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The Symmetric M-Matrix and Symmetric Inverse M-Matrix Completion Problems

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Abstract

The symmetric *M*-matrix and symmetric *M*0-matrix completion problems are solved and results of Johnson and Smith [Linear Algebra Appl. 290 (1999) 193] are extended to solve the symmetric inverse *M*-matrix completion problem: (1) A pattern (i.e., a list of positions in an $n \times n$ matrix) has symmetric *M*-completion (i.e., every partial symmetric *M*-matrix specifying the pattern can be completed to a symmetric *M*-matrix) if and only if the principal subpattern *R* determined by its diagonal is permutation similar to a pattern that is block diagonal with each diagonal block complete, or, in graph theoretic terms, if and only if each component of the graph of *R* is a clique. (2) A pattern has symmetric *M*0-completion if and only if the pattern is permutation similar to a pattern that is block diagonal with each diagonal block either complete or omitting all diagonal positions, or, in graph theoretic terms, if and only if every principal subpattern corresponding to a component of the graph of the pattern either omits all diagonal positions, or includes *all* positions. (3) A pattern has symmetric inverse *M*-completion if and only if its graph is block-clique and no diagonal position is omitted that corresponds to a vertex in a graph-block of order >2. The techniques used are also applied to matrix completion problems for other classes of symmetric matrices.

Keywords

Matrix completion, Symmetric M-matrix, Symmetric inverse M-matrix, Partial matrix

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Comments

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The Symmetric M-Matrix and Symmetric Inverse M-Matrix Completion Problems

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Abstract A list of positions in an n × n real matrix (a pattern) is said to have Π -completion if every partial Π-matrix that specifies exactly these positions can be completed to a Π-matrix. The symmetric M-matrix and symmetric M_0-matrix completion problems are solved and results of Johnson and Smith [JS2] are extended to complete the solution to the symmetric inverse M-matrix problem: A pattern has symmetric M-completion if and only if each component of the graph of the principal subpattern determined by the diagonal positions is a clique. A pattern has symmetric M_0-completion if and only if every principal subpattern corresponding to a component of the graph of the pattern either omits all diagonal positions, or includes all positions. A pattern has symmetric inverse M-completion if and only if its graph is block-clique and no diagonal position is omitted that corresponds to a vertex in a block of order > 2. The techniques used are also applied to the completion problem for other classes of symmetric matrices.

1. Introduction

A *partial matrix* is a matrix in which some entries are specified and others are not. A *completion* of a partial matrix is a specific choice of values for the unspecified entries. A *pattern* for n × n matrices is a list of positions of an n × n matrix, that is, a subset of {1,...,n} × {1,...,n}. A partial matrix *specifies the pattern* if its specified entries are exactly those listed in the pattern. Note that in this paper a pattern does not need to include all diagonal positions. For a particular class Π of matrices, the Π-matrix completion problem for patterns asks which patterns of positions have the property that any partial Π-matrix that specifies the pattern can be completed to a Π-matrix. In this paper we concern ourselves only with completion problems for patterns. Such completion problems have been studied for M-matrices [H2], M_0-matrices [H4], inverse M-matrices [JS1], [H1], [H3], symmetric inverse M-matrices [JS2], and many other classes. Results on matrix completion problems and techniques are surveyed in [H4].

A partial matrix is *symmetric* if whenever b_{ij} is specified then so is b_{ji} and $b_{ji} = b_{ij}$. For a class Σ of symmetric matrices, the pattern must be *positionally symmetric*, i.e., if position (i,j) is in the pattern, then so is (j,i). Thus, the Σ-matrix completion problem asks which positionally symmetric patterns have the property that any partial Σ-matrix that specifies the pattern can be completed to a Σ-matrix. For a class Σ of symmetric matrices, the term "pattern" means "positionally symmetric pattern."

The matrix A $\mathbf{R}^{nxn}$ is called *positive stable* (respectively, *semistable*) if all the eigenvalues of A have positive (nonnegative) real part. An *M-matrix* (respectively, *M_0-matrix*) is a positive stable (semistable) matrix with nonpositive off-diagonal entries. There are many equivalent characterizations of M- and M_0-matrices [HJ]: A matrix with nonpositive off-diagonal entries is an M-matrix (M_0-matrix) if and only every principal minor is positive (nonnegative). A matrix with nonpositive off-diagonal entries is an M-matrix if and only if it is nonsingular and its inverse is entrywise nonnegative.

The matrix B is an *inverse M-matrix* if B is the inverse of an M-matrix. Equivalently, an inverse M-matrix is a nonsingular, entrywise nonnegative matrix B such that B^{-1} has nonpositive off-diagonal entries. A substantial amount is known about M-matrices, M_0-matrices and inverse M-matrices [HJ], [J], [LN].

For α a subset of $\{1,...,n\}$, the *principal submatrix* $A[\alpha]$ is obtained from the $n \times n$ matrix A by deleting all rows and columns not in α. Similarly, the *principal subpattern* $Q[\alpha]$ is obtained from the pattern Q by deleting all positions whose first or second coordinate is not in α. The *principal subpattern determined by the diagonal positions* is $Q[\delta]$ where $\delta = \{i \mid (i,i) \quad Q\}$.

From the characterization of M- (M_0-) matrices as matrices with nonpositive off-diagonal entries and positive (nonnegative) principal minors, it is clear that any principal submatrix of an M- (M_0-) matrix is an M- (M_0-) matrix. Any principal submatrix of an inverse M-matrix is an inverse M-matrix [J]. So in order for a partial matrix to have a completion that is an M- (M_0-, inverse M-) matrix, it is clearly necessary that every fully specified principal submatrix be an M- (M_0-, inverse M-) matrix and any sign conditions on the entries be respected.

Thus, a *partial inverse M-matrix* is an entrywise nonnegative partial matrix such that any fully specified principal submatrix is an inverse M-matrix. A *partial symmetric inverse M-matrix* is a partial inverse M-matrix that is symmetric. Hereafter, we will follow the notation of [JS2] in referring to a symmetric inverse M-matrix as a SIM matrix. We say a pattern *has SIM completion* if every partial SIM matrix specifying the pattern can be completed to a SIM matrix.

A *partial M- (M_0-) matrix* is a partial matrix such that any fully specified principal submatrix is an M- (M_0-) matrix and all specified off-diagonal entries are nonpositive. A *partial symmetric M- (M_0-) matrix* is a partial M- (M_0-) matrix that is symmetric. We say a pattern *has symmetric M (M_0-) completion* if every partial symmetric M- (M_0-) matrix specifying the pattern can be completed to a symmetric M- (M_0-) matrix.

In view of the fact that the property of being a SIM (symmetric M-, symmetric M_0-) matrix is inherited by principal submatrices, it is hardly surprising that the property of having SIM (symmetric M-, symmetric M_0-) completion is inherited by principal subpatterns [H4]. Thus, if a principal subpattern $Q[\alpha]$ of the pattern Q does not have SIM (symmetric M-, symmetric M_0-) completion, neither does Q.

In recent years, graphs and directed graphs (digraphs) have been used very effectively to study matrix completion problems. Patterns that are positionally symmetric can be studied by means of their graphs. Our graph terminology follows [H4], and the reader is referred to that paper for the precise definitions of graph theoretic terms. Let A be a (fully specified) symmetric $n \times n$ matrix. The *nonzero-graph* of A is the graph having as vertex set $\{1,...,n\}$, and, as its set of edges, the set of (unordered) pairs $\{i,j\}$ such that both i and j are vertices with $i \neq j$ and $a_{ij} \neq 0$. For a positionally symmetric pattern Q, the *pattern-graph* of Q is the graph having $\{1,...,n\}$ as its vertex set and, as its set of edges, the set of (unordered) pairs $\{i,j\}$ such that position (i,j) (and therefore also (j,i)) is in Q. If G is the pattern-graph of Q, then the pattern-graph of a principal subpattern $Q[\alpha]$ is $\langle\alpha\rangle$, the subgraph induced by α. The principal subpattern $Q[\alpha]$ and the induced subgraph $\langle\alpha\rangle$ are said to correspond; in particular, the vertex v and diagonal position (v,v) correspond.

A *component* of a graph is a maximal connected subgraph. A *cut-vertex* of a connected graph is a vertex whose deletion disconnects the graph; more generally, a cut-vertex is a vertex whose deletion disconnects a component of the graph. A graph is *nonseparable* if it is connected and has no cut-vertices. A *block* of a graph is a subgraph that is nonseparable and is maximal with respect to this property. A (sub)graph is called a *clique* if it contains all possible edges between its vertices. A graph is *block-clique* if every block is a clique. Block-clique graphs are called “1 chordal” in [JS2]. It is well known that a graph G is block-clique if and only if the induced subgraph of any cycle of G is a clique [JS1].

A digraph is *transitive* if the existence of a path from v to w implies the arc (v,w) is in the digraph. Recall that the nonzero-digraph of any inverse M-matrix is transitive [LN].

Johnson and Smith [JS2] determined that a pattern that includes all diagonal positions has SIM completion if and only if the graph of the pattern is block-clique. For some classes Π, such as symmetric M-matrices, the case of a pattern that includes all diagonal positions is the only part of the problem that needs to be studied, because a pattern has Π-completion if and only if the principal subpattern defined by the diagonal positions included in the pattern has Π-completion [H4]. However, this reduction does not work for SIM matrix completions or symmetric M_0-matrix completions [H4]. Thus the classification of patterns having completion for these classes is not complete without consideration of patterns omitting some diagonal positions. This is done for the class of SIM matrices in the next section. The patterns that have symmetric M- and symmetric M_0-completion are classified in the third section.

2. Classification of patterns having SIM completion

In a digraph D, an alternate path to a single arc is a path $(v_1,v_2), (v_2,v_3), ..., (v_{k-1},v_k)$ with $k > 2$, such that (v_1,v_k) is an arc of D. A pattern-digraph is called path-clique if the induced subdigraph of any alternate path to a single arc is a clique and the diagonal position (v_i,v_i) is included in the pattern for every vertex v_i in the path. In [H3] it was shown that a necessary condition for a pattern to have inverse M-completion is that its pattern-digraph be path-clique. In this section we show that the analogous condition remains necessary for SIM completion. The analogous definition for an alternate path to a single edge would be a path $\{v_1,v_2\}, \{v_2,v_3\}, ..., \{v_{k-1},v_k\}$ with $k > 2$, such that $\{v_1,v_k\}$ is an edge of G. But such an alternate path and the edge combine to form the cycle $v_1, v_2..., v_k, v_1$. Thus the condition analogous to "path-clique" for a pattern-graph is that for any cycle $v_1, v_2..., v_k, v_1$, the induced subgraph $\langle\{v_1, v_2..., v_k\}\rangle$ is a clique and the diagonal position (v_i,v_i) is included in the pattern for every vertex v_i in the cycle. The necessity of this condition for SIM completion is established in Theorem 2.1 below, and the characterization of patterns having SIM completion is given in Theorem 2.2.

2.1 Theorem Let Q be a pattern and let G be its pattern-graph. If Q has SIM completion, then the subgraph induced by the vertices of any cycle in G is a clique and the diagonal positions corresponding to the vertices of the cycle are all included in Q.

Proof: Let Q be a pattern and let G be its pattern-graph. Suppose G does not have the required property. Let Γ be a cycle of minimal length among cycles such that the subgraph induced by the vertices of the cycle is not a clique or at least one of the diagonal positions corresponding to a vertex in the cycle is omitted from Q. By renaming vertices if necessary, assume $\Gamma = 1, 2, \ldots, k, 1$ with $k > 2$.

Suppose first that $\langle\{1,...,k\}\rangle$ does not contain any chord of Γ. If $k > 3$, then $Q[\{1,...,k\}]$ does not contain any complete principal subpattern of size larger than 2×2, because all chords are omitted. If $k = 3$, then Γ is the cycle 1,2,3,1 and $\langle\{1,2,3\}\rangle$ is a clique, so necessarily Q omits at least one of (1,1), (2,2), (3,3). Thus, in either case, $Q[\{1,...,k\}]$ does not contain any complete principal subpattern of size larger than 2×2. Define a $k \times k$ partial matrix B specifying $Q[\{1,..,k\}]$ by $b_{ii} = 2$ for $i=1,...,k$, $b_{i\,i+1} = 1 = b_{i+1\,i}$ for $i=1,...,k-1$, and $b_{1k} = 0 = b_{k1}$. Since the only completely specified principal submatrices are 2×2 or smaller, and these are SIM matrices, B is a partial SIM matrix. But B cannot be completed to a SIM matrix because the nonzero-digraph of any completion of B is not transitive, since $b_{12} = ... = b_{k-1\,k} = 1$ and $b_{1k} = 0$. So $Q[\{1,...,k\}]$ does not have SIM completion.

Now suppose $k > 3$ and $\langle\{1,...,k\}\rangle$ contains a chord of Γ. Each of the two pieces of Γ on either side of the chord, together with the chord, forms a shorter cycle. By the minimal length assumption, Q must include all diagonal positions corresponding to vertices in these two shorter cycles. Hence, Q includes all diagonal positions (1,1), ..., (k,k). Then $\langle\{1,..,k\}\rangle$ is not a clique, and thus is not block-clique. So $Q[\{1,...,k\}]$ is a pattern that includes all diagonal positions and whose graph is not block-clique, and therefore does not have SIM completion [JS2].

In either case Q does not have SIM completion because Q[{1,...,k}] does not. ■

Theorem 2.1 provides the necessary material for one direction of the next theorem. In [JS2], it was shown that a pattern that includes all diagonal positions and whose pattern-graph G is obtained from two cliques G_1 and G_2 by identifying a vertex of G_1 with a vertex of G_2 has SIM completion, and thus a pattern including the diagonal whose pattern-graph is block-clique has SIM completion. In [H4] this was extended to the case where G_1 and G_2 are not necessarily cliques, but the principal subpattern corresponding to each has SIM completion, yielding the result that a pattern has SIM completion if and only if the principal subpattern corresponding to each block of its pattern-graph does.

2.2 Theorem A pattern has SIM completion if and only if its pattern-graph is block-clique and for every vertex v in a block of order > 2, the diagonal position (v,v) is included in the pattern.
Proof: Let Q be a pattern that has SIM completion and let G be its pattern-graph. By Theorem 2.1, G has the property that the induced subgraph of any cycle is a clique, so G is block-clique. Furthermore, Q must include the diagonal position corresponding to any vertex in a cycle. Since any vertex in a block of order > 2 occurs in a cycle, Q must include the diagonal position corresponding to any vertex in a block of order > 2.

For the converse, suppose Q is a pattern such that its pattern-graph G is block-clique and for every vertex v in a block of G of order > 2, the diagonal position (v,v) is included in Q. Any pattern for 2 × 2 matrices has SIM completion [H4]. Thus the principal subpattern of Q corresponding to each block of G is complete or has SIM completion. Then Q has SIM completion, by [JS2], [H4]. ■

The next example exhibits patterns that do not include all diagonal positions, one having SIM completion and other not having SIM completion. In the diagrams of the pattern-graphs, if a diagonal position (v,v) is included in the pattern, then vertex v is indicated by a solid black dot ( ); if (v,v) is omitted, then vertex v is indicated by a hollow circle ( ) (this follows the notation of [H3] and [H4]).

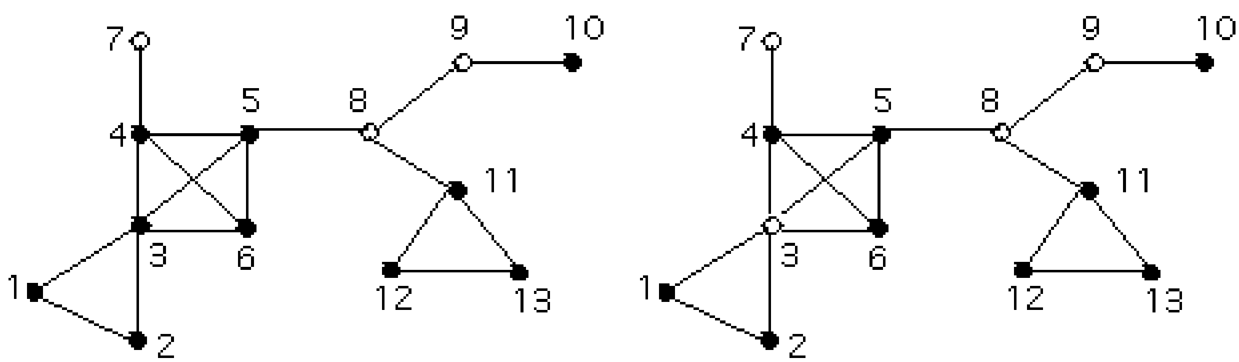

Figure 1 (a) Q_1 has SIM completion (b) Q_2 does not have SIM completion

2.3 Example The pattern Q_1 = {(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,3), (4,4), (4,5), (4,6), (4,7), (5,3), (5,4), (5,5), (5,6), (5,8), (6,3), (6,4), (6,5), (6,6), (7,4), (8,5), (8,9), (8,11), (9,8), (9,10), (10,9), (10,10), (11,8), (11,11), (11,12), (11,13), (12,11), (12,12), (12,13), (13,11), (13,12), (13,13)}, whose pattern-graph is shown in Figure 1(a), has SIM completion. The pattern Q_2 obtained from Q_1 by deleting the diagonal position (3,3), whose pattern-graph is shown in Figure 1(b), does not.

3. Symmetric M- and symmetric M_0-matrices

It is well-known that a partial M- (respectively, M_0-) matrix with all diagonal entries specified can be completed to an M- (M_0-) matrix only if the zero completion of A is an M- (M_0-) matrix [JS1], [H4] (the *zero completion* of A is a the result of setting all unspecified entries to 0).

Lemma 3.1 If a pattern Q has symmetric M- or symmetric M_0-completion and includes positions (a,a), (b,b), (c,c), (a,b), (b,a) (b,c), and (c,b), then Q also includes (a,c) and (c,a).
Proof: Suppose Q includes positions (a,a), (b,b), (c,c), (a,b), (b,a) (b,c), and (c,b), and does not include (a,c) and (c,a). Then the partial matrix $A = \begin{bmatrix} 1 & -0.9 & ? \\ -0.9 & 1 & -0.9 \\ ? & -0.9 & 1 \end{bmatrix}$ is a partial symmetric M-matrix specifying Q[{a,b,c}] that cannot be completed to an M_0-matrix because the zero-completion of A has determinant -0.62. Thus Q does not have either symmetric M- or symmetric M_0-completion. ■

Theorem 3.2 Let Q be a pattern with the property that if (v,w) is in Q then (v,v) and (w,w) are both in Q. Then Q has symmetric M- (symmetric M_0-) completion if and only if each component of its pattern-graph is a clique.
Proof: If each component of the graph is a clique, then the pattern has symmetric M- and symmetric M_0-completion because the pattern is permutation similar to a block-diagonal pattern with each diagonal block being complete or a single omitted diagonal position [H4].

For the converse, let Q have symmetric M- or symmetric M_0-completion. Let H be a component of the pattern-graph of Q. Since H is connected, for any two vertices v and u of H, there is a path from v to u. Since Q contains the diagonal positions corresponding to all vertices in H, we can use Lemma 3.1 and induction on the length of the path from v to u to conclude that H must contain {v,u}. Thus H is a clique. ■

As noted earlier, a pattern has symmetric M-completion if and only if the principal subpattern determined by the diagonal positions does [H4]. The principal subpattern determined by the diagonal positions satisfies the hypotheses of Theorem 3.2, so this completes the classification of patterns having symmetric M-completion.

Corollary 3.3 A pattern Q has symmetric M-completion if and only if each component of the pattern-graph of the principal subpattern determined by the diagonal positions of Q is a clique.

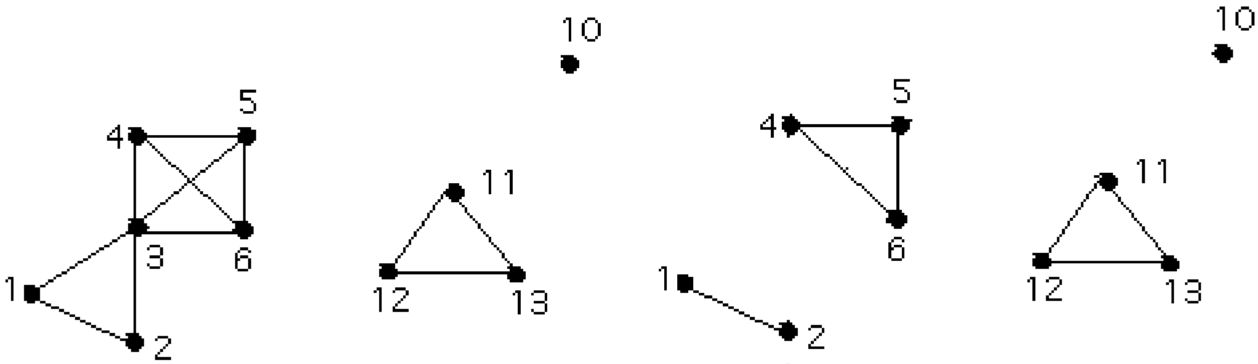

Figure 2 (a) R_1 (b) R_2

Example 3.4 The pattern Q_1 in Example 2.3 does not have symmetric M-completion because one of the components of the pattern-graph of R_1, the principal subpattern determined by the diagonal, is not a clique. The pattern Q_2 in Example 2.3 has M-completion, because each component of the pattern-graph R_2, the principal subpattern determined by the diagonal positions, is a clique. The pattern-graphs of R_1 and R_2 are shown in Figure 2(a) and (b), respectively.

For a symmetric class Σ, when patterns omitting diagonal positions are examined, one of two things frequently happens:
(1) a pattern has Σ-completion if and only if the principal subpattern determined by the diagonal positions included in the pattern does, or
(2) the pattern $\{(a,a), (a,b), (b,a)\}$ does not have Σ-completion.
(Note that neither of these things happened for SIM matrices.) If (1) applies, then the Σ-completion problem reduces to determining which patterns that include the diagonal have Σ-completion, as we have just done for symmetric M-matrices. If (2) applies, then the Σ-completion problem reduces to determining which patterns that include the diagonal or omit all diagonal positions have Σ-completion (this statement will be proved in Lemma 3.6). This is the situation for symmetric M_0-matrices:

Example 3.5 The partial matrix $A = \begin{bmatrix} 0 & -1 \\ -1 & ? \end{bmatrix}$, which specifies the pattern $Q = \{(1,1),(1,2), (2,1)\}$, is a partial symmetric M_0-matrix. But A cannot be completed to an M_0-matrix because the determinant of any completion of A equals -1.

Thus, neither Q_1 nor Q_2 from Example 2.3 has symmetric M_0-completion, because both contain the principal subpattern $S = \{(4,4), (4,7), (7,4)\}$.

Lemma 3.6 Let Σ be a class of symmetric matrices such that $\{(a,a), (a,b), (b,a)\}$ does not have Σ-completion. Then if a pattern has Σ-completion, every principal subpattern corresponding to a component of its pattern-graph includes all diagonal positions or omits all diagonal positions.
Proof: Let Q be a pattern, let G be its pattern graph, let H be a component of G, and let S be the principal subpattern corresponding to H. Suppose S includes (v,v) and omits (w,w). Since H is a component, it is connected, and there is a path $\{u_1,u_2\}, \{u_2,u_3\}, \ldots, \{u_{k-1},u_k\}$ from vertex $v = u_1$ to vertex $w = u_k$. Let t be the number such that S includes $(u_1, u_1), \ldots, (u_t, u_t)$ and S does not include (u_{t+1}, u_{t+1}) (such a t must exist since S includes $(v,v) = (u_1, u_1)$ and omits $(w,w) = (u_k, u_k)$). Then $Q[\{u_t, u_{t+1}\}] = \{(u_t, u_t),(u_t, u_{t+1}),(u_{t+1},u_t)\}$. Thus by the hypothesis, $Q[\{u_t, u_{t+1}\}]$ does not have Σ-completion, and so neither does Q. ■

Theorem 3.7 Let Q be a pattern and let G be its pattern-graph. Then Q has symmetric M_0-completion if and only if every principal subpattern corresponding to a component of G either omits all diagonal position, or includes *all* positions.
Proof: Suppose every principal subpattern of Q corresponding to a component H of G either omits all diagonal positions or includes all positions. By [H4], a pattern that omits all diagonal positions has symmetric M_0-completion. A pattern that includes *all* positions trivially has symmetric M_0-completion. Since Q is permutation similar to a block-diagonal pattern in which each diagonal block has symmetric M_0-completion, Q has symmetric M_0-completion [H4].

The converse follows from Example 3.5, Lemma 3.6 and Theorem 3.2. ■

Corollary 3.8 Any pattern that has symmetric M_0-completion also has symmetric M-completion, but the converse is false. A pattern that includes all diagonal positions and has symmetric M-completion also has symmetric M_0-completion.

We can also use Lemma 3.6 to complete the classification of other classes of symmetric matrices. The matrix A is *positive semidefinite* if A is symmetric and $x^TAx \geq 0$, or equivalently, A is symmetric and every principal minor is nonnegative. In [H4] this idea, together with results from [GJSW] and [JS3] was used to show that a pattern has positive semidefinite completion if and only if every principal subpattern corresponding to a component H of G either omits all diagonal positions, or includes all diagonal positions and H is chordal.

The matrix A is *doubly nonnegative (DN)* if A is positive semidefinite and entrywise nonnegative. The matrix A is *completely positive (CP)* if $A = BB^T$ for some entrywise nonnegative $n \times m$ matrix B. Drew and Johnson [DJ] determined that a pattern that includes the diagonal has DN- (CP-) completion if and only if its pattern-graph is block-clique. Example 3.1 of [H4] also works to show that {(a,a), (a,b), (b,a)} does not have DN- or CP-completion.

Corollary 3.9 Let Q be a pattern and let G be its pattern-graph. Then Q has DN- (CP-) completion if and only if every principal subpattern corresponding to a component H of G either omits all diagonal positions, or includes all diagonal positions and H is block-clique.

Furthermore, Lemma 3.6 can be extended to positionally symmetric patterns even when the class of matrices is not symmetric.

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