 1)$. In the same vein, $F_L^1(\widetilde{k}, 1) = F_L^1(K, L) > F_L^2(K, L) = F_L^2(\widetilde{k}, 1)$.

(ii) By contradiction, suppose $F(K, L) = F^1(K, L)$ (a similar argument holds for $F(K, L) = F^2(K, L)$). Take from F^1 factors $\Delta K (> 0)$ and $\Delta L (> 0)$ satisfying $\frac{\Delta K}{\Delta L} > \frac{F_L^1(\widetilde{k},1) - F_L^2(\widetilde{k},1)}{F_K^2(\widetilde{k},1) - F_K^1(\widetilde{k},1)} > 0$ and transfer them into F^2. The last inequality follows from part (i). Then, for ΔK and ΔL sufficiently small, the change in total output is given by $\left(F_K^2(\widetilde{k}, 1) - F_K^1(\widetilde{k}, 1)\right) \Delta K + \left(F_L^2(\widetilde{k}, 1) - F_L^1(\widetilde{k}, 1)\right) \Delta L > \left(F_L^1(\widetilde{k}, 1) - F_L^2(\widetilde{k}, 1) + F_L^2(\widetilde{k}, 1) - F_L^1(\widetilde{k}, 1)\right) \Delta L = 0$. Then F^1 is not optimal. A contradiction. ■

Lemma 10 *Let $(K_1^*, L_1^*) \in H(K, L)$. (i) If $L > L_1^* > 0$ then $\frac{K_1^*}{L_1^*} < \widetilde{k} < \frac{K - K_1^*}{L - L_1^*}$. (ii) If $L_1^* = 0$, then $\frac{K}{L} > \widetilde{k}$. (iii) If $L_1^* = L$, then $\frac{K}{L} < \widetilde{k}$*

Proof. (i) By contradiction suppose $\frac{K_1^*}{L_1^*} \geq \widetilde{k}$. Then, by Assumption 1.2, $F^2(K_1^*,L_1^*) \geq F^1(K_1^*,L_1^*)$. If $F^2(K_1^*,L_1^*) > F^1(K_1^*,L_1^*)$ then (K_1^*,L_1^*) is not optimal (a contradiction). If $F^2(K_1^*,L_1^*) = F^1(K_1^*,L_1^*)$ then $\frac{K_1^*}{L_1^*} = \widetilde{k}$. But from Lemma 9 $F(K_1^*, L_1^*) > F^1(K_1^*,L_1^*)$ (a contradiction). A similar argument can be used to prove that $\widetilde{k} < \frac{K-K_1^*}{L-L_1^*}$.

(ii) Suppose, by contradiction, that $\frac{K}{L} \leq \widetilde{k}$. Since $L_1^* = 0$ then $K_1^* = 0$ and $F(K,N) = F^2(K,N)$. However, by Assumption 1.2, $F^2(K,N) \leq F^1(K,N)$. If the inequality is strict then $L_1^* = 0$ and $K_1^* = 0$ is not optimal (a contradiction). If a equality prevail, then $K/N = \widetilde{k}$ so that by Lemma 9 $F(K,N) > F^2(K,N)$ (A contradiction).

(iii) Similar to part (ii). ■

Lemma 11 *$G(K_1, L_1; K, L)$ is concave in (K_1, L_1). It is strictly concave in $(K_1, L_1) \in$ $intD(K,L)$ for $K_1/L_1 \neq (K - K_1)/(L - L_1)$.*

Proof. The Hessian associated to G is given by

$$H = \begin{bmatrix} F^1_{KK} + F^2_{KK} & F^1_{KL} + F^2_{KL} \\ F^1_{LK} + F^2_{LK} & F^1_{LL} + F^2_{LL} \end{bmatrix}$$

where $F^i_{XY} = \frac{\partial}{\partial Y}\left(\frac{\partial F^i(X,Y)}{\partial X}\right)$. The terms in the diagonal are strictly negative by Assumption 1.1. Thus, G is concave if $|H| \geq 0$ and strictly concave if $|H| > 0$. By Young's Theorem and the fact that G is C^2, $|H| = \left(F^1_{KK} + F^2_{KK}\right)\left(F^1_{LL} + F^2_{LL}\right) - \left(F^1_{KL} + F^2_{KL}\right)^2$. Linear homogenous functions have the property that $F^i_{KK}F^i_{LL} - \left(F^i_{KL}\right)^2 = 0$. Therefore, $|H| = F^1_{KK}F^2_{LL} + F^2_{KK}F^1_{LL} - 2F^1_{KL}F^2_{KL}$. Other properties of linear homogenous functions are $F^i_{LL} = \left(\frac{K_i}{L_i}\right)^2 F^i_{KK}$ and $F^i_{KL} = -\frac{K_i}{L_i}F^i_{KK}$. Using these two properties into the previous expression we get

$$\begin{aligned} |H| &= F^1_{KK}\left(\frac{K_2}{L_2}\right)^2 F^2_{KK} + F^2_{KK}\left(\frac{K_1}{L_1}\right)^2 F^1_{KK} - 2\frac{K_1}{L_1}\frac{K_2}{L_2}F^1_{KK}F^2_{KK} \\ &= F^1_{KK}F^2_{KK}\left[\left(\frac{K_2}{L_2}\right)^2 + \left(\frac{K_1}{L_1}\right)^2 - 2\frac{K_1}{L_1}\frac{K_2}{L_2}\right] \\ &= F^1_{KK}F^2_{KK}\left[\left(\frac{K_2}{L_2}\right) - \left(\frac{K_1}{L_1}\right)\right]^2 > 0 \text{ for } \left(\frac{K_2}{L_2}\right) \neq \left(\frac{K_1}{L_1}\right) \end{aligned}$$

■

Lemma 12 *$H(K,N)$ is singlenton.*

Proof. Let $(K_1^*, L_1^*) \in H(K,L)$. (i) If $L > L_1^* > 0$ then $\frac{K_1^*}{L_1^*} \neq \frac{K-K_1^*}{L-L_1^*}$ from Lemma 10. Using this fact and the previous Lemma 11, it follows that (K_1^*, L_1^*) is a unique local maximizer. Since G is concave it also follows that (K_1^*, L_1^*) is the unique global maximizer. (ii) If $L_1^* = 0$, then $F(K,N) = F^2(K,N)$. This must be the only solution. Suppose not. Then $\exists$ $(K_1^{*\prime}, L_1^{*\prime}) \neq (K_1^*, L_1^*)$ such that $F(K,N) = G(K_1^{*\prime}, L_1^{*\prime}) = G(K_1^*, L_1^*)$. From the previous result $L_1^{*\prime} \notin (0, L)$. Otherwise $(K_1^{*\prime}, L_1^{*\prime})$ would be the unique maximizer. Then there are two alternatives (a) $K_1^{*\prime} = K$ and $L_1^{*\prime} \in \{0, L\}$ which implies $L_1^{*\prime} = L$; (b) $0 \leq K_1^{*\prime} \leq K$ and $L_1^{*\prime} = L$ which implies $K_1^{*\prime} = K$; In both cases it follows that $G(K_1^{*\prime}, L_1^{*\prime})$ $= F^1(K,L) = F^2(K,L) = F(K,L)$. A contradiction. ■

Let $K_1(K,N)$ be the first component of $H(K,N)$ and $L_1(K,N)$ the second component. Define $K_2(K,N) \equiv K - K_1(K,N)$ and $L_2(K,N) \equiv L - L_1(K,N)$. When (K,N) is well defined by the context we just write K_1^* instead of $K_1(K,N)$, etc.

Define

$$\begin{aligned} \underline{k} &= \frac{K_1(\widetilde{k},1)}{L_1(\widetilde{k},1)} \\ \overline{k} &= \frac{K_2(\widetilde{k},1)}{L_2(\widetilde{k},1)} \end{aligned}$$

$\underline{k}$ and $\overline{k}$ are well defined since $L > L_1(\widetilde{k},1) > 0$ (Lemma 9). Also note that $\underline{k}$ can be equal to zero due to the fact that capital is not essential to produce with technique 1.

The following result follows from Lemma 9 and efficiency considerations:

Lemma 13 *(i)* $\underline{k} < \overline{k}$ *(ii)* $F_L^1(\underline{k},1) = F_L^2(\overline{k},1)$ *and* $F_K^1(\underline{k},1) \leq F_K^2(\overline{k},1)$ *with equality if* $\underline{k} > 0$.

Proof. (i) Follows directly from the fact that $\underline{k} < \widetilde{k} < \overline{k}$ and (ii) follow from the fact that the solution is interior. (The marginal product of labor across techniques must be equal since labor is essencial to produce with either technique. The marginal product of capital need not to be equal in an efficient allocation since capital is not essencial to produce with the TT technique.) ■

Lemma 14 *Let* $\frac{K}{L}$ *be such that* $\overline{k} > \frac{K}{L} > \underline{k}$. *Then, (i)* $\frac{K_1(K,L)}{L_1(K,L)} = \underline{k}$, $\frac{K_2(K,L)}{L_2(K,L)} = \overline{k}$, *(ii)* $L_1(K,L) = \left(\frac{\overline{k}-K/L}{\overline{k}-\underline{k}}\right) L$ *and* $L_2(K,L) = \left(\frac{K/L-\underline{k}}{\overline{k}-\underline{k}}\right) L$, *(iii)* $F(K,L) = F_K^2(\overline{k},1)K + F_L^2(\overline{k},1)L$

Proof. (i) We just need to check that the proposed allocation satisfies the Kunh-Tucker first-order conditions (Theorem 7.16, Sundaram). Sufficiency of the first order conditions is assured by concavity of G. Uniqueness is assured by strict concavity of G at the proposed allocation. The K-T first-order conditions of the problem under the proposed allocation are:

(a) $F_L^1(\frac{K_1(K,L)}{L_1(K,L)},1) = F_L^2(\frac{K_2(K,L)}{L_2(K,L)},1)$

(b) $F_K^1(\frac{K_1(K,L)}{L_1(K,L)},1) \leq F_K^2(\frac{K_2(K,L)}{L_2(K,L)},1)$ with equality if $K_1(K,L) > 0$.

(c) $K = K_1(K,L) + K_2(K,L)$

(d) $L = L_1(K,L) + L_2(K,L)$

By construction, $\frac{K_1(K,L)}{L_1(K,L)} = \underline{k}$ and $\frac{K_2(K,L)}{L_2(K,L)} = \overline{k}$ satisfy (a) and (b) since $\underline{k}$ and $\overline{k}$ are the solutions for $K/L = \widetilde{k}$. Note also that $K_1(K,L) + K_2(K,L) = \underline{k}L_1(K,L) + \overline{k}L_2(K,L) = \underline{k}(1 - \frac{K/L-\underline{k}}{\overline{k}-\underline{k}})L + \overline{k}\frac{K/L-\underline{k}}{\overline{k}-\underline{k}}L = K$ and $L_1 + L_2 = L$.

(ii) $K = K_1(K,L) + K_2(K,L) = \underline{k}L_1(K,L) + \overline{k}(L - L_1(K,L))$. Then $L_1(K,L) = \frac{\overline{k}-K/L}{(\overline{k}-\underline{k})}L$

(iii)$F(K,L) = F^1(\underline{k},1)L_1(K,L) + F^2(\overline{k},1)(L - L_1(K,L)) = \left(F^1(\underline{k},1) - F^2(\overline{k},1)\right) L_1(K,L) + F^2(\overline{k},1)L =$

$\left(F_K^1(\underline{k},1)\underline{k} - F_K^2(\overline{k},1)\overline{k}\right) L_1(K,L) + F^2(\overline{k},1)L =$

$\left(F_K^1(\underline{k},1)\underline{k} - F_K^2(\overline{k},1)\overline{k}\right) \frac{\overline{k}-K/L}{(\overline{k}-\underline{k})}L + F^2(\overline{k},1)L =$

There are two cases: (a) $\underline{k} > 0$. Then $F_K^1(\underline{k},1) = F_K^2(\overline{k},1)$ so that $F(K,L) = F_K^2(\overline{k},1)K$ + $\left(F^2(\overline{k},1) - F_K^2(\overline{k},1)\overline{k}\right) L$;(b) $\underline{k} = 0$. Then the same result follows. ∎

Lemma 15 *(i)* $F(K,L) = F^1(K,L)$ *for* $K/L \leq \underline{k}$ *(ii)* $F(K,L) = F^2(K,L)$ *for* $K/L \geq \overline{k}$.

Proof. We only prove part (ii). To prove part (i) similar arguments can be used. Note that $\frac{K_2^*}{L_2^*} \geq \overline{k}$. Otherwise $F(K_2^*, L_2^*) > F^2(K_2^*, L_2^*)$ so that (K_2^*, L_2^*) would not be optimal. Suppose by contradiction (of ii), that $L_1^* > 0$. Then $F_L^1(\frac{K_1^*}{L_1^*},1) = F_L^2(\frac{K_2^*}{L_2^*},1)$. If $\frac{K_2^*}{L_2^*} = \overline{k}$ then $\frac{K_1^*}{L_1^*} = \underline{k}$ but then $(K_1 + K_2)/L = \underline{k}\frac{L_1}{L} + \overline{k}\frac{L_2}{L} \leq \overline{k} < K/L$ so that there are unemployed resources. This cannot be optimal. Therefore, $\frac{K_2^*}{L_2^*} > \overline{k}$. Since marginal products of labor need to be equal, it follow that $\widetilde{k} > \frac{K_1^*}{L_1^*} > \underline{k}$, and from Lemma 14 $F(K_1^*, L_1^*) > F^1(K_1^*, L_1^*)$ so that (K_1^*, L_1^*) is not optimal. Therefore, (ii) must be true. ∎