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Slot-specific Priorities with Capacity Transfers

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This paper studies two-sided, many-to-one matching markets in which agents match to institutions that may have multiple slots available to accept contracts. In many real-world institutions, there are restrictions for some slots (if not all) either on slot priorities or on the transferability of unfilled slots over others (or both). We construct a rich family of practical choice rules, slot-specific priorities with capacity transfers (SSPwCT), that utilize both independent slot priorities and transferability of vacant slots. We show that the cumulative offer mechanism (COM) is stable, strategy-proof and respects improvements with regards to SSPwCT choice rules. Transferring the capacity of one more unfilled slot, while all else is constant, leads to strategy-proof Pareto improvement of the COM.We also provide comparative static results for expansion of branch capacity and addition of new contracts. Our results have implications for resource allocation problems with diversity considerations.

Keywords

Market design, matching, affirmative action

Disciplines

Economics | Economic Theory | Marketing

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1 Introduction

The slot-specific priorities framework of *Kominers and Sönmez* (2016) is an influential model of agent-institution matching in which each institution has a set of slots that can be assigned to different agents. Slots have their own (potentially independent) rankings for contracts, where each contract is between an agent and an institution and specifies some terms and conditions. Within each institution, a linear order – referred to as the *precedence order* – determines the sequence in which slots are filled. This framework provides a powerful tool for market designers to handle diversity and affirmative action in two-sided matching models. Institutions choose contracts by filling their slots sequentially. Implementing affirmative action might cause agents' priorities to vary across the slots of a given institution. In the context of cadet-branch matching (*Switzer and Sönmez*, 2013 and *Sönmez*, 2013), for example, some slots for each service branch grant higher priority for cadets who are willing to serve additional years of service. *Kominers and Sönmez* (2016) develop a general framework for handling these sorts of slot-specific priority structures. Their model embeds classical priority matching settings (*Balinski and Sönmez*, 1999 and *Abdulkadiroğlu and Sönmez*, 2003), models of affirmative action (*Kojima*, 2012, *Hafalir et al.*, 2013, and *Aygün and Bó*, 2019), and cadet-branch matching framework.

The slot-specific priorities framework provides flexible solutions to important real-world matching problems. *Aygün and Bó* (2019), for example, design slot-specific priorities choice rules for the Brazilian college admission problem, where students have multidimensional privileges. In 2012 Brazilian public universities were mandated to use affirmative action policies for candidates from racial and income minorities. The law established that certain fractions of the accepted students in each program should have studied in public high schools, come from a low-income family, and/or belong to racial minorities. This objective was implemented by partitioning the positions in each program, earmarking them for different combinations of these affirmative action characteristics. *Aygün and Bó* (2019) analyze several of these choice rules, present their shortcomings and correct their shortcomings by designing slot-specific priorities choice rules for programs.

More recently, *Pathak et al.* (2020) utilize the slot-specific priorities framework to design triage protocol for ventilator rationing. The authors analyze the consequences of rationing medical resources via a reserve system. In a reserve system, ventilators are partitioned into multiple categories. These reserve categories can differ based on the groups of patients to receive higher priority and the ethical principle used. *Pathak et al.* (2020) propose sequential reserve matching rules, which are first introduced in the slot-specific priorities framework of *Kominers and Sönmez* (2016).

Slot-specific priorities choice rules are great tools for market designers to solve practical problems. However, when slot priorities are restricted by institutional constraints the full potential of this framework cannot be achieved. *Westkamp*'s (2013) model also permits priorities to vary across slots. His model allows more general forms of interactions across slots than *Kominers and Sönmez* (2016) allow. However, *Westkamp* (2013) does not allow the variation in contractual terms, which is necessary for applications such as cadet-branch matching (*Switzer and Sönmez*, 2013 and *Sönmez*, 2013), admissions for publicly funded educational institutions and government sponsored jobs in India (*Aygün and Turhan*, 2017, 2018, 2019, and 2020), and airline upgrade allocation (*Kominers and Sönmez*, 2016).

In many real-world applications, either slot priorities (for some slots if not all) or the ability to transfer vacant slots may be restricted due to institutional constraints especially in diversity and affirmative action in school choice, college admissions, government-sponsored job matching, and also faculty hiring. When both are restricted, some slots might remain unfilled. This, in turn, might lead to a Pareto inferior outcome. A real-world example of this case can be seen in India. For admissions to publicly funded educational institutions and government-sponsored jobs in India, each institution reserves 15 percent of its slots for people from Scheduled Castes (SC), 7.5 percent for people from Scheduled Tribes (ST), and 27 percent for people from Other Backward Classes (OBC). The remaining slots are called open-category slots and are available to everyone, including those from SC, ST, and OBC. In each institution, for slots that are reserved for SC, ST and OBC, only applicants who declare they belong to these respective categories are considered. If there is low demand from either SC or ST applicants, some slots remain vacant. Vacant SC/ST slots can potentially be utilized by two ways: (1) other candidates can be made eligible for SC/ST slots, but at a lower priority than all SC/ST applicants, or (2) vacant SC/ST slots can be reverted into, say, open-category slots. Currently, none of these possibilities are allowed. Each year many SC/ST slots remain vacant.

There are instances where slot priorities are restricted but where vacant slots can be transferred. *Baswana et al.* (2018) designed and implemented a new joint seat allocation process for technical universities in India. Since 2008, following a Supreme Court decision, unfilled OBC slots are required to be made available to GC candidates of publicly funded educational institutions. However, institutions are prohibited from modifying the priorities of OBC slots. This possibility was offered to Indian authorities but ultimately rejected. On the other hand, reverting vacant OBC slots into GC slots is allowed. *Baswana et al*. (2018) report their interaction with the Indian policy makers as follows:

> "Business rule 5 required unfilled OBC seats to be made available to Open category candidates. *The approach we initially suggested involved construction of augmented Merit Lists making Open category candidates eligible for OBC seats but at a lower priority than all OBC candidates*, and modification of virtual preference lists so that general candidates now apply for both the OPEN and the OBC virtual programs. We

> showed that running our algorithm on these modified inputs would produce the candidate optimal allocation satisfying the business rules. However, *the authorities feared that this approach may cause issues* with computing the closing rank correctly (see Design Insight 6), or have some other hidden problem. An authority running centralized college or school admissions is typically loathe to modify, add complexity to, or replace software that is tried and tested."

It might also be the case that reverting vacant slots of some type into another type is prohibited while modifying slot priorities is allowed. In this paper, we utilize both of these powerful tools –i.e., capacity transfers across slots and independent slot priorities– to formulate a larger set of practical choice rules. Our aim is to expand the toolkit of market designers to be able to implement comprehensive selection criteria, especially when there are institutional restrictions. We construct institutional choices as follows: Each institution has two types of slots: *original* slots and *shadow* slots. Each shadow slot has an initial capacity of 0 and is associated with an original slot. Each slot (original and shadow) has a linear order (potentially independent) over contracts. Institutions have precedence orders over original and shadow slots. Each institution is also endowed with a *location vector* for shadow slots. The exact order at which slots are processed is determined by the precedence orders over original and shadow slots together with the location vector. The interaction between associated original and shadow slots is as follows: If an original slot is assigned a contract, then the capacity of the associated shadow slot remains 0, i.e., the shadow slot will be inactive. If an original slot cannot be filled, the institution has the option to transfer its capacity to the associated shadow slot. In this case, the capacity of the shadow slot becomes 1. A capacity transfer vector of the institution determines for which original slots such reversion is allowed and for which ones it is not.

When the transferability of all original slots is prohibited, our model reduces to that of *Kominers and Sönmez* (2016). Given the exact precedence order (i.e., precedence orders over original and shadow slots, location vector of shadow slots), capacity transfer vector, and slot priorities over contracts, slots are filled *sequentially* in a straightforward manner. We call this family of choice rules that are constructed this way *Slot-specific Priorities with Capacity Transfers Choice Rules* (SSPwCT). We show how markets with SSPwCT choice rules can be cleared by the COM. To do so, we borrowed the novel observability theory of *Hatfield et al.* (2019). In the Appendix, we prove that each choice rule in the SSPwCT family satisfies the irrelevance of rejected contracts (IRC) condition (Proposition 1), the observable substitutability (Proposition 2), the observable size monotonicity (Proposition 3), and the non-manipulation via contractual terms (Proposition 4). The COM is the unique stable and strategy-proof mechanism in the SSPwCT environment (Theorem 1) that also respects improvements (Theorem 2).

Finally, we provide several comparative static regarding the outcome of the COP with respect to SSPwCT choice rules. We first show that when an institution reverts one more original slot into a shadow slot in the case of a vacancy, if all else is fixed, we obtain a strategy-proof Pareto improvement under the COM (Theorem 3). Then, building on *Kominers'* (2019) analysis, we show that the outcome of the COP is (weakly) improved for all agents when (1) an original slot is added to an institution, while all else remains fixed (Theorem 4), (2) new contracts are added at the bottom of slots' priority orders right before the null contract (Theorem 5), and (3) the new contracts of a single agent are added anywhere in the slots' priority orders (Theorem 6).

SSPwCT choice rules may be used in cadet-branch matching in USMA and ROTC (*Sönmez and Switzer,* 2013 and *Sönmez,* 2013), resource allocation in India with comprehensive affirmative action (*Boswana et al.,* 2019, *Aygün and Turhan,* 2018 and 2020), Chilean school choice and college admissions with affirmative action (*Rios et al.,* 2018 and *Correa et al., 2019*) , and Brazilian college admissions with multidimensional reserves (*Aygün and Bó*, 2019).

Related Literature

The matching with contracts model is introduced by *Fleiner* (2003) and *Hatfield and Milgrom* (2005). Important work on matching with contracts, among many others, include *Hatfield and Kojima* (2010), *Aygün and Sönmez* (2013), *Afacan* (2017), *Hatfield and Kominers* (2019), and *Hatfield et al.* (2017, 2019).

The technicality of this paper is built on *Hatfield et al.* (2019), in which the authors characterize when stable and strategy-proof matching is possible in many-to-one matching setting with contracts. We show that each choice rule in the SSPwCT class satisfies the three conditions from this paper, as well as the IRC of *Aygün and Sönmez* (2013). Another closely related paper to ours is *Hatfield et al.* (2017). The authors introduce a model of institutional choice in which each institution has a set of divisions and flexible allotment capacities that vary as a function of the set of available contracts. Each institution is modeled as having an allotment function that determines how many positions are allocated to each division, given the set of available contracts.

Kominers (2019) gives a novel proof of the entry comparative static via the respecting improvement property. The argument in his proof extends to yield comparative static results in matching with slot-specific priorities. We further extend *Kominer*'s (2019) results to the SSPwCT environment. Papers that study entry comparative static include *Kelso and Crawford* (1982), *Gale and Sotomayor* (1985), *Crawford* (1991), *Hatfield and Milgrom* (2005), *Biró et al.* (2008), *Ostrovsky* (2008), *Hatfield and Kominers* (2013), and *Chambers and Yenmez* (2017), among others.

Lexicographic choice rules are also studied in *Chambers and Yenmez* (2018) and *Doğan et al.* (2018), among others. *Chambers and Yenmez* (2018) consider lexicographic choice rules from an axiomatic perspective and show that these rules satisfy acceptance and path independence. *Doğan* *et al.* (2018) provide characterizations of lexicographic choice rules and of the deferred acceptance mechanism that operate based on a lexicographic choice structure.

There is extensive literature on matching with distributional constraints. A partial list includes *Abdulkadiroğlu and Sönmez* (2003), *Abdulkadiroğlu* (2005), *Biró et al.* (2010), *Kojima* (2012), *Hafalır et al.* (2013), *Westkamp* (2013), *Echenique and Yenmez* (2015), *Kamada and Kojima* (2015, 2017, and 2018), *Fragiadakis and Troyan* (2017), *Aygün and Turhan (2018, 2019, 2020), Jagadeesan* (2019), *Nguyen and Vohra* (2019), and *Afacan* (2020).

2 Matching with Contracts Framework

There is a finite set of *agents* $\mathscr{I} = \{i_1, ..., i_n\}$ and a finite set of *branches* $\mathscr{B} = \{b_1, ..., b_m\}$. There is a finite set of *contracts* $\mathscr{X}$. Each contract $x \in \mathscr{X}$ is associated with an agent $\mathbf{i}(x)$ and a branch $\mathbf{b}(x)$. There may be many contracts for each agent-branch pair. We call a set of contracts $X \subseteq \mathscr{X}$ an *outcome*, with $\mathbf{i}(X) \equiv \underset{x\in X}{\cup}\{\mathbf{i}(x)\}$ and $\mathbf{b}(X) \equiv \underset{x\in X}{\cup}\{\mathbf{b}(x)\}$. For any $i \in \mathscr{I} \cup \mathscr{B}$, we let $X_i \equiv \{x \in X \mid i \in \{\mathbf{i}(x), \mathbf{b}(x)\}\}$.

Each agent $i \in \mathscr{I}$ has unit demand over contracts in $\mathscr{X}_i$ and an outside option $\emptyset_i$. The strict preference of agent i over $\mathscr{X}_i \cup \{\emptyset_i\}$ is denoted by P_i. A contract $x \in \mathscr{X}_i$ is *acceptable for i* (with respect to P_i) if $xP_i\emptyset_i$. Agent preferences over contracts are naturally extended to preferences over outcomes. For each individual $i \in \mathscr{I}$ and set of contracts $X \subseteq \mathscr{X}$, we denote by $\underset{P_i}{max}X$ the maximal element of X according to preference order P_i, and we assume that $\underset{P_i}{max}X = \emptyset$ if $\emptyset_i P_i x$ for all $x \in X$. Each individual always chooses the best available contract according to his preferences, so that choice rule $C^i(X)$ of an individual $i \in \mathscr{I}$ from contract set $X \subseteq \mathscr{X}$ is defined by $C^i(X) = \underset{P_i}{max}X$.

Each branch $b \in \mathscr{B}$, on the other hand, has multi-unit demand and is endowed with a choice rule C^b that describes how b would choose from any offered set of contracts. We assume throughout that for all $X \subseteq \mathscr{X}$ and for all $b \in \mathscr{B}$, the choice rule C^b:

1. Only selects contracts to which b is a party, i.e., $C^b(X) \subseteq X_b$, and
2. Selects at most one contract with any given agent, i.e., $C^b(X)$ is feasible.

For any $X \subseteq \mathscr{X}$ and $b \in \mathscr{B}$, we denote by $R^b(X) \equiv X \setminus C^b(X)$ the set of contracts that b *rejects* from X.

2.1 A Model of Branch Choices: Slot-specific Priorities with Capacity Transfers (SSPwCT) Choice Rules

Each branch $b \in \mathscr{B}$ has two types of seats: **original** seats and **shadow** seats. Let $O_b = \{o_b^1, o_b^2, ..., o_b^{n_b}\}$ and $E_b = \{e_b^1, e_b^2, ..., e_b^{n_b}\}$ be branch b's set of original seats and shadow seats, respectively, where

n_b denotes the physical capacity of branch b. Each seat in both O_b and E_b has priority orders over contracts in $\mathscr{X}_b \cup \{\emptyset\}$ denoted by Π^o_b for $o \in O_s$ and Π^e_b for $e \in E_s$ (the weak orders are denoted by Γ^o and Γ^e) and can be assigned at most one contract in $\mathscr{X}_b \equiv \{x \in \mathscr{X} \mid \mathbf{b}(x) = b\}$. Let $\Pi_b = (\Pi^o_b, \Pi^e_b)$ denote the priority profile of branch b and $\Pi = (\Pi_b)_{b \in \mathscr{B}}$ denote the priority profiles of all branches. We denote by $\max_{\pi^o} X$ the maximal element of X according to priority ordering Π^o and by $\max_{\pi^e} X$ the maximal element of X according to priority ordering Π^e. We assume $\max_{\pi^o} X = \emptyset$ if $\emptyset \Pi^o x$ for all $x \in X$ and $\max_{\pi^e} X = \emptyset$ if $\emptyset \Pi^e x$ for all $x \in X$.

Each branch $b \in \mathscr{B}$ has two linear precedence orders, one over original seats, $\triangleright^O_b$, and one over shadow seats, $\triangleright^E_b$. We denote $O_b = \{o^1_b, o^2_b ..., o^{n_b}_b\}$ with $o^l_b \triangleright^O_b o^{l+1}_b$ and $E_b = \{e^1_b, e^2_b ..., e^{n_b}_b\}$ with $e^l_b \triangleright^E_b e^{l+1}_b$ unless otherwise stated. The interpretation of $\triangleright^O_b$ is that if $o \triangleright^O_b o'$, then, whenever possible, branch b fills seat o before o'. Each shadow seat is associated with an original seat. If the original seat remains empty, then branch b can decide whether to transfer its capacity to its associated shadow seat, which initially has no capacity, through a capacity transfer scheme q_b defined below.

A *capacity transfer scheme* is an integer-valued vector $q_b = (q^1_b, q^2_b, ..., q^{n_b}_b)$ such that for every $k = 1, ..., n_b$:

$$q^k_b = \begin{cases} 0 & \text{if branch } b \text{ does not transfer capacity from } o^k_b \text{ to } e^k_b \text{ when } o^k_b \text{ is not filled.} \\ 1 & \text{if branch } b \text{ transfers capacity from } o^k_b \text{to } e^k_b \text{ when } o^k_b \text{ is not filled.} \end{cases}$$

Since a capacity transfer from o^k_b to e^k_b is possible *only when* o^k_b is not filled, the physical capacity of branch b is never violated. We define an *indicator function* for the original seats as follows:

$$\mathbf{1}_{o^l_b} = \begin{cases} 0 & \text{if seat } o^l_b \text{ remains empty.} \\ 1 & \text{if seat } o^l_b \text{ is filled.} \end{cases}$$

Given precedence orders $\triangleright^O_b$ and $\triangleright^E_{\mathrm{b}}$, a *location vector* for the shadow seats of branch b is an integer-valued vector $L_b = (l_1, ..., l_{n_b})$ where l_k is the number of original seats that precede shadow seat e^k_b that satisfy the following condition:

$$L_b = \{(l_1, ..., l_{n_b}) \mid k \leq l_k \leq n_b \ \forall k = 1, ..., n_b \text{ and } l_k \geq l_{k-1} \ \forall k = 2, ..., n_b\}.$$

The condition in the definition of L_b ensures that for every shadow seat, the number of preceding original seats is greater than the number of preceding shadow seats. Hence, a shadow seat will never come before its associated original seat in this order. The location vector L_b together with precedence orders $\triangleright^O_b$ and $\triangleright^E_b$ gives us the exact order in which the original and shadow seats are filled. Let $\triangleright_b \equiv (\triangleright^O_b, \triangleright^E_b, L_b)$ denote the exact order of branch b's slots. We illustrate this with an example below.

Example 1. Consider a branch with three original seats with $o_b^1 \rhd_b^O o_b^2 \rhd_b^O o_b^3$ and three shadow seats with $e_b^1 \rhd_b^E e_b^2 \rhd_b^E e_b^3$ together with the location vector $L_b = (1,3,3)$. The order $\rhd_b$ in which the original and shadow seats are filled is as follows:

o_b^1 | e_b^1 | o_b^2 | o_b^3 | e_b^2 | e_b^3

Description of SSPwCT Choice Rules

For branch $b \in \mathscr{B}$, $C^b(\cdot, \rhd_b, q_b, \Pi_b) : 2^X \to 2^X$ denotes the choice rule of branch b given the precedence order of slots $\rhd_b$, the capacity transfer function q_b, and the priority profile of slots Π_b. Given a set of contracts $X \subseteq \mathscr{X}$, $C^b(X, \rhd_b, q_b, \Pi_b)$ denotes the set of chosen contracts for branch b from the set of contracts X.

To formulate the choice rule, we first rename the slots as $S = (s^1, s^2, \ldots, s^{2n_b})$ where s^k is either an original or a shadow seat, depending on $\rhd_b^O$, $\rhd_{\mathrm{b}}^E$, and L_b. In Example 1 above with $L_b = (1,3,3)$, we can rename slots as follows: $s_b^1 = o_b^1$, $s_b^2 = e_b^1$,$s_b^3 = o_b^2$, $s_b^4 = o_b^3$, $s_b^5 = e_b^2$, and $s_b^6 = e_b^3$. It is important to note that $\Pi_b^{s_b^1} = \Pi_b^{o_b^1}$, $\Pi_b^{s_b^2} = \Pi_b^{e_b^1}$, etc...

We are now ready to describe the choice procedure. Given $X \subseteq \mathscr{X}$:

- Start with the original seat s_b^1. Assign the contract x^1 that is $\Pi_b^{s_b^1} - \textit{maximal}$ among the contracts in X.

- If s_b^2 is either an original or a shadow seat such that $\mathbf{1}_{o_b^1} = 0$ and $q_b^1 = 1$, assign the contract x^2 that is $\Pi_b^{s_b^2} - \textit{maximal}$ among the contracts in $X \setminus X_{\mathbf{i}(\{x^1\})}$. Otherwise, assign the empty set.

- This process continues in sequence. If s_b^k is an original seat or a shadow seat such that $\mathbf{1}_{o_b^r} = 0$, where o_b^r is the original seat that is associated with the shadow seat s_b^k, and $q_b^r = 1$, then assign contract x^k that is $\Pi_b^{s_b^k} - \textit{maximal}$ among the contracts in $X \setminus X_{\mathbf{i}(\{x^1, \ldots, x^{k-1}\})}$. Otherwise, assign the empty set.

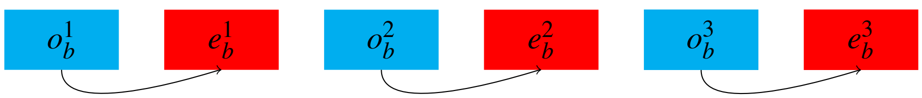
Given n_b, $(\rhd_b, q_b, \Pi_b)$ parametrizes the family of SSPwCT choice rules for branch b.

Examples of SSPwCT Choice Rules

Example 2. Consider $b \in \mathscr{B}$ with $n_b = 3$, $L_b = (1,2,3)$, and $q_b = (1,1,1)$. The capacity transfer scheme allows branch b to transfer capacities from original seats to shadow seats whenever they remain unfilled. Given the location vector and capacity transfer scheme, the choice procedure for branch b is as follows. Given an offer set $X \subseteq \mathscr{X}$:

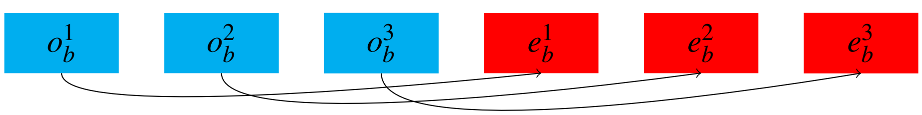
- Assign o_b^1 the contract x^1 that is $\Pi_b^{o_b^1}-$*maximal* among the contracts in X.
- If $\mathbf{1}_{o_b^1}=0$, assign e_b^1 the contract x^2 that is $\Pi_b^{e_b^1}-$*maximal* among the contracts in $X\setminus X_{\mathbf{i}(\{x^1\})}$. Otherwise, assign e_b^1 the empty set.
- Assign o_b^2 the contract x^3 that is $\Pi_b^{o_b^2}-$*maximal* among the contracts in $X\setminus X_{\mathbf{i}(\{x^1,x^2\})}$.
- If $\mathbf{1}_{o_b^2}=0$, assign e_b^2 the contract x^4 that is $\Pi_b^{e_b^2}-$*maximal* among the contracts in $X\setminus X_{\mathbf{i}(\{x^1,x^2,x^3\})}$. Otherwise, assign e_b^2 the empty set.
- Assign o_b^3 the contract x^5 that is $\Pi_b^{o_b^3}-$*maximal* among the contracts in $X\setminus X_{\mathbf{i}(\{x^1,x^2,x^3,x^4\})}$.
- If $\mathbf{1}_{o_b^3}=0$, assign e_b^3 the contract x^6 that is $\Pi_b^{e_b^3}-$*maximal* among the contracts in $X\setminus X_{\mathbf{i}(\{x^1,x^2,x^3,x^4,x^5\})}$. Otherwise, assign e_b^3 the empty set.

The following picture depicts the order of slots and the capacity transfer scheme where arrows indicate that the capacity is transferred if the original seat remains empty:



In the previous example each shadow seat appears right after its corresponding original seat. It is common in practice for the institution to try to fill its original seats first before the shadow seats. We provide such an example below.

Example 3. Consider branch $b\in\mathscr{B}$ with $n_b=3$, $L_b=(3,3,3)$, and $q_b=(1,1,1)$. The following picture depicts the order of slots and the capacity transfer scheme where arrows indicate that the capacity is transferred if the original seat remains empty:



Note that for any location vector, if the capacity transfer scheme is a vector of zeros, the SSPwCT choice rules are equivalent to the slot-specific priorities choice rules in *Kominers and Sönmez* (2016). Their model embeds variety of matching models, including cadet-branch matching, controlled school choice with majority quotas and with minority reserves, and models where priorities are heterogenous across all seats of a given institution. One exception is the model with dynamic reserves choice rules from *Aygün and Turhan* (2018). The family of SSPwCT choice rules embeds the models of both *Kominers and Sönmez* (2016) and *Aygün and Turhan* (2018) and, thus, provides a unifying framework for significant market design applications.

2.2 Stability

The standard solution concept in matching with contracts environments is stability. An **outcome** is a set of contracts $Y \subseteq \mathscr{X}$ that is "*feasible*". In other words,

- Y contains at most one contract for each agent, i.e., $|Y_i| \leq 1$ for each $i \in \mathscr{I}$.
- Y contains at most n_b contracts for each branch b, i.e., $|Y_b| \leq n_b$ for each $b \in \mathscr{B}$.

An outcome is **stable** if neither agents nor branches wish to walk away from their assignment unilaterally, and agents and branches cannot benefit by recontacting outside of the assigned outcome. Formally, an outcome Y is *stable* if:

- It satisfies individual rationality: $C^i(Y) = Y_i$ for all $i \in \mathscr{I}$ and $C^b(Y) = Y_b$ for all $b \in \mathscr{B}$.
- It satisfies unblockedness: A branch $b \in \mathscr{B}$ and a blocking set $Z \neq C^b(Y)$ such that $Z_b = C^b(Y \cup Z)$ and $Z_i = C^i(Y \cup Z)$ for all $i \in \mathbf{i}(Z)$ do not exist.

In other words, individual rationality requires that no individual be assigned a contract that she finds unacceptable, and unblockedness requires that no individual desires a branch at which she has a justified claim, under the priority, precedence, and capacity transfer structure.

2.3 Cumulative Offer Mechanism (COM)

A mechanism $\mathscr{M}(\cdot, C)$, where $C = (C^b)_{b \in \mathscr{B}}$ is a given profile of choice rules for branches, is a mapping from preference profiles of agents $P = (P_i)_{i \in I}$ to outcomes. A mechanism $\mathscr{M}(\cdot, C)$ is *stable* if $\mathscr{M}(P, C)$ is a stable outcome for every preference profile P. A mechanism $\mathscr{M}(\cdot, C)$ is *strategy-proof* if for every preference profile P, and for each individual $i \in \mathscr{I}$, there is no reported preference $\widetilde{P}_i$, such that

$$\mathscr{M}((\widetilde{P}_i, P_{-i}), C) P_i \mathscr{M}(P, C).$$

A particularly important class of mechanisms are the COMs. These types of mechanisms are defined with respect to a preference profile P and a strict ordering $\vdash$ of the elements of $\mathscr{X}$. In a COM, $\mathscr{C}^{\vdash}$, agents propose contracts according to a strict ordering $\vdash$ of the elements of $\mathscr{X}$. In every step, some agent who does not currently have a contract held by any institution proposes his most preferred contract that has not yet been proposed. Then, each branch chooses its most preferred set of contracts according to its choice rule and holds this set until the next step. When multiple agents are able to propose in the same step, the agent who actually proposes is determined by the ordering $\vdash$. The mechanism terminates when no agent is able to propose; at that point, each institution is assigned the set of contracts it is holding. We give the formal definition of the COM in Appendix A.

We are now ready to present our first main result.

Theorem 1. *The COM with respect to SSPwCT choice rules is stable and strategy-proof.*

The proof of our Theorem 1 follows directly from Theorem 4 of *Hatfield et al.* (2019). This theorem states that if all choice rules satisfy IRC, observable substitutability, observable size monotonicity and non manipulability via contractual terms, then the COP is stable and strategy proof. As shown in the Appendix, every choice rule in the SSPwCT family satisfies all of these conditions. Hence, the COP in this environment is stable and strategy-proof.

3 Respecting Improvements

Respect for improvements is an intuitive and desirable property in practice. It suggest that agents should have no incentive to try to lower their standings in branches' priority orders. This natural property becomes crucial, especially in merit-based systems where branches' priority orderings are determined through exam scores. To formally define it in our framework, fix the precedence order $\triangleright_b \equiv (\triangleright_b^O, \triangleright_b^E, L_b)$ and the capacity transfer function q_b of branch b.

Definition 1. We say that a choice rule $C^b(\cdot, \triangleright_b, q_b, \overline{\Pi}_b)$ of branch b is an *improvement over* $C^b(\cdot, \triangleright_b, q_b, \Pi_b)$ for agent $i \in \mathscr{I}$ if for all slots $s \in O_b \cup E_b$ the following conditions hold:

1. for all $x \in \mathscr{X}_i$ and $y \in \left(\mathscr{X}_{\mathscr{I} \setminus \{i\}} \cup \{\emptyset\}\right)$, if $x \Pi_b^s y$ then $x \overline{\Pi}_b^s y$; and
2. for all $y, z \in \mathscr{X}_{\mathscr{I} \setminus \{i\}}$, $y \Pi_b^s z$ if and only if $y \overline{\Pi}_b^s z$.

That is, $C^b(\cdot, \triangleright_b, q_b, \overline{\Pi}_b)$ is an improvement over $C^b(\cdot, \triangleright_b, q_b, \Pi_b)$ for agent i if $\overline{\Pi}_b$ is obtained from Π_b by increasing the priority of some of i's contracts while leaving the relative priority of other agents' contracts unchanged. We say that a profile of branch choices $\overline{C} \equiv (C^b(\cdot, \triangleright_b, q_b, \overline{\Pi}_b))_{b \in \mathscr{B}}$ is an **improvement** over $C \equiv (C^b(\cdot, \triangleright_b, q_b, \Pi_b))_{b \in \mathscr{B}}$ for agent $i \in \mathscr{I}$ if, for each branch $b \in \mathscr{B}$, $C^b(\cdot, \triangleright_b, q_b, \overline{\Pi}_b)$ is an improvement over the choice rule $C^b(\cdot, \triangleright_b, q_b, \Pi_b)$.

We say that a mechanism φ **respects improvements** for $i \in I$ if for any preference profile $P \in \times_{i \in I} \mathscr{P}^i$

$$\varphi_i(P; \overline{C}) R^i \varphi_i(P; C)$$

whenever $\overline{C} \equiv (C^b(\cdot, \triangleright_b, q_b, \overline{\Pi}_b))_{b \in \mathscr{B}}$ is an improvement over $C \equiv (C^b(\cdot, \triangleright_b, q_b, \Pi_b))_{b \in \mathscr{B}}$.

Theorem 2. *The COM with respect to SSPwCT choice rules respects improvements.*

Proof. See Appendix C.

4 Comparative Statics

In this section, we first look at the effect of increasing the transferability of original seats on agents' welfare under the COP with respect to SSPwCT choice rules. We, then, extend two comparative static results of *Kominers* (2019) to the SSPwCT family. *Kominers* (2019) provides a new proof of the entry comparative static, by way of the respect for improvements property. The author sheds light on a strong relationship between several different entry comparative statics and the respecting improvement property in many-to-one matching markets with contracts. We analyze the effect of expanding the capacity of a single branch on agents' welfare agent-proposing under the COP in an SSPwCT environment. We also examine the effect of adding contracts on agents' welfare under the COP in our setting.

4.1 Increasing Transferability

SSPwCT is a large family of choice rules. If transferability of all original slots is prohibited, we obtain the slot-specific priorities choice rules of *Kominers and Sönmez* (2016). Allowing transferability of an original slot, while everything else remains fixed, is welfare-improving for agents. We analyze the monotonicity of improvements on agents' welfare by only changing the transferability of original slots, while all else remains fixed.

Let $\widetilde{q}_b$ and q_b be two capacity transfer schemes for branch b. We say that $\widetilde{q}_b$ is **more flexible** than q_b if $\widetilde{q}_b > q_b$, i.e., $\widetilde{q}_b^k \geq q_b^k$ for all $k = 1, ..., n_b$ and $\widetilde{q}_b^l > q_b^l$ for some $l = 1, ..., n_b$. Suppose that $\rhd_b$ and Π_b are fixed. Then, the SSPwCT choice rule $C^b(\cdot, \rhd_b, \widetilde{q}_b, \Pi_b)$ can be interpretted as an expansion of the SSPwCT choice rule $C^b(\cdot, \rhd_b, q_b, \Pi_b)$ if $\widetilde{q}_b$ is more flexible than q_b. We are now ready to present our result.

Theorem 3. *Suppose that Z is the outcome of the COM at (P, C), where $P = (P_{i_1}, ..., P_{i_n})$ is the profile of agent preferences and $C = (C^{b_1}, ..., C^{b_m})$ is the profile of branches' SSPwCT choice rules. Fix a branch $b \in \mathscr{B}$. Suppose that $\widetilde{C}^b$ takes as an input capacity transfer scheme that is more flexible than that of C^b, holding all else constant. Then, the outcome of the COM at $(P, (\widetilde{C}^b, C_{-b}))$, $\widetilde{Z}$, Pareto dominates Z.*

Proof. See Appendix C.

Theorem 3 is intuitive and indicates that making an untransferable original slot of a branch transferable leads to strategy-proof Pareto improvement of the COM. One should note that expanding a branch's choice rule changes the stability definition. However, Theorem 3 provides a normative foundation for such a change, as it increases agents' welfare. This result does not contradict the findings of *Alva and Manjunath* (2019), because transferring the capacity of an original slot to an associated shadow slot changes the feasible set in their context.

4.2 Expanding Capacity

We consider what happens to the COP outcome in an SSPwCT environment when an original slot added to branch b. Fix choice rules of all branches other than b. Abusing notation, we extend the set of original slots in branch b to $\widetilde{O}_b = O_b \cup \{\widetilde{o}\}$, where $\widetilde{o}$ is the newly added original slot. We require the priorities of slots in O_b to be unchanged. We write $\widetilde{\Pi}_b = (\Pi_b, \pi^{\widetilde{o}})$ where $\Pi_b = (\Pi^o_b, \Pi^e_b)$ is the priority profile of slots in $O_b \cup E_b$ and $\pi^{\widetilde{o}}$ is the priority ordering of new original slot $\widetilde{o}$.

As pointed out by *Kominers* (2019), adding a new original slot $\widetilde{o}$ is similar to raising the positions of all agents' contracts that were previously unacceptable in $\widetilde{o}'s$ priority, holding all other slots' priorities fixed. Hence, by our Theorem 2, adding $\widetilde{o}$ leads to an improvement for all agents. We state this result formally as follows:

Theorem 4. *Suppose that Z is the outcome of the COP in the market with the set of slots $\{O_b \cup E_b\}_{b\in\mathscr{B}}$ and $\widetilde{Z}$ is the outcome of the COP in the market with the set of slots $\{O_b \cup E_b\}_{b\in\mathscr{B}} \cup \{\widetilde{o}_b\}$ where $\widetilde{o}_b$ is an original slot added to branch b. Then, each agent $i \in \mathscr{I}$ (weakly) prefers her assignment under $\widetilde{Z}$ to her assignment under Z. That is,*

$$\widetilde{Z}_i R_i Z_i.$$

Our Theorem 4 expands Theorem 2 of *Kominers* (2019) to the SSPwCT family.

4.3 Adding Contracts

Kominers (2019) shows that adding new contracts at the bottom of some slots' priority orders (that is, right before the null contract $\emptyset$) improves outcomes for all agents. We follow his terminology and extend his result to the SSPwCT family. Suppose that a new set of contracts $\widetilde{X}$ is introduced so that the contract set expands to $X \cup \widetilde{X}$. Let $\widetilde{P} = (\widetilde{P}_i)_{i\in\mathscr{I}}$ and $\widetilde{\Pi} = (\widetilde{\Pi}_b)_{b\in\mathscr{B}}$ denote the vector of agent preferences and slot priorities over $X \cup \widetilde{X}$, respectively. Contracts in $\widetilde{X}$ do not affect agents' preferences and slots' priorities over X. That is,

1. $x\widetilde{P}_i x'$ if and only if $xP_i x'$ for all $i \in \mathscr{I}$ and $x, x' \in X$, and

2. $x\widetilde{\Pi}^s_b x'$ if and only if $x\Pi^s_b x'$ for all slots $s \in \{O_b \cup E_b\}_{b\in\mathscr{B}}$ and $x, x' \in X$.

If $x\widetilde{\Pi}^s_b \widetilde{x}$ for all slots $s \in \{O_b \cup E_b\}_{b\in\mathscr{B}}$, $x \in X$, and $\widetilde{x} \in \widetilde{X}$, then we can say $\widetilde{\Pi}$ as an improvement over Π under the contract set $X \cup \widetilde{X}$: Each slot $s \in \{O_b \cup E_b\}_{b\in\mathscr{B}}$ ranks all the contracts in $\widetilde{X}$ as unacceptable in Π^s_b. Hence, $\widetilde{\Pi}$ can be obtained from Π by improving the ranking of contracts in $\widetilde{X}$ above the outside option. Then, our Theorem 2 implies the following result.

Theorem 5. *Let Z be the outcome of the COP in the market with the set of contracts X. Let $\widetilde{Z}$ be the outcome of the COP in the market with the set of contracts $X \cup \widetilde{X}$. Then, each agent $i \in \mathscr{I}$ (weakly) prefers her assignment under $\widetilde{Z}$ to her assignment under Z. That is,*

$$\widetilde{Z}_i R_i Z_i.$$

Our Theorem 5 expands Theorem 3 of *Kominers* (2019) to the SSPwCT family.

Our last comparative static result involves adding new contracts for a single agent $i \in \mathscr{I}$ anywhere in the slots' priorities. We show that agent i is better off under the COP. Suppose that a new contract $\widetilde{x}$ is introduced so that the contract set expands to $X \cup \{\widetilde{x}\}$. Let $\widetilde{P} = (\widetilde{P}_i)_{i \in \mathscr{I}}$ and $\widetilde{\Pi} = (\widetilde{\Pi}_b)_{b \in \mathscr{B}}$ denote the vector of agent preferences and slot priorities over $X \cup \{\widetilde{x}\}$, respectively. The newly added contract $\widetilde{x}$ does not affect the agents' preferences or slots' priorities over X. That is, (1) $x\widetilde{P}_i x'$ if and only if $xP_i x'$ for all $i \in \mathscr{I}$ and $x, x' \in X$, and (2) $x\widetilde{\Pi}^s_b x'$ if and only if $x\Pi^s_b x'$ for all slots $s \in \{O_b \cup E_b\}_{b \in \mathscr{B}}$ and $x, x' \in X$. Adding the new contract $\widetilde{x}$ improves agent $\mathbf{i}(\widetilde{x})$ because agent $\mathbf{i}(\widetilde{x})$ has (weakly) higher ranked contracts at every slot, while the relative rankings of other contracts remain unchanged. Then, our Theorem 2 implies the following result.

Theorem 6. *Let Z be the outcome of the COP in the market with the set of contracts X. Let $\widetilde{Z}$ be the outcome of the COP in the market with the set of contracts $X \cup \{\widetilde{x}\}$. Then, each agent $\mathbf{i}(\widetilde{x})$ (weakly) prefers her assignment under $\widetilde{Z}$ to her assignment under Z. That is,*

$$\widetilde{Z}_{\mathbf{i}(\widetilde{x})} R_{\mathbf{i}(x)} Z_{\mathbf{i}(x)}.$$

Our Theorem 6 expands Theorem 4 of *Kominers* (2019) to the SSPwCT family.

5 Conclusions

In this paper, we introduce a practical family of SSPwCT choice rules and show how markets with these choice rules can be cleared by the COM. We show that the COM is stable, strategy-proof, and respects improvements with regards to SSPwCT choice rules. Moreover, we show that transferring the capacity of one more unfilled slot, if all else remains constant, leads to strategy-proof Pareto improvement of the COM. We provide two additional comparative static results. We show that both expansion of branch capacities and adding new contracts (weakly) increase agents' welfare under the COP.

The SSPwCT choice rules expands the toolkit available to market designers and may be used in real-world matching markets to accommodate diversity concerns. We believe SSPwCT choice rules may be used in cadet-branch matching in USMA and ROTC, resource allocation problems in India with comprehensive affirmative action, Chilean school choice with affirmative action constraints, and Brazilian college admissions with multidimensional reserves.

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Appendices

A. Formal Description of the Cumulative Offer Process

The COP **associated with proposal order** Γ is described by the following algorithm

1. Let $l = 0$. For each $s \in S$, let $D_s^0 \equiv \emptyset$, and $A_s^1 \equiv \emptyset$.

2. For each $l = 1, 2, ...$
   Let i be the $\Gamma_l - maximal$ agent $i \in I$, such that $i \notin i(\underset{s\in S}{\cup} D_s^{l-1})$ and $\underset{P^i}{max}(X \setminus (\underset{s\in S}{\cup} A_s^l))_i \neq \emptyset_i$—that is, the first agent in the proposal order who wants to propose a new contract, if such an agent exists. (If no such agent exists, then proceed to Step 3 below.)
   (**a**) Let $x = \underset{P^i}{max}(X \setminus (\underset{s\in S}{\cup} A_s^l))_i$ be i's most preferred contract that has not been proposed.
   (**b**) Let $s = s(x)$. Set $D_s^l = C^s(A_s^l \cup \{x\})$ and set $A_s^{l+1} = A_s^l \cup \{x\}$. For each $s' \neq s$, set $D_{s'}^l = D_{s'}^{l-1}$ and $A_{s'}^{l+1} = A_{s'}^l$.

3. Return the outcome

$$Y \equiv (\underset{s\in S}{\cup} D_s^{l-1}) = (\underset{s\in S}{\cup} C^s(A_s^l)),$$

   which consists of contracts held by institutions at the point when no agents want to propose an additional contract.

Here, the sets D_s^{l-1} and A_s^l denote the set of contracts **held by** and **available to** institution s at the beginning of the COP step l. We say that a contract z is **rejected** during the COP if $z \in A_{s(z)}^l$ but $z \notin D_{s(z)}^{l-1}$ for some l.

B. Properties of SSPwCT Choice Rules

Definition. Given a set of contracts X, a choice rule C^b satisfies the IRC if $x \notin C^b(Y \cup x)$ implies , for all $Y \subseteq X$ and $x \in X \setminus Y$.

Proposition 1. *Every choice rule in the SSPwCT family satisfies the IRC.*

Following the terminology in *Hatfield et al.* (2019), we give definitions of three properties that, if satisfied, guarantee the existence of a stable and strategy-proof mechanism. We first introduce the necessary concepts. An **offer process** for a given branch $b \in B$, with choice rule C^b, is a finite sequence of distinct contracts $(x^1, x^2, ..., x^M)$, such that for all $m = 1, ..., M$, $x^m \in X_b$. We say that an offer process $(x^1, x^2, ..., x^M)$ for b is **observable** if for all $m = 1, ..., M$, $i(x^m) \notin i(C^b(\{x^1, ..., x^{m-1}\}))$. In the case of the COP, if T denotes the last step of the COP with respect to $\vdash$ and P, A^T is the set of observable contracts.

Definition. (*Hatfield et al.,* 2019) A choice rule C^b is *observable substitutable* if there is no observable offer process $(x^1, \ldots x^M)$ for b such that $x^t \notin C^b(\{x^1, \ldots, x^t, \ldots x^{M-1}\}$ but $x^t \in C^b(\{x^1, \ldots, x^t, \ldots, x^M\})$.

Our next result shows that every SSPwCT choice rule satisfies observable substitutability.

Proposition 2. *Every choice rule in the SSPwCT family is observably substitutable.*

Two other properties are required to guarantee strategy-proofness. We define these below.

Definition. (*Hatfield et al.,* 2019) A choice rule C^b satisfies *observable size monotonicity* if there is no observable offer process $(x^1, \ldots, x^M)$ for b such that $\mid C^b(\{x^1, \ldots, x^M\}) \mid < \mid C^b(\{x^1, \ldots, x^{M-1}\}) \mid$.

This condition weakens the size monotonicity condition in that it only needs to be satisfied for observable offer processes.

Proposition 3. *Every choice rule in the SSPwCT family is observably size monotonic.*

Definition. (*Hatfield et al.,* 2019) A choice rule C^b of a branch $b \in B$ is *non-manipulable via contractual terms* if there is no ordering $\vdash$ and preference profile P for agents (under which only contracts with b are acceptable) with some individual $i \in \mathscr{I}$, with a preference relation $\widetilde{P}^i$, under which only contracts with b are acceptable, such that $\mathscr{C}^{\vdash}(\widetilde{P}^i, P_{-i}) P_i \mathscr{C}^{\vdash}(P^i, P_{-i})$.

Our next result shows that every SSPwCT choice rule satisfies the non-manipulability via contractual terms condition.

Proposition 4. *Every choice rule in the SSPwCT family is non-manipulable via contractual terms.*

Proofs of Propositions 1-4

Before we start proving the results, we will introduce some notation:

- If $X = \{x^1, \ldots, x^M\}$ is an observable offer process, we say $X^m = \{x^1, .., x^m\}$.
- $H_{s_b^t}(X^m)$ denotes the set of contracts available to seat s_b^t in the computation of $C^b(X^m)$, i.e. $H_{s_b^t}(X^m) = X^m \setminus X_{i((\max_{\Pi^{s_b^{t-1}}}(H_{s_b^{t-1}}(X^m))) \cup \ldots \cup (\max_{\Pi^{s_b^1}}(H_{s_b^1}(X^m)))))}$.
- $F_{s_b^t}(X^m) = \underset{n \leq m}{\cup} H_{s_b^t}(X^n)$, i.e., $F_{s_b^t}(X^m)$ is the set of all contracts that were available to s_b^t at some point of offer process $X^m = \{x^1, \ldots, x^m\}$.
- $R_{s_b^t}(X^m)$ is the set of contracts rejected by seat s_b^t at step m of the observable offer process X^m.

Note that $X^M = X$.

Lemma 1. *For all $s_b^t \in S_b$, for all $m \in \{1,...,M\}$ where M is the last step of observable offer process $X = \{x^1,...,x^M\}$, and for all observable offer processes Y, such that $X \subseteq Y$:*

1. $\max_{\Pi^{s_b^t}}(H_{s_b^t}(X^m)) = \max_{\Pi^{s_b^t}}(F_{s_b^t}(X^m))$.
2. $\max_{\Pi^{s_b^t}}(F_{s_b^t}(X^m)) \subseteq \max_{\Pi^{s_b^t}}(H_{s_b^t}(X^{m-1})) \cup (H_{s_b^t}(X^m) \setminus H_{s_b^t}(X^{m-1}))$.
3. $\max_{\Pi^{s_b^t}}(H_{s_b^t}(X^{m-1})) \subseteq H_{s_b^t}(X^m)$.
4. $R_{s_b^t}(F_{s_b^t}(X^m)) \subseteq R_{s_b^t}(F_{s_b^t}(Y))$ *if* $F_{s_b^t}(X^m) \subseteq F_{s_b^t}(Y)$.

Proof of Lemma 1 We follow *Hatfield et al.* (2019) at every step of the proof except step 4. We use mathematical induction on pairs (m, s_b), ordered in the following way:

$$(1,s_b^1),(1,s_b^2),...,(1,s_b^{|n_b|}),(2,s_b^1),(2,s_b^2),...,(2,s_b^{|n_b|}),...,(M,s_b^1),(M,s_b^2),...,(M,s_b^{|n_b|})$$

Step 1: for $m = 1$ and any $s_b^t \in S_b$ we have:

1. Since it is the first step, $F_{s_b^t}(X^m) = H_{s_b^t}(X^m)$. Hence, it follows directly that $\max_{\Pi^{s_b^t}}(H_{s_b^t}(X^1)) = \max_{\Pi^{s_b^t}}(F_{s_b^t}(X^1))$.

2. Since $m-1 = 0$ but the offer process starts at $m = 1$, $X^0 = \emptyset$ and $H_{s_b^t}(X^0) = \emptyset$. Hence,

$$\max_{\Pi^{s_b^t}}(F_{s_b^t}(X^1)) \subseteq \max_{\Pi^{s_b^t}}(H_{s_b^t}(X^0)) \cup (H_{s_b^t}(X^1) \setminus H_{s_b^t}(X^0))$$

holds because $\max_{\Pi^{s_b^t}}(F_{s_b^t}(X^1)) \subseteq H_{s_b^t}(X^1)$ holds. Note that either $\max_{\Pi^{s_b^t}}(F_{s_b^t}(X^1)) = \emptyset$ or $\max_{\Pi^{s_b^t}}(F_{s_b^t}(X^1)) = H_{s_b^t}(X^1)$.

3. Since $\max_{\Pi^{s_b^t}}(H_{s_b^t}(X^0)) = \emptyset$, we have $\max_{\Pi^{s_b^t}}(H_{s_b^t}(X^0)) \subseteq H_{s_b^t}(X^1)$.

4. It is the same proof for the general case (m, s_b). Please see the last paragraph of this lemma.

Inductive assumption: Assume that 1-4 hold for:

- Every (m', s_b^t) with $m' < m$ and $s_b^t \in S_b$.
- Every $(m, s_b^{t'})$ with $t' < t$.

Induction:

3. Take some contract z chosen by some seat s_b^t at step $m-1$, i.e. $z = \underset{\Pi^{s_b^t}}{max}(H_{s_b^t}(X^{m-1}))$. It must be the case that no contract of agent $i(z)$ is chosen by a seat that precedes s_b^t at step $m-1$ of the observable offer process, i.e.,

$$(\{x^1,...,x^{m-1}\})_{i(z)} \subseteq \underset{t'<t}{\cap} R_{s_b^{t'}}(H_{s_b^{t'}}(X^{m-1})).$$

However, by inductive assumptions 1 and 2, we know that for all $t' < t$,

$$\underset{\Pi^{s_b^{t'}}}{max}(H_{s_b^{t'}}(X^m)) \subseteq \underset{\Pi^{s_b^{t'}}}{max}(H_{s_b^{t'}}(X^{m-1})) \cup (H_{s_b^{t'}}(X^m) \setminus H_{s_b^{t'}}(X^{m-1})).$$

Hence, since $(\{x^1,...,x^{m-1}\})_{i(z)} \subseteq H_{s_b^{t'}}(X^{m-1})$, it must be that

$$(\{x^1,...,x^{m-1}\})_{i(z)} \subseteq \underset{t'<t}{\cap} R_{s_b^{t'}}(H_{s_b^{t'}}(X^m)).$$

This implies that no contract that belongs to $i(z)$, except possibly x^m, is chosen by a seat that precedes s_b^t at step m of the offer process. However, X is an observable offer process, so if $z = \underset{\Pi^{s_b^{t'}}}{max}(H_{s_b^{t'}}(X^{m-1}))$ for some $s_b^{t'}$, then $i(z) \neq i(x^m)$. Thus,

$$(\{x^1,...,x^m\})_{i(z)} \subseteq \underset{t'<t}{\cap} R_{s_b^{t'}}(H_{s_b^{t'}}(X^m)).$$

This implies $z = \underset{\Pi^{s_b^t}}{max}(H_{s_b^t}(X^{m-1})) \subseteq H_{s_b^t}(X^m)$.

2. By inductive assumption 4, if we take $Y = X^m$, since $X^{m-1} \subseteq X^m$,

$$R_{s_b^t}(F_{s_b^t}(X^{m-1})) \subseteq R_{s_b^t}(F_{s_b^t}(X^m)).$$

This implies that

$$\underset{\Pi^{s_b^t}}{max}(F_{s_b^t}(X^m)) \subseteq \underset{\Pi^{s_b^t}}{max}(F_{s_b^t}(X^{m-1})) \cup (F_{s_b^t}(X^m) \setminus F_{s_b^t}(X^{m-1})).$$

However, by definition, $F_{s_b^t}(X^m) = \underset{n\leq m}{\cup} H_{s_b^t}(X^n)$ and $F_{s_b^t}(X^{m-1}) = \underset{n\leq m-1}{\cup} H_{s_b^t}(X^n)$. Therefore,

$$(F_{s_b^t}(X^m) \setminus F_{s_b^t}(X^{m-1}) = (\underset{n\leq m}{\cup} H_{s_b^t}(X^n)) \setminus (\underset{n\leq m-1}{\cup} H_{s_b^t}(X^n))$$

$$= ((H_{s^t_b}(X^m) \setminus (\bigcup_{n \leq m-1} H_{s^t_b}(X^n))) \subseteq H_{s^t_b}(X^m) \setminus H_{s^t_b}(X^{m-1}).$$

Also, by inductive assumption 1, we know $\underset{\Pi^{s^t_b}}{max}(H_{s^t_b}(X^{m-1})) = \underset{\Pi^{s^t_b}}{max}(F_{s^t_b}(X^{m-1}))$. Hence,

$$\underset{\Pi^{s^t_b}}{max}(F_{s^t_b}(X^m)) \subseteq \underset{\Pi^{s^t_b}}{max}(F_{s^t_b}(X^{m-1})) \cup (F_{s^t_b}(X^m) \setminus F_{s^t_b}(X^{m-1}))$$

in conjunction with $F_{s^t_b}(X^m) \setminus F_{s^t_b}(X^{m-1}) \subseteq H_{s^t_b}(X^m) \setminus H_{s^t_b}(X^{m-1})$ imply

$$\underset{\Pi^{s^t_b}}{max}(F_{s^t_b}(X^m)) \subseteq \underset{\Pi^{s^t_b}}{max}(H_{s^t_b}(X^{m-1})) \cup (H_{s^t_b}(X^m) \setminus H_{s^t_b}(X^{m-1})).$$

1. As shown above, we know

$$\underset{\Pi^{s^t_b}}{max}(F_{s^t_b}(X^m)) \subseteq \underset{\Pi^{s^t_b}}{max}(H_{s^t_b}(X^{m-1})) \cup (H_{s^t_b}(X^m) \setminus H_{s^t_b}(X^{m-1}))$$

and by the proof of (3) in the induction step we have $\underset{\Pi^{s^t_b}}{max}(H_{s^t_b}(X^{m-1})) \subseteq H_{s^t_b}(X^m)$. Combining these two relations, we obtain $\underset{\Pi^{s^t_b}}{max}(F_{s^t_b}(X^m)) \subseteq H_{s^t_b}(X^m)$. However, by definition, $H_{s^t_b}(X^m) \subseteq F_{s^t_b}(X^m)$. Therefore, we have $\underset{\Pi^{s^t_b}}{max}(F_{s^t_b}(X^m)) = \underset{\Pi^{s^t_b}}{max}(H_{s^t_b}(X^m))$.

4. We need to prove that $R_{s^t_b}(F_{s^t_b}(X^m)) \subseteq R_{s^t_b}(F_{s^t_b}(Y))$ if $F_{s^t_b}(X^m) \subseteq F_{s^t_b}(Y)$. There are four possible cases.

Case 1: s^t_b is an original seat. Since $F_{s^t_b}(X^m) \subseteq F_{s^t_b}(Y)$ and s^t_b can hold at most one contract, it follows directly that $R_{s^t_b}(F_{s^t_b}(X^m)) \subseteq R_{s^t_b}(F_{s^t_b}(Y))$.

Case 2: s^t_b is a shadow seat and is empty in both computations. $R_{s^t_b}(F_{s^t_b}(X^m)) = F_{s^t_b}(X^m)$, $R_{s^t_b}(F_{s^t_b}(Y)) = F_{s^t_b}(Y)$, and $F_{s^t_b}(X^m) \subseteq F_{s^t_b}(Y)$. Hence, $R_{s^t_b}(F_{s^t_b}(X^m)) \subseteq R_{s^t_b}(F_{s^t_b}(Y))$.

Case 3: s^t_b is a shadow seat and it holds a contract in both computations. Since $F_{s^t_b}(X^m) \subseteq F_{s^t_b}(Y)$, it follows directly that $R_{s^t_b}(F_{s^t_b}(X^m)) \subseteq R_{s^t_b}(F_{s^t_b}(Y))$.

Case 4: s^t_b is a shadow seat and it holds a contract from X^m but not from Y. $R_{s^t_b}(F_{s^t_b}(Y)) = F_{s^t_b}(Y)$ and $R_{s^t_b}(F_{s^t_b}(X^m) \subseteq F_{s^t_b}(X^m)$. Hence, $R_{s^t_b}(F_{s^t_b}(X^m)) \subseteq R_{s^t_b}(F_{s^t_b}(Y))$.

Note that the case where s^t_b is a shadow seat that holds a contract in Y but not in X^m is not possible. By the first inductive assumption, we know

$$\max_{\Pi^{s_b^t}}(H_{s_b^t}(X^{m-1})) \subseteq H_{s_b^t}(X^m).$$

If s_b^t is not holding a contract in X^m, it is because its associated original seat, o_b^t, is holding a contract. Moreover, if o_b^t is holding a contract at the last step of $X^m \cap Y$, it will continue to hold it in the first step in $Y \setminus X^m$. Because the contract it held in the last step of $X^m \cap Y$ will be among those o_b^t can choose from. The contract o_b^t holds in the first step of $Y \setminus X^m$ will be among those o_b^t can choose from in the second step of $Y \setminus X^m$. This will continue until the last step of $Y \setminus X^m$. Since s_b^t's associated original seat, o_b^t, is holding a contract in Y, s_b^t must remain empty in Y. This ends our induction.

Proof of Proposition 1 Take any branch $b \in \mathscr{B}$ with a location vector L_b, capacity transfer scheme q_b, and precedence orders $\triangleright_O^b$ and $\triangleright_E^b$. We have to show that for every set of contracts $Y \subset \mathscr{X}$ and for every contract $z \in \mathscr{X} \setminus Y$, such that $z \notin C_b(Y \cup \{z\})$, $C_b(Y \cup \{z\}) = C_b(Y)$. We proceed by mathematical induction.

In the first step of the computation of $C_b(Y \cup \{z\})$, seat s_b^1 is assigned the contract x^1 that is $\Pi_b^{s_b^1} - maximal$ in $Y \cup \{z\}$. Similarly, in the first step of the computation of $C_b(Y)$, seat s_b^1 is assigned the contract w^1 that is $\Pi_b^{s_b^1} - maximal$ in Y. However, since $z \notin C_b(Y \cup \{z\})$ we know that

$$x^1 \equiv \max_{\Pi^{s_b^1}}(Y \cup \{z\}) = \max_{\Pi^{s_b^1}}(Y) \equiv w^1$$

For steps $2, ..., k$ we assume that the same contracts are assigned in the computation of $C_b(Y)$ and $C_b(Y \cup \{z\})$, i.e., $x^2 = w^2$,..., $x^k = w^k$.

In step $k+1$ of the computation of $C_b(Y \cup \{z\})$, if seat s_b^{k+1} is either an original seat or a shadow seat such that $\mathbf{1}_{o_b^r} = 0$, where o_b^r is the original seat that is associated with the shadow seat s_b^{k+1}, and $q_b^r = 1$, seat s^{k+1} is assigned the contract x^{k+1} that is $\Pi_b^{s_b^{k+1}} - maximal$ in $Y \cup \{z\} \setminus Y_{i(\{x^1,...,x^k\})}$. Similarly, in step $k+1$ of the computation of $C_b(Y)$, if seat s_b^{k+1} is either an original seat or a shadow seat such that $\mathbf{1}_{o_b^r} = 0$, where o_b^r is the original seat that is associated with the shadow seat s_b^{k+1}, and $q_b^r = 1$, seat s^{k+1} is assigned the contract w^{k+1} that is $\Pi_b^{s_b^{k+1}} - maximal$ in $Y \setminus Y_{i(\{w^1,...,w^k\})}$. However, since $z \notin C_b(Y \cup \{z\})$ and $x^2 = w^2$,..., $x^k = w^k$, we know that

$$x^{k+1} \equiv \max_{\Pi^{s_b^{k+1}}}(Y \cup \{z\} \setminus Y_{i(\{x^1,...,x^k\})}) = \max_{\Pi^{s_b^{k+1}}}(Y \setminus Y_{i(\{x^1,...,x^k\})}) = \max_{\Pi^{s_b^{+1}}}(Y \setminus Y_{i(\{w^1,...,w^k\})}) \equiv w^{k+1}.$$

In any other case, in step $k+1$ of the computations of $C_b(Y \cup \{z\})$ and $C_b(Y)$, seat s^{k+1} is assigned $\emptyset$. This is true because in both computations the location vector L_b, capacity transfer scheme q_b, and precedence orders $\rhd_O^b$ and $\rhd_E^b$ are the same. Since the same contracts are assigned in every step of $C_b(Y \cup \{z\})$ and $C_b(Y)$, it must be that $C_b(Y \cup \{z\}) = C_b(Y)$.

Proof of Proposition 2 We have to prove that if $x^t \notin C^b(\{x^1, ..., x^{m-1}\})$, then $x^t \notin C^b(\{x^1, ..., x^m\})$ where $\{x^1, ..., x^m\}$ is an observable offer process. First, note that by Lemma 1

$$\max_{\Pi^{s_b}}(H_{s_b}(X^m)) \subseteq \max_{\Pi^{s_b}}(H_{s_b}(X^{m-1})) \cup (H_{s_b}(X^m) \setminus H_{s_b}(X^{m-1}))$$

for every $s_b \in \mathscr{S}_b$.

Since $x^t \notin C^b(\{x^1, ..., x^{m-1}\})$, it must be that $x^t \neq \max_{\Pi^{s_b}}(H_{s_b}(X^{m-1}))$ for every seat s_b that is holding a contract at step $m-1$ of the COP, and that $x^t \notin (H_{s_b}(X^m) \setminus H_{s_b}(X^{m-1}))$.

For all original or shadow seats s_b that hold a contract both in steps $m-1$ and m, if $x^t \neq \max_{\Pi^{s_b}}(H_{s_b}(X^{m-1}))$ and $x^t \notin (H_{s_b}(X^m) \setminus H_{s_b}(X^{m-1}))$, then by Lemma 1, $x^t \notin \max_{\Pi^{s_b}}(H_{s_b}(X^m))$.

For the original seats that are holding a contract in m but not in $m-1$, it must be that $\emptyset \equiv \max_{\Pi^{s_b}}(H_{s_b}(x^{m-1}))$. Thus, given $x^t \neq \max_{\Pi^{s_b}}(H_{s_b}(X^{m-1}))$ and $x^t \notin (H_{s_b}(X^m) \setminus H_{s_b}(X^{m-1}))$ by Lemma 1 $x^t \neq \max_{\Pi^{s_b}}(\{H_{s_b}(X^m))$.

By Lemma 1, we also know that a shadow seat that is not holding contracts $m-1$ cannot become capable of holding contracts at m, because its associated original seat can always hold the contract it was holding at step $m-1$. Since $x^t \notin \max_{\Pi^{s_b}}(H_{s_b}(X^m))$ for every seat s_b that is holding a contract at step m of the COP, it must be that $x^t \notin C^b(\{x^1, ..., x^{m-1}, x^m\})$.

Proof of Proposition 3 We prove that $| C^b(\{x^1, ..., x^{m-1}\}) | \leq | C^b(\{x^1, ..., x^m\}) |$, where $\{x^1, ..., x^m\}$ is an observable offer process. By Lemma 1, we know that:

1. An original seat that is holding a contract at step $m-1$ of the observable offer process will also hold a contract at step m, because it can always be able to hold the contract it held in the previous step.

2. The only reason a shadow seat that is holding a contract at step $m-1$ would stop holding a contract in m is its associated original seat starts holding a contract.

This implies that at every step of the observable offer process there are at least the same number of seats holding contracts as in the previous step, and each contract that is held is an element of C^b. Hence, it follow directly that $| C^b(\{x^1, ..., x^{m-1}\}) | \leq | C^b(\{x^1, ..., x^m\}) |$.

Proof of Proposition 4 Our proof is built on Proposition 2 of *Hatfield et al.* (2019). Suppose that C^b satisfies observable substitutability. To show non-manipulability via contractual terms, it suffices to prove that there is no preference profile P, and preference for some individual $i \in I$, $\widetilde{P}_i$, under which only contracts with b are acceptable, of the form:

$$P_i : z^1 P_i z^2 P_i ... P_i z^N$$

$$\widetilde{P}_i : z^0 \widetilde{P}_i z^1 \widetilde{P}_i z^2 \widetilde{P}_i ... \widetilde{P}_i z^N$$

such that either:

1. $\mathscr{C}_i^{\vdash}(P) = \emptyset_i$ while $\mathscr{C}_i^{\vdash}(\widetilde{P}_i, P_{-i}) P_i \emptyset_i$, or,
2. $\mathscr{C}_i^{\vdash}(\widetilde{P}_i, P_{-i}) = \emptyset_i$ while $\mathscr{C}_i^{\vdash}(P) \widetilde{P}_i \emptyset_i$.

We first prove a claim that will support.

Claim 1. Let $P_i : z^1 P_i z^2 P_i \succ ... P_i z^N$ and $\widetilde{P}_i : z^0 \widetilde{P}_i z^1 \widetilde{P}_i z^2 \widetilde{P}_i ... \widetilde{P}_i z^N$. Let $X = \{x^1, ..., x^M\}$ and $\widetilde{X} = \{\widetilde{x}^1, ..., \widetilde{x}^{\widetilde{M}}\}$ denote the observable offer processes induced by $P = (P_i, P_{-i})$ and $\widetilde{P} = (\widetilde{P}_i, P_{-i})$, respectively, with ordering $\vdash$. We claim that if $z^0 \notin C^b(\widetilde{X})$ then, $R_{s_b^k}(X) \subseteq R_{s_b^k}(\widetilde{X})$ and $F_{s_b^k}(X) \subseteq F_{s_b^k}(\widetilde{X})$ for all $k = 1, ..., 2n_b$.

Proof of Claim 1: We will first show that $R_{s_b^k}(F_{s_b^k}(X^m) \subseteq R_{s_b^k}(F_{s_b^k}(\widetilde{X}))$ and $F_{s_b^k}(X^m) \subseteq F_{s_b^k}(\widetilde{X})$ for all $k = 1, ..., 2n_b$ and all $m \in \{1, ..., M\}$. We proceed by mathematical induction on pairs (m, s_b^t) in the following order:

$$(1, s_b^1), (1, s_b^2), ..., (1, s_b^{2n_b}), (2, s_b^1), (2, s_b^2), ..., (2, s_b^{2n_b}), ..., (M, s_b^1), (M, s_b^2), ..., (M, s_b^{2n_b}).$$

Step 1: We prove the claim for $(1, s_b^1)$. Under offer process X, x^1 is the highest ranked contract by some individual who is not i or $x^1 = z^1$. Under offer process $\widetilde{X}$, if $i(x^1) \neq i$, x^1 must be offered at some step of $\widetilde{X}$, because it is the highest ranked contract by some individual other than i. Since z^0 is rejected by our assumption in the claim, if x^1 names individual i, then i must offer his second-favorite contract with respect to $\widetilde{P}_i$, i.e., he must offer $z^1 = x^1$ at some point of offer process $\widetilde{X}$. Hence, $x^1 \in \widetilde{X}$. However, since $F_{s_b^1}(\widetilde{X}) = \widetilde{X}$, $F_{s_b^1}(X^1) \subseteq F_{s_b^1}(\widetilde{X})$ where $X^1 = \{x^1\}$. Hence, by Lemma 1, $R_{s_b^1}(F_{s_b^1}(X^1)) \subseteq R_{s_b^1}(F_{s_b^1}(\widetilde{X}))$.

Inductive assumption: Assume $R_{s_b^t}(F_{s_b^t}(X^m) \subseteq R_{s_b^t}(F_{s_b^t}(\widetilde{X}))$ and $F_{s_b^t}(X^m) \subseteq F_{s_b^t}(\widetilde{X})$ for:

- Every pair (m', s_b^t) with $m' < m$ and $t = 1, ..., 2n_b$.

- Every pair (m, s_b^t) with $t < k$.

Induction: By the inductive assumption, we know $R_{s_b^t}(F_{s_b^t}(X^{m-1})) \subseteq R_{s_b^t}(F_{s_b^t}(\widetilde{X}))$ for all $t \in \{1,...,2n_b\}$. Additionally, since X is an observable offer process it must be that

$$(\{x^1,...,x^{m-1}\})_{\mathbf{i}(x^m)} \subseteq R_{s_b^t}(H_{s_b^t}(X^{m-1}))$$

for all $t \in \{1,...,2n_b\}$. Moreover, since $H_{s_b^t}(X^{m-1}) \subseteq F_{s_b^t}(X^{m-1})$ by definition, it directly follows that $R_{s_b^t}(H_{s_b^t}(X^{m-1})) \subseteq R_{s_b^t}(F_{s_b^t}(X^{m-1}))$. By combining these inclusion relations, we have

$$(\{x^1,...,x^{m-1}\})_{\mathbf{i}(x^m)} \subseteq R_{s_b^t}(H_{s_b^t}(X^{m-1})) \subseteq R_{s_b^t}(F_{s_b^t}(X^{m-1})) \subseteq R_{s_b^t}(F_{s_b^t}(\widetilde{X})),$$

where the last inclusion is by the inductive assumption. Thus,

$$(\{x^1,...,x^{m-1}\})_{\mathbf{i}(x^m)} \subseteq R_{s_b^t}(F_{s_b^t}(\widetilde{X}))$$

for all $t = 1,...,2n_b$.

By Lemma 1, we know that $\underset{\Pi^{s_b^t}}{max}(H_{s_b^t}(X^m)) = \underset{\Pi^{s_b^t}}{max}(F_{s_b^t}(X^m))$ for all $m = 1,...,M$ and $t = 1,...,2n_b$. Therefore, there must exist some $\overline{m} \leq \widetilde{M}$ such that

$$(\{x^1,...,x^{m-1}\})_{\mathbf{i}(x^m)} \subseteq R_{s_b^t}(H_{s_b^t}(\widetilde{X}^{\overline{m}}))$$

for all $t = 1,...,2n_b$. However, since $\widetilde{X}$ is an observable offer process and represents all offers made under $\widetilde{P}_i$, and since all contracts from $\mathbf{i}(x^m)$ are rejected, there must exist some step $\widetilde{m}$ at which x^m is proposed in $\widetilde{X}$. Hence, since $F_{s_b^1}(\widetilde{X}) = \widetilde{X}$, we have $x^m \in F_{s_b^1}(\widetilde{X})$.

By the inductive assumption, we also know that for $(m-1, s_b^1)$, $X^{m-1} = F_{s_b^1}(X^{m-1}) \subseteq F_{s_b^1}(\widetilde{X})$. Moreover, since $F_{s_b^1}(X^m) = X^m$ and $X^m = X^{m-1} \cup \{x^m\}$, $F_{s_b^1}(X^m) = X^{m-1} \cup \{x^m\}$. Thus, we know that $F_{s_b^1}(X^m) \subseteq F_{s_b^1}(\widetilde{X})$. By Lemma 1, this implies $R_{s_b^1}(F_{s_b^1}(X^m)) \subseteq R_{s_b^1}(F_{s_b^1}(\widetilde{X}))$. This ends the proof for the case (m, s_b^1).

Next, we will show that the claim holds for (m, s_b) with $s_b^1 \triangleright_b s_b$. By the inductive assumption, we know $F_{s_b}(X^{m-1}) \subseteq F_{s_b}(\widetilde{X})$. But, since $F_{s_b}(X^m) = F_{s_b}(X^{m-1}) \cup H_{s_b}(X^m)$,

$$F_{s_b}(X^m) \subseteq F_{s_b}(\widetilde{X}) \cup H_{s_b}(X^m).$$

Hence, to prove that the claim holds, it is sufficient to show that $H_{s_b}(X^m) \subseteq F_{s_b}(\widetilde{X})$ for all $s_b \in S_b$.

Let $y \in H_{s_b}(X^m)$. This implies, by the structure of the $b's$ choice rule, that $y \in \cap_{t_b < s_b} R_{t_b}(H_{t_b}(X^m))$

and since $H_{t_b}(X^m) \subseteq F_{t_b}(X^m)$,

$$y \in \cap_{t_b < s_b} R_{t_b}(F_{t_b}(X^m)).$$

But, by the inductive assumption for pairs (m, t_b) with $t_b < s_b$, $R_{t_b}(F_{t_b}(X^m)) \subseteq R_{t_b}(F_{t_b}(\widetilde{X}))$. Hence, $y \in \cap_{t_b < s_b} R_{t_b}(F_{t_b}(\widetilde{X}))$.

By Lemma 1, we know that $\underset{\Pi^{s_b}}{max}(H_{s_b}(\widetilde{X})) = \underset{\Pi^{s_b}}{max}(F_{s_b}(\widetilde{X}))$ for all $s_b \in S_b$. Therefore, if there is a $t_b \leq s_b$ such that $y = \underset{\Pi^{t_b}}{max}(H_{t_b}(\widetilde{X}))$, then

$$y = \underset{\Pi^{t_b}}{max}(F_{t_b}(\widetilde{X})).$$

However, this contradicts $y \in \cap_{t_b < s_b} R_{t_b}(F_{t_b}(\widetilde{X}))$. Thus,

$$y \in \cap_{t_b < s_b} R_{t_b}(H_{t_b}(\widetilde{X})).$$

This means that there must exist some step $m^{'}$ of $\widetilde{X}$ such that $y \in F_{s_b}(\widetilde{X}^{m^{'}}) \subseteq F_{s_b}(\widetilde{X})$. This ends the proof of $H_{s_b}(X^m) \subseteq F_{s_b}(\widetilde{X})$. So, $F_{s_b}(X^m) \subseteq F_{s_b}(\widetilde{X})$. Moreover, by Lemma 1, since $F_{s_b}(X^m) \subseteq F_{s_b}(\widetilde{X})$, we have

$$R_{s_b}(F_{t_b}(X^m)) \subseteq R_{s_b}(F_{t_b}(\widetilde{X})).$$

This ends our induction.

We know $F_{s_b}(X^m) \subseteq F_{s_b}(\widetilde{X})$ and $R_{s_b}(F_{t_b}(X^m)) \subseteq R_{s_b}(F_{t_b}(\widetilde{X}))$ for all $s_b \in S_b$ and all $m \in \{1, \ldots, M\}$. We need to prove that $R_{s_b}(X) \subseteq R_{s_b}(\widetilde{X})$. We will proceed by contradiction. Suppose $R_{s_b}(X) \subseteq R_{s_b}(\widetilde{X})$ does not hold, i.e., $\exists y \in X$ such $y \in R_{s_b}(X) \setminus R_{s_b}(\widetilde{X})$. As $y \notin R_{s_b}(\widetilde{X})$, and by the induction we just proved for $y \in X = F_{s_b^1}(X) \subseteq F_{s_b^1}(\widetilde{X}) = \widetilde{X}$, it must be that $y = \underset{\Pi^{s_b}}{max}(H_{s_b}(\widetilde{X}))$ for some $s_b \in S_b$. But, by Lemma 1,

$$\underset{\Pi^{s_b}}{max}(H_{s_b}(\widetilde{X})) = \underset{\Pi^{s_b}}{max}(F_{s_b}(\widetilde{X})).$$

Thus, $y = \underset{\Pi^{s_b}}{max}(F_{s_b}(\widetilde{X}))$, so $y \notin R_{s_b}(F_{s_b}(\widetilde{X}))$. However, since $F_{s_b}(X) \subseteq F_{s_b}(\widetilde{X})$, by the induction proved above and Lemma 1,

$$R_{s_b}(F_{s_b}(X)) \subseteq R_{s_b}(F_{s_b}(\widetilde{X})).$$

This implies that $y \notin R_{s_b}(F_{s_b}(X))$. So, $y \notin R_{s_b}(H_{s_b}(X))$. However, this means that some s_b holds y under X, which contradicts $y \in R_{s_b}(X)$.

Now we proceed to prove Proposition 4.

From Proposition 2 of *Hatfield et al.* (2019), it is sufficient to prove that:

1. If $\mathscr{C}_i^{\vdash}(P) = \emptyset$, then either $\mathscr{C}_i^{\vdash}(\widetilde{P}_i, P_{-i}) = \emptyset$ or $\mathscr{C}_i^{\vdash}(\widetilde{P}^i, P_{-i}) = \{z^0\}$.

2. If $\mathscr{C}_i^{\vdash}(\widetilde{P}_i, P_{-i}) = \emptyset$ then $\mathscr{C}_i^{\vdash}(P) = \emptyset$.

Proof We will first show (1). If $\mathscr{C}_i^{\vdash}(\widetilde{P}^i, P_{-i}) = \{z^0\}$, then we are done. Suppose that $\{z^0\} \neq \mathscr{C}_i^{\vdash}(\widetilde{P}_i, P_{-i})$. Then, by Claim 1, we have

$$R_{s_b^k}(X) \subseteq R_{s_b^k}(\widetilde{X}).$$

Moreover, since $\mathscr{C}_i^{\vdash}(P) = \emptyset$, $\{z^1, ..., z^N\} \subseteq R_{s_b^k}(X)$ for all $s_b^k \in S_b$. Combining the two inclusions, we get that $\{z^1, ..., z^N\} \subseteq R_{s_b^k}(\widetilde{X})$ for all $k = 1, ..., 2n_b$. Hence, we conclude that $\mathscr{C}_i^{\vdash}(\widetilde{P}_i, P_{-i}) = \emptyset$.

We now proceed to show (2). By *Hatfield et al.* (2019), since the choice rule of branch b satisfies observable substitutability and observable size monotonicity, the outcome of the COP is order independent. Hence, the outcome of the COP is not changed if we suppose that the order $\vdash$ is such that all the other agents's contracts precede agent i's contracts. That is, $\vdash$ is such that if $x \in \mathscr{X}_i$ and $y \in \mathscr{X} \setminus \mathscr{X}_i$, then $y \vdash x$. If this is so, there must exist some m^* such that:

- $x^m = \widetilde{x}^m$ for all $m < m^*$.
- $x^{m^*} = z^1$.
- $\widetilde{x}^{m^*} = z^0$.

That is, m^* is the first step of the COP at which agent i proposes a contract. Note that at each step after m^*, exactly one contract is newly rejected and the offer process must end with z^N being rejected, because we assumed $\mathscr{C}_i^{\vdash}(\widetilde{P}_i, P_{-i}) = \emptyset$. Since the choice rule of branch b is observably substitutable and observably size monotonic, and since all contracts from i are rejected and z^N follows all contracts with agents other than i, the following must hold:

1. $\mid R_{s_b}(\widetilde{X}^{m'}) \setminus R_{s_b}(\widetilde{X}^{m'-1}) \mid = 1$ for all $m' \in \{m^*, m^*+1, ..., \widetilde{M}\}$.
2. $z^N \in (R_{s_b}(\widetilde{X}^{\widetilde{M}}) \setminus R_{s_b}(\widetilde{X}^{\widetilde{M}-1}))$.

For offer process X, we have that $\mid R_{s_b}(X^{m'}) \setminus R_{s_b}(X^{m'-1}) \mid = 1$, for all $m' \in \{m^*, m^*+1, ..., M-1\}$.

We know that $X^{m^*-1} = \widetilde{X}^{m^*-1}$. Therefore, $\mid C^b(X^{m^*-1}) \mid = \mid C^b(\widetilde{X}^{m^*-1}) \mid$.

However, since $\mid R_{s_b}(\widetilde{X}^{m'}) \setminus R_{s_b}(\widetilde{X}^{m'-1}) \mid = 1$ for all $m' \in \{m^*, m^*+1, ..., M'\}$ and exactly one contract is proposed at each step, we have

$$\mid C^b(\widetilde{X}^{m^*-1}) \mid = \mid C^b(\widetilde{X}^{M'}) \mid .$$

Note that $\widetilde{X}^{\widetilde{M}} = \widetilde{X}$. For offer process X, since $\mid R_{s_b}(X^{m'}) \setminus R_{s_b}(X^{m'-1}) \mid = 1$ for all $m' \in \{m^*, m^*+1, ..., M-1\}$,

$$\mid C^b(X^{m^*-1}) \mid = \mid C^b(X^{M-1}) \mid .$$

Hence, $\mid C^b(X^{M-1}) \mid = \mid C^b(\widetilde{X}^{\widetilde{M}}) \mid$. But, since $X^M \subseteq \widetilde{X}^{\widetilde{M}}$, by observable size monotonicity

$$\mid C^b(X^M) \mid \leq \mid C^b(\widetilde{X}^{\widetilde{M}}) \mid .$$

So, it follows that $\mid C^b(X^M) \mid \leq \mid C^b(X^{M-1}) \mid$. This implies that $R_{s_b}(X^M) \setminus R_{s_b}(X^{M-1}) \neq \emptyset$.

We have to prove that $\mathscr{C}_i^{\vdash}(P) = \emptyset$, i.e., that $z^N \in R_{s_b}(X^M) \setminus R_{s_b}(X^{M-1})$. Suppose, toward a contradiction, that there exists $y \in R_{s_b}(X^M) \setminus R_{s_b}(X^{M-1})$ such that $y \neq z^N$. This means that y is the least preferred contract from an individual other than i, i.e., $\mathbf{i}(y) \neq i$. However, by Claim 1, we know that

$$R_{s_b}(X) \subseteq R_{s_b}(\widetilde{X}).$$

Therefore, there must exist some step $m^{'} \geq m^*$ such that

$$y \in R_{s_b}(\widetilde{X}^{m^{'}}) \setminus R_{s_b}(\widetilde{X}^{m^{'}-1}).$$

However, $\mid R_{s_b}(\widetilde{X}^{m^{'}}) \setminus R_{s_b}(\widetilde{X}^{m^{'}-1}) \mid = 1$ and y is the least preferred acceptable contract of individual $\mathbf{i}(y)$. That means $\mathscr{C}_i^{\vdash}(\widetilde{P}_i, P_{-i})$ would end at step $m^{'}$ with the rejection of y. But this would contradict $\mathscr{C}_i^{\vdash}(\widetilde{P}_i, P_{-i}) = \emptyset$. Then, since $z^N \in R_{s_b}(X^M) \setminus R_{s_b}(X^{M-1})$, we have

$$\mathscr{C}_i^{\vdash}(P) = \emptyset.$$

C. Proofs of Theorems

Proof of Theorem 2 Assume, toward a contradiction, that the COM with regard to SSPwCT does not respect improvements. Then, there exists an agent $i \in I$, a preference profile of agents $P \in \times_{i \in I} \mathscr{P}^i$, and choice profiles $\overline{C}$ and C such that $\overline{C}$ is an improvement over C for agent i and

$$\mathscr{C}_i(P; C) P^i \mathscr{C}_i(P; \overline{C}).$$

Let $\mathscr{C}_i(P; C) = x$ and $\mathscr{C}_i(P; \overline{C}) = \bar{x}$. Consider a preference $\widetilde{P^i}$ of agent i according to which the only acceptable contract is x, i.e., $\widetilde{P^i} : x - \emptyset_i$. Let $\widetilde{P} = (\widetilde{P^i}, P_{-i})$. We will first prove the following claim:

Claim: $\mathscr{C}_i(\widetilde{P}; C) = x \Longrightarrow \mathscr{C}_i(\widetilde{P}; \overline{C}) = x$**.**

Proof of the Claim: Consider the outcome of the COM under choice profile C given the preference profile of agents $\widetilde{P}$. Proposition 3 of *Hatfield et al.* (2019) shows that if the choice rule of each branch satisfies observable substitutability, then the outcome of the COM is order independent. We have already established that each SSPwCT choice rule satisfies observable substitutability. Hence, we can utilize Proposition 3 of *Hatfield et al.* (2019). We can first completely ignore agent i and run the COP until it stops. Let Y be the outcome. At this point, agent i makes an offer for his only contract x. This might create a chain of rejections, but it does not reach agent i since we assumed $\mathscr{C}_i(\widetilde{P};C) = x$. Let the k^{th} slot with respect to precedence order $\vartriangleright_{\mathbf{b}(x)}$ be the slot that chose contract x.

Now consider the COP under choice profile $\overline{C}$. Again, we completely ignore agent i and run the COP until it stops. The same outcome Y is obtained, because the only difference between the two COPs is agent i's position in the priority rankings. At this point, agent i makes an offer for his only contract x. If x is chosen by the same slot, i.e., k^{th} slot with respect to $\vartriangleright_{\mathbf{b}(x)}$, then the same rejection chain (if there was one in the COP under the choice profile C) will occur and it does not reach agent i; otherwise, we would have a contradiction with the case under choice profile C. The only other possibility is the following: since agent i's ranking is now (weakly) better under $\overline{\Pi}_{\mathbf{b}(x)}$ compared to $\Pi_{\mathbf{b}(x)}$, his contract x might be chosen by slot l which precedes slot k with respect to $\vartriangleright_{\mathbf{b}(x)}$. Then, it must be the case that by selecting x slot l must reject some other contract it was holding. Let us call this contract y. If no contract of agent $\mathbf{i}(y) = j$ is chosen between slots l and k, then the slots between l and k choose the same contracts under both priority profiles. In this case, y is chosen by slot k. Thus, if a rejection chain starts, it will not reach agent i; otherwise, we could have a contradiction, due to the fact that x was chosen at the end of the COP under choice profile C. A different contract of agent j cannot be chosen between groups l and k; otherwise, the observable substitutability of branch $\mathbf{b}(x)$'s SSPwCT choice rule would be violated. Therefore, if any contract of agent j is chosen by slots between l and k, it must be y. If y is chosen by a slot that precedes k, then it must replace a contract—we call this contract z. By the same reasoning, no other contract of agent $\mathbf{i}(z)$ can be chosen before slot k; otherwise, we would violate the observable substitutability of branch $\mathbf{b}(x)$'s SSPwCT choice rule. Proceeding in this fashion causes the same contract in slot k to be rejected and initiates the same rejection chain that occurs under choice profile C. Since the same rejection chain does not reach agent i under choice profile C, it will not reach agent i under choice profile $\overline{C}$, which ends our proof for the claim.

Since $\mathscr{C}_i(P;C) = x$ and $\mathscr{C}_i(P;\overline{C}) = \overline{x}$ such that $xP^i\overline{x}$, if agent i misreports and submits $\widetilde{P^i}$ under choice profile $\overline{C}$, then she can successfully manipulate the COM. This is a contradiction as we have already established that the COM is strategy-proof.

Proof of Theorem 3 Suppose that Z is the outcome of the COM at (P,C), where $P = (P_{i_1},...,P_{i_n})$ is the profile of agent preferences and $C = (C^{b_1},...,C^{b_m})$ is the profile of branches' SSPwCT choice rules. Consider a branch $b \in \mathscr{B}$. Suppose that $\widetilde{C}^b$ and C^b take as an input capacity transfer schemes $\widetilde{q}_b$ and q_b, respectively, where $\widetilde{q}_b^k = q_b^k$ for all $k = 1,...,s-1,s+1,...,n_b$. For slot s, let $\widetilde{q}_b^s = 1$ and $q_b^s = 0$. That is, the capacity of the original seat s is transferred to the associated shadow seat under capacity function $\widetilde{q}_b$, but not under the capacity function q_b. We need to prove that the outcome of the COM at $(P,(\widetilde{C}^b,C_{-b}))$, $\widetilde{Z}$, Pareto dominates Z.

In the computation of COP, if the original slot s is filled, then we have $\widetilde{Z} = Z$ because, under both $\widetilde{q}_b$ and q_b the shadow slot associated with the original slot s will become inactive.

We now consider the case where the original slot s remains vacant in the computation of COP under (P,C). Then, under $(P,(\widetilde{C}^b,C_{-b}))$, the shadow slot associated with the original slot s – we call it $\widetilde{s}$ – will be active, i.e., it will have a capacity of 1. There are two cases to consider. If the shadow slot $\widetilde{s}$ remains vacant in the computation of COP under $(P,(\widetilde{C}^b,C_{-b}))$, then we again have $\widetilde{Z} = Z$, as the only difference between the two COPs, under (P,C) and $(P,(\widetilde{C}^b,C_{-b}))$, is the capacity of the shadow slot $\widetilde{s}$.

The non-trivial case is the one where the shadow slot $\widetilde{s}$ is assigned a contract in COP under $(P,(\widetilde{C}^b,C_{-b}))$. We now define an *improvements chain* algorithm that starts with the outcome Z.

Step 1. Let x_1 be the contract that is assigned to slot $\widetilde{s}$ in the SSPwCT choice procedure of branch $\mathbf{b}(x_1)$. If agent $\mathbf{i}(x_1)$ is assigned $\emptyset$ under Z, then the improvement process ends and we have $\widetilde{Z} = Z \cup \{x_1\}$. Otherwise, set as z_1 the contract that agent $\mathbf{i}(x_1)$ is assigned under Z. Note that $x_1 P_{\mathbf{i}(x_1)} z_1$.

Step 2. Let x_2 be the contract that is chosen by the slot vacated by z_1 (or the shadow seat that is associated with it). If agent $\mathbf{i}(x_2)$ is assigned $\emptyset$ under Z, then the improvement process ends and we have $\widetilde{Z} = Z \cup \{x_1,x_2\} \setminus \{z_1\}$. Otherwise, set as z_2 the contract that agent $\mathbf{i}(x_2)$ is assigned under Z. Note that $x_2 P_{\mathbf{i}(x_2)} z_2$.

Step n. Let x_n be the contract that is chosen by the slot vacated by z_{n-1} (or the shadow seat that is associated with it). If agent $\mathbf{i}(x_n)$ is assigned $\emptyset$ under Z, then the improvement process ends and we have $\widetilde{Z} = Z \cup \{x_1,...,x_n\} \setminus \{z_1,...,z_{n-1}\}$. Otherwise, set as z_n the contract that agent $\mathbf{i}(x_n)$ is assigned under Z. Note that $x_n P_{\mathbf{i}(x_n)} z_n$.

In every step of the improvement chain algorithm a contract is replaced by a more preferred contract. Since there are finitely many contracts the improvement chain algorithm must end. Therefore, we reach $\widetilde{Z}$, which Pareto dominates Z, in finitely many step. This ends our proof.