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Assignment Maximization

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Disciplines

Economic Theory | Education Policy

ASSIGNMENT MAXIMIZATION

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JEL classification: D47, C78, D63.

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1. Introduction

In discrete assignment problems, a large variety of properties have been proposed and implemented. These include Pareto efficiency,[1] various notions of fairness and stability,[2] distributions of agents within schools,[3] and doctors across hospitals,[4] among many others. In this paper, we consider instead the objective of maximizing the number of individuals assigned to objects that they deem as acceptable.

This is an important and natural design objective in many practical domains, such as school choice. Abdulkadiroğlu et al. (2005), for example, describe the change in New York City's high schools' matching program. One of the main problems identified was that the normal process would leave a large proportion of the students unmatched, and would end up assigning them via an administrative process to schools which were not

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[1] Shapley and Scarf (1974)

[2] Roth (1984a); Balinski and Sönmez (1999); Abdulkadiroglu and Sönmez (2003); Kojima et al. (2013)

[3] Ehlers et al. (2014); Fragiadakis and Troyan (2017)

[4] Kamada and Kojima (2015)

necessarily among those stated in their preferences. In fact, they show that 30,000 out of 100,000 students were assigned in this way in 2002.[5] Data from the New Orleans OneApp, another centralized school choice program, show that an average of 20% of the applicants remained unmatched after the main assignment round (Harris et al., 2015). In these examples as well as in many others, school districts and governments don't simply leave students without schools while seats are left empty. Instead, additional rounds of the school choice mechanisms and other ad-hoc administrative assignments are used to match the remaining students. These additional rounds, however, in general eliminate many of the incentive and fairness characteristics that justify the use of those mechanisms in the first place (Dur and Kesten, 2014). In this paper, we start from the assumption that the policy maker wants to minimize the number of students left unmatched, and evaluate to what extent the usual welfare, incentive, and fairness objectives can be achieved, and how.

Having to go through the additional processes used to assign students who are not matched in the main process can also cause frustration and emotional stress, as shown in the quote below:

> *"(...)The High School application process is a nerve wrecking nightmare and extremely unfair to single parents, new immigrant families and any other families who simply cannot put in the countless hours it takes to attend Open Houses, tours and fairs. We got lucky and our daughter got into a school of her choice, but my heart goes out to the families who have to go through this process twice." (Tine Kindermann)* [6]

From the perspective of policymakers, leaving students unassigned, even temporarily, may have serious consequences. In 2013, for example, the city of São Paulo (Brazil) was ordered by a state court to pay restitution to 943 parents who had to put their children in temporary private childcare, as a result of remaining unmatched by the city's assignment process.[7] Maximizing the number of assignments might in fact be the primary objective of the assignment process, as indicated by the following quote from the Frankfurt secondary school district and North Rhine-Westphalia secondary school district:

> *"The organization of the "Frankfurt School Mechanism" is shared between State, city and school. Its primary goal is to give as many applicants as possible one of their preferred schools. Each school decides for itself which students to admit..." (Basteck et al., 2015)*

School choice is not the only practical problem where assignment maximization matters. It is perhaps the most important objective in organ exchange programs, as evidenced by

[5]Even after a change in the mechanism, proposed by the authors, the number of students who remained unmatched was still about 7,600, requiring additional elicitation of preferences over what are supposedly undesired schools.

[6]*Source: http://www.nytimes.com/2011/05/08/nyregion/in-applying-for-high-school-some-8th-graders-find-a-maze.html (NYT selected comments, accessed 09/11/2017.)*

[7]Source: http://g1.globo.com/sao-paulo/noticia/2013/05/mae-ganha-direito-de-indenizacao-apos-ficar-sem-vaga-para-o-filho-em-creche.html (in Portuguese, accessed *09/11/2017.)*

the recent literature on those types of mechanisms. For both kidney exchange (Roth et al., 2005) and lung exchange *(Ergin et al., 2017)*, the objective of maximizing the number of matchings (and therefore transplants), is put first and foremost in the design of their mechanisms.

Another area in which maximizing the number of assignments is relevant and has raised significant interest on the part of market designers is in the matching of asylum seekers to countries or states. Andersson and Ehlers (2018), for example, propose an algorithm to find maximum mutually acceptable matchings[8] which are also stable. Other examples of applications in which matching maximization is relevant include the matching of babies to nurseries (Sasaki and Ura, 2016) and public housing.

In this study, we consider the economic problems faced by a policymaker who wants to produce maximal matchings deterministically, when agents have strict preferences over their outcomes. Consider the problem of assigning students to schools. The reason why efficiency and stability (or equivalently, fairness) may conflict with maximizing the number of matches is that some schools may be deemed unacceptable to some students. As a result, there may be some Pareto efficient and/or stable matchings that do not maximize assignments. Consider, for example, the case in which there are two schools (A and B), each with only one seat, and two students (1 and 2). Student 1 only deems A as acceptable, whereas student 2 simply prefers A to B. In this case, student 2 being matched to A and 1 remaining unmatched is a Pareto efficient assignment. Moreover, if student 2 has higher priority at school A than student 1, that is also the unique stable assignment. Therefore, there may typically be Pareto efficient and stable matchings that can be significantly improved upon in terms of the number of assignments.[9]

We set the maximization of the number of assignments as our primary design goal. We show that maximizing the number of assignments is incompatible not only with strategy-proofness, but also with fairness (Proposition 1), and that no mechanism is maximal in equilibrium (Proposition 7). While these can be interpreted as strong negative results, we present a large set of proposals and analyses.

First, we design a family of mechanisms, denoted *Efficient Assignment Maximizing Mechanisms* (EAMs), that are Pareto efficient and maximal in terms of the number of assignments (Theorem 1). Due to the impossibility above, EAMs are not strategy-proof, but we characterize the unique Nash equilibrium outcome, which is Pareto efficient (Proposition 6). Moreover, EAMs are not dominated (in terms of the number of assignments) by any other mechanism in equilibrium (Theorem 3).

While assignment maximality and fairness are incompatible, we show that a weaker version of fairness is compatible. We say that an outcome is *fair for unassigned students* if there is no situation in which an unassigned student justifiably envies the assignment

[8]A refugee family and a landlord are *mutually acceptable* if they have a language in common and the number of beds offered by the household exceeds the number of beds needed by the refugee family.

[9]The efficiency cost of stability has been pointed out before in the literature. See Abdulkadiroglu and Sönmez (2003) and Kesten (2010).

of some other agent. We define another family of mechanisms, denoted *Fair Assignment Maximizing mechanisms* (FAMs), which maximize the number of assignments and are fair for unassigned students (Theorem 2). Interestingly, a tradeoff between fairness and efficiency also emerges for this weaker notion of fairness (Proposition 5). Moreover, while EAMs are also Pareto efficient in equilibrium, we show that FAMs produce at least the same number of assignments as the problem's stable matchings in equilibrium (Theorem 4). We also show that, for any maximal mechanism, the induced preference-reporting game in which some proportion of agents are truthful and non-strategic is such that, as the number of truthful agents increases, the number of students matched in equilibrium weakly increases (Proposition 8).

We also provide results regarding how well-known mechanisms compare in terms of the number of assignments made. We show that there is no dominance relation between four mechanisms used in practice and the literature (Proposition 2): Gale-Shapley Deferred Acceptance (DA),[10] Boston Mechanism (BM), Top-Trading Cycles (TTC), and Serial Dictatorship (SD).

To test the relevance of our theoretical results and see how much EAMs/FAMs improve upon well-known mechanisms in terms of number of assignments, we conduct a simulation analysis comparing the number of assignments produced by five different mechanisms – DA, BM, TTC, SD, and EAMs/FAMs. Two of the mechanisms being simulated are not strategy-proof: BM and EAMs/FAMs. As a result, the values obtained for these simulations cannot be (and are not) interpreted as outcome predictions. But they are very informative about the impact of not considering the objective of maximizing assignments, and also give a range for equilibrium outcomes. Simulations show that the difference between EAMs/FAMs and other mechanisms in terms of number of assignments is large and significant. Moreover, they show that for any choice of parameters, the number of matched students in DA, BM, TTC, and SD are very similar. Since equilibrium outcomes of AMM are equivalent to SD (Proposition 6), that equilibrium outcomes of FAM never match less students than stable mechanisms (Theorem 4), that equilibria of BM are stable (Ergin and Sönmez, 2006), and that the cardinality of the matching of equilibrium outcomes of EAMs/FAMs increase monotonically with the number of naïve students (Proposition 8), we have the indication that also when considering incentives those simulations gives us the range of outcomes for BM and EAMs/FAMs as well.

While for most of the remainder of the text we will frame the problems in terms of school choice, most of our analysis applies to the general problem of producing maximal allocations when agents have strict preferences over objects (or classes of objects) and unit demand.

The remainder of the paper is organized as follows. The relevant literature is reviewed in the next section 2, and then in section 3, we introduce the model, the mechanisms we propose, and their properties. In section 4, we show the equilibrium behavior and

[10] Or any other stable mechanism.

outcomes induced by those mechanisms, and in section 5, we present the result of computer simulations comparing mechanisms outcomes. Proofs absent from the main text can be found in the appendix.

2. Related Literature

While algorithms for finding maximum matchings are well-known (Kuhn, 1955; Berge, 1957), the research on the incentives induced by the use of these procedures is limited, and typically rely on random mechanisms. One exception is Afacan and Dur (2018), which follows-up to this paper and shows that no strategy-proof and individually rational mechanism systematically matches more students than either of Boston, Gale-Shapley deferred acceptance, and serial dictatorship mechanisms. Krysta et al. (2014) consider the problem of producing maximal matchings in a house allocation problem. They show that there is no mechanism that is deterministic, maximal, and strategy-proof, and provide instead a random mechanism that is strategy-proof and yields approximately-maximal outcomes. Bogomolnaia and Moulin (2015) evaluate the trade-off between maximality and envy-freeness, a notion of fairness that is stronger than the ones we consider in this paper. Bogomolnaia and Moulin (2004) consider the random assignment when agents have dichotomous preferences. When that is the case, Pareto efficiency is equivalent to maximality of the matching, and moreover, since agents are indifferent between all "acceptable" allocations, maximality doesn't result in incentive problems even in deterministic mechanisms. Noda (2018) studies the matching size achieved by strategy-proof mechanisms in a general model of matching with constraints.

Assignment maximization has been the primary objective in the organ exchange literature, as it means the maximum number of transplants. This literature was initiated by the seminal work on kidney exchange of Roth et al. (2004). In a subsequent study, in order to accommodate several physical and geographical restrictions in operating transplants, Roth et al. (2005) introduce the idea of pairwise kidney exchange where exchanges can only be made between two pairs. They suggest implementing the priority-based maximal matching algorithm from the combinatorial optimization literature (Korte and Vygen, 2011). The first stages of both EAM and FAM are adaptations of the priority-based maximal matching algorithm. Some other studies on organ exchanges include Sönmez et al. (2018), Andersson and Kratz (2018), Chun et al. (2018), Ergin et al. (2017), Nicoló and Rodríguez-Alvarez (2017), and Ergin et al. (2018).

Refugee reassignment is another real-world application in which maximality might be a primary design objective. Andersson and Ehlers (2018) study the problem of finding housing for refugees once they have been granted asylum. The authors propose an easy-to-implement mechanism that finds an efficient stable maximum matching. They show that such a matching guarantees that housing is efficiently provided to a maximum number of refugees and that no unmatched refugee-landlord pair prefers each other.

Our "fairness for unassigned students" is a weakening of the usual stability of Gale and Shapley (1962), therefore, the current study is also related to the surging literature on the weakening of stability in different ways. Among others, Dur et al. (2018), Afacan et al. (2017), Morrill and Ehlers (2018), and Troyan and Kloosterman (2018) are recent papers from that literature.

3. Model

A **school choice problem** consists of the following elements:

- A finite set of students $I = \{i_1, ..., i_n\}$,
- a finite set of schools $S = \{s_1, ..., s_m\}$,
- a strict priority structure for schools $\succ = (\succ_s)_{s \in S}$ where $\succ_s$ is a linear order over I,
- a capacity vector $q = (q_{s_1}, ..., q_{s_m})$ where q_s is the number of available seats at school s,
- a profile of strict preference of students $P = (P_i)_{i \in I}$, where P_i is student i's preference relation over $S \cup \{\emptyset\}$ and $\emptyset$ denotes the option of being unassigned. We denote the set of all possible preferences for a student by $\mathcal{P}$. Let R_i denote the *at-least-as-good-as* preference relation associated with P_i, that is: $sR_is' \Leftrightarrow sP_is'$ or $s = s'$. A school s is **acceptable** to i if $sP_i\emptyset$, and **unacceptable** otherwise. Let $Ac(P_i) = \{c \in S : \ cP_i\emptyset\}$.

In the rest of the paper, we consider the tuple $(I, S, \succ, q)$ as the commonly known primitive of the problem and refer to it as the **market**. We suppress all those from the problem notation and simply write P to denote the problem. A **matching** is a function $\mu : I \to S \cup \{\emptyset\}$ such that for any $s \in S$, $|\mu^{-1}(s)| \leq q_s$. A student i is **assigned** under μ if $\mu(i) \neq \emptyset$. For any $k \in I \cup S$, we denote by μ_k the assignment of k. Let $|\mu|$ be the total number of students assigned under μ.

A matching μ is **individually rational** if, for any student $i \in I$, $\mu_i R_i \emptyset$. A matching μ is **non-wasteful** if for any school s such that $sP_i\mu_i$ for some student $i \in I$, $|\mu_s| = q_s$. A matching μ is **fair** if there is no student-school pair (i, s) such that $sP_i\mu_i$, and for some student $j \in \mu_s$, $i \succ_s j$. A matching μ is **stable** if it is individually rational, non-wasteful, and fair.

In the rest of the paper, we will consider only individually rational matchings. Therefore, whenever we refer to a matching, unless explicitly stated, we refer to an individually rational matching. Let $\mathcal{M}$ be the set of matchings.

A matching μ **dominates** another matching μ' if, for any student $i \in S$, $\mu_i R_i \mu'_i$, and for some student j, $\mu_j P_j \mu'_j$. A matching μ is **efficient** if it is not dominated by any other matching. Note that efficiency implies both individual rationality and non-wastefulness.

We say that a matching μ **size-wise dominates** another matching μ' if $|\mu| > |\mu'|$. A matching μ is **maximal** if it is not size-wise dominated.[11]

A **mechanism** ψ is a systematic way of selecting a matching for every problem, that is, it is a function from $\mathcal{P}^{|I|}$ to $\mathcal{M}$. A mechanism ψ is [stable, efficient, fair, individually rational] if, for any problem $P \in \mathcal{P}^{|I|}$, $\psi(P)$ is [stable, efficient, fair, individually rational]. A Mechanism ψ is **strategy-proof** if there exist no problem P, and student i with a false preference P_i' such that $\psi_i(P_i', P_{-i}) P_i \psi_i(P)$.[12]

At first sight, the natural objective of a designer would be to find a mechanism that is fair, maximal, and strategy-proof.

Proposition 1. *Regarding maximal mechanisms:*

(i) *No fair mechanism is maximal.*

(ii) *No strategy-proof mechanism is maximal.*

The result in item (ii) in Proposition 1 was obtained by Krysta et al. (2014). Both items combined set the stage for the rest of the paper. Not only there is no strategy-proof mechanism that is maximal, but even without considering incentives, there exists a fundamental incompatibility between fairness and maximality.

Since we will focus on the number of students matched to schools, we also make use of a method for comparing mechanisms with respect to that dimension. A mechanism ψ **size-wise dominates** another mechanism ϕ if, for any problem P, $\phi(P)$ does not size-wise dominate $\psi(P)$, while, for some problem P', $\psi(P')$ size-wise dominates $\phi(P')$. A mechanism ψ is maximal if it is not size-wise dominated by any other mechanism.

3.1. **A Size-Wise Domination Comparison Among Well-Known Mechanisms.** Here we compare well-known mechanisms in terms of the number of assigned students. Namely, we consider the Gale-Shapley deferred acceptance (DA), Top Trading Cycles (TTC), Boston (BM), and serial dictatorship (SD) mechanisms. Their definitions are given in the Appendix.

Proposition 2. *There is no size-wise domination between any pair of mechanisms among the DA, TTC, BM, and SD.*

As a consequence of the rural hospitals theorem (Roth, 1984b), every stable matching assigns the same number of students to schools, and so we have the following more general result.

Corollary 1.

(i) *There is no size-wise domination between any pair of mechanisms among the class of*

[11] Notice that the notions of size domination and maximality we use is in th set of *agents* (or nodes) involved in a matching. In most of the literature in graph theory, the cardinality of a matching is measured in the set of edges of the graph that are part of the matching. While when considering the set of edges there is a difference between maximal and maximum cardinality matchings, in our setup these are equivalent: maximal matchings are always maximum.

[12] P_{-i} is the preference profile of all students except student i.

stable mechanisms, the TTC, the BM, and the SD.
(ii) None of stable, TTC, the BM, and SD mechanisms are maximal.

The results above take place based on the fact that some students might not rank all of the schools as acceptable. When that is not the case, the only reason for a student to be unassigned under these mechanisms is that all schools have been filled up, and therefore they all assign the same number of students.

Remark 1. If every school is acceptable to every student, then DA, TTC, BM, and SD all match the same number of students in any problem, consisting of the total sum of schools' capacities.

3.2. **A Class of Efficient Maximal Mechanisms.** In what follows, we first introduce two concepts which will be critical to the class of mechanisms in this section.

Definition 1. A matching μ admits an **improvement chain** at problem P if there are distinct students and schools $\{i_1, ..., i_n, c_1, c_2, .., c_{n+1}\}$ such that $|\mu_{c_{n+1}}| < q_{c_{n+1}}$ and for every $k = 1, ..n$,
(i) $\mu_{i_k} = c_k$,
(ii) $c_{k+1} P_{i_k} c_k$.

Definition 2. A matching μ admits an **improvement cycle** in problem P if there are distinct students and schools $\{i_1, ..., i_n, c_1, c_2, .., c_n, c_{n+1}\}$ such that $c_{n+1} = c_1$ and for every $k = 1, ..n$,
(i) $\mu_{i_k} = c_k$,
(ii) $c_{k+1} P_{i_k} c_k$.

We are now ready to introduce the class of mechanisms. Given a problem P and an enumeration of the students in I $(i_1, ..i_n)$,
Step 0. Let $\xi^0 = \mathcal{M}$.
Step 1.
Substep 1.1. Define the set $\xi^1 \subseteq \xi^0$ as follows:

$$\xi^1 = \begin{cases} \{\mu \in \xi^0 : \ \mu_{i_1} \neq \emptyset\} & \text{If } \exists \mu \in \xi^0 \text{ such that } \mu_{i_1} \neq \emptyset \\ \xi^0 & \text{otherwise} \end{cases}$$

In general, for every $k \leq n$,
Substep 1.k. Define the set $\xi^k \subseteq \xi^{k-1}$ as follows:

$$\xi^k = \begin{cases} \{\mu \in \xi^{k-1} : \ \mu_{i_k} \neq \emptyset\} & \text{If } \exists \mu \in \xi^{k-1} \text{ such that } \mu_{i_k} \neq \emptyset \\ \xi^{k-1} & \text{otherwise} \end{cases}$$

Step 1 ends with the selection of a matching $\mu \in \xi^n$.
Step 2.

Substep 2.1. If the matching μ does not admit an improving chain or cycle, then the algorithm ends with the final outcome of μ. Otherwise, pick a chain or cycle, and obtain a new matching by assigning each student in the chosen chain (cycle) to the school she prefers in the chain (cycle), and move to the next substep.

In general:

Substep 2.k. Let $\tilde{\mu}$ be the matching obtained in the previous round. If $\tilde{\mu}$ does not admit an improving chain or cycle then the algorithm ends with the final outcome of $\tilde{\mu}$. Otherwise, pick such a chain or cycle, and obtain a new matching by assigning each student in the chosen chain (cycle) to the school he prefers in the chain (cycle), and move to the next substep.

As everything is finite and, in every substep of Step 2, students are all weakly better off with at least one being strictly better off, Step 2 terminates after finitely many substeps. The matching obtained in the final round of Step 2 is the outcome of the algorithm. This algorithm defines a class of mechanisms, each of which is associated with different selections of the student ordering, the matching in the end of Step 1, and chains and cycles in the course of Step 2. We refer to this class of mechanisms as "Efficient Assignment Maximizing" (EAM) mechanisms.

The first step of the EAM mechanisms is a "priority mechanism", introduced by Roth et al. (2005) in the context of the pairwise kidney exchange problem. The authors show that this process finds a maximal matching. Though it may seem counterintuitive that this simple process yields a maximal matching, the intuition behind it is simple. At each step, the set of outcomes is restricted to outcomes that will match the student being considered to an acceptable school. Each one of these may lead to at most one other student remaining unmatched. Therefore, following the enumeration above and trying to match each student leads to a maximal matching.[13]

The matching produced, however, may not be efficient. To fix this, the second stage implements improving chains and cycles. As these chains and cycles are welfare-improving, the second stage preserves the maximality of the first stage outcome while benefiting the students. Consequently, every EAM mechanism is maximal and efficient.

Theorem 1. *Every EAM mechanism is maximal and efficient.*

From Proposition 1, fairness and maximality are incompatible. This, along with Theorem 1, implies that no EAM mechanism is fair. However, since maximality aims to assign as many students as possible, we may be able to satisfy a weaker notion of fairness. We say that a matching μ is **fair for unassigned students** if there is no student-school pair (i, s) where $\mu_i = \emptyset$ and $i \succ_s j$ for some $j \in \mu_s$. A mechanism ψ is fair for unassigned students if, for any problem P, $\psi(P)$ is fair for unassigned students.

[13]Notice that EAM mechanisms don't use the priorities as a constraint in the construction of the matching. This is not uncommon when these constraints are incompatible with the main objective, which in this case is maximality. In the context of refugee resettlement, for example, Delacrétaz and Teytelboym (2017) argue that in that setup priorities may be safely ignored.

Proposition 3. *No EAM mechanism is fair for unassigned students.*

Proof. Let $I = \{i, j\}$ and $S = \{a, b\}$, each with unit capacity. Let ψ be any EAM mechanism where the student ordering starts with i. Let the priorities be such that $\succ_a$: j, i and $\succ_b$: i, j. Let us first consider the following preferences: $P_i :\ a, \emptyset$ and $P_j :\ a, \emptyset$. Then, $\psi_i(P) = a$ and $\psi_j(P) = \emptyset$, violating fairness for unassigned students.

Next, consider any EAM mechanism, say ϕ, such that the student ordering starts with j. Let us now consider the preferences where $P_i : b, \emptyset$ and $P_j :\ b, \emptyset$. Then, $\phi_i(P) = \emptyset$ and $\phi_j(P) = b$, violating fairness for unassigned students. □

In the next subsection we show, however, that this weaker notion of fairness is compatible with assignment maximization, and we provide a mechanism that produces those outcomes.

3.3. **A Class of Maximal and Fair for Unassigned Students Mechanisms.** Below is a description of how each mechanism in this class works. Given a problem P,

Step 1. Pick an EAM mechanism ψ, and let $\psi(P) = \mu$.

Step 2.

Substep 2.1. If μ is fair for unassigned students then the algorithm terminates with the final outcome of μ. Otherwise, pick a student-school pair (i, s) such that $sP_i\emptyset$, $\mu_i = \emptyset$, and $i \succ_s j$ for some $j \in \mu_s$. Place student i at school s, and let the lowest priority student in μ_s be unassigned (note that since μ is maximal, we have $|\mu_s| = q_s$), while keeping everyone else's assignment the same. Let μ' be the obtained matching, and move to the next substep.

In general,

Substep 2.k. Let $\tilde{\mu}$ be the matching obtained in the previous step. If $\tilde{\mu}$ is fair for unassigned students, the algorithm terminates with the outcome $\tilde{\mu}$. Otherwise, pick a student-school pair (i, s) such that $sP_i\emptyset$, $\tilde{\mu}_i = \emptyset$, and $i \succ_s j$ for some $j \in \tilde{\mu}_s$. Place student i at school s, and let the lowest priority student in $\tilde{\mu}_s$ be unassigned, while keeping everyone else's assignment the same. Note that as in each substep the number of assigned students is preserved, $\tilde{\mu}$ is maximal. Hence, we have $|\tilde{\mu}_s| = q_s$. Let $\hat{\mu}$ be the obtained matching, and move to the next substep.

As, in every substep, a higher priority student is placed at a school while a lower priority one is displaced from the school, and both the students and schools are finite, the algorithm terminates in finitely many rounds. The above procedure defines a class of mechanisms, each of which is associated with different selections of the first stage EAM mechanism as well as the student-school pairs in the course of Step 2. We refer to this class of mechanisms as "Fair Assignment Maximizing" (FAM) mechanisms.

The procedure above is similar to the Deferred Acceptance with Arbitrary Input (DAAI) in Blum et al. (1997). Its fundamental difference from our proposal is that in the second step of a FAM, only unmatched students may fulfill their justified envies, whereas under the DAAI, students who are matched may also fulfill their justified envies. In fact, while outcomes of the DAAI mechanism are always stable, outcomes of a FAM may not be.

Theorem 2. *Every FAM mechanism is fair for unassigned students and maximal.*

Proof. Let ψ be a FAM mechanism, and μ be the outcome of its first step. As μ is the outcome of an EAM mechanism, and in Step 2 of ψ, no student is assigned to one of his unacceptable choices, ψ is individually rational. Because μ is maximal and the number of assigned students is preserved as $|\mu|$ in the course of Step 2, ψ is maximal. Moreover, as ψ does not stop until no student-school pair violates fairness for unassigned students, ψ is fair for unassigned students as well. □

The fair for unassigned students notion rules out priority violations of only unassigned students. One may wonder whether we can go beyond that while keeping maximality. The answer turns out to be very negative in the sense that if we rule out priority violations of just one particular student at one particular school, independent of the student's assignment, then we lose maximaliy, as formally shown below.

Proposition 4. *For any student i and school s, there always exists a problem P in which no matching μ is maximal and fair for that student-school pair in the sense that if $sP_i\mu_i$, then for each $j \in \mu_s$, $j \succ_s i$.*

Proof. Let us consider a problem where $I = \{i, j\}$ and $S = \{s, s'\}$, each with unit capacity. Let the preferences be such that $P_i :\ s, s', \emptyset$ and $P_j :\ s, \emptyset$. Priorities are such that $i \succ_s j$. Here, the unique maximal matching is μ where $\mu_i = s'$ and $\mu_j = s$. However, $sP_i\mu_i$ and $i \succ_s j$ where $j \in \mu_s$. □

Proposition 4 reveals that in the fairness ground, we can at most achieve fairness for unassigned students under maximality, and our FAM achieves that.

An important downside of the FAM class is the lack of efficiency, in that no FAM mechanism is efficient. However, this is not a problem specific to the FAM class as there exists a general incompatibility between efficiency and fair for unassigned students, as shown below.

Proposition 5. *No mechanism is efficient and fair for unassigned students.*

Proof. Let $I = \{i, j, k\}$ and $S = \{a, b\}$, each with unit capacity. Consider the following preferences and priorities:

$P_i :\ a, b, \emptyset$; $P_j :\ b, a, \emptyset$; $P_k :\ b, \emptyset$.

$\succ_a$: j, i, k; $\succ_b$: i, k, j.

Let ψ be an efficient mechanism, and $\psi(P) = \mu$. By the efficiency of μ, exactly one student is left unassigned.

Case 1. Suppose $\mu_k = \emptyset$. Then, by efficiency of μ, $\mu_i = a$ and $\mu_j = b$. However, as $k \succ_b j$, μ cannot be fair for unassigned students.

Case 2. Suppose $\mu_j = \emptyset$. Then, by efficiency of μ, $\mu_i = a$ and $\mu_k = b$. However, as $j \succ_a i$, μ cannot be fair for unassigned students.

Case 3. Suppose $\mu_i = \emptyset$. By efficiency of μ, $\mu_j = a$ and $\mu_k = b$. However, as $i \succ_b k$, μ cannot be fair for unassigned students. $\square$

4. Incentives and Equilibrium Analysis

As shown in Proposition 1, there is no mechanism which is maximal and strategy-proof. Hence, in particular, none of the EAM and FAM mechanisms are strategy-proof.

Corollary 2. *None of the EAM and FAM mechanisms are strategy-proof.*

In this section we show, however, that the mechanisms in the classes EAM and FAM have surprisingly regular properties in terms of equilibrium outcomes. We also present some results comparing equilibrium outcomes between mechanisms. Consider the preference reporting game induced by a mechanism ψ. At problem P, a preference submission $P' = (P'_i)_{i\in I}$ is a (Nash) **equilibrium** of ψ if for every student i, $\psi_i(P')\, R_i \psi_i(P''_i, P'_{-i})$ for any $P''_i \in \mathcal{P}$. Let Ω be the set of mechanisms that admit an equilibrium in any problem $P \in \mathcal{P}^{|I|}$. In the rest of this section, we consider only the mechanisms in Ω.

The first result relates to the equilibria of EAM and FAM mechanisms.

Proposition 6. *Every EAM and FAM mechanism is in Ω. Moreover, for any problem, an EAM mechanism has a unique equilibrium outcome that is equivalent to the outcome of the serial dictatorship where the student ordering is the same as that used in that EAM mechanism.*

Proposition 6 shows, therefore, that equilibrium outcomes of EAM are not only Pareto efficient, but will match as many students as a commonly used strategy-proof mechanism.

A mechanism ψ is **maximal in equilibrium** if, at any problem P and any equilibrium submission P' under ψ, $\psi(P')$ is maximal.

Proposition 7. *No mechanism is maximal in equilibrium.*

Corollary 3. *No EAM and FAM mechanism is maximal in equilibrium.*

Our next question is how mechanisms compare, in terms of the number of assignments, in equilibrium. For that, we define the concept of **size-wise domination in equilibrium.**

Definition 3. For a given market $(I, S, \succ, q)$, a mechanism ψ **size-wise dominates** another mechanism ϕ **in equilibrium** if, for any problem P and for every equilibria P', P'' under ψ and ϕ, respectively $|\psi(P')| \geq |\phi(P'')|$, and there exists a problem P^* such that for every equilibria $\hat{P}, \tilde{P}$ under ψ and ϕ, respectively $|\psi(\hat{P})| > |\phi(\tilde{P})|$.

What is needed, therefore, for a mechanism ψ to size-wise dominate mechanism ϕ in equilibrium in a given market, is that in every problem ψ assigns at least as many students as ϕ regardless of the equilibrium selection that is made, and that there is at least one problem in which those differences are strict.

One reason why this definition is appropriate for these comparisons is that when mechanisms have multiple equilibria, those may match different numbers of students, and therefore some pairwise comparisons may go in one direction or the other. Our definition makes sure that whether a mechanism dominates another doesn't depend on which pair of equilibria is being chosen, making the comparison well-defined.

Theorem 3. *In any market $(I, S, \succ, q)$, no EAM mechanism is size-wise dominated by an individually rational mechanism in equilibrium.*

Notice that the fact that size-wise domination is defined in terms of a given market makes Theorem 3 stronger: it is not enough to show that the result is true for a specific market. The Theorem instead shows that for any set of students, schools, capacities and priorities there is no individually rational mechanism that dominates any EAM in equilibrium.

While we do not have a similar result to above for the FAM mechanisms, we are able to compare the number of assigned students under the FAM in equilibrium and the weakly dominant strategy equilibrium of the DA, which is truth-telling.

Theorem 4. *Regarding the FAM mechanisms:*

(i) For any problem P and any stable matching for P μ^*, for every equilibrium P' of a FAM mechanism ψ, $|\psi(P')| \geq |\mu^*|$.

(ii) There exist a FAM mechanism ψ, problem P, and an equilibrium profile P' of ψ at P such that $|\psi(P')| > |\mu^{**}|$, where μ^{**} is any stable matching for P.

One may interpret the results in this section as an indication that there isn't much gain in using maximal mechanisms such as EAM and FAM, since when agents respond to their incentives, outcomes are similar to those produced by other non-maximal mechanisms. Below we show, however, that there are improvements in terms of the cardinality of the matching, as long as some fraction of the students are sincere.

Proposition 8. *For any maximal mechanism ψ, problem P, and student i with false preferences P_i' such that $\psi_i(P_i', P_{-i}) P_i \psi_i(P)$, we have $|\psi(P)| \geq |\psi(P_i', P_{-i})|$. Moreover, there exist a problem $\tilde{P}$ and student i with false preferences $\bar{P}_i$ such that $\psi_i(\bar{P}_i, \tilde{P}_{-i}) \tilde{P}_i \psi_i(\tilde{P})$ and $|\psi(\tilde{P})| > |\psi(\bar{P}_i, \tilde{P}_i)|$.*

In a preference-reporting game induced by a maximal mechanism where the only active players are strategic students in the sense that the rest is always sincere, Proposition 8 leads to the following corollary.

Corollary 4. *Under any maximal mechanism, as the set of sincere students increases, in any problem, the number of students matched in equilibrium either stays the same or increases.*

Corollary 4 shows, therefore, that when using maximal mechanisms such as EAM and FAM, the cardinality of the equilibrium outcomes is monotonically increasing in the set of sincere students, going in the case of EAM from the outcome of SD towards maximum matchings. These make a stronger case for the use of these mechanisms as opposed to SD or stable mechanisms when the cardinality of the matching matters.

5. Simulations

While we have shown that the EAM family of mechanisms[14] dominate any individually rational mechanism under true preferences and that they also produce good outcomes in equilibrium, one may wonder whether in practice the magnitude of the difference in the actual number of students assigned justifies the proposal of a new mechanism. To provide an answer to that question, in this section we describe and analyze simulation results in which we compare the number of students matched under five mechanisms: EAM, DA, BM, TTC, and SD.

The construction of the problems to be simulated follows a method similar to that applied in Hafalir et al. (2013). Each problem contains a set of students $I = \{i_1, \dots, i_n\}$, a set of schools $S = \{s_1, \dots, s_m\}$ and their capacities $Q = \{q_1, \dots, q_m\}$. Students have strict preferences $\{P_{i_1}, \dots, P_{i_n}\}$ over $S \cup \{\emptyset\}$ and schools have strict priorities $\{P_{s_1}, \dots, P_{s_m}\}$ over $I \cup \{\emptyset\}$. Those ordinal preferences and priorities are derived from utilities that each student and school have over the other side of the market. Let us first consider a student $i \in I$. Her utility from being assigned to school $s \in S$ is the following:

$$U_i(s) = \begin{cases} \alpha \Theta^s + (1-\alpha)\, \Theta_i^s & \text{if } \alpha \Theta^s + (1-\alpha)\, \Theta_i^s \geq \lambda_i \\ -\infty & \text{otherwise} \end{cases}$$

The interpretation of the parameters goes as follows. The utility that a student i derives from being assigned to a school s is a combination of a value that is shared by all students (Θ^s) and an idiosyncratic value that is unique to a student-school pair (Θ_i^s). The value of Θ^s could therefore be the widespread understanding of the quality of the school and Θ_i^s incorporate, for example, the distance of the school to the student's house and whether the extra-curricular activities fit the student's taste. For each problem, and for each values of $s \in S$ and $(s, i) \in S \times I$, Θ^s and Θ_i^s are independently drawn from the normal distribution with mean zero and variance 1. The value of α, which represents the correlation of preferences between students, is exogenously set in the range $[0, 1]$.

Remark 1 showed that when every student deems every school as acceptable and no student is unacceptable to any school, every mechanism among those being evaluated assign the same number of students. We therefore allow for students to have outside options and for schools to deem some students unacceptable.

[14] For simplicity, in this section we refer only to EAM mechanisms. Since the number of assignments is the same under any EAM and FAM mechanism, however, unless explicitly stated, all the results below hold for both families of mechanisms.

Each student i has an outside option which yields utility λ_i. Therefore, a student would only accept being matched to a school if the utility that she derives from that school exceeds the value of λ_i.[15] The value of those outside options are also a combination of common and idiosyncratic values:

$$\lambda_i = \gamma\overline{\Theta} + (1-\gamma)\,\overline{\Theta_i}$$

For each problem and $i \in I$, $\overline{\Theta}$ and $\overline{\Theta_i}$ are independently drawn from the normal distribution with a mean of zero and variance 1. The exogenous parameter $\gamma \in [0,1]$ represents how correlated the value of the outside options are between students.

Schools' priorities over students follow a similar model. The ordinal priorities of school s over the students are derived from utility functions:

$$U_s(i) \;=\; \begin{cases} \beta\Theta^i + (1-\beta)\,\Theta_s^i & \text{if } \beta\Theta^i + (1-\beta)\,\Theta_s^i \geq \lambda_s \\ -\infty & \text{otherwise} \end{cases}$$

Here once again, for each problem, each value of Θ^i and Θ_s^i is independently drawn from the normal distribution with a mean of zero and variance 1. The concept of acceptability here, however, is not related to the presence of some "outside option" for the school. We interpret λ_s, instead, as an eligibility criterion. In exam schools, for example, it could be a minimum exam score for admission. For schools which give distance-based priority it could be a maximal distance requirement, and so on. For each $s \in S$, λ_s is drawn independently from the normal distribution with mean λ^* and variance 1. Therefore, when $\lambda^* = -\infty$, no student is unacceptable to any school. Moreover, $\beta \in [0,1]$ is an exogenous parameter which represents the degree of correlation between schools' priority rankings. Notice that the case in which students may be unacceptable to schools is not considered in the theoretical analysis, and therefore those simulations should be taken as an additional experiment on the outcomes of those mechanisms under true preferences.

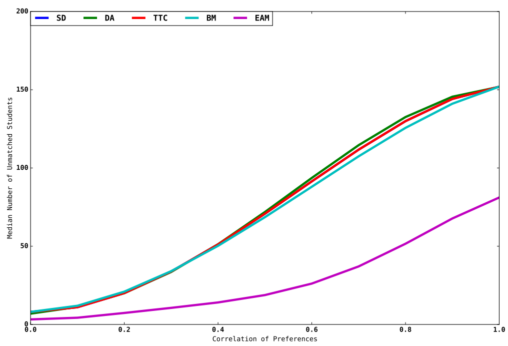
In each simulation performed, we set the values of the parameters $(n, m, Q, \alpha, \gamma, \beta, \lambda^*)$ and generated 100 problems, each representing different draws for values of the random variables. More specifically, in all simulations shown below, $n = 400$, $m = 20$ and every school had capacity $q = 20$. Every combination of the values of the parameters α, β and γ, in steps of 0.1, were used. In other words, every $(\alpha, \beta, \gamma) \in [0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1]^3$ was simulated.

For each problem generated, we produced the matching outcome for each of the five mechanisms : EAM, DA, BM, TTC, and SD, and recorded the number of students who remained unassigned.[16]

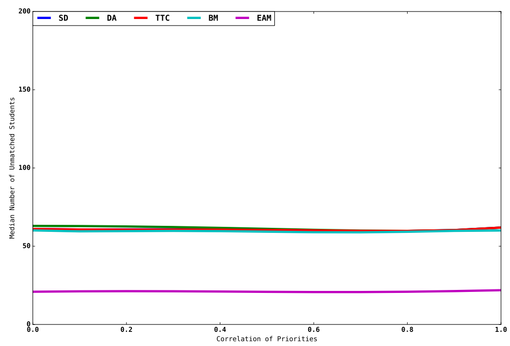
[15]Although it may seem extreme to define the utility of being matched to any school with value below λ_i to be $-\infty$, that choice is inconsequential when we translate those utilities to ordinal preferences. That is, for any i, s such that $U_i(s) = -\infty$, it will simply be the case that school s is unacceptable to i: $\emptyset\ P_i\ s$.

[16]For SD, following the principle behind the equilibrium results of EAM, the ordering of students that was used was drawn from a uniform distribution, independently of the schools' priorities.

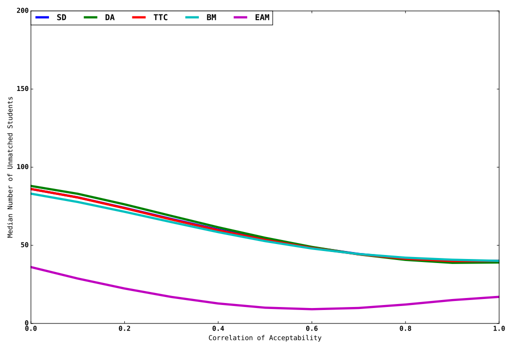
5.1. **Case 1: No Student is Unacceptable to Any School.** In this case we set the value of λ^* to be low enough such that no student is deemed unacceptable to any school.[17] This is often the case in school choice problems. Figure 1 shows the median value of the number of unmatched students across simulations, for each value of the indicated correlation parameter.[18] Two facts clearly stand out. One is that the median number of unmatched students, for any choice of fixed parameter among α, β and γ, is very similar between the DA, BM, TTC, and SD mechanisms. The second is how significant the difference is in the number of unmatched students between EAM and all the other mechanisms. When combining all the simulations performed in case 1, the DA, BM, TTC, and SD mechanisms had a median number of unmatched students of 60 or 61, while for EAM the value was 21, a reduction of 65% in the number of students unmatched.



Median Number of Unmatched Students as a function of α



Median Number of Unmatched Students as a function of β



Median Number of Unmatched Students as a function of γ

FIGURE 1. Median Number of Unmatched Students as a function of correlation parameters

In fact, when performing two-sided T-tests testing the null hypotheses that the number of unmatched students is the same between any two mechanisms, we are not able to reject the null hypothesis of them being equal at the 0.01 significance level for a wide range of parameters for the DA, BM, TTC, and SD mechanisms. That is not the case for any

[17]More specifically, the value of λ^* was set to -1.797×10^{308}, the lowest technically possible.

[18]For the purpose of presentation, the graphs in this section were generated by polynomial fitting of the simulation results.

value of those parameters for any two-sided comparison between EAM and the other four mechanisms. Table 5.1 shows the precise results for all combinations of two mechanisms.

The values of the median and variance of the number of students unmatched for each mechanism and each value of α, β and γ can be found in the appendix.

		SD	DA	TTC	BM	AMM
SD	α					
	β	$---$				
	γ					
DA	α	$[0.0, 1.0]$				
	β	$[0.0, 1.0]$	$---$			
	γ	$[0.0, 1.0]$				
TTC	α	$[0.0, 1.0]$	$[0.0, 1.0]$			
	β	$[0.0, 1.0]$	$[0.0, 1.0]$	$---$		
	γ	$[0.0, 1.0]$	$[0.0, 1.0]$			
BM	α	$[0.0, 0.6] \cup [0.9, 1.0]$	$[0.0, 0.5] \cup [1.0]$	$[0.0, 0.6] \cup [0.9, 1.0]$		
	β	$[0.0, 1.0]$	$[0.0, 1.0]$	$[0.0, 1.0]$	$---$	
	γ	$[0.2, 1.0]$	$[0.5, 1.0]$	$[0.3, 1.0]$		
AMM	α	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	
	β	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$---$
	γ	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	

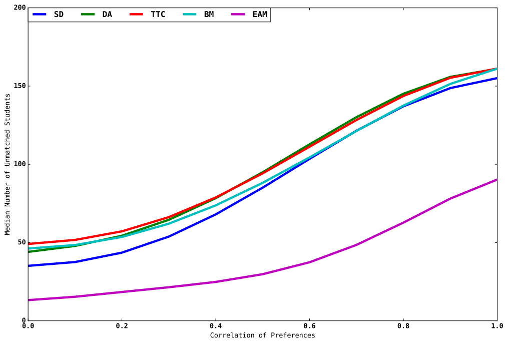
TABLE 1. Ranges of values for α, β and γ for which we cannot reject the null hypothesis that the number of unassigned students is the same between the two mechanisms, at the 0.01 significance level (Case 1)

In light of Proposition 6 and Theorem 4, which characterize the equilibrium outcome of EAMs and establishes a lower-bound on the number of assignments in equilibrium for FAMs, the simulation results are also informative about equilibrium results. Ergin and Sönmez (2006) showed that, under the assumptions that we used, every Nash equilibrium in undominated strategies for the BM is stable and therefore have the same number of assignments as DA.

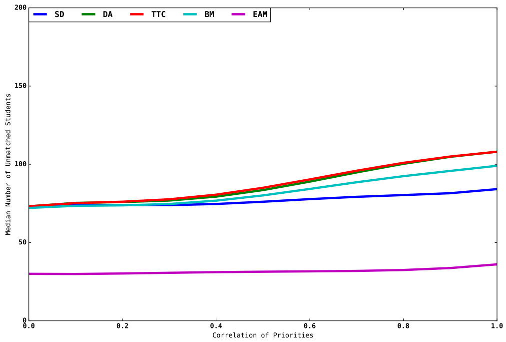
So, to sum up, in equilibrium, EAMs have the same number of assignments as SD, BM the same as DA, and FAMs have at least the same number as DA. The results in table 5.1 imply, therefore, that there is no statistically significant difference between equilibrium outcomes of DA, BM, TTC, SD and AMMs, in terms of the number of assignments, for any of the combinations of parameters considered. Moreover, those results together with Theorem 4 do not allow us to reject the hypothesis that equilibrium outcomes of FAMs are also indistinguishable from those outcomes as well.

5.2. **Case 2: Students May be Unacceptable.** In this case we set $\lambda^* = -1$, that is, schools may find some students unacceptable. Figure 2 shows the median value of the number of unmatched students across simulations, for each value of the indicated correlation parameter.

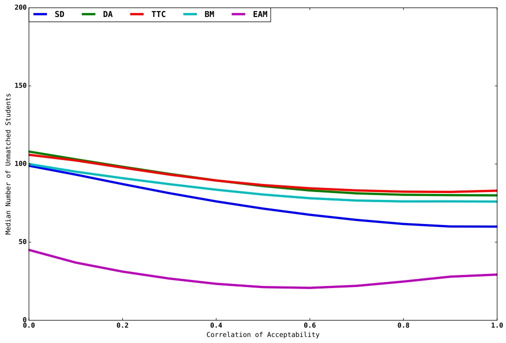
Similarly to case 1, EAM mechanisms perform significantly better than all other mechanisms in terms of the number of students matched, in all configurations of parameters



Median Number of Unmatched Students as a function of α



Median Number of Unmatched Students as a function of β



Median Number of Unmatched Students as a function of γ

FIGURE 2. Median Number of Unmatched Students as a function of correlation parameters

evaluated. When combining all the simulations performed in case 2, the mechanisms had more distinct performances, with SD, DA, TTC, BM, and EAM having a median number of unmatched students 77, 91, 91, 85, 32, respectively. Table 2 shows, for each pair of distinct mechanisms, the ranges of values for α, β and γ for which we cannot reject the null hypothesis that the number of unassigned students is the same between the two mechanisms at the 0.01 significance level.

The values of the median and variance of the number of students unmatched for each mechanism and each value of α, β and γ can be found in the appendix.

		SD	DA	TTC	BM	AMM
SD	α					
	β	$---$				
	γ					
DA	α	$\emptyset$				
	β	$[0.0, 0.3]$	$---$			
	γ	$\emptyset$				
TTC	α	$\emptyset$	$[0.4, 1.0]$			
	β	$[0.0, 0.2]$	$[0.0, 1.0]$	$---$		
	γ	$\emptyset$	$[0.0, 1.0]$			
BM	α	$[0.7, 0.8]$	$[0.1, 0.3] \cup [1.0]$	$[1.0]$		
	β	$[0.0, 0.3]$	$[0.0, 0.5]$	$[0.0, 0.3]$	$---$	
	γ	$\emptyset$	$[0.6, 1.0]$	$[0.9, 1.0]$		
AMM	α	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	
	β	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$---$
	γ	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	

TABLE 2. Ranges of values for α, β and γ for which we cannot reject the null hypothesis that the number of unassigned students is the same between the two mechanisms at the 0.01 significance level (Case 2)

References

Abdulkadiroglu, Atila and Tayfun Sönmez, "School choice: A mechanism design approach," *The American Economic Review*, 2003, *93* (3), 729–747. 2, 9

Abdulkadiroğlu, Atila, Parag A Pathak, and Alvin E Roth, "The new york city high school match," *American Economic Review*, 2005, pp. 364–367. 1

Afacan, M.O, Z.H. Aliogullari, and Mehmet Barlo, "Sticky Matching in School Choice," *Economic Theory*, 2017, *64*, 509–538. 2

Afacan, Mustafa O and Umut M Dur, "Strategy-proof Size Improvement: Is it Possible?," *mimeo*, 2018. 2

Andersson, Tommy and Jörgen Kratz, "Pairwise kidney exchange over blood group barrier," *mimeo*, 2018. 2

______ **and Lars Ehlers**, "Assigning Refugees to Landlords in Sweden: Efficient Stable Maximum Matchings," *forthcoming, Scandinavian Journal of Economics*, 2018. 1, 2

Balinski, Michel and Tayfun Sönmez, "A tale of two mechanisms: student placement," *Journal of Economic theory*, 1999, *84* (1), 73–94. 2

Basteck, Christian, Katharina Huesmann, and Heinrich Nax, "Matching Practices for secondary schools – Germany," Technical Report, MiP Country Profile 21 2015. 1

Berge, Claude, "Two theorems in graph theory," *Proceedings of the National Academy of Sciences*, 1957, *43* (9), 842–844. 2

Blum, Yosef, Alvin E Roth, and Uriel G Rothblum, "Vacancy chains and equilibration in senior-level labor markets," *Journal of Economic theory*, 1997, *76* (2), 362–411. 3.3

Bogomolnaia, Anna and Herve Moulin, "Random Matching under Dichotomous Preferences," *Econometrica*, 2004, *72* (1), 257–279. 2

______ **and** ______, "Size versus fairness in the assignment problem," *Games and Economic Behavior*, 2015, *90*, 119–127. 2

Chun, Youngsub, Eun Jeong Heo, and Sunghoon Hong, "Kidney exchange with immunosuppressants," *mimeo*, 2018. 2

David, Komiers Scott Duke Delacrétaz and Alexander Teytelboym, "Refugee resettlement," *mimeo*, 2017. 13

Dur, Umut and Onur Kesten, "Sequential versus simultaneous assignment systems and two applications," *Economic Theory*, 2014, pp. 1–33. 1

______**, Arda Gitmez, and Özgur Yilmaz**, "School choice with partial fairness," *forthcoming, Theoretical Economics*, 2018. 2

Ehlers, Lars, Isa E Hafalir, M Bumin Yenmez, and Muhammed A Yildirim, "School choice with controlled choice constraints: Hard bounds versus soft bounds," *Journal of Economic Theory*, 2014, *153*, 648–683. 3

Ergin, Haluk and Tayfun Sönmez, "Games of school choice under the Boston mechanism," *Journal of public Economics*, 2006, *90* (1), 215–237. 1, 5.1

Ergin, Haluk I, Tayfun Sonmez, and M Utku Ünver, "Dual-Donor Organ Exchange," *Econometrica*, 2017, *85*, 1645–1671. 1, 2

———, ———, and ———, "Efficient and incentive compatible liver exchange," *mimeo*, 2018. 2

Fragiadakis, Daniel and Peter Troyan, "Improving matching under hard distributional constraints," *Theoretical Economics*, 2017, *12* (2), 863–908. 3

Gale, David and Lloyd S Shapley, "College admissions and the stability of marriage," *The American Mathematical Monthly*, 1962, *69* (1), 9–15. 2

Hafalir, Isa E, M Bumin Yenmez, and Muhammed A Yildirim, "Effective affirmative action in school choice," *Theoretical Economics*, 2013, *8* (2), 325–363. 5

Harris, Douglas N, Jon Valant, and Betheny Gross, "The New Orleans OneApp," *Education Next*, 2015, *15* (4). 1

Kamada, Yuichiro and Fuhito Kojima, "Efficient matching under distributional constraints: Theory and applications," *American Economic Review*, 2015, *105* (1), 67–99. 4

Kesten, Onur, "School choice with consent," *Quarterly Journal of Economics*, 2010, *125* (3). 9

Kojima, Fuhito, Parag A Pathak, and Alvin E Roth, "Matching with couples: Stability and incentives in large markets," *The Quarterly Journal of Economics*, 2013, *128* (4), 1585–1632. 2

Korte, Bernhard and Jens Vygen, *Algorithms and Combinatorics* 2011. 2

Krysta, Piotr, David Manlove, Baharak Rastegari, and Jinshan Zhang, "Size versus truthfulness in the house allocation problem," *Proceedings of the fifteenth ACM conference on Economics and computation - EC '14*, 2014. 2, 3

Kuhn, Harold W, "The Hungarian method for the assignment problem," *Naval research logistics quarterly*, 1955, *2* (1-2), 83–97. 2

Morrill, Thayer and Lars Ehlers, "(Il)legal Assignments in School Choice," *mimeo*, 2018. 2

Nicoló, Antonio and Carmelo Rodríguez-Alvarez, "Age-based preferences in paired kidney exchanges," *Nicoló, Antonio, and Rodríguez-Alvarez, Carmelo*, 2017, *102*, 508–524. 2

Noda, Shunya, "Large matching in large markets with flexible supply," *mimeo*, 2018. 2

Peter, Delacrétaz David Troyan and Andrew Kloosterman, "Essentially stable matchings," *mimeo*, 2018. 2

Roth, Alvin E, "The evolution of the labor market for medical interns and residents: a case study in game theory," *Journal of political Economy*, 1984, *92* (6), 991–1016. 2

Roth, Alvin E., "The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory," *Journal of Political Economy*, 1984, *92* (6), 991–1016. 3.1, 5.2

Roth, Alvin E, Tayfun Sönmez, and M Utku Ünver, "Kidney exchange," *Quarterly*

Journal of Economics, 2004, *119*, 105–129. 2

———, ———, and ———, "Pairwise kidney exchange," *Journal of Economic theory*, 2005, *125* (2), 151–188. 1, 2, 3.2

Sasaki, Yasuo and Masahiro Ura, "Serial dictatorship and unmatch reduction: A problem of Japan's nursery school choice," *Economics Letters*, 2016, *147*, 38–41. 1

Shapley, Lloyd and Herbert Scarf, "On cores and indivisibility," *Journal of mathematical economics*, 1974, *1* (1), 23–37. 1

Sönmez, Tayfun, M Utku Ünver, and Özgur Yilmaz, "How (not) to integrate blood subtyping technology to kidney exchange," *Journal of Economic Theory*, 2018, *176*, 193–231. 2

Appendix

Description of mechanisms.

The Deferred Acceptance Mechanism (DA). Step 1. Each student applies to her favorite acceptable school. Each school tentatively accepts the students among its applicants one at a time following its priority order up to its capacity, and rejects the rest.

In general,

Step k. Each rejected student in the previous step applies to her next favorite acceptable school. Each school tentatively accepts the students among its current step applicants and the tentatively accepted ones in the previous step one at a time following its priority order, and rejects the rest.

The algorithm terminates whenever any student is tentatively accepted by a school or has all acceptable applications rejected. The tentative assignments in the terminal step become the final DA assignments.

The Top Trading Cycles Mechanism (TTC). Step 1. Each student points to her favorite acceptable school. Each school points to the highest priority student. As both the sets of students and schools are finite, there exists a cycle. Assign each student in a cycle to the school he is pointing to, and decrease the capacity of each school appearing in a cycle by one.

In general,

Step k. Each unassigned student points to her favorite acceptable school with remaining capacity. Each school with an empty seat points to the highest priority unassigned student. As there are finitely many unassigned students and schools with remaining capacity, there exists a cycle. Assign each student in a cycle to the school he is pointing to, and decrease the remaining capacity of each school appearing in a cycle by one.

The algorithm terminates whenever any student is assigned or all of his acceptable schools exhaust their capacities.

Boston Mechanism (BM). Step 1. Each student applies to her best acceptable school. Each school permanently accepts the students among its applicants one at a time following its priority order up to its capacity, and rejects the rest.

In general,

Step k. Each rejected student applies to her next best acceptable school. Each school with remaining capacity permanently accepts the students among its current step applicants one at a time following its priority order up to its remaining capacity, and rejects the rest.

The algorithm terminates whenever any student is assigned or all of his acceptable schools exhaust their capacities.

Serial Dictatorship (SD). Step 0. Enumerate the students $I = \{i_1, .., i_n\}$.

Step 1. Start with the first student i_1, and let him choose his top acceptable school with an available seat. Decrease the capacity of his assigned school by one while keeping the capacity of every other school the same. If there is no acceptable school with an available seat, then leave him unassigned.

In general,

Step k. Let student i_k choose his top acceptable school among those with an available seat. Decrease the capacity of his assigned school by one while keeping the capacity of every other school the same. If there is no acceptable school with an available seat then leave him unassigned.

The algorithm terminates by the end of Step n. The above description indeed defines a class of mechanisms, each member of which is associated with a different enumeration in Step 0. We call any mechanism in this class serial dictatorship (SD).

Proofs.

Proposition 1. (i). Let ψ be a fair mechanism. Consider a problem where $I=\{i,j\}$ and $S=\{a,b\}$, each with unit capacity. Let the preferences and priorities be as follows:

$$P_i:\ a,b,\emptyset;\ P_j:\ a,\emptyset.$$

$$\succ_a=\succ_b=i,j.$$

The unique maximal matching is μ' where $\mu'_i=b$ and $\mu'_j=a$. However, μ' is not fair, showing that no fair mechanism is maximal.

(ii). Assume for a contradiction that ψ is a strategy-proof and maximal mechanism. Consider a problem where $I=\{i,j\}$ and $S=\{a,b\}$, each with unit capacity. Let the priorities be such that $\succ_a=\succ_b$: i,j. Consider the problem P where $P_i:a,b,\emptyset$ and $P_j:\ a,\emptyset$.

As ψ is maximal, $\psi_i(P)=b$ and $\psi_j(P)=a$. Let $P'_i:a,\emptyset$ and $P'=(P'_i,P_j)$. Due to the strategy-proofness of ψ, $\psi_i(P')=\emptyset$ and $\psi_j(P')=a$. The latter is because ψ is maximal.

Let us now consider $P''_j:\ a,b,\emptyset$ and $P''=(P'_i,P''_j)$. As ψ is maximal, $\psi_i(P'')=a$ and $\psi_j(P'')=b$. This, along with the fact that $\psi_j(P')=a$, implies that student j profitably reports false preferences P'_j whenever the true preferences are P''. This, however, contradicts the strategy-proofness of ψ, which finishes the proof.

Proposition 2. Let us consider a problem consisting of $I=\{i,j,k\}$ and $S=\{a,b,c\}$, each with unit capacity. Let the preferences and priorities be as follows:

$$P_i:a,\emptyset;\ P_j:\ a,b,c,\emptyset;\ P_k:\ b,a,c,\emptyset.$$

$$\succ_a:\ k,j,i;\ \succ_b:\ i,j,k;\ \succ_c:\ j,i,k.$$

In the above problem, the DA and BM produce the same matching, say μ, and it is such that $\mu_i=\emptyset$, $\mu_j=a$, and $\mu_k=b$. That is, $|\mu|=2$. On the other hand, the TTC outcome, say μ', is such that $\mu'_i=a$, $\mu'_j=c$, and $\mu'_k=b$. That is, $|\mu'|=3$. Hence, neither the DA nor the BM dominate the TTC.

Let us now consider $I=\{i,j,k,h\}$ and $S=\{a,b,c,d\}$, each with unit capacity. Let the preferences and priorities be as follows:

$$P_i:a,b,\emptyset;\ P_j:\ a,\emptyset;\ P_k:\ d,b,c,\emptyset;\ P_h:d,\emptyset.$$

$$\succ_a:\ k,j,i;\ \succ_b:\ i,j,k;\ \succ_c:\ j,i,k;\ \succ_d:\ h,i,j,k.$$

The DA and BM outcomes are the same, say μ, where $\mu_i=b$, $\mu_j=a$, $\mu_k=c$, and $\mu_h=d$. On the other hand, the TTC outcome, say μ', is such that $\mu'_i=a$, $\mu'_j=\emptyset$, $\mu'_k=b$, and $\mu'_h=d$. Hence, $|\mu|>|\mu'|$, showing that the TTC does not dominate either of the DA and the BM.

For the non-existence of a domination relation between the DA and the BM, consider $I=\{i,j,k\}$ and $S=\{a,b,c\}$, each with unit capacity. Let the preferences and priorities be as follows:

$$P_i:a,c,\emptyset;\ P_j:\ b,a,\emptyset;\ P_k:\ b,\emptyset.$$

$$\succ_a: \ k, j, i; \succ_b: \ k, i, j; \succ_c: \ j, i, k.$$

In the above problem, the DA outcome, say μ, is such that $\mu_i = c$, $\mu_j = a$, and $\mu_k = b$. On the other hand, the BM outcome, say μ', is such that $\mu'_i = a$, $\mu'_j = \emptyset$, and $\mu'_k = b$. Hence, $|\mu| > |\mu'|$, showing that the BM does not dominate the DA. Next, for the converse, consider the following preferences and priorities:

$$P_i : b, a, c, \emptyset; \ P_j : \ a, \emptyset; \ P_k : \ b, \emptyset.$$

$$\succ_a: \ k, i, j; \succ_b: \ k, i, j; \succ_c: \ j, i, k.$$

In the above problem, the DA outcome, say μ, is such that $\mu_i = a$, $\mu_j = \emptyset$, and $\mu_k = b$. On the other hand, the BM outcome, say μ', is such that $\mu'_i = c$, $\mu'_j = a$, and $\mu'_k = b$. Hence, $|\mu| < |\mu'|$, showing that the DA does not dominate the BM. This finishes the proof.

For the non-existence of a domination relation between the SD and the other mechanisms, consider $I = \{i, j\}$ and $S = \{a, b\}$, each with unit capacity. Let the preferences and priorities be as follows:

$$P_i : a, b; \ P_j : a, \emptyset.$$

$$\succ_a: j, i; \succ_b: j, i.$$

Let us consider the SD mechanism where student i comes first in the student ordering. Then, the SD outcome μ is such that $\mu_i = a$ and $\mu_j = \emptyset$. On the other hand, all the DA, TTC, and BM outcomes are the same, say μ', and it is such that $\mu'_i = b$ and $\mu'_j = a$. Hence, the SD mechanism does not size-wise dominate the DA, TTC, and BM.

Let us now consider the following preferences, with the same priorities as above.

$$P_i : a, \emptyset, \ P_j : a, b, \emptyset.$$

At the above problem, the SD outcome μ is such that $\mu_i = a$ and $\mu_j = b$. All the DA, TTC, and BM outcomes are the same, say μ', and it is such that $\mu'_i = \emptyset$ and $\mu'_j = a$. Hence, none of DA, TTC, and BM size-wise dominate the SD mechanism.

In the above market, the symmetric arguments easily show that there is no size-wise domination relation between the other SD mechanism where student j comes first, and the other mechanisms. This finishes the proof.

Theorem 1. We will use the following Lemma:

Lemma. *A maximal matching μ is efficient if and only if it does not admit an improving chain or cycle.*

Proof. **"Only If" Part.** Let μ be an efficient matching. If it admits an improving chain $\{i_1, ..i_n, c_1, .., c_{n+1}\}$, then we can define a new matching by assigning each agent i_k to c_{k+1} while keeping the assignments of the others the same. By the improving chain definition, that new matching dominates μ, contradicting our starting supposition that μ is efficient. The same argument shows for the case of improving cycle.

"If" Part. Let μ be a maximal matching such that it does not admit an improving chain or cycle. We want to show that it is efficient. Assume for a contradiction that there exists a matching μ' that dominates μ.

Let $W = \{i \in I : \ \mu'_i P_i \mu_i\}$. By the supposition, $W \neq \emptyset$. Note that for any student i with $\mu_i \neq \emptyset$, we have $\mu'_i \neq \emptyset$. This, along with the maximality of μ, implies that $|\mu'| = |\mu|$. Hence, for any student i with $\mu_i = \emptyset$, $\mu'_i = \emptyset$.

Let us enumerate the students in $W = \{i_1, .., i_n\}$ and write $\mu'_{i_k} = c_k$ for any $k = 1, .., n$. If $|\mu_{c_k}| < q_{c_k}$ for some k, then the pair $\{i_k, c_k\}$ would constitute an improving chain, which would yield a contradiction.

Let us suppose that $|\mu_{c_k}| = q_{c_k}$ for any $k = 1, .., n$. As school c_1 does not have excess capacity at μ, and $\mu'_{i_1} = c_1$, we have another student in W, say i_2, such that $\mu_{i_2} = c_1$. Then, consider student i_2, and as c_2 does not have excess capacity at μ and $\mu'_{i_2} = c_2$, we have another student in W, say i_3, such that $\mu_{i_3} = c_2$. If we continue to apply the same arguments to the other students in W, as W is finite, we would eventually obtain an improving cycle, which yields a contradiction. □

We can now proceed to the proof of the Theorem. Let ψ be an EAM mechanism, and μ and μ' be its first stage and final outcome, respectively. As students are not assigned to one of their unacceptable schools in the course of Step 1 of ψ, μ is individually rational.

We next show that μ is maximal. Assume for a contradiction that it is not maximal. This means that there exists another matching $\mu'' \neq \mu$ such that $|\mu''| > |\mu|$. Let $\{i_1, .., i_n\}$ be the agent-enumeration that is used under ψ.

As $|\mu''| > |\mu|$, there exists some agent $i_k \in I$ such that $\mu''_{i_k} \neq \emptyset$ and $\mu_{i_k} = \emptyset$. Let $i_{k'}$ be the first agent according to the above enumeration such that $\mu''_{i_{k'}} \neq \emptyset$ and $\mu_{i_{k'}} = \emptyset$. This means that for each $k < k'$, either $\mu_{i_k} \neq \emptyset$ or $\mu_{i_k} = \emptyset$ and $\mu''_{i_k} = \emptyset$. Let $B(\mu, k') = \{i \in N : \ \mu_{i_k} \neq \emptyset$ for any $k < k'\}$. That is, it is set of agents who come before agent $i_{k'}$ in the above enumeration and are assigned under matching μ.

Let us now consider agent $i_{k'}$. By the definition of ψ, $\mu_{i_{k'}} = \emptyset$ because it is not possible to match agent $i_{k'}$ to some of his acceptable objects while keeping all the agents in $B(\mu, k')$ assigned to one of their acceptable objects. This means that in order for agent $i_{k'}$ to receive one of his acceptable objects, one of the assigned agents under μ from $B(\mu, k')$ has to be unassigned. This arguments holds for each other agent who is assigned under μ'', but not under μ. This implies that μ is maximal.

In Step 2 of ψ, new matchings are obtained by implementing improving chains and cycles (if any). By their definitions, in the course of Step 2, no student receives a worse school than his assignment μ. This, along with the individual rationality of μ, implies that μ' is maximal. The efficiency of μ' directly comes from the Lemma above.

Proposition 6. Let ψ be an EAM mechanism. By its definition, the first student in the ordering in Step 0 of the EAM obtains his top choice by reporting it as the only acceptable choice, irrespective of the other students' preference submissions. By the same reasoning, the second student in the ordering can obtain his top choice among the schools with available seats (after the capacity of the first student assignment is decreased by one) by reporting that school as his only acceptable choice, irrespective of the other students' preference submissions. Once we repeat the same arguments for every other student, we not only find an equilibrium of ψ, but also conclude that it is the unique equilibrium outcome, which coincides with the outcome of serial dictatorship with the ordering as the same as that in Step 0 of ψ.

Let ϕ be a FAM mechanism. Let μ be a stable matching at P. Consider the preferences submission P' under which for any student i, the only acceptable school is μ_i. Any unassigned student at μ reports no school acceptable at P'. It is easy to verify that $\phi(P') = \mu$.

Next, we claim that P' is an equilibrium submission under ϕ. Suppose for a contradiction that there exist a student i and P''_i such that $\phi_i(P''_i, P'_{-i}) P_i \phi_i(P')$. For ease of writing, let $\phi_i(P''_i, P'_{-i}) = s$ and $\phi_i(P') = s'$. As μ is stable, $|\mu_s| = q_s$. This, along with the definition of P' and $\phi_i(P''_i, P'_{-i}) = s$, implies that there exists a student $j \neq i$ such that $\mu_j = s$ and $\phi_j(P''_i, P'_{-i}) = \emptyset$. Moreover, from the stability of μ, we also have $j \succ_s i$. These altogether contradict the fairness for unassigned students of ϕ, showing that P' is equilibrium of ϕ.

Proposition 7. Let $I = \{i, j\}$ and $S = \{a, b\}$, each with unit capacity. Assume for a contradiction that $\psi \in \Omega$ such that it is maximal in equilibrium.

Consider the preferences where $P_i : \ a, \emptyset$ and $P_j : \ a, b, \emptyset$. In any equilibrium at P, ψ places student i and student j at school a and b, respectively.

Consider the problem P' where $P'_i : a, \emptyset$ and $P'_j : \ a, \emptyset$. If there exists an equilibrium of ψ at P' under which student j is assigned to school a, then this submission constitutes an equilibrium at P as well. This, however, contradicts ψ being maximal in equilibrium. Hence, under any equilibrium at P', student i is assigned to school a while student j is unassigned.

Let us now consider the problem P'' where $P''_i : \ a, b, \emptyset$ and $P''_j = P'_j : a, \emptyset$. As ψ is maximal in equilibrium, under any equilibrium at P'', student i and student j have to be placed at school b and school a, respectively.

We next claim that any equilibrium at P' is also an equilibrium at P''. To see this, let $\tilde{P}$ be equilibrium at P'. As ψ is maximal in equilibrium, either of agents has to receive object a at $\psi(\tilde{P})$. Without loss of generality, let us assume that agent i receives object a at $\psi(\tilde{P})$. This automatically implies that $\psi_j(\tilde{P}) = \emptyset$. But then, because $\tilde{P}$ is equilibrium under ψ at problem P', for no preference $\bar{P}_j \in \mathcal{P}$, $\psi_j(\tilde{P}_i, \bar{P}_j) = a$. From here, we conclude that $\tilde{P}$ is equilibrium of ψ at problem P'' as well.

The above analysis shows that there exists an equilibrium at P'' under which student i is assigned to school a, and student j is unassigned. This, however, contradicts ψ being maximal in equilibrium, finishing the proof.

Theorem 3. In the proof, we will use the following lemma.

Lemma. *Let ψ be an EAM and ϕ be an individually rational mechanism. In any market $(I, S, \succ, q)$ and problem P, if $|\psi(P')| < |\phi(P'')|$ where P' and P'' are equilibria under ψ and ϕ, respectively, then there exists a student i such that $\psi_i(P') P_i \phi_i(P'') P_i \emptyset$.*

Proof. In a market $(I, S, \succ, q)$ and problem P, let $|\psi(P')| < |\phi(P'')|$ where P' and P'' are equilibria under ψ and ϕ, respectively. This implies that for some school s, $|\psi_s(P')| < |\phi_s(P'')| \leq q_s$. Hence, let $i \in \phi_s(P'') \setminus \psi_s(P')$. By the individual rationality of ϕ and P'' being equilibrium under ϕ, we have $sP_i\emptyset$, where $\phi_i(P'') = s$. As the unique equilibrium outcome of ψ coincides with the (truthtelling) outcome of a SD mechanism (Proposition 5), we have $\psi(P') = SD(P)$. Hence, school s has an excess capacity under $SD(P)$. Moreover, from above, $\psi_i(P') = SD_i(P) \neq s$. Hence, by the non-wastefulness of SD, i must be matched to a school strictly better than s and therefore $\psi_i(P') = SD_i(P) P_i \phi_i(P'') P_i \emptyset$, which finishes the proof. $\square$

Let now $(I, S, \succ, q)$ be a market and ψ be an EAM mechanism. Assume for a contradiction that an individually rational mechanism ϕ size-wise dominates ψ in equilibrium. This in particular implies that for some problem P, $|\psi(P')| < |\phi(P'')|$ for every equilibria P' and P'' under ψ and ϕ, respectively. In what follows, we will fix one such pair P', P''. We prove the result in two steps.

Step 1. By the Lemma above, there exists a student i such that $\psi_i(P') P_i \phi_i(P'') P_i \emptyset$. Let $\bar{P}_i$ be the preference relation that keeps the relative rankings of the schools the same as under P_i, while reporting any school that is worse than $\psi_i(P')$ as unacceptable. In other words, $\bar{P}_i$ truncates P_i below $\psi_i(P')$. Let us write $\bar{P} = (\bar{P}_i, P_{-i})$. Recall that the unique equilibrium outcome of ψ always coincides with the truthtelling outcome of a SD mechanism (Proposition 5). Moreover, by the construction of $\bar{P}$, $SD(P) = SD(\bar{P})$. This in turn implies that $\psi(P') = \psi\left(\bar{P}'\right)$ for every equilibrium $\bar{P}'$ under ψ in problem $\bar{P}$.

We next consider problem $\bar{P}$. If there exists no student j such that $\psi_j\left(\bar{P}'\right) \bar{P}_j \phi_j\left(\bar{P}''\right) \bar{P}_j \emptyset$ for some equilibria $\bar{P}'$ and $\bar{P}''$ under ψ and ϕ, respectively, then we move to Step 2. Otherwise, we pick such student j. Note that because of the definition of $\bar{P}_i$ states that any outcome below $\psi_i\left(\bar{P}'\right)$ is unacceptable for i and ϕ is individually rational, $\psi_j\left(\bar{P}'\right) \bar{P}_j \phi_j\left(\bar{P}''\right) \bar{P}_j \emptyset$ cannot hold for $j = i$, therefore $j \neq i$. Then, as the same as above, let $\bar{P}_j$ be the preference list that truncates P_j below $\psi_j\left(\bar{P}'\right)$. Let us write $\tilde{P} = (\bar{P}_i, \bar{P}_j, P_{-\{i,j\}})$. By the same reason as above, $\psi(P') = \psi\left(\tilde{P}'\right)$ for any equilibrium $\tilde{P}'$ under ψ in problem $\tilde{P}$.

We next consider problem $\tilde{P}$. If there exists no student k such that $\psi_k\left(\tilde{P}'\right) \tilde{P}_k \phi_k\left(\tilde{P}''\right) \tilde{P}_k \emptyset$ for some equilibria $\tilde{P}'$ and $\tilde{P}''$ under ψ and ϕ, respectively, then we move to Step 2. Otherwise, we pick such a student k. By the same reason as above, student k is different than both i and j. Then, we follow the same arguments above and obtain a new preference profile. In each iteration, we have to consider a different student. But then, since there are finitely many students, this case cannot hold forever. Hence, we eventually obtain a problem, say $\hat{P}$, in which there exists no student h such that $\psi_h(\hat{P}')\hat{P}_h\phi_h\left(\hat{P}''\right)\hat{P}_h\emptyset$ for some equilibria $\hat{P}'$ and $\hat{P}''$ under ψ and ϕ, respectively, and move to Step 2. We also have $\psi(P') = \psi(\hat{P}')$ for any equilibrium $\hat{P}'$ under ψ in problem $\hat{P}$.

Step 2. By the Lemma above, in problem $\hat{P}$, we have $\left|\psi\left(\hat{P}'\right)\right| \geq \left|\phi\left(\hat{P}''\right)\right|$ for any equilibria $\hat{P}'$ and $\hat{P}''$ under ψ and ϕ, respectively. If it holds strictly for some equilibria, then we reach a contradiction. Suppose $\left|\psi(\hat{P}')\right| = \left|\phi\left(\hat{P}''\right)\right|$ for any equilibria $\hat{P}'$ and $\hat{P}''$.

We now claim that $\hat{P}''$ is an equilibrium under ϕ in problem P. Suppose it is not, and let student k have a profitable deviation, say $\ddot{P}_k$, from $\hat{P}''_k$. This means that $\phi_k\left(\ddot{P}_k, \hat{P}''_{-k}\right) P_k \phi_k\left(\hat{P}''\right)$. But then, by construction above, $\hat{P}_k$ preserves the relative rankings under P_k. This implies that $\phi_k\left(\ddot{P}_k, \hat{P}''_{-k}\right) \hat{P}_k \phi_k\left(\hat{P}''\right)$, contradicting $\hat{P}''$ being an equilibrium under ϕ in problem $\hat{P}$.

Recall that $\psi(P') = \psi(\hat{P}')$. Hence, this, along with $\left|\psi(\hat{P}')\right| = \left|\phi\left(\hat{P}''\right)\right|$ and our above finding, implies that in problem P, $|\psi(P')| = \left|\phi\left(\hat{P}''\right)\right|$ where P' and $\hat{P}''$ are equilibria under ψ and ϕ, respectively. Therefore, we constructed an equilibrium pair for problem P where ψ matches as many students as ϕ, contradicting our assumption that this does not hold in problem P.

Theorem 4. (i). First, by the rural hospital theorem (Roth, 1984b), the number of assignments in any stable matching is the same as that of DA. Let ψ be a FAM mechanism. Assume for a contradiction that there exist a problem P and an equilibrium profile P' under ψ such that $|\psi(P')| < |DA(P)|$. For ease of writing, let $DA(P) = \mu$ and $\psi(P') = \mu'$.

We now claim that for some student i, $\mu_i = s$ for some school s whereas $\mu'_i = \emptyset$ and, moreover, $|\mu'_s| < q_s$. To prove this claim, let us define $W = \{i \in I : \ \mu_i = s \text{ and } \mu'_i = \emptyset\}$. By our supposition that $|DA(P)| > |\psi(P')|$, we have $W \neq \emptyset$. Suppose that for each $i \in W$ with $\mu_i = s$, $|\mu'_s| = q_s$. But then this implies that $|\mu'| \geq |\mu|$, contradicting our initial supposition, which finishes the proof of the claim.

Let $i \in I$ such that $\mu_i = s$, $\mu'_i = \emptyset$, and $|\mu'_s| < q_s$. Now, consider the following preferences P'':

$$P''_k = \begin{cases} P'_k & \text{If } k \neq i \\ s, \emptyset & \text{If } k = i \end{cases}$$

First, observe that there exists a (individually rational) matching at P'' that assigns $|\mu'| + 1$ many students (to see this, keep the assignment of everyone except student i the same as at μ', and place student i at school s). Therefore, due to the maximality of ψ, we have $|\psi(P'')| \geq |\mu'| + 1$. If student i is assigned to school s at $\psi(P'')$ then this contradicts P' being equilibrium under ψ. Hence, $\psi_i(P'') = \emptyset$. But then, by the definition of P'', $\psi(P'')$ is individually rational at P'. This, along with the maximality of ψ, implies that $|\psi(P')| \geq |\psi(P'')|$, contradicting our previous finding that $|\psi(P'')| \geq |\psi(P')| + 1$, which finishes the proof of the first part.

(ii). Let us consider $I = \{i, j, k, h\}$ and $S = \{a, b, c\}$, each with unit capacity. The preferences and the priorities are given below.

$$P_i: \ a, b, \emptyset; \ P_j: \ c, a, \emptyset; \ P_k: \ c, a, \emptyset; \ P_h: \ c, \emptyset.$$
$$\succ_a: \ k, i, j, h; \succ_b: \ k, h, j, i; \succ_c: \ k, h, i, j.$$

Let ψ be a FAM mechanism with the student ordering k, j, i, h. Mechanism ψ is such that it produces matching μ at P where $\mu_i = b$, $\mu_j = a$, $\mu_k = c$, and $\mu_h = \emptyset$. For any $P_i' \in \mathcal{P}$ with $bP_i'\emptyset$, let $\psi(P_i', P_{-i}) = \mu'$ where $\mu_i' = b$, $\mu_j' = \emptyset$, $\mu_k' = a$, and $\mu_h' = c$. Moreover, for any $P_i' \in \mathcal{P}$ with $\emptyset P_i' b$, $\psi(P_i', P_{-i}) = \mu''$ where $\mu_i'' = \emptyset$, $\mu_j'' = \emptyset$, $\mu_k'' = a$, and $\mu_h'' = c$. And, for any $P_h' \in \mathcal{P}$, let $\psi(P_{-h}, P_h') = \mu$.

Note that student j can never get school c under ψ by misreporting because otherwise student h would be unassigned, and he has higher priority at school c. It is immediate to see that the above matchings can be obtained in the course of FAM through particular selection. All of these show that under ψ, truth-telling is an equilibrium at P, and $|\psi(P)| = 3$. On the other hand, $DA(P)$ is such that $DA_i(P) = a$, $DA_k(P) = c$, and $DA_h(P) = DA_j(P) = \emptyset$. Hence, $|\psi(P)| > |DA(P)|$, finishing the proof of the second part.

Proposition 8. Let $P' = (P_i', P_{-i})$, $\psi(P) = \mu$, and $\psi(P_i', P_{-i}) = \mu'$. Assume for a contradiction that $|\mu'| > |\mu|$. By our supposition, $\mu_i' P_i \mu_i$. This, along with the fact that $P_j = P_j'$ for each $j \neq i$, μ' is individually rational in problem P. But then, $|\mu'| > |\mu|$ contradicts the fact that μ is maximal in problem P.

Let us now consider a problem where $\{i, j\} \subseteq N$, $\{a, b\} \subseteq S$, each with unit capacity. Let the preferences be such that $P_i: \ a, \emptyset$, $P_j: \ a, \emptyset$, and each other student (if any) finds any school unacceptable. Without loss of generality, let us assume that the outcome of ψ in that problem, say μ, is such that $\mu_i = a$, and each other student is unassigned.

Let us next consider problem where $P_i': \ a, b, \emptyset$, while each other student's preferences are the same as above. Under the true preferences here, ψ produces μ' where $\mu_i' = b$, $\mu_j' = a$, and each other student is unassigned. However, student i can beneficially misreport his preferences by submitting P_i above as, under this false profile, ψ produces matching μ above. Finally, note that $|\mu'| > |\mu|$, finishing the proof.

Simulation results.

α	SD	DA	TTC	BM	AMM
0.0	8 (28.49)	7 (28.65)	8 (28.5)	8 (28.41)	3 (28.55)
0.1	11 (32.93)	11 (33.13)	11 (32.92)	12 (32.79)	5 (32.85)
0.2	20 (41.86)	20 (42.17)	20 (41.87)	21 (41.68)	7 (41.06)
0.3	34 (50.85)	34 (51.19)	34 (50.84)	34 (50.56)	10 (49.18)
0.4	51 (57.47)	51 (57.82)	51 (57.46)	50 (57.14)	14 (55.81)
0.5	71 (65.87)	72 (66.14)	71 (65.88)	69 (65.58)	20 (66.31)
0.6	91 (71.85)	93 (71.99)	91 (71.88)	88 (71.71)	26 (76.3)
0.7	112 (75.68)	115 (75.64)	112 (75.68)	107 (75.71)	36 (84.76)
0.8	130 (76.52)	133 (76.31)	130 (76.47)	126 (76.66)	52 (89.18)
0.9	144 (76.96)	145 (76.81)	144 (76.95)	141 (77.12)	68 (92.52)
1.0	152 (75.82)	152 (75.85)	152 (75.85)	152 (75.85)	81 (92.84)

Table 3. Median and standard deviation for the number of unmatched students, varying α from 0.0 to 1.0 (Case 1)

β	SD	DA	TTC	BM	AMM
0.0	61.0 (81.28)	63.0 (82.22)	61.0 (81.3)	60.0 (80.69)	21.0 (76.73)
0.1	61.0 (80.11)	63.0 (81.11)	61.0 (80.16)	60.0 (79.53)	21.0 (75.38)
0.2	60.0 (80.31)	62.0 (81.21)	60.0 (80.3)	59.0 (79.68)	21.0 (75.25)
0.3	61.5 (81.42)	63.0 (82.17)	61.0 (81.39)	60.0 (80.8)	22.0 (76.9)
0.4	61.0 (81.67)	62.0 (82.32)	61.0 (81.66)	60.0 (81.03)	21.0 (77.28)
0.5	60.0 (80.08)	60.0 (80.59)	60.0 (80.07)	59.0 (79.46)	20.0 (75.22)
0.6	60.0 (80.89)	61.0 (81.22)	60.0 (80.88)	59.0 (80.3)	21.0 (76.42)
0.7	60.0 (80.57)	60.0 (80.77)	60.0 (80.57)	59.0 (79.95)	21.0 (75.95)
0.8	60.0 (80.13)	60.0 (80.22)	60.0 (80.13)	59.0 (79.53)	21.0 (75.38)
0.9	60.0 (80.4)	60.0 (80.44)	60.0 (80.41)	60.0 (79.82)	21.0 (75.77)
1.0	62.0 (82.15)	62.0 (82.14)	62.0 (82.14)	60.0 (81.55)	22.0 (77.85)

Table 4. Median and standard deviation for the number of unmatched students, varying β from 0.0 to 1.0 (Case 1)

γ	SD	DA	TTC	BM	AMM
0.0	86.0 (47.46)	88.0 (48.12)	86.0 (47.49)	83.0 (46.52)	36.0 (20.18)
0.1	81.0 (49.51)	83.0 (50.19)	81.0 (49.55)	78.0 (48.51)	29.0 (21.78)
0.2	73.0 (53.1)	76.0 (53.76)	73.0 (53.09)	71.0 (52.13)	22.0 (27.64)
0.3	68.0 (57.83)	69.0 (58.48)	67.0 (57.85)	65.0 (56.85)	17.0 (36.49)
0.4	60.0 (63.05)	62.0 (63.63)	60.0 (63.02)	59.0 (62.03)	13.0 (45.51)
0.5	53.0 (70.3)	54.0 (70.93)	53.0 (70.29)	52.0 (69.38)	10.0 (58.25)
0.6	49.0 (79.21)	49.0 (79.77)	48.0 (79.15)	48.0 (78.35)	9.0 (71.71)
0.7	45.0 (89.01)	45.0 (89.54)	45.0 (89.0)	45.0 (88.31)	10.0 (86.04)
0.8	41.0 (101.12)	40.0 (101.66)	41.0 (101.13)	41.5 (100.61)	12.0 (101.22)
0.9	40.0 (110.93)	39.0 (111.43)	40.0 (110.94)	41.0 (110.51)	15.0 (113.0)
1.0	40.0 (120.95)	39.0 (121.41)	40.0 (120.94)	40.0 (120.54)	17.0 (123.78)

Table 5. Median and standard deviation for the number of unmatched students, varying γ from 0.0 to 1.0 (Case 1)

Case 1: No student is unacceptable to any school.

α	SD	DA	TTC	BM	AMM
0.0	35 (32.29)	44 (35.9)	49 (34.26)	46 (32.99)	13 (33.26)
0.1	38 (38.43)	48 (40.8)	52 (39.22)	49 (38.4)	16 (40.12)
0.2	43 (44.95)	54 (46.15)	57 (44.71)	53 (44.27)	18 (46.55)
0.3	54 (53.29)	65 (53.13)	66 (51.85)	62 (51.93)	21 (54.53)
0.4	68 (60.06)	78 (58.78)	79 (57.7)	74 (58.2)	25 (61.98)
0.5	85 (65.82)	95 (63.75)	94 (62.92)	88 (63.76)	30 (69.18)
0.6	104 (71.33)	113 (68.62)	112 (68.08)	105 (69.41)	38 (78.28)
0.7	121 (74.43)	130 (71.38)	128 (71.15)	121 (72.62)	48 (84.72)
0.8	137 (74.87)	145 (71.72)	143 (71.65)	137 (73.14)	62 (88.18)
0.9	149 (75.16)	156 (72.12)	156 (72.11)	152 (73.18)	79 (91.14)
1.0	155 (74.32)	161 (71.58)	161 (71.58)	161 (71.58)	90 (90.68)

Table 6. Median and standard deviation for the number of unmatched students, varying α from 0.0 to 1.0 (Case 2)

β	SD	DA	TTC	BM	AMM
0.0	73 (77.73)	73 (79.29)	73 (77.36)	72 (76.55)	30 (77.02)
0.1	75 (77.36)	76 (78.89)	76 (76.89)	74 (76.19)	30 (77.02)
0.2	73 (77.85)	75 (78.97)	75 (76.89)	73 (76.47)	30 (77.82)
0.3	74 (78.05)	77 (78.58)	78 (76.33)	75 (76.37)	31 (78.26)
0.4	75 (75.7)	80 (75.24)	81 (73.1)	77 (73.55)	31 (75.9)
0.5	76 (77.95)	83 (75.98)	85 (74.03)	80 (75.04)	31 (78.78)
0.6	78 (75.93)	89 (72.36)	90 (70.86)	84 (72.24)	32 (77.14)
0.7	79 (75.45)	95 (70.05)	96 (69.09)	89 (70.87)	32 (76.94)
0.8	80 (74.65)	100 (67.84)	101 (67.37)	92 (69.41)	32 (76.76)
0.9	82 (74.47)	105 (66.57)	105 (66.45)	96 (68.71)	34 (76.87)
1.0	84 (74.9)	108 (66.4)	108 (66.4)	99 (68.69)	36 (77.65)

Table 7. Median and standard deviation for the number of unmatched students, varying β from 0.0 to 1.0 (Case 2)

γ	SD	DA	TTC	BM	AMM
0.0	99.0 (42.3)	108.0 (42.4)	106.0 (41.52)	100.0 (40.95)	45.0 (21.06)
0.1	93.0 (44.3)	103.0 (44.46)	102.0 (43.42)	95.0 (42.82)	37.0 (24.0)
0.2	87.0 (47.8)	98.0 (47.72)	98.0 (46.52)	91.0 (45.91)	31.0 (29.75)
0.3	82.0 (52.92)	94.0 (52.47)	93.0 (51.1)	87.0 (50.73)	27.0 (39.01)
0.4	76.0 (58.68)	90.0 (58.06)	90.0 (56.53)	84.0 (56.27)	23.0 (49.05)
0.5	71.0 (66.66)	85.0 (65.49)	86.0 (63.82)	80.0 (63.78)	21.0 (61.25)
0.6	67.0 (75.38)	83.0 (73.8)	84.0 (72.01)	78.0 (72.28)	21.0 (74.42)
0.7	65.5 (85.64)	82.0 (83.49)	84.0 (81.64)	77.0 (82.2)	23.0 (88.56)
0.8	61.0 (95.31)	80.0 (92.64)	82.0 (90.66)	76.0 (91.56)	23.0 (100.71)
0.9	60.0 (107.81)	80.0 (104.5)	82.0 (102.54)	76.0 (103.71)	29.0 (115.26)
1.0	60.0 (114.66)	80.0 (111.05)	83.0 (109.05)	76.0 (110.38)	29.0 (122.64)

Table 8. Median and standard deviation for the number of unmatched students, varying γ from 0.0 to 1.0 (Case 2)

Case 2: Students may be unacceptable.