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Pan Liu
Beijing Normal University

Torben M. Andersen
University of Aarhus

Joydeep Bhattacharya
Iowa State University, joydeep@iastate.edu

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Keywords

time inconsistency, commitment, present bias

Disciplines

Behavioral Economics | Economic Policy | Public Economics

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Pan Liu[*]
Beijing Normal University

Torben M. Andersen[†]
University of Aarhus

Joydeep Bhattacharya[‡]
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[*]Corresponding author: Business School, Beijing Normal University, Beijing, 100875, China. E-mail: liupan@bnu.edu.cn

[†]Corresponding author: Department of Economics and Business, Aarhus University, Fuglesangs Allé 4, DK-8210 Aarhus V, Denmark. E-mail: tandersen@econ.au.dk

[‡]Department of Economics, Iowa State University, Ames IA 50011-1070, USA. Phone: (515) 294 5886, Fax: (515) 294 0221; E-mail: joydeep@iastate.edu

1 Introduction

People in their fifties, for whom retirement is looming, often find themselves in the following predicament. Even as they develop the best intentions of wanting to save for the post retirement years, they confront current consumption demands that simply cannot wait. Many are self aware enough to realize that the supposed "can't-wait" immediacy of current consumption may *reappear* in the early retirement years. At that time, the trip to Italy would, again, likely not wait. And it will be paid for by eating into retirement savings (or borrowing against them) thereby jeopardizing consumption during the *late* retirement years. This paper studies the problem of people who are in the above-described quandary and are looking for outside help.[1] Specifically, it asks, how should such middle-aged people, with varying degrees of self-awareness or sophistication about their present-bias, invest their savings so as to, both, finance current gratification *and* thwart their early-retirement self from impoverishing their late-retirement self?

At first glance, it would appear that self aware people in such situations ought to seek out commitment devices that lock in saving to prevent "overspending" or borrowing by their future, recently-retired selves.[2] Governments, acting in a paternalistic fashion, routinely offer such commitment machinery to help such people out. These include, for example, restrictions (or bans) on early withdrawals from defined contribution (DC) retirement schemes (Beshears et al. 2015a). A germane example is the U.K. where, up until recently, residents on money-purchase pension schemes were forced to take an annuity – the income guaranteed by pension providers in exchange for receiving all or part of the funds in their pension pot. Additionally, a 55% tax rate was imposed on anyone who took out more than 25% of the savings in their pension pot.[3] In the U.S., retirement savings accounts are partially illiquid: withdrawals before age $59\frac{1}{2}$ incur a 10% tax

[1]There is another branch of the literature that studies the role of *inside* commitment devices (broadly, choice-set restrictions) such as "personal budgeting" which involves "grouping expenses into categories and constraining each with an implicit or explicit cap applied to a specified time period" (Galperti, 2019). Our focus is entirely on the use of *outside* commitment machinery whereby agents lock in their saving into specific assets of varying liquidity.

[2]Rarely are these devices provided by the market. As Kocherlakota (2001) argues, good commitment assets, by their very nature, are hard to sell and hard to use as collateral.

[3]In Singapore, those turning 55 after 2012 may roughly withdraw upto S$5,000 of their Central Provident Fund (CPF) balances; remainder is paid out as an annuity beginning at the drawdown age of 64. See Beshears et al. (2015a) for a comprehensive look at the liquidity provisions embedded in the various employer-based defined contribution (DC) retirement savings systems of rich nations.

penalty. Critics argue such restrictions on the liquidity of retirement savings hurt those who must "respond to pre-retirement events that raise the marginal utility of consumption, like medical emergencies or income shocks" (Beshears et al. 2015b). Others point out that such limitations reduce the scope for "behavioral mistakes". In this paper, we study the optimal usage of commitment devices by self-aware, present-biased agents and the restrictions on early withdrawal of their retirement savings.

To that end, we employ ideas about present-biasedness and associated self-awareness popular in the literature. From Chetty (2015), we adopt a) the notion that individuals are comprised of multiple selves, possibly in conflict with one another, and b) the construct of a rift between a self's "true preferences" (experienced utility), that which she uses to determine how much she should save, versus her "choice" or "behavioral" preferences (decision utility), that which determines how much she actually saves. The latter can help rationalize the gap between actual and best-intention saving if, for example, the choice preferences of the current self attach a lower weight on future utility than her true preferences do – this is *present-bias* from the standpoint of the true self. Likewise, there may be disagreements – *preference reversal* – between the choice preferences of the current middle-aged self and her future retired selves. Time-inconsistent preferences (quasi-hyperbolic discounting) help explain the gap between what the current, decision-making self wishes a future self to save and what that self, when her turn to decide arrives, actually does.

Much depends on the self-awareness of the current self. Following O'Donoghue and Rabin (2001), we allow for partial naivete (sophistication) where the current self has beliefs about the time preference of future selves that are, in principle, different from the *actual* preference of the latter. This means the agent is aware she will have to wrestle with self-control problems in the future but is not fully aware of their magnitude. The more aware an yet-to-retire self is of the impending preference reversal, the more sophisticated she is, and the stronger her desire to protect the consumption possibilities of her late-retirement self. This is precisely when commitment devices are sought.

We cast these ideas in the context of a simple lifecycle model in which a time-inconsistent agent lives for three periods, middle-aged, old, and very old. She has access to two safe saving instruments, a liquid one-period asset with return, $R_1 = R > 1$ and a two-period asset with return $R_2 = R^2$ (if liquidated after two periods) and a return $(1-\theta)\,R$ (if re-traded, i.e., liquidated after one period) where θ is an early-withdrawal (tax) penalty imposed by the government. (Alternatively, one may think of θ as a short-cut measure of interest loss due to re-trading or credit frictions.) By choosing a portfolio of these assets, the middle-aged self makes a commitment towards retirement consumption. The strength of this commitment depends on whether her future, just-retired self will/can undo her plans by liquidating some of her two-period asset holding. The following margins are at play. First, the yet-to-retire self is herself present-biased but is also concerned about retirement consumption. Second, depending on how sophisticated she is, she incorporates her *perception* of the choices to be made by her just-retired, present-biased self which likely differ from the actual choices the latter will make. The yet-to-retire self may find it desirable to give up some current utility so as to pass on more wealth to her just-retired self. This would indulge the latter's present bias, thwart premature liquidation, and in the process, protect her late-retirement self. Or, should she allow her just-retired self to liquidate early? Which is the better strategy, from the perspective of the middle-aged self's choice *and* true utility? What, then, is the "optimal" θ? While higher penalties may reduce premature withdrawals, could they discourage saving thereby sabotaging the aim of raising net savings? More generally, is a "stronger" commitment technology (larger costs of deviating from full commitment) always welfare improving?

Intuitively, the more naive a person is, the less their demand for commitment. Such a person is mostly unaware of their impending preference reversal and do not feel a strong need to protect against it. For such a person, higher θ (higher early-withdrawal penalties) may improve their choice utility. We go on to show that, for partially-naive individuals, a higher θ is not necessarily welfare improving from a *true utility* perspective. Clearly, credit frictions provide an important societal benefit because their presence makes it harder for future selves to borrow/liquidate, and hence finance a deviation from previously made plans. This is why more sophisticated agents like higher θ. What is somewhat striking is that, for low θ, agents with some naivete may be happier from a true utility perspective than their fully sophisticated counterparts!

Our paper is part of a long line of research – Strotz (1956), Phelps and Pollack (1968), Laibson (1997, 1998) and the review by Bryan, Karlan, and Nelson (2010) – studying the demand for commitment among present-biased individuals; specifically, what is the optimal savings rule in models where people are tempted to consume earlier than what their best-intention plans suggest? In particular, Laibson (1997) and others have emphasized how sophistication can lead people to invest heavily in illiquid assets as a commitment device. Even though evidence of non-experimental evidence for a demand for commitment is somewhat sketchy (Laibson, 2015), there is general appreciation that these notions, nevertheless, can improve our understanding of saving behavior. Amador, Werning, and Angeletos (2006) explain low demand for commitment by pointing to the fact that uncertainty about future consumption needs (due to taste and income shocks) generates a countervailing demand for flexibility. Beshears et al. (2015b) extend their results to show that sophisticated agents invest more in commitment accounts that are more illiquid even when there is a demand for flexibility due to uninsurable taste shocks. Our paper is also thematically connected to John (2019) which highlights the argument that commitment may be welfare improving if an agent can "anticipate how her future selves will behave" and that "adopting a commitment device that is ill-suited to one's preferences may backfire and become a threat to welfare." In our case, too strict a level of commitment may be incompatible with one's beliefs about the present-biasedness of their future selves.

Our paper generates a demand for flexibility not via unforeseen (but insurable) shocks but by allowing for agents to form beliefs about the present-biasedness of their future selves. As such, in our setup, it matters how sophisticated an agent is, meaning how aware she is of the possibility that she may be about to relinquish control over her retirement consumption to her immediate future self who, in turn, may prioritize her own utility over that of an even later self thereby hurting the current decisionmaker. By tying her hands completely, she may get hurt, not because she may get hit with a taste/income shock, but because her future self may untie those hands unless the cost of doing so is prohibitive. In a way, our paper informs the larger literature on low uptake of commitment assets (Laibson, 2015) by adding another possible reason for this observed behavior: agents may shy away from buying commitment not because they seek flexibility to handle taste or budget shocks but because they may not have accurate beliefs about the extent of present biasedness of their future selves.

The plan for the rest of the paper is as follows. Section 2 describes the model economy, the primitives and the agent choice sets. Section 3 considers the perceived and actual choices by the middle-aged and the old, while Section 4 studies the effect of the strength of the commitment technology on true and choice welfare. Section 5 concludes. Proofs are contained in the appendices.

2 The model

Consider an economy with a unit mass of agents who live through three time periods – middle-aged (m), old (o) and very old (v). Agents receive an endowment, w, only as middle-aged and must save to provide for consumption when old and very old.

2.1 Preferences

Agents may have both myopic and quasi-hyperbolic preferences (exhibiting time-inconsistent behavior), and as such, we distinguish between their "true" and "choice" utility see e.g. Chetty (2015). Agents' behavior is dictated by their choice utility, but their actual well-being is governed by their true lifetime utility. Let c_m denote consumption as middle-aged, c_o denote consumption as old, and c_v be consumption as very old. The felicity function $u(\cdot)$ is assumed to fulfill standard assumptions, including $u_c(\cdot) > 0$, $u_{cc}(\cdot) < 0$ and Inada conditions. For some specific results we assume a CES form, i.e., $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}, \sigma > 0$.

The "true" life-time preferences, with a "*", defined over consumption in each period of life is given by

$$\Omega^* \equiv u(c_m) + \beta^*\delta^* \left[u(c_o) + \delta^* u(c_v)\right] \tag{1}$$

where $\delta^* \in [0, 1]$ is the true discount factor and β^* is a parameter. The life-time choice preferences when middle-aged are given as

$$\Omega^m \equiv u(c_m) + \beta\delta \left[u(c_o) + \delta u(c_v)\right] \tag{2}$$

and of when old as

$$\Omega^{oa} \equiv u(c_o) + \beta\delta u(c_v) \tag{3}$$

where δ is the actual discount factor. If $\delta = \delta^*$ and $\beta = \beta^*$, then true and choice preferences coincide. If $\delta < \delta^*$ but $\beta = \beta^* = 1$, the agent's choice suffers from myopia. If $\delta = \delta^*$ and $\beta < 1$, her preferences exhibit preference reversal. The parameter $\beta < 1$ thus represents quasi-hyperbolic preferences which generate time inconsistency (preference reversal) since the marginal rate of substitution (MRS) between consumption as old and very old from the view point of the middle-aged is $\frac{\partial c_v}{\partial c_o}\ |_{\text{middle-aged}} = -\frac{u_c(c_o)}{\delta u_c(c_v)}$ while it is $\frac{\partial c_v}{\partial c_o}\ |_{\text{old}} = -\frac{u_c(c_o)}{\beta\delta u_c(c_v)}$ from the point of view of the old – see Laibson (1997). Clearly, for $\delta = \delta^*$ and $\beta < \beta^* < 1$, her choice preferences show no myopia (but does show preference reversal) and shows more present bias than her true self. In short, for $\delta < \delta^*$ and $\beta = 1$, there is myopia and no preference reversal, and for $\delta = \delta^*$ and $\beta < 1$ preference reversal but no myopia. *Henceforth*, without loss of generality, we set $\beta^* = 1$. In that case, the true preferences can written as

$$\begin{aligned} \Omega^* &\equiv u(c_m) + \delta^* \left[u(c_o) + \delta^* u(c_v)\right] \\ \Omega_o^* &\equiv u(c_o) + \delta^* u(c_v) \\ \Omega_v^* &\equiv u(c_v) \end{aligned}$$

Here on, our yardstick of welfare will be Ω^*.

Following O'Donoghue and Rabin (2001), we allow for partial naivete (sophistication) where the middle-aged have beliefs/perceptions (β^p) about the time-preference of the old that are, in principle, different from the old's actual preference (β):

$$\Omega^{op} \equiv u\left(c_o\right) + \beta^p \delta u\left(c_v\right), \tag{4}$$

where $\beta \leq \beta^p \leq 1$. When $\beta^p = \beta$, the agent is fully sophisticated and when $\beta^p = 1$, the agent is fully naive. When $\beta < \beta^p < 1$, the agent is aware that she has future self-control problems, but underestimates their magnitude. Importantly, the middle-aged perceive that their behavior as old is governed by (4) while in actuality it is determined by (3). Our description thus far can also be understood in terms of a model of multiple selves where future selves may behave differently than what the current self anticipates and finds optimal.

Of interest are two margins. The first is in the current, the consumption allocation between the middle-aged and the future ($u(c_m)$ vs. $u(c_o) + \delta u(c_v)$). The second is in the future, the consumption allocation between old and very old age ($u\left(c_o\right)$ vs. $u\left(c_v\right)$). The first margin grapples with present bias captured by $\beta\delta < \delta^*$. The second one, in addition, must incorporate the idea of incorrect beliefs or partial naivete, $\beta^p \geq \beta$. Figure 1 illustrates.

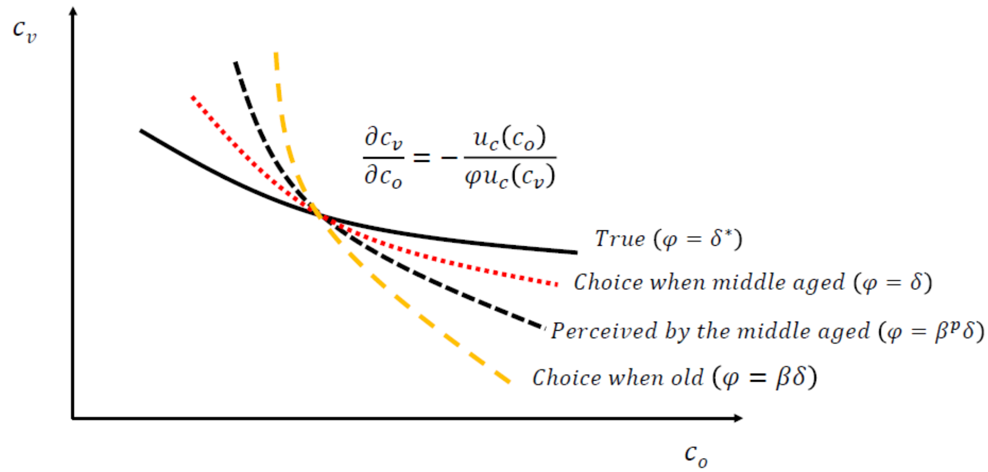

Figure 1: Indifference curves (c_o, c_v) – true, perceived, and choice preferences

The true preferences of the middle aged over c_o and c_v are shown by the solid black indifference curve based on the discount factor δ^*. The old's actual choices regarding c_o and c_v are made with a discount factor $\beta\delta < \delta^*$ and lie on the yellow dashed line which is steeper than the solid curve. The middle-aged's actual choices follow the red dotted line. She *perceives* that, as old, her decisions regarding c_o and c_v will be made based on the black dashed indifference curve since $\beta^p\delta \geq \beta\delta$ ($\beta^p\delta < \delta^*$).

2.1.1 Commitment technology/illiquidity

Much like in Diamond and Dybvig (1983), the middle-aged agent can save in two safe assets – a one-period asset with a gross, one-period return R_1 and a two-period asset with a two-period return of R_2. Without loss of generality, assume $R_1 = R$ and $R_2 = R^2$. The middle-aged makes a personal commitment on consumption as old and very old, respectively, via holdings of these two assets. How strong the commitment is will depend on whether the later self as old will/can undo it by partial liquidation of the two-period asset. The government also helps determine the strength of the commitment technology (parameterized by θ; the higher θ, the stronger the commitment technology). For concreteness, think of θ a tax penalty on early liquidation of the two-period asset such that the after-tax, one-period return on the two-period asset is $(1-\theta)R$. Clearly, if $\theta \to 1$ there is full commitment. For completeness, note that the old, if they so wish, can make additional savings at the rate R.

Our formalization is sparse and stylized so as to permit singular focus on the issue at hand. While we describe θ as a tax penalty above, one may re-interpret our formulation as an exogenously specified characterization of market frictions (with θ capturing some sort of dead-weight loss). First, think of annuities. Suppose π_o is probability of reaching old age and π_v is the probability for an old agent to reach very old age. Then, we may define $\Omega^m \equiv u(c_m)+\delta\beta\pi_o\left[u(c_o)+\beta\pi_v u(c_v)\right]$ while Ω^{op} continues to be defined by (4) and Ω^{oa} by (3) with π_o and π_v added. An annuity delivering one unit of consumption in case of survival has a so-called fair price $q=\frac{\pi}{R}<\frac{1}{R}$ where R is the gross real return to a safe, one-period bond. The gross rate of return on the annuity is the inverse of the price, i.e., $\frac{R}{\pi}$. If there are annuities that are bought in middle-age and pay out 1 unit when old and 1 unit when very old, under complete markets, the price of the former is $\frac{\pi_o}{R}$, and that of the latter is $\frac{\pi_o\pi_v}{R^2}$. Finally, an annuity that is bought when old and pays 1 unit when very old is priced at $\frac{\pi_v}{R}$. In this case, one can easily redefine $R_1=\frac{R}{\pi_o}$ and $R_2=\frac{R^2}{\pi_o\pi_v}$ and θ, in that case, would be a penalty (sold at a discount) imposed by the annuity market for early liquidation of the two-period annuity. Alternatively, consider a setup in which an old agent may borrow against future wealth at the rate $R_b>R$ where the gap (R_b-R) is akin to θ; the higher the gap (R_b-R), the harder it is for the old to borrow and undo the plans of the middle-aged, hence easier for the latter to achieve commitment.

2.1.2 Budget constraints

The budget constraint for the middle-aged reads $w=c_m+s_o+s_v=c_m+s_m$ where $s_v(s_o)$ is saving in the two (one) period asset and total saving, s_m is $s_m\equiv s_o+s_v$. For the old, the holdings of assets (s_o and s_v) are predetermined. Define b as the portion of the very old savings (s_v) liquidated by the old. In this case, the budget constraint is

$$
\begin{aligned}
&(5)\quad c_o = Rs_o+(1-\theta)Rb; \qquad 0<b<s_v\\
&(6)\quad c_v = R^2(s_v-b)
\end{aligned}
$$

The old, in addition to s_v, may also choose to save for very old age ($b\le 0$)

$$
\begin{aligned}
&(7)\quad c_o = Rs_o+Rb,\\
&(8)\quad c_v = R^2(s_v-b).
\end{aligned}
$$

Combining, we have

$$(9)\quad \begin{cases} c_o + (1-\theta)\,\frac{c_v}{R} = Rs_o + (1-\theta)\,Rs_v \text{ for } c_o > Rs_o \;\;(b>0) \\ c_o + \frac{c_v}{R} = Rs_o + Rs_v \text{ for } c_o \leq Rs_o \;\;(b \leq 0) \end{cases}.$$

This defines the budget set in the (c_o, c_v) space, illustrated in Figure 2. For all $c_o \leq Rs_o$, the budget line is AC with slope $-R$ but for $c_o > Rs_o$, the budget line is CD with a steeper slope $-\frac{R}{1-\theta}$ – there is a kink at C.

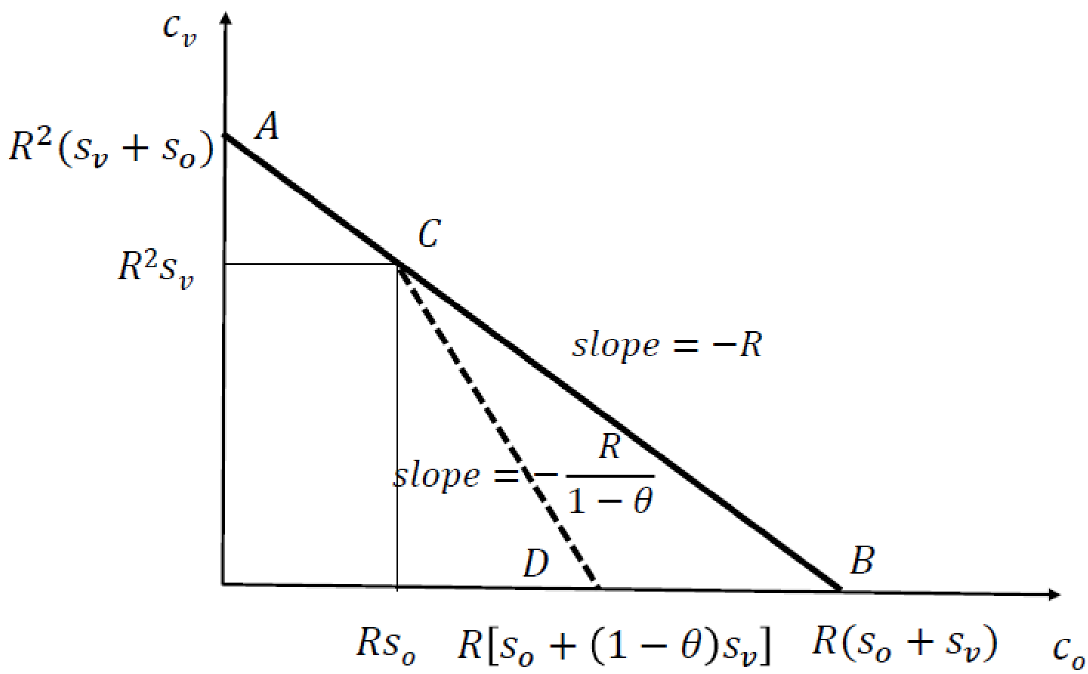

Figure 2: Budget set - consumption when old and very old (c_o, c_v)

Notice, early liquidation as old implies an allocation on the segment CD, which lies entirely inside of the segment AB. Such a move, to foreshadow, would hurt the middle-aged by lowering life-time welfare. The middle-aged would like to thwart this and commit to choices of s_o and s_v so as to keep the old on the segment ACB if possible (by "moving" point C toward B). However, the actual budget constraint for the old is ACD with θ determining the cost of deviating from the plan set by the middle-aged. The middle-aged understands the role of θ and also perceives $(\beta^p \leq 1)$ her old self's desire to be on the segment CD. Her perception may, of course, be wrong. These struggles form the subject matter of the rest of the paper.

3 Middle-aged savings and asset allocation

To determine the saving decision made as middle-aged, the usual backward induction method is applied. Bear in mind that partial naivete means decisions made by the middle-aged depend on her incorrect *perception* of the preferences her old self will use to make choices. Further below, we also study the *actual* decisions made when old.

3.1 What the middle-aged perceives her old self will choose

Define b^p as the middle-aged's perception of how much the old will liquidate. Given yet-to-be-determined choices (s_o, s_v), the middle-aged perceives that her old self faces the following decision problem:

$$\max_{c_o, c_v} \ \Omega^{op} = u(c_o) + \beta^p \delta u(c_v)$$

$$\text{s.t.} \begin{array}{l} c_o + (1-\theta)\frac{c_v}{R} = Rs_o + (1-\theta) Rs_v \text{ for } c_o > Rs_o \ (b^p > 0) \\ \qquad c_o + \frac{c_v}{R} = Rs_o + Rs_v \text{ for } c_o \leq Rs_o \quad (b^p \leq 0). \end{array}$$

The solution is summarized as follows:

$$(10) \quad \begin{array}{lll} b^p > 0 & \text{if } \dfrac{u_c(Rs_o)}{\beta^p \delta u_c(R^2 s_v)} > \frac{R}{1-\theta} & b^p : \dfrac{u_c(Rs_o + (1-\theta) Rb^p)}{\beta^p \delta u_c(R^2(s_v - b^p))} = \frac{R}{1-\theta} \\ b^p = 0 & \text{if } R \leq \dfrac{u_c(Rs_o)}{\beta^p \delta u_c(R^2 s_v)} \leq \frac{R}{1-\theta} & b^p = 0 \\ b^p < 0 & \text{if } \dfrac{u_c(Rs_o)}{\beta^p \delta u_c(R^2 s_v)} < R & b^p : \dfrac{u_c(Rs_o + Rb^p)}{\beta^p \delta u_c(R^2(s_v - b^p))} = R \end{array}$$

For given (s_o, s_v), there exist critical levels for s_o – $\underline{s}_o$ and $\overline{s}_o$ – such that the middle-aged believes that her old self will i) liquidate ($b^p > 0$) if $s_o < \underline{s}_o$, ii) neither liquidate nor save ($b^p = 0$) if $\underline{s}_o \leq s_o \leq \overline{s}_o$, and iii) save ($b^p < 0$) if $s_o > \overline{s}_o$. The critical saving levels are determined by

$$(11) \quad u_c(R\underline{s}_o) \equiv \frac{R}{1-\theta}\beta^p \delta u_c\left(R^2(s_m - \underline{s}_o)\right),$$

$$(12) \quad u_c(R\overline{s}_o) \equiv R\beta^p \delta u_c\left(R^2(s_m - \overline{s}_o)\right).$$

Of paramount importance will be the situation in which the middle-aged chooses $s_o < \underline{s}_o$ ("too low" a saving in the one-period asset). Think of $\underline{s}_o$ as the upper bound level of middle-age saving which keeps her old self from liquidating some of s_v.

Note for later use,

$$(13)\quad \frac{\partial \underline{s}_o(\cdot)}{\partial \theta} = \frac{\frac{R}{(1-\theta)^2}\beta^p \delta u_c\left(R^2\left(s_m - \underline{s}_o\right)\right)}{Ru_{cc}\left(R\underline{s}_o\right) + \frac{R^3}{1-\theta}\beta^p \delta u_{cc}\left(R^2\left(s_m - \underline{s}_o\right)\right)} < 0,$$

$$(14)\quad \frac{\partial \underline{s}_o(\cdot)}{\partial s_m} = \frac{\frac{R^3}{1-\theta}\beta^p \delta u_{cc}\left(R^2\left(s_m - \underline{s}_o\right)\right)}{Ru_{cc}\left(R\underline{s}_o\right) + \frac{R^3}{1-\theta}\beta^p \delta u_{cc}\left(R^2\left(s_m - \underline{s}_o\right)\right)} \in (0,1),$$

$$(15)\quad \frac{\partial \underline{s}_o(\cdot)}{\partial \beta^p} = \frac{\frac{R}{1-\theta}\delta u_c\left(R^2\left(s_m - \underline{s}_o\right)\right)}{Ru_{cc}\left(R\underline{s}_o\right) + \frac{R^3}{1-\theta}\beta^p \delta u_{cc}\left(R^2\left(s_m - \underline{s}_o\right)\right)} < 0.$$

Both higher θ and higher β^p reduce $\underline{s}_o$, the lower bound saving in the one-period asset. The intuition is that a higher θ makes it more costly for the old to liquidate part of s_v; this help to prevent the old from deviating at even lower levels of s_o than before. A higher β^p (more strongly naive) means the less the middle-aged believes the old will liquidate.

3.2 The best-intention plans of the middle-aged

According to the choice utility of middle-aged, eq. (2), the best-intention plans of the middle-aged take s_m as predetermined for the old and maximize $[u(c_o) + \delta u(c_v)]$. Such plans are described by

$$(16)\quad u_c\left(R\widehat{s}_o\right) \equiv R\delta u_c\left(R^2\left(s_m - \widehat{s}_o\right)\right).$$

The ideal is to have consumption $\widehat{c}_o = R\widehat{s}_o$ as old and $\widehat{c}_v = R^2\left(s_m - \widehat{s}_o\right)$ as very old. Put differently, if the middle-aged were to hold $(s_o, s_v) = (\widehat{s}_o,\ s_m - \widehat{s}_o)$ satisfying (16), her welfare would be highest. Notice, because it is the best-intention plan, $\widehat{s}_o(\cdot)$ does not depend on θ or β^p. Also,

$$\frac{\partial \widehat{s}_o(\cdot)}{\partial s_m} = \frac{R^2 \delta u_{cc}\left(R^2\left(s_m - \widehat{s}_o\right)\right)}{u_{cc}(R\widehat{s}_o) + R^2 \delta u_{cc}\left(R^2\left(s_m - \widehat{s}_o\right)\right)} \in (0,1).$$

Leaving a higher s_m for the old would raise $\widehat{c}_o = R\widehat{s}_o$ and $\widehat{c}_v = R^2\left(s_m - \widehat{s}_o\right)$ since $\frac{\partial \widehat{s}_o(\cdot)}{\partial s_m} \in (0,1)$. The crucial question is, does the middle-aged perceive that were she to choose

$s_o = \widehat{s}_o$ and $\widehat{s}_v = (s_m - \widehat{s}_o)$, the old would not try to undo it? In other words, is her best-intention plan $(s_o, s_v) = (\widehat{s}_o,\ s_m - \widehat{s}_o)$ implementable or foolproof?

To begin with, $\widehat{s}_o$, since it will be chosen based on her choice preferences which include her perceptions, must satisfy $\underline{s}_o \leq s_o \leq \overline{s}_o$ as defined in (11) and (12); otherwise, the old will revise her plans by premature liquidation or by adding on saving. We say $\widehat{s}_o$ is perceived to be *implementable* if $\underline{s}_o \leq \widehat{s}_o \leq \overline{s}_o$. If $\widehat{s}_o < \underline{s}_o$, the plan is not implementable since the old will wish to liquidate early. Similarly, if $\widehat{s}_o > \overline{s}_o$, the plan is again not implementable since the old will wish to save additional amounts. Both early liquidation and extra saving decisions depend crucially on β^p and θ. Using (11), (12), and (16), it follows

$$\begin{aligned} \widehat{s}_o(\cdot) &\gtreqless \underline{s}_o(\cdot) \text{ for } \theta \gtreqless 1-\beta^p, \\ \widehat{s}_o(\cdot) &\leq \overline{s}_o(\cdot) \text{ (equality holds iff } \beta^p = 1). \end{aligned}$$

It can be shown that

Lemma 1 *For given s_m, the optimal asset allocation, and thus the level of savings for old age, $s_o(s_m)$, is*

$$s_o(s_m) = \begin{cases} \underline{s}_o(s_m) \text{ for } \theta \leq 1-\beta^p \\ \widehat{s}_o(s_m) \text{ for } \theta > 1-\beta^p \end{cases} \tag{17}$$

and $s_v(s_m) = s_m - s_o(s_m)$.

Proof. See Appendix A. ■

Lemma 1 lays out the optimal choices of asset holdings by the middle-aged as functions of θ and β^p. Recall, s_o must exceed $\underline{s}_o$ otherwise the old will prematurely liquidate. For $\theta > 1-\beta^p$, luckily, there is no implementation problem. Early liquidation is so expensive that the old is perceived to not to want to deviate; in this case, $s_o(s_m) = \widehat{s}_o(s_m)$ obtains, the best intention plans are implementable.

For $\theta \leq 1-\beta^p$ perceived liquidation as old constrains the options open to the middle-aged. Now, s_o must be set so high $(= \underline{s}_o)$ as to, again, prevent any liquidation when old $(\widehat{s}_o(s_m) \leq \underline{s}_o(s_m)$ for $\theta \leq 1-\beta^p)$. Choosing a lower s_o (including $\widehat{s}_o(s_m)$) would induce some liquidation which is not optimal. The perceived threat of plan revision by the old along with low-strength outside commitment really hurts the middle aged. This is precisely where high θs help. Notice, how the perceived present bias as old, β^p, plays a somewhat similar role as θ. A sufficiently low β^p – stronger perceived present bias – makes the implementation constraint binding.

To interpret the case $\theta \leq 1 - \beta^p$, assume for a moment that the old is perceived to liquidate $b^p > 0$. As discussed earlier, this hurts the middle-aged in utility terms – a loss of utility from being forced "off" the budget constraint ACB in Figure 2. Intuitively, the middle aged may agree to trade off some current utility if it allows her to thwart her future self from liquidating. This could, for example, be achieved if the middle-aged "endows" the old with more s_o so as to counteract the old's present bias and put brakes on her desire to liquidate. To that end, consider the possibility that the middle-aged allocates more to old age $s_o + x$ such that $R(s_o + x) = Rs_o + (1-\theta) Rb^p$ ensuring the old gets exactly the same consumption as when she liquidates b^p. This extra x is generated by taking it away from s_v. In other words, the new package has $(s_o + x, s_v - x)$ with $x = (1-\theta) b^p$. Under this package, the old has $c_o = Rs_o + (1-\theta) Rb^p$ and the very old has $c_v = R^2 (s_v - x) = R^2 (s_v - (1-\theta) b^p) > R^2 (s_v - b^p)$. In short, the new package leaves the old with the same consumption and the very old with higher consumption compared to the consumption bundle achieved by liquidating at cost θ. Clearly, choice life-time utility of the middle-aged is higher with the new package. Essentially, the intuition is that it is cheaper for the middle-aged to do the liquidating *on behalf of the old* (to help with the latter's present bias) than it is for the old to liquidate (1 vs $1-\theta$).

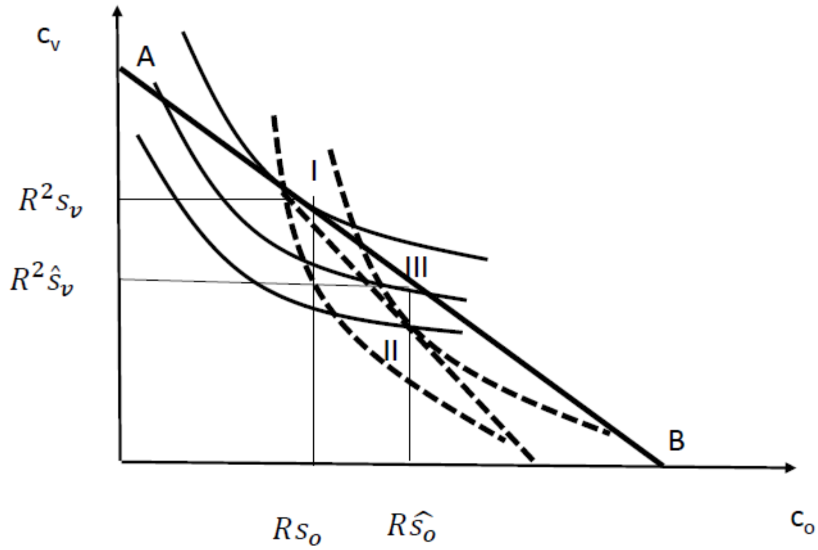

Figure 3: Implementation under perceived present-bias as old

Figure 3 illustrates this argument. Assume, the middle-aged chooses the initial package (s_o, s_v) implying a consumption bundle $(Rs_o, R^2 s_v)$ because that places him on the segment AB at point I. Such a package is not implementable as the old reoptimizes and move to point II (which, because of the old's present bias) has more c_o and less c_v than under the initial package. The new package, $(s_o + x, s_v - x)$ with $x = (1-\theta)\, b^p$, places the old at III, on a higher indifference curve from the perspective of the old's lifetime choice utility. It offers the middle aged both higher true and choice utility than at II. Hence, the middle-aged chooses s_o so as to be on the AB segment. Importantly, this choice of s_o is based on the perception of the preferences of the old, and it may, of course, be wrong.

3.2.1 The savings decision by the middle-aged

Having settled on how a given savings level is allocated between the two assets $s_o\,(s_m)$ and $s_v\,(s_m) = s_m - s_o\,(s_m)$, we turn to the optimal savings decision (s_m) for the middle-aged. Lifetime choice utility is

$$\Omega^m = u\,(w - s_m) + \beta\delta \left[u\,(Rs_o\,(s_m)) + \delta u \left(R^2\,(s_m - s_o\,(s_m)) \right) \right]$$

where $s_o(s_m)$ is determined from (17). The optimal s_m, thus, satisfies

(18)
$$u_c(w - s_m) = \beta\delta^2 R^2 u_c\left(R^2(s_m - s_o(s_m))\right) + \beta\delta R\left[u_c(Rs_o(s_m)) - \delta R u_c\left(R^2(s_m - s_o(s_m))\right)\right]\frac{\partial s_o(s_m)}{\partial s_m}.$$

Note, for $\theta > 1 - \beta^p$, we have $s_o(s_m) = \widehat{s}_o(s_m)$. Using (16), eq. (18) reduces to

(19) $\quad u_c(w - \widehat{s}_m) = \beta\delta^2 R^2 u_c\left(R^2(\widehat{s}_m - \widehat{s}_o(\widehat{s}_m))\right)$ for $\theta > 1 - \beta^p$,

where $\widehat{s}_m$ denotes the savings level as middle-aged when $\widehat{s}_o(s_m)$ can be implemented. This is also the savings level preferred by the middle-aged. When $\theta \leq 1 - \beta^p$, we have $s_o(s_m) = \underline{s}_o(s_m)$. Using (14) and noting $u_c(R\underline{s}_o) \equiv \frac{R}{1-\theta}\beta^p\delta u_c\left(R^2(s_m - \underline{s}_o)\right) \leq \delta R u_c\left(R^2(s_m - \underline{s}_o)\right)$, (18) implies

(20) $\quad u_c(w - \underline{s}_m) \leq \beta\delta^2 R^2 u_c\left(R^2(\underline{s}_m - \underline{s}_o(\underline{s}_m))\right)$ for $\theta \leq 1 - \beta^p$,

where $\underline{s}_m$ denotes the savings level by the middle-aged when the implementation constraint ($s_o = \underline{s}_o(s_m)$) is binding.

For the two key variables, θ and β^p, it is hard to find general comparative static results but it is possible to establish the following:

Lemma 2 *When $\theta < 1 - \beta^p$ and the implementation constraint is binding, the saving as middle-aged ($\underline{s}_m$) depends on the liquidating cost (θ) and the perceived present biased of the old (β^p), but the effects are generally ambiguously signed. For CRRA-utility $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$ $(\sigma > 0)$, we have*

(21) $$\underline{s}_m = \frac{w}{1 + R^2\left(\beta\delta R^2\right)^{-\frac{1}{\sigma}}\left[1 + \delta\left(\frac{R\delta\beta^p}{1-\theta}\right)^{\frac{1}{\sigma}-1}\right]^{-\frac{1}{\sigma}}\left[R + \left(\frac{R\delta\beta^p}{1-\theta}\right)^{\frac{1}{\sigma}}\right]^{\frac{1}{\sigma}-1}},$$

and

$$\textit{sign}\left(\frac{\partial \underline{s}_m}{\partial\theta}\right) = \textit{sign}\left(\frac{\partial \underline{s}_m}{\partial\beta^p}\right) = \textit{sign}\ (1 - \sigma).$$

When $\theta \geq 1 - \beta^p$ saving as middle-aged ($\widehat{s}_m$) is unaffected by θ and β^p.

Proof. See Appendix B. ■

From standard, two-period saving models, it is well-known that $\sigma \lesseqgtr 1$ determines whether the substitution or income effect of a change in the (effective) rate of return, $(1-\theta)\,R$, is dominating. We recover a similar condition here for θ and β^p although the effects of an interest change is, in general, ambiguous. The complication arises here because $\underline{s}_m$ depends on the saving decision made as old. Recall, when θ rises reducing the effective interest rate, the old will attempt to front load consumption by cutting saving if $\sigma < 1$. Hence, if relative risk aversion is below one ($\sigma < 1$), a lower θ and a lower β^p both reduce saving made as middle-aged. A low θ increases the incentive of the old to liquidate so as to front load consumption, and the middle-aged responds to this by saving less. Similary, a lower β^p increases the perception (the firmness of the belief) that the old will front load consumption, and therefore the middle-aged saves less. Either change reduces the tensions arising from the implementation problem prompting the middle-aged to transfer less resources to the later self.

From Lemma 2, we have the following proposition.

Proposition 1 *The commitment problem arises for $\theta \leq 1-\beta^p$ and*
(i) if $\sigma < 1$, the commitment problem reduces total saving of the middle-aged, that is $\underline{s}_m < \widehat{s}_m$;
(ii) if $\sigma > 1$, the commitment problem increases total saving of the middle-aged, that is $\underline{s}_m > \widehat{s}_m$.

Interestingly, the inability to prevent the future self from borrowing ($\theta \leq 1-\beta^p$) makes the middle-aged either save more or less, depending on the intertemporal elasticity of substitution ($1/\sigma$). The intuition is that the commitment problem forces the middle-aged to choose an asset allocation which is front-loaded ($\underline{s}_o(s_m) > \widehat{s}_o(s_m)$) compared to what the middle-aged finds optimal. As a consequence, on the one hand, the middle-aged finds it less attractive to save, since too much goes to consumption as old rather than as very old. On the other hand, the middle-aged finds it more attractive to save, since she wants to protect the very old by leaving more consumption to the very old. The whole effect on savings depends on which one is dominant.

3.3 The actual choices by the old

The preceding was based on the *perception* of the middle-aged regarding preferences of her old self: Eq. (4). Of course, the choices as old are determined by the old's actual choice preferences (3). Recall, limiting cases are when the middle-aged is either sophisticated ($\beta^p = \beta$) or completely naive ($\beta^p = 1$). Notice, since $\beta \leq \beta^p \leq 1$, the middle-aged never *over*estimates the level of present bias. If $\beta < \beta^p$, the old has an incentive to front-load consumption and liquidate some savings intended for the very old, but there will never be additional savings.

The actual decision taken as old given s_o (either $\widehat{s}_o(\widehat{s}_m)$ or $\underline{s}_o(\underline{s}_m)$) and $s_v = s_m - s_o$ (either $\widehat{s}_m - \widehat{s}_o(\widehat{s}_m)$ or $\underline{s}_m - \underline{s}_o(\underline{s}_m)$), is the solution to the problem ($b^a \geq 0$ always holds)

$$\begin{aligned} \max_{b^a} \Omega^{oa} &= u(c_o^a) + \beta\delta u(c_v^a) \\ c_o^a &= Rs_o + (1-\theta) Rb^a \\ c_v^a &= R^2(s_v - b^a) \end{aligned}$$

where the superscript a refers to the actual choices made by the old. It follows straightforwardly that

$$(22) \quad \begin{array}{lll} b^a > 0 & \text{if } \frac{u_c(Rs_o)}{\beta\delta u_c(R^2 s_v)} > \frac{R}{1-\theta} & b^a : \frac{u_c(Rs_o+(1-\theta)Rb^a)}{\beta\delta u_c(R^2(s_v - b^a))} = \frac{R}{1-\theta} \\ b^a = 0 & \text{if } R \leq \frac{u_c(Rs_o)}{\beta\delta u_c(R^2 s_v)} \leq \frac{R}{1-\theta} & b^a = 0 \end{array}.$$

From (16) and (11), we know that if $\theta > 1 - \beta^p$, $\frac{u_c(R\widehat{s}_o)}{\beta^p\delta u_c(R^2\widehat{s}_v)} = \frac{R}{\beta^p}$ and if $\theta \leq 1 - \beta^p$, $\frac{u_c(R\underline{s}_o)}{\beta^p\delta u_c(R^2\underline{s}_v)} = \frac{R}{1-\theta}$. Thus

$$(23) \quad \begin{aligned} \frac{u_c(R\widehat{s}_o)}{\beta\delta u_c(R^2\widehat{s}_v)} &= \frac{\beta^p}{\beta}\frac{u_c(R\widehat{s}_o)}{\beta^p\delta u_c(R^2\widehat{s}_v)} = \frac{R}{\beta} \text{ if } \theta > 1-\beta^p; \\ \frac{u_c(R\underline{s}_o)}{\beta\delta u_c(R^2\underline{s}_v)} &= \frac{\beta^p}{\beta}\frac{u_c(R\underline{s}_o)}{\beta^p\delta u_c(R^2\underline{s}_v)} = \frac{\beta^p}{\beta}\frac{R}{1-\theta} \text{ if } \theta \leq 1-\beta^p. \end{aligned}$$

Combining (22) and (23), it follows that:

$$\begin{array}{llll} \text{(I)} & \theta > 1-\beta \Longrightarrow \theta > 1-\beta^p & s_o = \widehat{s}_o(\widehat{s}_m) & b^a = 0 \text{ since } \frac{R}{\beta} < \frac{R}{1-\theta} \\ \text{(II)} & 1-\beta^p < \theta < 1-\beta & s_o = \widehat{s}_o(\widehat{s}_m) & b^a > 0 \text{ since } \frac{R}{\beta} \geq \frac{R}{1-\theta} \\ \text{(III)} & \theta < 1-\beta^p & s_o = \underline{s}_o(\underline{s}_m) & b^a > 0 \text{ since } \frac{\beta^p}{\beta}\frac{R}{1-\theta} > \frac{R}{1-\theta} \end{array}.$$

It follows that

Lemma 3 *(I) $\theta > 1-\beta$: $b^a = b^p = 0$, i.e., actual liquidating is zero as perceived, and preferred saving levels as middle aged $\widehat{s}_o(\widehat{s}_m)$ and $\widehat{s}_v(\widehat{s}_m)$ can be implemented. Hence, actual* *consumption when old is as planned/perceived as middle-aged,* $c_o^a = c_o^p, c_v^a = c_v^p$.

(II) $1 - \beta^p < \theta \leq 1 - \beta$*:* $b^a > b^p = 0$*, the implementation constraint is perceived to be non-binding and the middle-aged chooses an asset allocation* $(s_o, s_v) = (\widehat{s}_o(\widehat{s}_m), \widehat{s}_v(\widehat{s}_m))$ *and saving level* $\widehat{s}_m$ *under the perception that there will be no liquidation by the old. However, in actuality, the old liquidates, and therefore,* $b^a > b^p = 0$*,* $c_o^a > c_o^p$*, and* $c_v^a < c_v^p$*.*

(III) $\theta \leq 1 - \beta^p$*:* $b^a > b^p = 0$*, the implementation constraint is perceived to be binding and the middle-aged chooses an asset allocation* $(s_o, s_v) = (\underline{s}_o(\underline{s}_m), \underline{s}_v(\underline{s}_m))$ *and saving level* $\underline{s}_m$*. The old liquidates and therefore,* $b^a > b^p = 0$*,* $c_o^a > c_o^p$*, and* $c_v^a < c_v^p$*.*

The possible outcome regimes are shown in Figure 4. In regime I, there is no deviation from the perceived choices. Regime II and III share the feature that the middle-aged's perception that her old self will not liquidate is proven to be wrong: in actuality, her old self does liquidate some of the saving intended for the very old. The two regimes differ in the following way. In case II, the middle-age perceives the liquidating cost to be high enough to prevent the old from liquidating, so the ideal asset allocation was chosen (the "first best": $\widehat{s}_m, \widehat{s}_o(\widehat{s}_m)$ and $\widehat{s}_v(\widehat{s}_m)$) as in case I. In regime III, the middle-age perceives that the most preferred plan is not implementable, and therefore chooses the "second best" ($\underline{s}_m, \underline{s}_o(\underline{s}_m)$ and $\underline{s}_v(\underline{s}_m)$); however, the incentive for the old to liquidate is underestimated, so there is deviation from her "second best" plan.

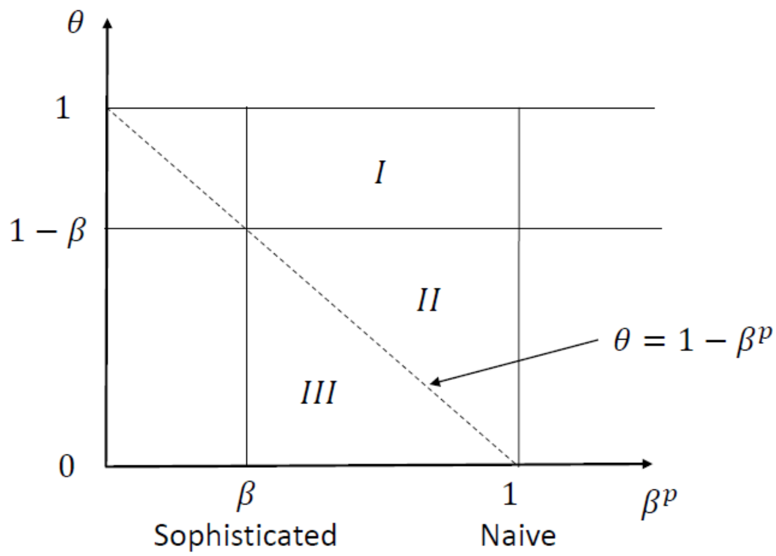

Figure 4: Regimes for actual choices - liquidating cost (θ) and perceived present bias (β^p)

3.4 Numerical illustration

To see more clearly how β^p and θ influence the actual outcomes, we present some numerical results for $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$ $(\sigma > 0)$; see Figure 5. The following parameter values are used: $\sigma = 0.3$, $\beta = 0.3$, $R = 2$, $w = 1$, $\delta^* = 0.8$, and $\delta = 0.7$; for these values, regime I ($\theta > 1 - \beta$) arises for $\theta > 0.7$. We consider five values of β^p; the lowest value ($\beta^p = 0.3$) corresponds to a fully sophisticated agent, and the highest value ($\beta^p = 1$) to a fully naive agent. Figure 5 shows how actual savings (and hence, consumption) as middle-aged, liquidation by the old, consumption as old and consumption as very old depend on the costs of liquidation (θ). Observe that as the strength of the commitment technology starting from $\theta = 0$ increases, first regime III prevails ($\theta < 1 - \beta^p$ for $\beta^p < 1$), then $1 - \beta^p < \theta < 1 - \beta$ we enter regime II, and eventually for $\theta > 1 - \beta$, we are in regime I.

With $\sigma < 1$, for a given θ, when both in regime III, saving as middle-aged is higher (and consumption lower), the more "naive" the agent (higher β^p). For a given degree of naivete, middle-aged saving is increasing in θ, and for sufficiently high θ, the commitment problem disappears (regime II and I where the commitment constraint is *perceived* not binding is reached). Note that, as shown in Proposition 1, when $\sigma < 1$, middle-aged saving is lower when $\theta < 1 - \beta^p$. As expected, liquidation is larger, the more naive the agent is. Interestingly, the level of liquidation is increasing in the liquidation cost up to some critical level (which, itself, is decreasing in β^p). This should be seen in combination with the fact that middle-aged saving is increasing in the liquidation costs (see Figure 5a). Consumption as old is generally higher (frontloading old vs very old), the more naive the agent is. The consumption level is non-monotone in the strength of the commitment technology. Recall the consumption effect is determined by $(1 - \theta)b_a$, hence even though b_a is increasing in θ, consumption may not be. Consumption as very-old is not unambiguously higher for less naive agents, but it is generally increasing in the strength of the commitment technology. The non-monotone responses of consumption to the strength of the commitment technology are interesting findings challenging the general perception that stronger commitments technologies counteracts the implications of present-biased preferences. The reason for the non-monotone responses is the strategic interaction between the different selves as explained above. The non-monotone responses are also important for interpretations of the empirical evidence on the importance of commitment technologies, see discussion above.

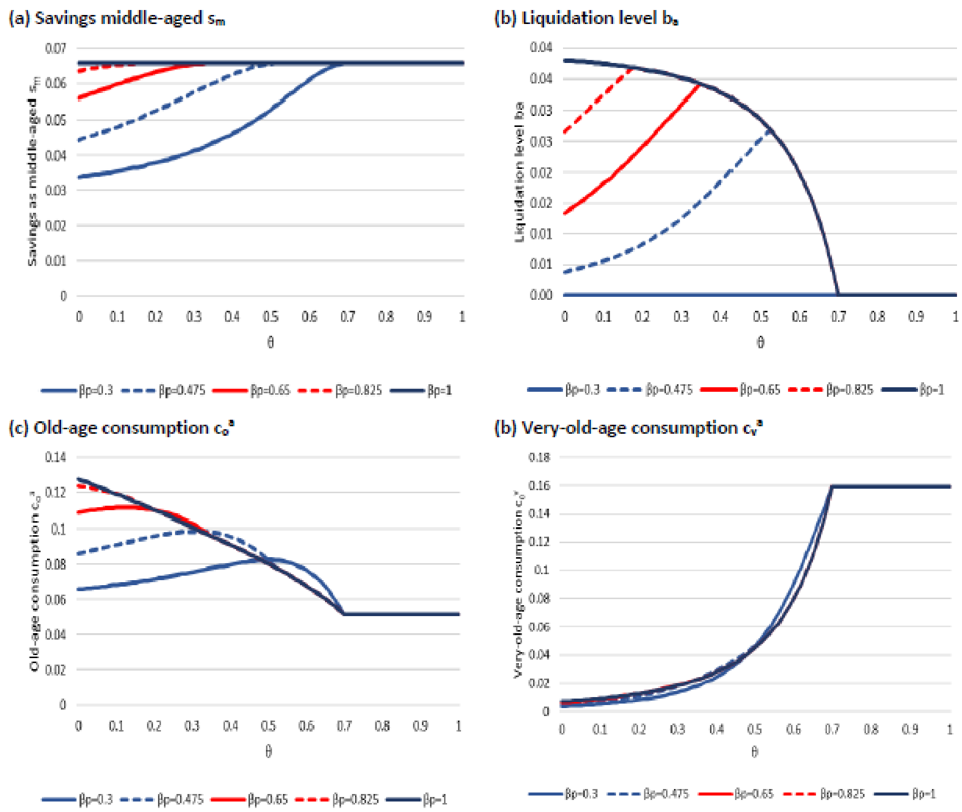

Figure 5: Savings, liquidation and consumption as a function of liquidation costs (θ) for different values of β^p.

4 Welfare

How does welfare depend on the strength of the commitment mechanism? This question can be addressed both in terms of choice utility (Ω^m) and true utility (Ω^*) as middle-aged. We consider them in turn. Recall, we have already established that a sufficiently high liquidating cost, $\theta \geq 1 - \beta$, implies that the middle-aged can make commitments which are not undone by the old (corresponding to case I in Figure 4). Neither Ω^m nor Ω^* depend on θ in this case. Put differently, for a sufficiently high liquidating cost, there is no commitment problem.

The following considers the outcomes for $\theta < 1 - \beta$ where both the implementation problem and the importance of the distinction between the perceived and actual behavior, cf. case II and III in Figure 4.

4.1 Commitment and choice utility as middle-aged

In regime II ($1 - \beta^p \leq \theta < 1 - \beta$) the implementation constraint is perceived to be non-binding so the choice utility as middle-aged (Ω^m) is unaffected by θ. Regime III ($\theta < 1 - \beta^p$) is slightly less straightforward. The choice utility as middle-aged (Ω^m) depending on the *perceived* choices (c_m^p, $c_o^p = R\underline{s}_o$, $c_v^p = R^2\underline{s}_v$) and (2) can be written as

$$\Omega^m(\theta) = u(w - \underline{s}_m(\theta)) + \beta\delta\left[u(R\underline{s}_o(\theta, \underline{s}_m(\theta))) + \delta u\left(R^2(\underline{s}_m(\theta) - \underline{s}_o(\theta, \underline{s}_m(\theta)))\right)\right],$$

where $\underline{s}_o(\theta, \underline{s}_m(\theta))$ is determined by (11) and $\underline{s}_m(\theta)$ by (18). By standard envelope arguments, it can be shown that

$$\frac{\partial\Omega^m}{\partial\theta} = \frac{\partial\Omega^m}{\partial\underline{s}_o}\frac{\partial\underline{s}_o}{\partial\theta} = \beta\delta R\left[u_c(c_o^p) - \delta R u_c(c_v^p)\right]\frac{\partial\underline{s}_o}{\partial\theta}.$$

Using (11),

$$\frac{\partial\Omega^m}{\partial\theta} = \left(\frac{\beta^p}{1-\theta} - 1\right)\beta\delta^2 R^2 u_c(c_v^p)\frac{\partial\underline{s}_o}{\partial\theta}.$$

Since $\frac{\beta^p}{1-\theta} - 1 < 0$ and $\frac{\partial\underline{s}_o}{\partial\theta} < 0$ cf. (13), it follows that $\frac{\partial\Omega^m}{\partial\theta} > 0$. Hence, we have

Proposition 2 *Choice utility as middle-aged (Ω^m) is I) increasing in the liquidating cost, $\frac{\partial\Omega^m}{\partial\theta} > 0$ for $\theta < 1 - \beta^p$, and II) unaffected by the liquidating cost, $\frac{\partial\Omega^m}{\partial\theta} = 0$ for $\theta \geq 1 - \beta^p$.*

The intuition is straightforward. In regime III ($\theta < 1 - \beta^p$) a higher liquidating cost increases the cost of deviating from the plan made as middle-aged, and therefore choice utility increases. For $\theta \geq 1 - \beta^p$ the old is perceived to follow the plans made as middle-aged and hence choice utility is unaffected by θ.

4.2 Commitment and true utility as middle-aged

Turn next to the effects of the strength of the commitment technology on true utility. Under true utility, (1), with $\beta^* = 1$, the marginal rates of substitution between the two margins – middle-aged and old, old and very-old – are the same and equal $R\delta^*$, and the optimal consumption allocation therefore satisfies

$$\frac{u_c\left(c_m^*\right)}{u_c\left(c_o^*\right)}=\frac{u_c\left(c_o^*\right)}{u_c\left(c_v^*\right)}=R\delta^*.$$

4.2.1 Regime I: $\theta \geq 1-\beta$

In this case the actual consumptions are as planned at middle-aged by the choice utility (2). The marginal rates of substitution between the two margins are given by

$$\begin{aligned}
\frac{u_c\left(w-\widehat{s}_m\right)}{u_c\left(R\widehat{s}_o\right)} &= \beta R\delta < R\delta^* = \frac{u_c\left(c_m^*\right)}{u_c\left(c_o^*\right)}, \\
\frac{u_c\left(R\widehat{s}_o\right)}{u_c\left(R^2\left(\widehat{s}_m-\widehat{s}_o\right)\right)} &= R\delta < R\delta^* = \frac{u_c\left(c_o^*\right)}{u_c\left(c_v^*\right)}.
\end{aligned}$$

i.e., there is front loading of consumption both between middle-aged and old, and old and very-old. This is a standard result. Hence, even though there is no commitment problem, life-time true utility is lowered due to myopia and preference reversal.

4.2.2 Regime II: $1-\beta^p \leq \theta < 1-\beta$

In this case, the marginal rates of substitution are

$$\begin{aligned}
\frac{u_c\left(w-\widehat{s}_m\right)}{u_c\left(R\widehat{s}_o+(1-\theta)Rb^a\right)} &> \frac{u_c\left(w-\widehat{s}_m\right)}{u_c\left(R\widehat{s}_o\right)} = \beta R\delta < R\delta^*, \\
\frac{u_c\left(R\widehat{s}_o+(1-\theta)Rb^a\right)}{u_c\left(R^2\left(\widehat{s}_v-b^a\right)\right)} &= \frac{\beta}{1-\theta}R\delta < R\delta < R\delta^*.
\end{aligned}$$

Now the present bias is worsened between old and very old, due to the liquidating made by the old. Recall, in this case the middle-aged did not perceive the commitment constraint to be binding (although it turns out to be the case) and therefore it does not affect the behavior of the middle-aged . Liquidating means front loading of consumption as old relative to the plan set as middle aged at the cost of consumption as very old; the costs can thus be assessed in terms of very old age consumption. If the liquidating cost increases it has counteracting effects discussed above via its direct effect and indirect effect on the amount liquidated. To consider this more specifically note that if the individual is fully sophisticated ($\beta^p = \beta$), this case does not arise. The following thus assumes $\beta^p > \beta$. Recall, in case II, the implementation constraint is not perceived to be binding (hence $s_o = \widehat{s}_o(\widehat{s}_m)$) and no liquidating is perceived ($b^p = 0$), but the old is actually going to liquidate ($b^a > 0$). In this case, the true utility (1) can be written as

$$\Omega^* = u(w - \widehat{s}_m) + \delta^* \left[u\left(R\widehat{s}_o(\widehat{s}_m) + (1-\theta) Rb^a \right) + \delta^* u\left(R^2 \left(\widehat{s}_m - \widehat{s}_o(\widehat{s}_m) - b^a \right) \right) \right],$$

where both $\widehat{s}_m$ and $\widehat{s}_o$ are not affected by θ, but b^a depends on θ, cf. (22). It follows that

$$\begin{aligned} \frac{d\Omega^*}{d\theta} &= \frac{\partial \Omega^*}{\partial \theta} + \frac{\partial \Omega^*}{\partial b^a}\frac{\partial b^a}{\partial \theta} = -\delta^* u_c\left(R\widehat{s}_o(\widehat{s}_m) + (1-\theta) Rb^a \right) Rb^a \\ &\quad + \delta^* \left[u_c\left(R\widehat{s}_o(\widehat{s}_m) + (1-\theta) Rb^a \right)(1-\theta) R - \delta^* R^2 u_c\left(R^2\left(\widehat{s}_m - \widehat{s}_o(\widehat{s}_m) - b^a \right) \right) \right] \frac{\partial b^a}{\partial \theta}, \end{aligned}$$

where by the use of (22),

$$\frac{d\Omega^*}{d\theta} = \left((\beta\delta - \delta^*) \frac{\partial b^a}{\partial \theta} - \frac{\beta\delta}{1-\theta} b^a \right) R^2 \delta^* u_c\left(R^2\left(\widehat{s}_m - \widehat{s}_o(\widehat{s}_m) - b^a \right) \right).$$

It can be easily shown that $\frac{d\Omega^*}{d\theta} > 0$ requires

$$\frac{\partial b^a}{\partial (1-\theta)} \frac{1-\theta}{b^a} > \frac{\beta\delta}{\delta^* - \beta\delta} > 0.$$

Two effects are at play. A higher liquidating cost may make the old liquidate more or less. If the old liquidates less ($\frac{\partial b^a}{\partial \theta} < 0$), it would increase consumption as very old and this is welfare improving. But a higher θ increases the costs of deviation which lowers welfare. We have

Proposition 3 *In regime II,* $1 - \beta^p \leq \theta < 1 - \beta$*, true utility* ($\Omega^*$) *is non-monotone in liquidating costs,*

$$\frac{d\Omega^*}{d\theta} \gtreqless 0 \textit{ for } \frac{\partial b^a}{\partial (1-\theta)} \frac{1-\theta}{b^a} \gtreqless \frac{\beta\delta}{\delta^* - \beta\delta} > 0.$$

For CRRA utility, there exist a unique $\widetilde{\theta} \in (1-\beta^p, 1-\beta)$ *such that* $\frac{\partial \Omega^*(\cdot)}{\partial \theta} \lesseqgtr 0$ *for* $\theta \lesseqgtr \widetilde{\theta}$ *under the sufficient condition* $\sigma > \frac{1 + \frac{1}{\beta^p} R (R\delta)^{-\frac{1}{\sigma}}}{1 - \left(\frac{\beta}{\beta^p} \right)^{\frac{1}{\sigma}}}$.

Proof. See Appendix C. ■

Also notice that although Ω^* is not directly affected by β^p in regime II, the level of naivete (β^p) matters by affecting the range $[1-\beta^p, 1-\beta]$ delimiting case II.

4.2.3 Regime III: $\theta < 1 - \beta^p$

In this case, the implementation constraint is binding and affects the behavior of the middle-aged. We have from (20) (11) and (22) that the marginal rates of substitution are

$$\begin{aligned}
\frac{u_c\left(w - \underline{s}_m\right)}{u_c\left(R\underline{s}_o + (1-\theta)\, Rb^a\right)} &> \frac{u_c\left(w - \underline{s}_m\right)}{u_c\left(R\underline{s}_o\right)} < \frac{\beta\,(1-\theta)}{\beta^p} R\delta < R\delta^*, \\
\frac{u_c\left(R\underline{s}_o + (1-\theta)\, Rb^a\right)}{u_c\left(R^2\left(\underline{s}_v - b^a\right)\right)} &= \frac{\beta}{1-\theta} R\delta < R\delta < R\delta^*.
\end{aligned}$$

There is still a present bias but between middle-aged and old it can go either way compared to regime II $\left(\frac{u_c(w-\widehat{s}_m)}{u_c(R\widehat{s}_o+(1-\theta)Rb^a)}\right)$. There are two countervailing effects. The commitment problem makes the middle-aged allocate more savings to the old ($\underline{s}_o\left(s_m\right) > \widehat{s}_o\left(s_m\right)$ for the same s_m) but also to change savings as middle-aged [see Proposition 2]. We, thus, have an interesting "conflict" between the two margins. The above argument suggests it is possible true utility can be higher in regime III compared to Regime II, even though there is a commitment problem in regime III.

In this case, the true utility (1) can be written as

$$\Omega^* = u\left(w - \underline{s}_m\right) + \delta^*\left[u\left(R\underline{s}_o + (1-\theta)\, Rb^a\right) + \delta^* u\left(R^2\left(\underline{s}_m - \underline{s}_o - b^a\right)\right)\right], \tag{24}$$

where $\underline{s}_m, \underline{s}_o$ and b^a are all affected by θ, cf.(18) (11) and (22):

$$\underline{s}_m = \underline{s}_m(\theta)\,;\, \underline{s}_o = \underline{s}_o\left(\theta, \underline{s}_m(\theta)\right); b^a = b^a\left(\theta, \underline{s}_m(\theta), \underline{s}_o\left(\theta, \underline{s}_m(\theta)\right)\right).$$

Using (24),

$$\begin{aligned}
\frac{d\Omega^*}{d\theta} &= -u_c\left(w - \underline{s}_m\right)\frac{\partial \underline{s}_m}{\partial\theta} + \delta^* u_c\left(R\underline{s}_o + (1-\theta)\, Rb^a\right) R\left[\frac{d\underline{s}_o(\cdot)}{d\theta} - b^a + (1-\theta)\frac{db^a}{d\theta}\right] \\
&\quad + \left(\delta^*\right)^2 u_c\left(R^2\left(\underline{s}_m - \underline{s}_o - b^a\right)\right) R^2\left(\frac{\partial \underline{s}_m}{\partial\theta} - \frac{d\underline{s}_o}{d\theta} - \frac{db^a}{d\theta}\right),
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
\frac{d}{d\theta}\underline{s}_o\left(\theta, \underline{s}_m(\theta)\right) &= \frac{\partial \underline{s}_o}{\partial\theta} + \frac{\partial \underline{s}_o}{\partial \underline{s}_m}\frac{\partial \underline{s}_m}{\partial\theta}, \\
\frac{d}{d\theta} b^a\left(\theta, \underline{s}_m(\theta), \underline{s}_o\left(\theta, \underline{s}_m(\theta)\right)\right) &= \frac{\partial b^a}{\partial\theta} + \frac{\partial b^a}{\partial \underline{s}_m}\frac{\partial \underline{s}_m}{\partial\theta} + \frac{\partial b^a}{\partial \underline{s}_o}\left(\frac{\partial \underline{s}_o}{\partial\theta} + \frac{\partial \underline{s}_o}{\partial \underline{s}_m}\frac{\partial \underline{s}_m}{\partial\theta}\right).
\end{aligned}$$

True life-time utility is affected by the liquidating cost through several channels. The effect identified in regime II is present via ($\frac{\partial b^a}{\partial \theta}$), but also savings responses ($\frac{\partial \underline{s}_m}{\partial \theta} \lesseqgtr 0$, $\frac{\partial \underline{s}_o}{\partial \theta} < 0$) and the implied repercussions ($\frac{\partial \underline{s}_o}{\partial \underline{s}_m}$, $\frac{\partial b^a}{\partial \underline{s}_m}$, $\frac{\partial b^a}{\partial \underline{s}_o}$) create complexities making analytical results difficult to obtain. In general, $\frac{d\Omega^*}{d\theta} \lesseqgtr 0$, but is possible to show that

Proposition 4 *For CRRA utility with full sophistication ($\beta^p = \beta$), $\frac{\partial \Omega^*}{\partial \theta} > 0$ for all $\theta \leq 1-\beta$. That is, in this case the savings effect can never dominate the commitment effect, and true utility is increasing in θ.*

Proof. See Appendix D. ■

Recall, with full sophistication the preference reversal is perceived, and therefore the commitment effect is dominating. We illustrate in Figure 6 how true welfare depends on the strength of the commitment technology for the different cases also used in Figure 6. First, if agents are sufficiently naive, a non-monotone relationship between true utility and the strength of the commitment technology arises. Stronger commitment does not always improve true utility. However, there exists a sufficiently high θ that eliminates the commitment problem and generates the highest utility. It is also seen that true utility is not necessarily higher, the less naive the agent is. The richness of the possible outcomes, even in the CRRA case, also shows why it is difficult to generate analytical results.

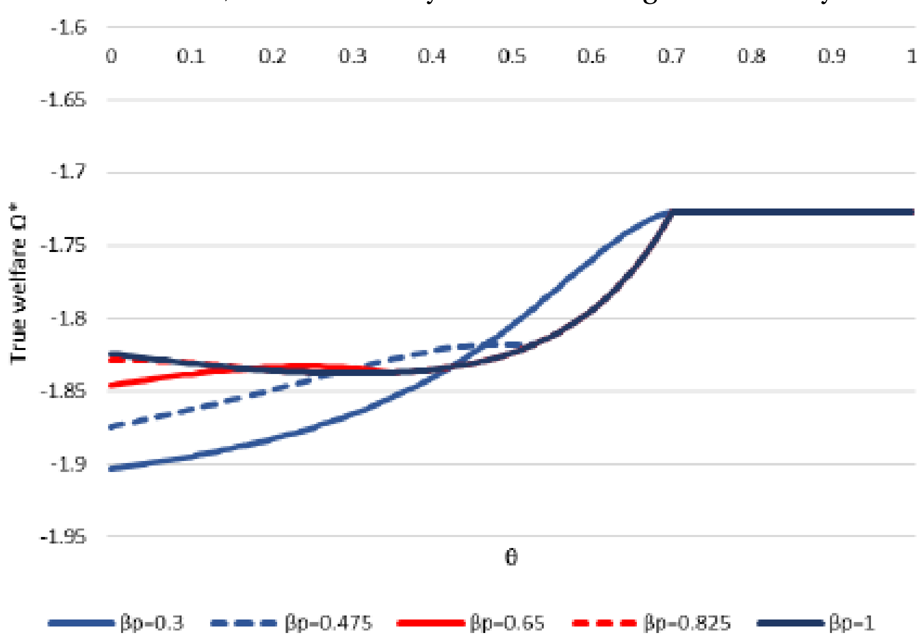

Figure 6: True welfare (Ω^*) and the liquidating cost (θ) for different degrees of perceived present bias (β^p)

5 Conclusion

There is a fair bit of evidence indicating that individuals fail to make decisions in a time-consistent manner. In the context of saving for retirement, researchers observe the all-too-often failure to provide for retirement. Individuals seem to realize these failures, but too late. Börsch-Supan, Hurd, and Rohwedder (2016) conducted an Internet survey among individuals aged 60 and older which show a substantial prevalence of regret over previous saving decisions – 60% of respondents wished that they had saved more earlier in life. High demand for commitment devices, even costly ones, provides more evidence to this finding (Rabin, 2013a,b; Beshears, et al., 2015). This paper studies the adoption and usage of commitment devices by time inconsistent agents. It ask, how should middle-aged people, with varying degrees of self-awareness or sophistication about their impending present-bias, invest their savings so as to, both, finance current gratification *and* thwart their early-retirement self from impoverishing their late-retirement self?

Our analysis generates several rich results. Savings levels are affected by the strength of the commitment device and the degree of agent sophistication. With a high level of intertemporal elasticity of substitution, saving is lower in the presence of commitment problems, and the downward saving bias is larger, the more naive the agent is. The liquidation of committed savings is non-monotone in the strength of the commitment device. Likewise old-age consumption level is non-monotone in the strength of the commitment technology. The non-monotone responses of saving and consumption to the strength of the commitment technology are interesting because they attack the general perception that stronger commitment technologies counteracts the implications of present-biased preferences. The reason for the non-monotone responses is the strategic interaction between the different selves. The non-monotone responses help reinterpret the empirical evidence on the importance of commitment technologies discussed in the introduction. The main take-away is that in terms of true utility, agents are not generally better off with stronger commitment. If agents are sufficiently naive, a non-monotone relationship between true utility and the strength of the commitment technology arises. It is also seen that true utility is not necessarily higher the less naive the agent is.

Our analysis sheds light on several practical issues pertaining to optimal pension design. The U.K. mandates discussed in the introduction have been successfully challenged as being patronizing and perpetrating the view that people cannot be trusted to invest the funds from their pension pot. The new rules imply that "affluent people approaching retirement should be free to blow their pension pot on a Lamborghini even if they end up relying on the state for support." (The Guardian, 2014) Sceptics at the time predicted that savers would withdraw unsustainable sums or blow their money on frivolities. It is too early to say that is not happening. It is true "the majority of withdrawals were at prudent levels." (The Telegraph, 2016) but it is not clear whether these withdrawals are being channeled into long or short term securities.

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Appendix

A Proof of Lemma 1

The middle-aged wants - for given s_m - an asset allocation $(s_o, s_v) = (\widehat{s}_o, s_m - \widehat{s}_o)$ determined by (16). If $\theta \geq 1 - \beta^p$, $\underline{s}_o \leq \widehat{s}_o \leq \overline{s}_o$, and it follows straightforward that it is perceived to be implementable and hence the optimal choice is $s_o(s_m) = \widehat{s}_o(s_m)$. However, if $\theta < 1 - \beta^p$, $\widehat{s}_o < \underline{s}_o$, the ideal allocation $(\widehat{s}_o, s_m - \widehat{s}_o)$ is not perceived to be implementable.

Case I: $s_o \leq \underline{s}_o(s_m)$

In this case, according to (10), the perceived liquidating amount as old, $b^p \geq 0$, is determined by

$$\frac{u_c\left(Rs_o + (1-\theta)Rb^p\right)}{\beta^p \delta u_c\left(R^2\left(s_m - s_o - b^p\right)\right)} = \frac{R}{1-\theta} \text{ for } s_o \leq \underline{s}_o(s_m)$$

implying that $b^p = b(s_o)$ where

$$\frac{\partial b^p}{\partial s_o} = -\frac{Ru_{cc}\left(c_o\right) + \frac{R^3}{1-\theta}\beta^p \delta u_{cc}\left(c_v\right)}{(1-\theta)Ru_{cc}\left(c_o\right) + \frac{R^3}{1-\theta}\beta^p \delta u_{cc}\left(c_v\right)} \leq -1$$

The utility to the middle-aged of consumption when old and very old is $u\left(Rs_o + (1-\theta)Rb^p\right) + \delta u\left(R^2\left(s_m - s_o - b^p\right)\right)$ and we have

$$\begin{aligned} & \frac{\partial}{\partial s_o}\left[u\left(Rs_o + (1-\theta)Rb^p\right) + \delta u\left(R^2\left(s_m - s_o - b^p\right)\right)\right] \\ = \quad & R\left[1 + (1-\theta)\frac{\partial b^p}{\partial s_o}\right]u_c\left(c_o\right) - R^2\left(1 + \frac{\partial b^p}{\partial s_o}\right)\delta u_c\left(c_v\right) > 0 \end{aligned}$$

since $1 + (1-\theta)\frac{\partial b^p}{\partial s_o} = \frac{\theta \frac{R^3}{1-\theta}\beta^p \delta u_{cc}(c_v)}{(1-\theta)Ru_{cc}(c_o) + \frac{R^3}{1-\theta}\beta^p \delta u_{cc}(c_v)} > 0$ and $1 + \frac{\partial b^p}{\partial s_o} \leq 0$.

Case II: $s_o > \underline{s}_o(s_m)$

When $\underline{s}_o \leq s_o \leq \overline{s}_o$, $b^p = 0$. The utility to the middle-aged of consumption when old and very old is $u\left(Rs_o\right) + \delta u\left(R^2\left(s_m - s_o\right)\right)$ and

$$\frac{\partial}{\partial s_o}\left[u\left(Rs_o\right) + \delta u\left(R^2\left(s_m - s_o\right)\right)\right] = Ru_c\left(Rs_o\right) - R^2 \delta u_c\left(R^2\left(s_m - s_o\right)\right) < 0$$

since $u_c\left(R\widehat{s}_o\right)=R\delta u_c\left(R^2\left(s_m-\widehat{s}_o\right)\right)$ and $s_o>\underline{s}_o>\widehat{s}_o$. Hence, for $\theta\leq 1-\beta^p$, the optimal choice is $\underline{s}_o\left(s_m\right)$.

B Proof of Lemma 2

When $\theta>1-\beta^p$, savings as middle-aged ($\widehat{s}_m$) is determined by (19) and (16), and thus $\widehat{s}_m$ is unaffected by θ and β^p.

When $\theta\leq 1-\beta^p$, savings as middle-aged ($\underline{s}_m$) is determined by (18) and (11). Define $\kappa\equiv\frac{\beta^p}{1-\theta}\leq 1$. Then (11) becomes $u_c\left(R\underline{s}_o\right)=R\delta\kappa u_c\left(R^2\left(\underline{s}_m-\underline{s}_o\right)\right)$, and assuming CRRA utility function $u\left(c\right)=\frac{c^{1-\sigma}-1}{1-\sigma}$ $(\sigma>0)$, we can solve

$$\text{(26)}\quad \underline{s}_o=\frac{R}{R+\left(R\delta\kappa\right)^{\frac{1}{\sigma}}}\underline{s}_m;\ \ \underline{s}_v=\frac{\left(R\delta\kappa\right)^{\frac{1}{\sigma}}}{R+\left(R\delta\kappa\right)^{\frac{1}{\sigma}}}\underline{s}_m.$$

Plug in (26) into (18), and we get

$$\underline{s}_m=\frac{w}{1+R^2\left(\beta\delta R^2\right)^{-\frac{1}{\sigma}}\left(1+\delta\left(R\delta\kappa\right)^{\frac{1}{\sigma}-1}\right)^{-\frac{1}{\sigma}}\left(R+\left(R\delta\kappa\right)^{\frac{1}{\sigma}}\right)^{\frac{1}{\sigma}-1}}.$$

It can be easily shown that

$$\begin{aligned}&\frac{\partial}{\partial\kappa}\left[\left(1+\delta\left(R\delta\kappa\right)^{\frac{1}{\sigma}-1}\right)^{-\frac{1}{\sigma}}\left(R+\left(R\delta\kappa\right)^{\frac{1}{\sigma}}\right)^{\frac{1}{\sigma}-1}\right]\\ \text{(27)}\quad =\;&-\frac{1-\sigma}{\sigma^2}\left(1+\delta\left(R\delta\kappa\right)^{\frac{1}{\sigma}-1}\right)^{-\frac{1}{\sigma}-1}\left(R+\left(R\delta\kappa\right)^{\frac{1}{\sigma}}\right)^{\frac{1}{\sigma}-2}\left(R\delta\kappa\right)^{\frac{1}{\sigma}-2}R^2\delta^2\left(1-\kappa\right).\end{aligned}$$

Obviously, $sign\left(\frac{\partial}{\partial\kappa}\underline{s}_m\right)=sign\left(1-\sigma\right)$ for $\theta<1-\beta^p$. Hence

$$sign\left(\frac{\partial s_m}{\partial\theta}\right)=sign\left(\frac{\partial s_m}{\partial\beta^p}\right)=sign\left(1-\sigma\right).$$

C Proof of Proposition 3

In general,

$$\text{sign}\left(\frac{d\Omega^*}{d\theta}\right)=\text{sign}\left(\left(\beta\delta-\delta^*\right)\frac{\partial b^a}{\partial\theta}-\frac{\beta\delta}{1-\theta}b^a\right)$$

With CRRA utility, we can calculate

$$
\begin{aligned}
& \left(\beta\delta-\delta^*\right)\frac{\partial b^a}{\partial\theta}-\frac{\beta\delta}{1-\theta}b^a \\
= & \left\{\left(\delta^*-\beta\delta\right)\frac{1-\sigma+\sigma\left(\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}+\frac{1}{1-\theta}R\left(R\delta\right)^{-\frac{1}{\sigma}}}{\sigma\left[1-\theta+R\left(R\delta\right)^{-\frac{1}{\sigma}}\left(\frac{\beta}{1-\theta}\right)^{-\frac{1}{\sigma}}\right]\left[1-\left(\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}\right]}-\frac{\beta\delta}{1-\theta}\right\}b^a \\
= & \frac{\left(\delta^*-\beta\delta\right)\left[1+\frac{1}{1-\theta}R\left(R\delta\right)^{-\frac{1}{\sigma}}\right]-\sigma\left[1-\left(\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}\right]\left[\delta^*+\left(R\delta\right)^{1-\frac{1}{\sigma}}\left(\frac{\beta}{1-\theta}\right)^{1-\frac{1}{\sigma}}\right]}{\sigma\left[1-\theta+R\left(R\delta\right)^{-\frac{1}{\sigma}}\left(\frac{\beta}{1-\theta}\right)^{-\frac{1}{\sigma}}\right]\left[1-\left(\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}\right]}b^a \\
= & \frac{\delta^*-\beta\delta-\sigma\frac{\left[1-\left(\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}\right]\left[\delta^*+\left(R\delta\right)^{1-\frac{1}{\sigma}}\left(\frac{\beta}{1-\theta}\right)^{1-\frac{1}{\sigma}}\right]}{1+\frac{1}{1-\theta}R\left(R\delta\right)^{-\frac{1}{\sigma}}}}{\sigma\left(\frac{\beta}{1-\theta}\right)^{-\frac{1}{\sigma}}\left[\left(1-\theta\right)\left(\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}+R\left(R\delta\right)^{-\frac{1}{\sigma}}\right]^2/\left[1+\frac{1}{1-\theta}R\left(R\delta\right)^{-\frac{1}{\sigma}}\right]}\widehat{s}_o
\end{aligned}
$$

Define the numerator

$$
\Psi\left(\theta\right)\equiv\delta^*-\beta\delta-\sigma\frac{\left[1-\left(\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}\right]\left[\delta^*+\left(R\delta\right)^{1-\frac{1}{\sigma}}\left(\frac{\beta}{1-\theta}\right)^{1-\frac{1}{\sigma}}\right]}{1+\frac{1}{1-\theta}R\left(R\delta\right)^{-\frac{1}{\sigma}}} \tag{28}
$$

then

$$
sign\left(\frac{d\Omega^*}{d\theta}\right)=sign\left(\Psi\left(\theta\right)\right).
$$

It follows straightforwardly that $\Psi\left(\theta\right)|_{\theta\to1-\beta}=\delta^*-\beta\delta>0$, implying that

$$
\left.\frac{d\Omega^*}{d\theta}\right|_{\theta\to1-\beta}>0.
$$

We first show that $\frac{\partial\Psi(\theta)}{\partial\theta}>0$ for all $\sigma>0$ (monotonicity).

I) $0<\sigma\leq1$, in (28), both $\left[1-\left(\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}\right]$ and $\left[\delta^*+\left(R\delta\right)^{1-\frac{1}{\sigma}}\left(\frac{\beta}{1-\theta}\right)^{1-\frac{1}{\sigma}}\right]$ are positive and decreasing in θ, and $\left[1+\frac{1}{1-\theta}R\left(R\delta\right)^{-\frac{1}{\sigma}}\right]$ is positive and increasing in θ, so we have $\frac{\partial\Psi(\theta)}{\partial\theta}>0$ for $0<\sigma\leq1$.

II) $\sigma > 1$, using

$$\frac{\partial}{\partial\left(\frac{\beta}{1-\theta}\right)}\Psi\left(\theta\right)=\frac{\begin{array}{c}\left[\delta^{*}\left(\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}-1}+\left(R\delta\right)^{1-\frac{1}{\sigma}}\left(\frac{\beta}{1-\theta}\right)^{-\frac{1}{\sigma}}\right]\left[1+\frac{1}{1-\theta}R\left(R\delta\right)^{-\frac{1}{\sigma}}\right]\\ +\sigma\frac{R}{\beta}\left(R\delta\right)^{-\frac{1}{\sigma}}\left[1-\left(\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}\right]\left[\delta^{*}-\delta\beta\left(\frac{\beta}{1-\theta}\right)^{-\frac{1}{\sigma}}\right]\end{array}}{\left[1+\frac{1}{1-\theta}R\left(R\delta\right)^{-\frac{1}{\sigma}}\right]^{2}},$$

since $0 < \frac{1}{\sigma} < 1$, and $0 < \frac{\beta}{1-\theta} < 1$, it follows that $\left(\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}} > \frac{\beta}{1-\theta} > \beta$, and hence

$$\delta^{*}-\delta\beta\left(\frac{\beta}{1-\theta}\right)^{-\frac{1}{\sigma}} > \delta^{*}-\delta > 0,$$

implying that $\frac{\partial}{\partial\left(\frac{\beta}{1-\theta}\right)}\Psi\left(\theta\right) > 0$, and thus $\frac{\partial\Psi(\theta)}{\partial\theta} > 0$ for $\sigma > 1$.

It follows that if $\Psi\left(1-\beta^{p}\right) \geq 0$, we always have $\frac{d\Omega^{*}}{d\theta} > 0$ over the range of $1-\beta^{p} < \theta < 1-\beta$. If $\Psi\left(1-\beta^{p}\right) < 0$, then there exists a $\widetilde{\theta}$ and a unique local minimum for true utility, say $\Omega^{*}(\widetilde{\theta})$, such that $\frac{\partial\Omega^{*}(\cdot)}{\partial\theta} \lesseqqgtr 0$ for $\theta \lesseqqgtr \widetilde{\theta}$. $\Psi\left(1-\beta^{p}\right) < 0$ is thus a necessary and sufficient condition for the existence of a turning point within this range of $1-\beta^{p} < \theta < 1-\beta$.

Turning to the sign of $\Psi\left(1-\beta^{p}\right)$ we have

$$\Psi\left(1-\beta^{p}\right)=\delta^{*}-\beta\delta-\sigma\frac{1-\left(\frac{\beta}{\beta^{p}}\right)^{\frac{1}{\sigma}}}{1+\frac{1}{\beta^{p}}R\left(R\delta\right)^{-\frac{1}{\sigma}}}\left[\delta^{*}+\left(R\delta\right)^{1-\frac{1}{\sigma}}\left(\frac{\beta}{\beta^{p}}\right)^{1-\frac{1}{\sigma}}\right],$$

and hence a sufficient condition that $\Psi\left(1-\beta^{p}\right) < 0$ is

$$\sigma\frac{1-\left(\frac{\beta}{\beta^{p}}\right)^{\frac{1}{\sigma}}}{1+\frac{1}{\beta^{p}}R\left(R\delta\right)^{-\frac{1}{\sigma}}} > 1,$$

or

$$\sigma > \frac{1+\frac{1}{\beta^{p}}R\left(R\delta\right)^{-\frac{1}{\sigma}}}{1-\left(\frac{\beta}{\beta^{p}}\right)^{\frac{1}{\sigma}}}.$$

D Proof of Proposition 4

Using (25), $\frac{\partial \Omega^*}{\partial \theta} > 0$ requires to show for all $\theta < 1 - \beta$,

$$\left\{ (\delta^*)^2 - \beta\delta^2 + \left[(\delta^* - \beta\delta)\,\delta \frac{\beta}{1-\theta} - (\delta^*)^2 + \beta\delta^2 \right] \frac{\partial \underline{s}_o}{\partial \underline{s}_m} \right\} \frac{\partial \underline{s}_m}{\partial \theta} + \delta^* \left(\frac{\beta}{1-\theta}\delta - \delta^* \right) \frac{\partial \underline{s}_o}{\partial \theta} > 0.$$

We only need to prove the result in the case where $\frac{\partial \underline{s}_m}{\partial \theta} < 0$, i.e., $\sigma > 1$ with CRRA utility. So it's equivalent to show

$$\frac{(\delta^*)^2 - \beta\delta^2 + \left[(\delta^* - \beta\delta)\,\delta \frac{\beta}{1-\theta} - (\delta^*)^2 + \beta\delta^2 \right] \frac{\partial \underline{s}_o}{\partial \underline{s}_m}}{\delta^* \left(\delta^* - \frac{\beta}{1-\theta}\delta \right)} < \frac{\frac{\partial \underline{s}_o}{\partial \theta}}{\frac{\partial \underline{s}_m}{\partial \theta}}.$$

Define the LHS to be $A(\theta)$, and the RHS to be $B(\theta)$. With CRRA utility, using (26),

$$\begin{aligned}
A(\theta) &= \frac{(\delta^*)^2 - \beta\delta^2 + \left[(\delta^* - \beta\delta)\,\delta \frac{\beta}{1-\theta} - (\delta^*)^2 + \beta\delta^2 \right] \frac{R}{R + \left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}}}{\delta^* \left(\delta^* - \frac{\beta}{1-\theta}\delta \right)} \\
&< \frac{(\delta^*)^2 - \beta\delta^2 + \left\{ \left[(\delta^*)^2 - \beta\delta^2 \right] \frac{\beta}{1-\theta} - \left[(\delta^*)^2 - \beta\delta^2 \right] \right\} \frac{R}{R + \left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}}}{(\delta^*)^2 \left(1 - \frac{\beta}{1-\theta} \right)} \\
&= \frac{\left[(\delta^*)^2 - \beta\delta^2 \right] \left[\left(R\delta\frac{\beta}{1-\theta} \right)^{\frac{1}{\sigma}} + \frac{\beta}{1-\theta} R \right]}{(\delta^*)^2 \left(1 - \frac{\beta}{1-\theta} \right) \left[R + \left(R\delta\frac{\beta}{1-\theta} \right)^{\frac{1}{\sigma}} \right]} \\
&< \frac{\left(R\delta\frac{\beta}{1-\theta} \right)^{\frac{1}{\sigma}} + \frac{\beta}{1-\theta} R}{\left(1 - \frac{\beta}{1-\theta} \right) \left[R + \left(R\delta\frac{\beta}{1-\theta} \right)^{\frac{1}{\sigma}} \right]}
\end{aligned}$$

Also, using (26) and (27),

$$\frac{\partial \underline{s}_o}{\partial \theta} = -\frac{\frac{1}{\sigma} \left(R\delta\frac{\beta}{1-\theta} \right)^{\frac{1}{\sigma}-1} R\delta \frac{\beta}{(1-\theta)^2}}{\left[R + \left(R\delta\frac{\beta}{1-\theta} \right)^{\frac{1}{\sigma}} \right]^2} R\underline{s}_m,$$

$$
\begin{aligned}
\frac{\partial \underline{s}_m}{\partial \theta} &= \frac{R^2\left(\beta\delta R^2\right)^{-\frac{1}{\sigma}}\frac{1-\sigma}{\sigma^2}\left(1+\delta\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}-1}\right)^{-\frac{1}{\sigma}-1}\left(R+\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}\right)^{\frac{1}{\sigma}-2}\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}-2}R^2\delta^2\left(1-\frac{\beta}{1-\theta}\right)}{1+R^2\left(\beta\delta R^2\right)^{-\frac{1}{\sigma}}\left(1+\delta\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}-1}\right)^{-\frac{1}{\sigma}}\left(R+\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}\right)^{\frac{1}{\sigma}-1}} \\
&\quad \times\frac{\beta}{\left(1-\theta\right)^2}\underline{s}_m
\end{aligned}
$$

Thus

$$
\begin{aligned}
B\left(\theta\right) &= \frac{-\frac{\frac{1}{\sigma}\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}-1}R\delta\frac{\beta}{(1-\theta)^2}}{\left[R+\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}\right]^2}R\underline{s}_m}{\frac{R^2(\beta\delta R^2)^{-\frac{1}{\sigma}}\frac{1-\sigma}{\sigma^2}\left(1+\delta\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}-1}\right)^{-\frac{1}{\sigma}-1}\left(R+\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}\right)^{\frac{1}{\sigma}-2}\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}-2}R^2\delta^2\left(1-\frac{\beta}{1-\theta}\right)}{1+R^2(\beta\delta R^2)^{-\frac{1}{\sigma}}\left(1+\delta\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}-1}\right)^{-\frac{1}{\sigma}}\left(R+\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}\right)^{\frac{1}{\sigma}-1}}\frac{\beta}{(1-\theta)^2}\underline{s}_m} \\
&= \frac{\sigma}{\sigma-1}\frac{\frac{\beta}{1-\theta}\left[1+R^2\left(\beta\delta R^2\right)^{-\frac{1}{\sigma}}\left(1+\delta\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}-1}\right)^{-\frac{1}{\sigma}}\left(R+\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}\right)^{\frac{1}{\sigma}-1}\right]}{R\left(\beta\delta R^2\right)^{-\frac{1}{\sigma}}\left(1+\delta\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}-1}\right)^{-\frac{1}{\sigma}-1}\left(R+\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}\right)^{\frac{1}{\sigma}}\left(1-\frac{\beta}{1-\theta}\right)} \\
&> \frac{\frac{\beta}{1-\theta}R^2\left(\beta\delta R^2\right)^{-\frac{1}{\sigma}}\left(1+\delta\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}-1}\right)^{-\frac{1}{\sigma}}\left(R+\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}\right)^{\frac{1}{\sigma}-1}}{R\left(\beta\delta R^2\right)^{-\frac{1}{\sigma}}\left(1+\delta\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}-1}\right)^{-\frac{1}{\sigma}-1}\left(R+\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}\right)^{\frac{1}{\sigma}}\left(1-\frac{\beta}{1-\theta}\right)} \\
&= \frac{\frac{\beta}{1-\theta}R+\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}}{\left(R+\left(R\delta\frac{\beta}{1-\theta}\right)^{\frac{1}{\sigma}}\right)\left(1-\frac{\beta}{1-\theta}\right)} > A\left(\theta\right)
\end{aligned}
$$

The proposition is proved.