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Fractional Factorial Designs with Admissible Sets of Clear Two-Factor Interactions

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Abstract
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Keywords
alias set, clear compromise plan, clear effect, even design, minimum aberration, requirement set

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Fractional Factorial Designs with Admissible Sets of Clear Two-Factor Interactions

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Abstract: We consider the problem of selecting two-level fractional factorial designs that allow joint estimation of all main effects and some specified two-factor interactions (2fi's) without aliasing from other 2fi's. This problem is to find, among all 2^{m-p} designs with given m and p, those resolution IV designs whose sets of clear 2fi's contain the specified 2fi's as subsets. A 2fi is clear if it is not aliased with any main effect or any other 2fi. We use a linear graph to represent the set of clear 2fi's for a resolution IV design, where each line connecting two vertices represents a clear 2fi between the two vertices. We call a 2^{m-p} resolution IV design admissible if its graph is not a real subgraph of any other graphs of 2^{m-p} resolution IV designs. We show that all even resolution IV designs are inadmissible. In fact, the number of admissible designs is much smaller than the number of non-isomorphic designs. This leads to a concise catalog of all admissible designs of 32 and 64 runs. We also use an algorithm to determine all admissible 128-run resolution IV designs, but only provide some representative designs here.

Key words and phrases: Alias set, clear compromise plan, clear effect, defining contrast subgroup, even design, linear graph, requirement set, resolution.

1 Introduction

Regular two-level fractional factorial designs are commonly used to identify important factors. Such a design is referred to as a 2^{m-p} design. It has m two-level factors, with 2^{m-p} runs, and is determined by its defining contrast subgroup, which consists of $2^p - 1$ defining words.

The number of letters in a word is its length. The length of the shortest word in the defining contrast subgroup is called the resolution of a design (Box and Hunter (1961)). For a design of resolution at least V, all main effects and two-factor interactions (2fi's) are estimable if all three-factor and higher order interactions are negligible. However, such designs often require more runs than one can afford, and for economical reasons, resolution IV designs are often used for estimating main effects and some 2fi's.

Wu and Hamada (2000, Chapter 4) proposed a general rule for selecting 2^{m-p} designs with maximum resolution IV based on the number of clear 2fi's. That is, among resolution IV designs with given m and p, those with the maximum number of clear 2fi's (called MaxC2 designs by Wu and Wu (2002)) are the best. A 2fi is clear if it is not aliased with any main effect or any other 2fi (Wu and Chen (1992)). Although MaxC2 designs have the maximum number of clear 2fi's, they may not allow estimation of some specified 2fi's without confounding with other 2fi's, while there exists such a design for the given m and p that satisfies the requirements. In this paper, we extend the general Wu-Hamada rule and consider the problem of selecting 2^{m-p} designs that allow joint estimation of all main effects and specified 2fi's clear of aliasing from other 2fi's. This problem is to find, among all 2^{m-p} resolution IV designs with given m and p, designs whose sets of clear 2fi's contain the specified 2fi's as subsets. Thus, instead of simply counting the number of clear 2fi's of a design, we need to consider the specific structure of its set of clear 2fi's.

For a 2^{m-p} resolution IV design d, we use a linear graph $G(d)$ to represent its set of clear 2fi's, where each vertex represents a factor and each line connecting two vertices represents a clear 2fi's between the two vertices (factors). We call $G(d)$ admissible if it is not a real subgraph of $G(d')$ for any other 2^{m-p} resolution IV design d'. In this case, we call design d *admissible*. Admissible designs simplify the search for resolution IV designs that satisfy certain requirements for some 2fi's to be clear. For example, the search for clear compromise plans (Ke, Tang, and Wu (2005)) can be carried out within the class of admissible designs. This greatly reduces the computational burden. The graphs representing the clear 2fi's of these admissible designs also give the structures of the clear 2fi's, which are useful for practitioners to arrange the factors in an experiment.

To obtain admissible designs, first we show that all even resolution IV designs are inadmissible; see the next section for a proof. (A design is even if its defining contrast subgroup consists entirely of even length words. Otherwise, its defining contrast subgroup must consist of both even and odd words and the design is called an even/odd design.) Among even/odd resolution IV designs with clear 2fi's, we apply a classical subgraph-isomorphism algorithm due to Ullmann (1976). As it turns out, many even/odd resolution IV designs are also inadmissible and can be eliminated, so the number of admissible designs is much smaller than the number of non-isomorphic designs. This leads to a concise catalog of all admissible designs of 32 and 64 runs, and a lengthy but substantially reduced list for 128 runs. In some cases, for given values of m and p, there is only one 2^{m-p} admissible design.

2 A Catalog of Admissible Designs of 32, 64, and 128 Runs

Let $k = m - p$ and $M(k)$ be the maximum value of m for which there exists a 2^{m-p} design of resolution V or more. To study admissible designs, we only need to consider resolution IV designs that have clear 2fi's. Based on the results of Chen and Hedayat (1998) it suffices to focus on $M(k)+1 \le m \le 2^{k-2}+1$ and $k \ge 5$. For example, this requires that $7 \le m \le 9$ for $k = 5$ (32-run designs), that $9 \le m \le 17$ for $k = 6$ (64-run designs), and that $12 \le m \le 33$ for $k = 7$ (128-run designs). (For $k = 7$, there exist orthogonal arrays of strength four for up to 15 factors (Mee 2004). These nonregular designs permit estimation of all 2fi's orthogonally and should be preferable to the regular designs with $m = 12$ to 15.)

Chen, Sun, and Wu (1993) made a complete enumeration of non-isomorphic resolution IV designs to produce their tables for 32 and 64 runs. (Two designs are said to be isomorphic if one can be obtained from the other by permuting the columns, switching the signs, or a combination of the above.) Since Chen, Sun, and Wu (1993) did not contain the complete list, we relied on the complete enumeration kindly provided by D. Sun. For 128-run designs, we relied on the list obtained by Block (2003); his enumeration of resolution IV designs was based on a shortcut isomorphism check. Xu (2007) verified that Block's (2003) enumeration is complete for even/odd resolution IV designs.

We now show that the search for admissible designs need not consider any even resolution IV designs.

Lemma 1: For any even resolution IV design with clear 2fi's, there exists an even/odd resolution IV (or higher) design for which the set of length-four words is a proper subset of the length-four words for the even design.

Proof: Let d denote any even resolution IV design with at least one clear 2fi. Let XY denote a clear 2fi. Observe that no length-four word contains XY. If X appears in d's defining contrast subgroup, then express the design with Y as a basic column and X as generated using an interaction of the basic columns. If the generator for X contains Y, drop Y from the generator, while if it does not contain Y, add Y to the generator. As a result of this change:

- Every word in d's defining contrast subgroup that contained XY is shortened by the deletion of Y in the new design, and every word that contained X but not Y in the even design is lengthened by the addition of Y in the new design.

- Since XY never appears in a length-four word of the even design, every length-four word that contained X in the even design becomes a length-five word containing XY in the new design.

- The new design is even/odd.

If the even design does not contain X, then simply add X to one of the generators. As a result, 2^{p-1} words in d's defining contrast subgroup will increase in length by the addition of X, while the other words will be unchanged. Once again, the new design is an even/odd design and has no new length-four words, so its set of length-four words is a proper subset of d's length-four words.

Lemma 2: For any resolution IV design with clear 2fi's, there exists a length-four word containing one factor X that appears in a clear 2fi XY.

Proof: Divide the factors into two sets, A = those appearing in length-four words and B = those not appearing in length-four words. A cannot be empty, since the design is resolution IV. If B is empty, then Lemma 2 is obviously true. If B is not empty, then any interaction involving a factor in A and a factor in B is clear.

Theorem: Every even resolution IV design will clear 2fi's is inadmissible, being dominated by an even/odd resolution IV design with less aberration.

Proof: By Lemmas 1 and 2, we know that for any even resolution IV design with clear 2fi's, there exists an even/odd resolution IV (or higher) design for which its set of length-four words is a proper subset of the set of length-four words for the even design. Therefore every clear 2fi in the even design is clear in an even/odd design with less aberration. Furthermore, let X denote a factor of the even design such that XY is clear but at least one 2fi involving X is not clear. (Such a factor necessarily exists, since otherwise, all 2fi's would be clear.) By altering the generator for X as described in the proof for Lemma 1, all the 2fi's involving X are clear in the even/odd design. Therefore, the original even design is inadmissible, since the set of clear 2fi's is increased for the even/odd design.

We apply a classical subgraph-isomorphism algorithm (Ullmann (1976)) to identify designs in these catalogs that are not admissible. By removing these inadmissible designs, we obtain all the admissible designs for 32-, 64-, and 128-run designs. The results are summarized in Tables 1 to 3. To identify the different admissible designs, we use the designation

Table 1: *Complete catalog of 32-run admissible designs*

Design	Additional Columns	C2	C
7_2.C1	7 27	*(8, 16, 27)	15
8_3.C1	7 11 29	*(16, 29)	13
9_4.C1	7 11 13 30	*(16, 30)	15

"m_p.Cx" rather than "m-p.x" to avoid confusion with the labels for designs found elsewhere, e.g., as in Chen, Sun, and Wu (1993).

In the following, we use the term "columns" (or "column") in the sense of Chen, Sun, and Wu (1993). In Tables 1 to 3, C2 represents clear 2fi's and C represents the number of clear 2fi's. We use some simple notation to denote specific structures of clear 2fi's. For example:

- *(8, 16, 27) represents the set of all 2fi's involving at least one of the factors (columns) 8, 16, or 27.
- For two sets of factors (columns) G_1 and G_2, $G_1 \times G_2$ is the set of all 2fi's between the factors in G_1 and those in G_2.
- $@(G_1, G_2, G_3) = (G_1 \times G_2) \bigcup (G_2 \times G3) \bigcup (G_3 \times G_1)$
- $\#(G_1, G_2, G_3) = (G_1 \times G_2) \bigcup (G_2 \times G3)$.

For example, @(4, 5, 8) represents the set of 2fi's 45, 58, and 84, while #(4, 5, 8) represents the set of 2fi's 45 and 58.

Tables 1 and 2 give all the 32- and 64-run admissible designs. Note that MaxC2 designs are always admissible. On the other hand, minimum aberration (MA) designs (Fries and Hunter (1980)) are generally inadmissible designs unless they are MaxC2 designs (see next section). For the cases of 2^{7-2}, 2^{8-3}, 2^{9-4}, 2^{9-3}, 2^{16-10}, 2^{17-11}, 2^{32-25}, and 2^{33-26} designs, the unique MaxC2 design is the only admissible design for each of the cases. Note also that all three admissible 2^{12-5} designs are MaxC2 designs and have the same linear graph of clear 2fi's. In general, it is easy to identify from the tables admissible designs that have the same linear graph.

Table 4 gives the number of non-isomorphic resolution IV designs N, the number of admissible designs N_{des}, and the number of admissible graphs N_{graph} for all the 32- and 64-run designs and 128-run designs with $12 \leq k \leq 16$. Note that the number of admissible

Table 4: *Number of admissible designs and graphs*

Design	N	N_{des}	N_{graph}
2^{7-2}	3	1	1
2^{8-3}	4	1	1
2^{9-4}	5	1	1
2^{9-3}	12	1	1
2^{10-4}	24	4	3
2^{11-5}	34	7	6
2^{12-6}	43	12	9
2^{13-7}	47	10	6
2^{14-8}	49	9	5
2^{15-9}	44	5	4
2^{16-10}	48	1	1
2^{17-11}	40	1	1
2^{12-5}	249	3	1
2^{13-6}	623	5	3
2^{14-7}	1535	30	15
2^{15-8}	3522	140	99
2^{16-9}	7500	682	584

designs or graphs is generally much smaller than the number of non-isomorphic resolution IV designs, especially for 128-run designs.

3 Minimum Aberration versus MaxC2 Designs

Table 5 shows the number of clear 2fi's for the MA designs, followed by the maximum number of clear 2fi's among all resolution IV designs, for sizes 32, 64, and 128. For designs where the number of factors m is only slightly large than $M(k)$, the MA designs also maximize the number of clear 2fi's; this is the case for two 32-run designs, four 64-run designs, and three 128-run designs. However, as m increases, the number of clear 2fi's for MA designs diminishes rapidly. This is because MA designs tend to have more uniform size alias sets (Cheng, Steinberg, and Sun (1999)). No MA design of size 64 (128) has any clear 2fi's for $m > 14$ (23). As m approaches $2^{k-2}+1$, the maximum number of clear 2fi's equals $2m-3$, and the set of clear 2fi's is always of the form $*(X, Y)$.

Table 5. *Number of clear 2fi's for MA versus MaxC2 designs*

m	No. 2fi's	32-runs	64-runs	128-runs
6	15	all		
7	21	15:15		
8	28	13:13	all	
9	36	8:15	30:30	
10	45	none	33:33	
11	55		34:34	all
12	66		36:36	60:60
13	78		20:36	66:66
14	91		8:25	73:73
15	105		0:27	63:77
16	120		0:29	60:69
17	136		0:31	46:75
18	153		none	33:81
19	171			36:78
20	190			24:84
21	210			26:84
22	231			25:48
23	253			12:45
24	276			0:45
⋮	⋮			⋮
33	528			0:63
34	561			none

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Table 2: *Complete catalog of 64-run admissible designs*

Design	Additional Columns	C2	C
9_3.C1	7 27 45	*(8, 16, 32, 27, 45)	30
10_4.C1	7 27 43 53	*(8, 53); (1, 2, 4, 7)×(16, 32, 27, 43)	33
10_4.C2	7 11 29 51	*(16, 32, 29, 51)	30
10_4.C3	7 11 29 46	*(16, 32, 29, 46)	30
10_4.C4	7 25 42 53	*53; @((4, 7), (16, 25), (32, 42)); 8×(4, 7); 2×(16, 25); 1×(32, 42)	27
11_5.C1	7 11 29 45 51	*51; (16, 32, 29, 45)×(1, 2, 4, 8, 7, 11)	34
11_5.C2	7 11 29 46 49	@((4, 8, 7, 11), (16, 32), (29, 46)); 49×(4, 8, 7, 11); 2×(16, 32); 1×(29, 46)	28
11_5.C3	7 11 19 29 62	*(32, 29, 62)	27
11_5.C4	7 11 13 14 51	*(16, 32, 51)	27
11_5.C5	7 25 42 52 63	*63; @(1, 32, 7, 16, 2, 25, 4, 8, 52); 42×(1, 4, 16); 2×52; 16×7; 32×25	25
11_5.C6	7 11 21 46 56	*46; @((1, 2, 4, 7), 56, 21, 11, 32); 11×(16, 56); 21×(8, 32)	25
11_5.C7	7 11 21 38 57	*57; @((8, 11), (16, 21), (32, 38))	22
12_6.C1	7 11 29 45 51 62	(1, 2, 4, 8, 7, 11)×(16, 32, 29, 45, 51, 62)	36
12_6.C2	7 11 21 46 54 56	@((32, 46, 54, 56),11, 21); (1, 2, 4, 7)×(32, 46, 54, 56); 16×11; 8×21	27
12_6.C3	7 11 21 41 51 63	@((16, 21), (32, 41), (51, 63)); (8, 11)×(16, 21); (4, 7)×(32, 41); (1, 2)×(51, 63)	24
12_6.C4	7 11 13 19 46 49	*46; #((4, 8, 7, 11), (32, 49), 13, (16, 19))	23
12_6.C5	7 11 21 41 54 56	*54; @(32, 7, 56, 11, 21); 4×32; 7×41; 2×56; 16×11; 8×21	21
12_6.C6	7 11 21 25 31 45	*(32, 45)	21
12_6.C7	7 11 13 19 21 57	*(32, 57)	21
12_6.C8	7 11 13 19 21 46	*(32, 46)	21
12_6.C9	7 11 13 14 19 53	*(32, 53)	21
12_6.C10	7 11 19 37 57 63	@((8, 16, 11, 19),(32, 37),(57, 63))	20
12_6.C11	7 11 19 29 37 59	*29; #((1, 2, 4, 7), 59, 37, (8, 16, 11, 19))	20
12_6.C12	7 11 13 19 37 57	*57; #(11, (32, 37),(16, 19), 13)	19
13_7.C1	7 11 13 30 46 49 63	(2, 4, 8, 7, 11, 13)×(16, 32, 30, 46, 49, 63)	36
13_7.C2	7 11 13 19 21 25 46	*(32, 46)	23
13_7.C3	7 11 13 14 19 21 57	*(32, 57)	23
13_7.C4	7 11 13 14 19 21 54	*(32, 54)	23
13_7.C5	7 11 13 19 35 49 63	*63; #((4, 8, 7, 11), 49, 13, (16, 32, 19, 35))	21
13_7.C6	7 11 19 29 37 41 47	*29; (16, 19)×(32, 37, 41, 47)	20

Table 2: (*Continued*)

Design	Additional Columns	C2	C
13_7.C7	7 11 13 19 21 41 54	*54; (32, 41)×(16, 7, 19, 21)	20
13_7.C8	7 11 13 19 21 46 54	*32; (46, 54)×(1, 2, 4, 7)	20
13_7.C9	7 11 13 19 37 49 63	*63; #((16, 19), 13, 49, 11, (32, 37)); 8×49	19
13_7.C10	7 11 13 19 37 57 63	@((16, 19), (32, 37), (57, 63)); 13×(16, 19); 11×(32, 37); 8×(57, 63)	18
14_8.C1	7 11 13 14 19 21 25 54	*(32, 54)	25
14_8.C2	7 11 13 14 19 21 22 57	*(32, 57)	25
14_8.C3	7 11 13 19 21 25 35 60	*60; (32, 35)×(13, 21, 25)	19
14_8.C4	7 11 13 14 19 21 41 54	*54; (32, 41)×(16, 19, 21)	19
14_8.C5	7 11 13 19 21 35 41 63	*63; #((16, 19), 41, 21, (32, 35))	18
14_8.C6	7 11 19 29 30 37 41 47	(16, 19, 29, 30)×(32, 37, 41, 47)	16
14_8.C7	7 11 13 19 21 41 54 63	(16, 7, 19, 21)×(32, 41, 54, 63)	16
14_8.C8	7 11 13 19 21 46 54 56	(1, 2, 4, 7)×(32, 46, 54, 56)	16
14_8.C9	7 11 13 14 19 37 57 63	@((16, 19), (32, 37), (57, 63))	12
15_9.C1	7 11 13 14 19 21 22 25 58	*(32, 58)	27
15_9.C2	7 11 13 14 19 21 35 41 63	*63; #((16, 19), 41, 21, (32, 35))	19
15_9.C3	7 11 13 14 19 21 25 35 60	*60; (32, 35)×(21, 25)	18
15_9.C4	7 11 13 19 21 25 35 60 63	(13, 21, 25)×(32, 35, 60, 63)	12
15_9.C5	7 11 13 14 19 21 41 54 63	(16, 19, 21)×(32, 41, 54, 63)	12
16_10.C1	7 11 13 14 19 21 22 25 26 60	*(32, 60)	29
17_11.C1	7 11 13 14 19 21 22 25 26 28 63	*(32, 63)	31

Table 3: *Selected 128-run admissible designs*

Design	Additional Columns	C
12_5.C1	7 57 90 108 119	
C2	*(8, 16, 32, 64, 57, 90, 108, 119)	60
12_5.C2	7 27 45 78 121	
C2	*(8, 16, 32, 64, 27, 45, 78, 121)	60
12_5.C3	7 27 45 86 120	
C2	*(8, 16, 32, 64, 27, 45, 86, 120)	60
13_6.C1	7 27 43 85 102 120	
C2	*(8, 64, 85, 102, 120); (1, 2, 4, 7)×(16, 32, 27, 43)	66
13_6.C2	7 27 43 53 78 120	
C2	*(8, 64, 53, 78, 120); (1, 2, 4, 7)×(16, 32, 27, 43)	66
13_6.C3	7 27 45 78 121 122	
C2	*(8, 16, 32, 64, 27, 45, 78)	63
13_6.C4	7 25 42 77 118 120	
C2	*(64, 77, 118, 120); @((4, 7), (16, 25), (32, 42)); 8×(4, 7); 2×(16, 25); 1×(32, 42)	60
13_6.C5	7 25 42 53 78 120	
C2	*(64, 53, 78, 120); @((4, 7), (16, 25), (32, 42)); 8×(4, 7); 2×(16, 25); 1×(32, 42)	60
14_7.C1	7 27 43 53 78 118 120	
C2	*(8, 53); @((1, 2, 4, 7), (16, 32, 27, 43), (64, 78, 118, 120))	73
14_7.C2	7 27 45 78 121 122 124	
C2	*(8, 16, 32, 64, 27, 45, 78)	70
14_7.C3	7 11 29 49 82 102 120	
C2	*(29, 102, 120); @((4, 8, 7, 11), (32, 49), (64, 82)); 16×(4, 8, 7, 11); 1×(32, 49); 2×(64, 82)	64
14_7.C4	7 27 29 46 78 118 120	
C2	*(8, 16); (1, 2, 4, 7, 27, 29)×(32, 64, 46, 78, 118, 120)	61
14_7.C5	7 11 29 51 83 102 120	
C2	*(16, 29, 102); (1, 2, 4, 8, 7)×(32, 64, 51, 83, 120)	61
15_8.C1	7 27 45 78 121 122 124 127	
C2	*(8, 16, 32, 64, 27, 45, 78)	77
15_8.C2	7 11 29 45 51 78 118 120	
C2	@((1, 2, 4, 8, 7), (16, 32, 29, 45), (64, 51, 78, 118, 120)); 11×(16, 32, 29, 45)	69
15_8.C3	7 11 25 45 50 86 110 120	
C2	*45; (64, 86, 110, 120)×(1, 2, 4, 8, 16, 32, 7, 11, 25, 50); #((1, 8), (32, 50), (4, 7), (16, 25))	66
15_8.C4	7 11 25 42 53 78 118 120	
C2	*53; (64, 78, 118, 120)×(1, 2, 4, 8, 16, 32, 7, 11, 25, 42); @((4, 7), (16, 25), (32, 42))	66

Table 3: (*Continued)*

Design	Additional Columns	C
15_8.C5	7 11 13 30 49 82 101 120	
C2	*30; (101, 120)×(1, 2, 4, 8, 32, 64, 7, 11, 49, 82); @((4, 8, 7, 11, 13), (32, 49), (64, 82)); 16×(4, 8, 7, 11); 2×(32, 49); 1×(64, 82)	66
16_9.C1	7 11 19 25 28 45 77 110 120	
C2	*(110, 120); (32, 64, 45, 77)×(1, 2, 4, 8, 16, 7, 11, 19, 25, 28)	69
16_9.C2	7 11 19 25 26 45 86 100 120	
C2	*(45, 86, 100); (32, 64, 120)×(1, 2, 8, 16, 7, 11, 19, 25, 26)	69
16_9.C3	7 11 19 25 41 53 78 118 120	
C2	*53; (64, 78, 118, 120)×(1, 2, 4, 8, 16, 32, 7, 11, 19, 25, 41); #(19, (32, 41), (4, 7), 25)	67
16_9.C4	7 11 29 45 51 53 78 118 120	
C2	@((1, 2, 4, 7), (16, 32, 29, 45), (64, 78, 118, 120)); @((64, 78, 118, 120), 8, 51, (16, 32, 29, 45), 11, 53)	66
16_9.C5	7 11 19 25 26 28 45 78 120	
C2	*(32, 64, 45, 78, 120)	65
17_10.C1	7 11 19 25 26 28 45 77 110 120	
C2	*(110, 120); (32, 64, 45, 77)×(1, 2, 4, 8, 16, 7, 11, 19, 25, 26, 28)	75
17_10.C2	7 11 19 25 26 28 31 45 78 120	
C2	*(32, 64, 45, 78, 120)	70
17_10.C3	7 11 13 19 25 26 46 85 100 120	
C2	*46; (32, 64, 85, 100, 120)×(1, 2, 8, 16, 7, 11, 19, 25, 26); #(4, (85, 120), (64, 100), 13)	69
17_10.C4	7 11 19 25 28 35 45 86 110 120	
C2	*45; (64, 86, 110, 120)×(1, 2, 4, 8, 16, 32, 7, 11, 19, 25, 28, 35); (25, 28)×(32, 35)	68
17_10.C5	7 11 13 19 46 49 50 85 109 120	
C2	*46; (64, 85, 109, 120)×(1, 2, 4, 8, 16, 7, 11, 19, 49, 50); 32×(4, 8, 7, 11); 13×(16, 19, 49, 50)	64
18_11.C1	7 11 19 25 26 28 31 45 77 110 120	
C2	*(110, 120); (32, 64, 45, 77)×(1, 2, 4, 8, 16, 7, 11, 19, 25, 26, 28, 31)	81
18_11.C2	7 11 13 19 21 25 26 46 92 103 120	
C2	(32, 64, 46, 92, 103, 120)×(1, 2, 8, 16, 11, 13, 19, 25, 26); @(4, 64, 92, 7, 46, 32, 21, 120, 103); @(64, 46, 120); 21×64; 4×46; 7×120	69
18_11.C3	7 11 13 19 21 25 41 63 78 118 120	
C2	(64, 63, 78, 118, 120)×(1, 2, 4, 8, 16, 32, 11, 13, 19, 21, 25, 41); (32, 41)×(7, 19, 21)	66
18_11.C4	7 11 13 19 25 26 46 49 85 109 120	
C2	*46; (64, 85, 109, 120)×(1, 2, 4, 8, 16, 7, 11, 19, 25, 26, 49); #(32, (4, 7), 49, 13)	66
18_11.C5	7 11 13 19 25 46 49 50 85 109 120	
C2	*46; (64, 85, 109, 120)×(1, 2, 4, 8, 16, 7, 11, 19, 25, 49, 50); 32×(4, 7); 13×(49, 50)	65

Table 3: (*Continued*)

Design	Additional Columns	C
19_12.C1	7 11 14 19 25 26 28 31 45 77 110 120	
C2	*120; (32, 64, 45, 77, 110)×(1, 2, 4, 8, 16, 7, 11, 19, 25, 26, 28, 31)	78
19_12.C2	7 11 14 19 25 26 28 31 45 77 117 120	
C2	(32, 64, 45, 77, 117, 120)×(1, 2, 4, 8, 16, 7, 11, 14, 19, 25, 26, 28, 31)	78
19_12.C3	7 11 19 25 26 28 31 35 45 86 110 120	
C2	*45; (64, 86, 110, 120)×(1, 2, 4, 8, 16, 32, 7, 11, 19, 25, 26, 28, 31, 35)	74
19_12.C4	7 11 19 25 26 28 35 45 50 86 110 120	
C2	*45; (64, 86, 110, 120)×(1, 2, 4, 8, 16, 32, 7, 11, 19, 25, 26, 35); 50×(4, 7); 28×(32, 35)	70
19_12.C5	7 11 19 21 25 26 28 35 45 86 110 120	
C2	(64, 45, 86, 110, 120)×(1, 2, 4, 8, 16, 32, 7, 11, 19, 25, 26, 28, 35); (32, 35)×(21, 28)	69
20_13.C1	7 11 14 19 22 25 26 28 31 45 77 117 120	
C2	(32, 64, 45, 77, 117, 120)×(1, 2, 4, 8, 16, 7, 11, 14, 19, 22, 25, 26, 28, 31)	84
20_13.C2	7 11 13 19 21 22 25 26 31 46 78 100 120	
C2	(32, 64, 46, 78, 100, 120)×(1, 2, 8, 16, 7, 11, 13, 19, 21, 25, 26, 31); #(4, 120, 100, 22)	75
20_13.C3	7 11 19 21 25 26 28 35 45 50 86 110 120	
C2	(64, 45, 86, 110, 120)×(1, 2, 4, 8, 16, 32, 7, 11, 19, 25, 26, 35); #((32, 35), 28, 45, 50)	64
20_13.C4	7 11 19 21 25 26 28 35 45 50 77 117 120	
C2	(64, 45, 77, 117, 120)×(1, 2, 4, 8, 16, 11, 19, 25, 26, 28, 35); #(7, 45, 50, 28, (32, 35))	60
20_13.C5	7 11 19 25 26 28 31 35 45 53 86 110 120	
C2	(64, 86, 110, 120)×(1, 2, 4, 8, 16, 32, 7, 11, 19, 25, 26, 28, 31, 45); #(35, 45, 32, 53)	59
21_14.C1	7 11 14 19 21 22 25 26 28 31 45 77 117 120	
C2	(32, 64, 45, 77, 117, 120)×(1, 2, 4, 8, 16, 7, 11, 14, 19, 22, 25, 26, 28, 31)	84
21_14.C2	7 11 19 25 26 28 35 45 50 67 73 86 103 120	
C2	(45, 86, 103, 120)×(1, 2, 8, 16, 11, 19, 25, 26); (4, 7)×(45, 50, 73, 86, 120); (32, 35)×(28, 45, 73, 86, 120); (64, 67)×(28, 45, 50, 86, 120)	62
21_14.C3	7 11 13 19 25 26 35 41 42 53 59 95 103 120	
C2	(64, 53, 95, 103, 120)×(1, 2, 8, 16, 11, 19, 25, 26, 41, 59); #(13, 59, (4, 7), 53, (32, 35))	57
21_14.C4	7 11 13 14 19 22 25 26 35 41 60 85 103 120	
C2	(64, 60, 85, 103, 120)×(1, 2, 8, 16, 11, 19, 22, 25, 26); #((64, 103), 13, 60, 7, (85, 120)); #((35, 41), 22, 32, 60)	55
21_14.C5	7 11 13 19 25 26 35 41 53 78 92 95 100 120	
C2	(64, 53, 100, 120)×(1, 2, 8, 16, 11, 19, 25, 26, 78); @((4, 32, 7, 35), 53, 95, (13, 41), 78); 95×(64, 100)	51
22_15.C1	7 11 19 29 30 35 41 42 44 47 53 59 78 118 120	
C2	(64, 78, 118, 120)×(1, 2, 4, 8, 32, 7, 11, 35, 41, 42, 44, 47)	48
22_15.C2	7 11 19 29 38 41 47 70 73 79 99 109 110 117 120	
C2	#(16, (32, 64, 38, 41, 47, 70, 73, 79), (29, 99, 120), (1, 2, 4, 7), 117)	48

Table 3: (*Continued*)

Design	Additional Columns	C
22_15.C3	7 11 13 19 25 26 35 41 53 78 85 92 95 100 120	
C2	@((1, 2, 8, 16, 11, 19, 25, 26), (64, 100), 78, 120); #((64, 100), 95, 53, (7, 35), 78); #((1, 2, 8, 16, 11, 19, 25, 26), 53, 78, (4, 32, 13, 41))	47
22_15.C4	7 11 13 14 19 21 22 37 41 49 59 62 89 100 120	
C2	@((64, 100), (2, 7, 11, 14, 19, 22, 59, 62), (89, 120)); (8, 16)×(64, 100); (13, 21)×(89, 120)	44
22_15.C5	7 11 13 19 22 41 60 69 70 74 84 91 93 94 120	
C2	@((32, 41), (7, 19, 22, 74, 91, 94), (60, 120)); (32, 41)×(16, 64, 69, 70); (60, 120)×(8, 11, 13, 93)	44
23_16.C1	7 11 13 14 19 25 26 31 35 41 53 67 73 85 100 120	
C2	@((1, 2, 8, 16, 11, 19, 25, 26), (53, 85, 120), 31, 100); 7×(53, 85, 120); 31×(32, 64, 35, 41, 67, 73)	45
23_16.C2	7 11 13 14 19 21 38 41 44 50 55 61 62 93 101 120	
C2	(64, 93, 101, 120)×(1, 2, 4, 7, 11, 38, 41, 44, 50, 55, 62)	44
23_16.C3	7 19 21 22 35 38 49 63 67 69 81 87 97 100 112 115	
C2	*(8, 63)	43
23_16.C4	7 19 21 22 35 37 38 42 67 81 87 92 100 103 112 117	
C2	*92; (8, 42)×(16, 64, 19, 21, 22, 67, 81, 100, 103, 112)	42
23_16.C5	7 11 13 14 19 22 38 41 60 76 87 91 99 111 116 120	
C2	@((64, 19, 76, 87, 91, 99, 111, 116, 120), (32, 38), 60, (16, 22), 41); 60×(1, 2, 7, 11, 13, 14, 41)	40
24_17.C1	7 19 21 22 35 37 38 49 67 69 81 87 92 100 103 112 117	
C2	*(8, 92)	45
24_17.C2	7 11 19 21 22 26 31 45 97 104 110 112 117 118 121 122 124	
C2	*45; (32, 64)×(2, 4, 7, 11, 19, 31, 110, 121, 124)	41
24_17.C3	7 11 13 19 25 26 28 35 41 44 47 59 61 62 78 118 120	
C2	(64, 78, 118, 120)×(1, 2, 4, 16, 32, 7, 19, 28, 35, 41)	40
24_17.C4	7 11 13 14 19 22 25 38 41 60 76 87 91 99 111 116 120	
C2	#((64, 19, 76, 87, 91, 99, 111, 116, 120), (32, 38, 41), 60, (1, 2, 16, 7, 11, 13, 14, 22, 25))	39
24_17.C5	7 11 19 26 41 49 63 86 88 101 104 107 112 115 121 122 124	
C2	#((64, 26, 41, 49), (4, 7, 124), (63, 86, 101), (8, 11, 112)); 101×(1, 16, 104, 121); 7×(32, 88)	36
25_18.C1	7 19 21 22 35 37 38 49 55 67 69 81 84 95 100 103 112 117	
C2	*(8, 95)	47
25_18.C2	7 11 19 21 25 26 31 45 97 104 110 112 117 118 121 122 124 127	
C2	*45; (32, 64)×(2, 4, 7, 11, 19, 110, 118, 124)	40

Table 3: (*Continued*)

Design	Additional Columns	C
25_18.C3	7 11 13 14 19 22 25 26 35 41 60 85 92 95 103 114 120 123	
C2	#((1, 2, 8, 16, 11, 13, 19, 25, 26), (64, 60, 103), 22,	
	(32, 35, 41, 85, 92, 95, 114, 120, 123))	39
25_18.C4	7 11 19 21 22 26 31 45 77 97 104 110 112 117 118 121 122 124	
C2	(32, 64, 45, 77)×(2, 4, 7, 11, 19, 31, 110, 121, 124)	36
25_18.C5	7 11 19 25 26 31 41 53 74 97 98 104 107 112 115 121 122 124	
C2	*53; #((31, 124), (32, 64), (4, 7) (41, 74))	36
26_19.C1	7 19 21 22 35 37 38 49 52 67 69 70 81 87 97 98 111 112 115	
C2	*(8, 111)	49
26_19.C2	7 19 21 22 35 37 38 49 52 59 67 69 70 81 87 97 111 112 115	
C2	*111; (8, 59)×(1, 4, 69, 70, 81, 87, 97)	39
26_19.C3	7 11 19 30 38 41 42 44 47 69 81 82 84 87 88 121 122 124 127	
C2	*30; #((8, 11, 88), 69, (32, 38), (16, 64, 19))	36
26_19.C4	7 19 25 26 41 42 49 56 59 67 95 97 98 104 107 112 121 122 124	
C2	*95; #((8, 112), 7, (32, 49), 124, (41, 42)); 4×(32, 49)	35
26_19.C5	7 11 13 19 21 22 35 37 38 49 67 81 82 84 87 103 105 112 115	
C2	*105; #((64, 67, 112, 115), 13, (49, 103), (8, 11))	35
27_20.C1	7 19 21 22 35 37 38 49 52 67 69 70 81 87 97 98 111 112 115 117	
C2	*(8, 111)	51
27_20.C2	7 19 21 30 35 37 38 49 50 55 67 69 70 76 81 87 98 112 117 118	
C2	*30; (8, 76)×(16, 32, 35, 37, 55, 112)	38
27_20.C3	7 11 19 30 38 41 42 44 47 67 69 81 82 84 87 88 121 122 124 127	
C2	*30; (32, 38)×(16, 64, 19); 69×(8, 11, 88)	35
27_20.C4	7 13 19 21 22 25 35 37 38 49 50 52 55 56 61 69 75 87 112 117	
C2	*75; #((8, 13, 56, 61), 87, 25, (64, 69, 112, 117))	35
27_20.C5	7 19 21 22 26 35 38 45 49 67 69 73 81 82 87 97 98 112 117 118	
C2	*45; #((7, 21), 73, 118, 8, (38, 117)); #(118, 26, (32, 97))	35
28_21.C1	7 19 21 22 35 37 38 49 50 52 67 69 70 81 87 97 98 100 111 112 115	
C2	*(8, 111)	53
28_21.C2	7 19 21 30 35 37 38 49 50 55 67 69 70 76 81 87 98 103 112 117 118	
C2	*30; (8, 76)×(16, 32, 37, 55, 112)	37
28_21.C3	7 19 21 22 25 35 37 38 49 50 52 55 56 67 69 81 84 87 97 112 126	
C2	*126; #((8, 25), (67, 87), 56, (69, 84))	35
28_21.C4	7 11 19 21 22 25 35 37 38 49 50 52 55 56 69 81 84 87 97 112 126	
C2	*126; 11×(64, 81, 97, 112); 56×(69, 84, 87)	34
28_21.C5	7 11 19 21 22 35 37 38 49 50 56 67 69 70 81 84 98 111 112 115 118	
C2	*111; #((4, 7), 56, 84, 8, (81, 98)); 11×84	34

Table 3: (*Continued*)

Design	Additional Columns	C
29_22.C1	7 19 21 22 35 37 38 49 50 52 67 69 70 81 82 84 87 97 98 100 111 112	
C2	*(8, 111)	55
29_22.C2	7 19 21 22 35 37 38 49 50 52 56 67 69 70 81 82 84 98 111 112 115 118	
C2	*111; (8, 56)×(7, 69, 81, 84)	36
29_22.C3	7 11 19 21 22 35 37 38 49 50 56 67 69 70 81 84 98 111 112 115 117 118	
C2	*111; #((4, 7), 56, 84, 8, (81, 98)); 11×84	35
29_22.C4	7 11 19 21 22 25 35 37 38 49 50 52 55 67 69 81 82 84 97 100 112 126	
C2	*126; #((8, 25), 100, 11, (69, 84, 97, 112))	35
29_22.C5	7 11 29 37 41 42 44 47 51 78 81 82 84 87 88 104 112 118 121 122 124 127	
C2	#((8, 32, 112), (29, 51, 78), (16, 64, 104), (11, 37, 118))	27
30_23.C1	7 19 21 22 35 37 38 49 50 52 55 67 69 70 81 82 84 87 97 98 100 111 112	
C2	*(8, 111)	57
30_23.C2	7 19 21 22 35 37 38 49 50 56 67 69 70 81 82 88 97 98 111 112 115 117 118	
C2	*111; (4, 7)×(8, 56, 88)	35
30_23.C3	7 19 21 22 35 37 38 49 50 52 56 67 69 70 81 82 88 97 98 111 112 115 118	
C2	*111; #((4, 52), 88, 7, (8, 56))	34
30_23.C4	7 19 21 30 35 37 38 44 49 52 55 58 67 69 81 82 84 87 98 100 103 112 117	
C2	(8, 30, 44, 58)×(69, 82, 87, 98)	16
30_23.C5	7 19 21 22 35 37 38 49 50 52 55 56 67 69 81 82 84 95 97 98 112 115 126	
C2	@((8, 56), (69, 84), (95, 126))	12
31_24.C1	7 19 21 22 35 37 38 49 50 52 55 67 69 70 81 82 84 87 97 98 100 111 112 115	
C2	*(8, 111)	59
31_24.C2	7 19 21 22 35 37 38 49 50 52 56 67 69 70 81 82 88 97 98 111 112 115 117 118	
C2	*111; #((4, 52), 88, 7, (8, 56))	35
31_24.C3	7 11 19 30 35 37 38 41 42 44 47 81 82 84 87 88 91 104 112 115 121 122 124 127	
C2	*30; (64, 104)×(37, 38)	34
31_24.C4	7 11 14 25 26 28 31 45 53 56 67 70 85 88 97 98 100 103 104 112 121 122 124 127	
C2	(45, 53, 85)×(11, 14, 67, 70)	12
32_25.C1	7 19 21 22 35 37 38 49 50 52 55 67 69 70 81 82 84 87 97 98 100 111 112 115 117	
C2	*(8, 111)	61
33_26.C1	7 19 21 22 35 37 38 49 50 52 55 67 69 70 81 82 84 87 97 98 100 111 112 115 117 118	
C2	*(8, 111)	63