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Bayesian Methods for Estimating the Reliability of Complex Systems Using Heterogeneous Multilevel Information

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Keywords
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ABSTRACT

We propose a Bayesian approach for assessing the reliability of multi-component systems. Our models allow us to evaluate system, subsystem, and component reliability using the available multilevel information. Data are collected over time, and include pass/fail, lifetime, censored, and degradation data. We illustrate the methodology through an example and discuss how to extend the approach to more complex systems.

KEY WORDS: Degradation, Hierarchical model, Lifetime, Multi-component system, System reliability

1. INTRODUCTION

This paper proposes methodology to integrate system, subsystem, and component data to assess system reliability as it changes over time. The approach addresses two common problems in reliability: estimating how reliability changes over time and using information from multiple levels in the system to make inference. Generalizing previous work (Johnson et al. 2003; Reese et al. 2009), we discuss models for pass/fail, lifetime, degradation, and expert opinion data at any system level.

Bayesian methods are appealing for this problem due to their natural incorporation of expert opinion, and a variety of approaches have been proposed in the literature. For example, considering only pass/fail data, Mastran (1976); Mastran and Singpurwalla (1978) describe a procedure to approximate the posterior mean reliability of a coherent system using test and prior data at both the component and system level. Martz et al. (1988); Martz and Waller (1990) propose a bottom-up approach for approximating the posterior distribution of reliability of series and parallel systems of independent Binomial subsystems and components. Tang et al. (1997) proposes methods to obtain the exact posterior distributions in special cases. Johnson et al. (2003) develops full simultaneous Bayesian for pass/fail data collected at any level of the system over time.

Extending beyond pass/fail data, Thompson and Chang (1975) and Chang and Thompson (1976) consider first the reliability of subsystems with one or more components in series, where each component has an independent exponential distribution, and then compute Bayesian credible intervals for arbitrary series-parallel system comprised of these subsystems. Winterbottom (1984) surveys classical and Bayesian results for estimating system reliability from Binomial and exponential component data in coherent systems. Robinson and Dietrich (1988) consider component-level data with exponential lifetimes that have decreasing failure rates as the system develops. Sharma and Bhutani (1994) estimate the availability of series and parallel systems where the components have exponential time to failure and repair. Bergman and Ringi (1997a) consider dependence between components induced by common operating environments; Bergman and Ringi (1997b) use data from non-identical environments. Hulting and Robinson (1994) is an exception to the above approaches, as they generalize the results of Martz et al. (1988) to make approximate inferences about the reliability of the system using multi-level information. They approximate the reliability for a series system using non-homogeneous Poisson processes to model the repair histories of repairable subsystems and time-to-failure data (modeled with a Weibull distribution) for nonrepairable subsystems.

In the last decade, Markov chain Monte Carlo (MCMC) has made fully Bayesian methods possible for addressing system reliability problems. For example, Johnson et al. (2003) and Hamada et al. (2004) propose fully Bayesian approaches for simultaneously estimating the reliability for a system and its subsystems/components described by a fault tree using pass/fail data. Wilson and Huzurbazar (2007) considers a system represented by a Bayesian network (BN), similarly with pass/fail data. Wilson et al. (2006) and Hamada et al. (2008) propose approaches for assessing system reliability with pass/fail data at the system, and pass/fail, lifetime, or degradation data at the components. Reese et al. (2009) considers lifetime data throughout the system. This paper discusses a unified fully Bayesian approach for simultaneously estimating system, subsystem, and component reliability when there are pass/fail, lifetime, degradation, or expert judgment data at any level of the system.

We develop this methodology using a series system with three components, which is represented as a fault tree in Figure 1. In Section 2, we introduce the models. In Section 3, we demonstrate the methodology by considering three scenarios applied to Figure 1. In each scenario, one component has pass/fail data collected over time, one has lifetime data, and one has degradation data; Scenario 1 has pass/fail data collected over time at the system, Scenario 2 has lifetime data at the system, and Scenario 3 has degradation data at the system. In Section 4, we discuss extensions of the methodology.

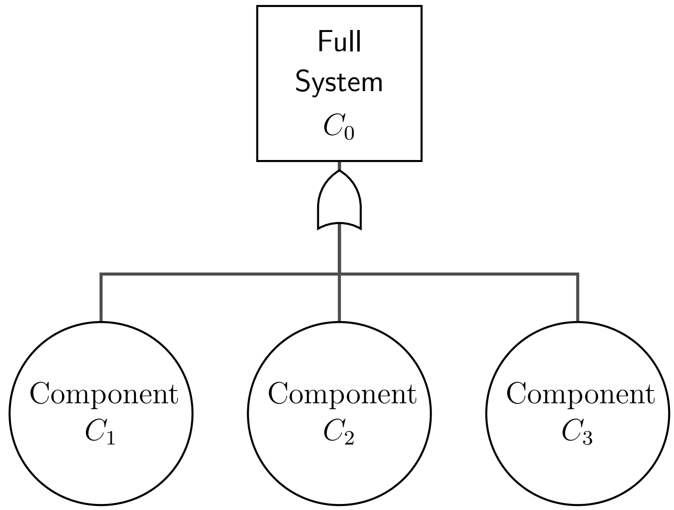

Figure 1: Three-component series system.

2. MODEL SPECIFICATION

In our development, we assume a coherent system represented by a fault tree. The fault tree describes the relationships between different level of failure events. We call those events requiring no further decomposition *basic events* and others simply *non-basic events*. Label each event C_i $(i = 0, 1, 2, 3, \ldots)$. In Figure 1, for example, C_0 denotes the system (a non-basic event) and C_1, C_2, C_3 denote the three components (basic events). For any event C_i, let $R_i(t \,|\, \Theta_i)$ denote its reliability function at time t given parameters Θ_i. Let T_i be the random variable associated with the lifetime of C_i, with probability density function $f_i(t \,|\, \Theta_i)$ and cumulative distribution function $F_i(t \,|\, \Theta_i)$. We assume that $R_i(t \,|\, \Theta_i)$ is differentiable with respect to t and Θ_i.

By definition, we have the following relations:

$$R_i(t \mid \Theta_i) = \Pr\{T_i > t \mid \Theta_i\} = 1 - F_i(t \mid \Theta_i), \tag{1}$$

and

$$f_i(t \mid \Theta_i) = \frac{\mathrm{d}\, F_i(t \mid \Theta_i)}{\mathrm{d}\, t} = \frac{\mathrm{d}\left(1 - R_i(t \mid \Theta_i)\right)}{\mathrm{d}\, t} = -\frac{\mathrm{d}\, R_i(t \mid \Theta_i)}{\mathrm{d}\, t}. \tag{2}$$

As a result, for C_i, either $R_i(t \mid \Theta_i)$ or $f_i(t \mid \Theta_i)$ is sufficient to determine the other. We call the lifetime distribution determined by the reliability function the *induced lifetime distribution*.

The first step of model development is specifying the reliability function $R_i(t \mid \Theta_i)$, which is specified directly or induced from the probability density function $f_i(t \mid \Theta_i)$. The second step is to use the system structure to determine the reliability functions for all of the non-basic events. Lifetime distributions for each event follow from the reliability functions. If the non-basic events have lifetime or pass/fail data, their likelihood functions are straightforward. If there are degradation data observed for the non-basic events, we specify the likelihoods with constraints determined by their reliability functions. Finally, the data for all events can be combined to model the system and estimate reliabilities.

For example, consider the system in Figure 1. The first step, modeling the three basic events, is detailed in Section 3. In the second step, since the system works if and only if all three components work, the reliability function of system is the product of the reliability functions of the three components. That is,

$$R_0(t \,|\, \Theta_0) = R_1(t \,|\, \Theta_1) \cdot R_2(t \,|\, \Theta_2) \cdot R_3(t \,|\, \Theta_3). \tag{3}$$

Another example of expressing the reliability of a system/subsystems in terms of basic events concerns the system in Figure 2. By virtue of the system structure, we can obtain the reliability functions of non-basic events as follows (parameters Θ_i's are suppressed):

$$R_4(t) = 1 - (1 - R_2(t))(1 - R_3(t)) = R_2(t) + R_3(t) - R_2(t)R_3(t), \tag{4}$$

$$R_0(t) = R_1(t)R_4(t) = R_1(t)R_2(t) + R_1(t)R_3(t) - R_1(t)R_2(t)R_3(t). \tag{5}$$

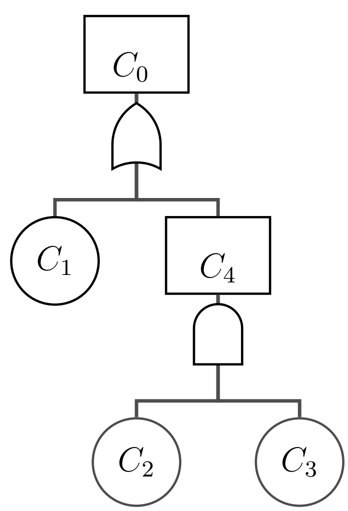

Figure 2: Another fault tree system: C_4 works if at least one of C_2 and C_3 works; the system C_0 works if and only if both C_1 and C_4 work.

2.1 Pass/fail data

Suppose at time s_{ij} ($j = 1, \ldots, n_i$), N_{ij} tests have been conducted on C_i, with b_{ij} passing the test. Denote the data vector as $\mathbf{b_i}$. The likelihood function for a basic event, using the Binomial distribution is given as

$$L_i(\mathbf{b_i} \,|\, \Theta_i) = \prod_{j=1}^{n_i} \binom{N_{ij}}{b_{ij}} \left[R_i(s_{ij} \,|\, \Theta_i)\right]^{b_{ij}} \left[1 - R_i(s_{ij} \,|\, \Theta_i)\right]^{N_{ij}-b_{ij}}. \qquad (6)$$

The reliability function in (6) can take many forms. Consider, for example, a Logit model, where we specify the reliability function as

$$R_i(t \,|\, \Theta_i) = \text{logit}^{-1}(\theta_i + \eta_i t), \quad \theta_i > 0,\ \eta_i < 0,\ \Theta_i = (\theta_i, \eta_i), \qquad (7)$$

where logit^{-1} is the inverse of Logit function which is defined as $\text{logit}(x) = \log x - \log(1-x),\ 0 < x < 1$.

The reliability function $R_i(t)$ for a non-basic event is determined by the system structure. In particular, $R_i(t)$ is a function of the reliabilities of basic events as illustrated by (3) for the system in Figure 1.

2.2 Lifetime data

For lifetime data, let $\mathbf{t_i} = t_{i1},\, t_{i2},\, \ldots,\, t_{im_i}$ be the lifetime data collected for event C_i. Assume the data are independent and identically distributed, with likelihood

$$L_i(\mathbf{t_i} \mid \Theta_i) = \prod_{j=1}^{m_i} f_i(t_{ij} \mid \Theta_i). \qquad (8)$$

The likelihood is easy to generalize to censored data, with details given in Section 4.

The probability density function $f_i(t)$ for a basic event depends on which model is used. As for a non-basic event, the density function can be derived from its reliability function. For the system in Figure 1, by using the relationship of (2), the density function for the lifetime of C_0 can be derived from (3).

2.3 Degradation Data

Degradation data measure some quantity about a component or subsystem that is indirectly related to reliability. In particular, degradation data are typically thought of a continuous quantity that changes over time, with failure occurring when the quantity passes some threshold.

Consider degradation data for event C_i, and suppose that we have measured a total v_i different units. Denote the time of the measurements, d_{ijk}, as q_{ijk}, where $j = 1, \ldots, v_i$ and $k = 1, \ldots, z_{ij}$. That is, for each of the v_i units, we measure the degradation quantity z_{ij} times. Let D_{ijk} denote the random variables associated with the degradation quantities d_{ijk}.

For basic events, the reliability function is derived from the degradation model: specifically, the reliability is the probability that the degradation quantity is above the threshold at time t. For example, consider the degradation process as in specified in Wilson et al. (2006):

$$D_{ijk} \sim \text{Normal}(\alpha_i - \beta_{ij}^{-1} q_{ijk},\, \sigma_i^2), \quad \alpha_i > 0,\ \beta_{ij} > 0,\ \sigma_i > 0. \tag{9}$$

That is, all events are identical at $t = 0$, but they each degrade at their own rates. Let $\mathbf{d_i}$ denote the degradation data for C_i. We can construct a likelihood function using (9):

$$L_i(\mathbf{d_i} \mid \boldsymbol{\beta_i}, \alpha_i, \sigma_i) = \prod_{j=1}^{v_i} \prod_{k=1}^{z_{ij}} \frac{1}{\sigma_i} \phi\left(\frac{d_{ijk} - \alpha_i + \beta_{ij}^{-1} q_{ijk}}{\sigma_i} \right), \tag{10}$$

where $\phi(\cdot)$ is the probability density function of standard normal distribution.

To connect the degradation model with the lifetime distribution and reliability function, let τ_i be the threshold of the degradation process. This event fails when the degradation quantity is less than τ_i. Then we have

$$T_{ij} = \inf\{t \geq 0 : \alpha_i - \beta_{ij}^{-1} t \leq \tau_i\} = (\alpha_i - \tau_i)\beta_{ij}, \quad \alpha_i > \tau_i > 0. \tag{11}$$

Suppose that we further assume that $\log \beta_{ij} \sim \text{Normal}(\mu_i, \psi_i^2)$. Then the lifetime of the event has a Log-normal distribution. That is, $\log T_{ij} \sim \text{Normal}(\mu_i + \log(\alpha_i - \tau_i),\, \psi_i^2)$. As a result, the reliability function at time t

is

$$R_i(t \mid \Theta_i) = 1 - \Phi\Big(\frac{\log t - \mu_i - \log(\alpha_i - \tau_i)}{\psi_i}\Big), \quad \Theta_i = (\mu_i, \psi_i, \alpha_i, \tau_i, \sigma_i), \tag{12}$$

where $\Phi(\cdot)$ is the cumulative distribution function of standard normal distribution.

For non-basic events, we first derive the induced lifetime distribution as described in Section 2. If we further assume the same degradation model as (9), the distribution of $(\alpha_i - \tau_i)\beta_{ij}$ in (11) is determined for the non-basic event. In our Bayesian model, we must choose our prior distribution for (α_i, τ_i, β_{ij}) such that the distribution of $(\alpha_i - \tau_i)\beta_{ij}$ is the same as the induced lifetime distribution for event C_i. One simple way to achieve this is to specify the following conditional probability density distribution function for β_{ij} in terms of the induced lifetime distribution $f_i(t \mid \Theta_i)$:

$$g_i(\beta_{ij} \mid \Theta_i) = (\alpha_i - \tau_i) f_i\big[\beta_{ij}(\alpha_i - \tau_i) \mid \Theta_i\big]. \tag{13}$$

This specification for β_{ij}, along with any proper prior distributions for α_i and τ_i, makes the distribution of $(\alpha_i - \tau_i)\beta_{ij}$ coincide with the induced lifetime distribution. Consequently the likelihood function for a non-basic event according to the above model specification still has the form of (10), but with constraints on β_{ij} from lower-level events.

2.4 Prior Information

Specifying prior distributions in a Bayesian context is also part of the modeling process. An advantage of Bayesian methodology is that we can incorporate "non-data" information into our models; for example, information from expert opinions, historical data, and from similar systems.

Initial specification of prior distributions for the parameters describing basic events follows standard Bayesian practice. However, some thought must be given to the specification of prior distributions for the degradation data. Suppose that we are working with the model (9). Consider the specification of the priors for the degradation quantity at time 0, α_i and the threshold τ_i, both of which are assumed to be positive. We consider two approaches.

A first approach, which is mentioned in Wilson et al. (2006), is to specify a Gamma prior distribution on α_i and then Beta distribution on τ_i/α_i given α_i. This approach is useful if we want to impose non-informative priors on τ_i or on both α_i and τ_i.

As an example, suppose that we specify the following Gamma prior distribution on α_i and uniform distribution on τ_i/α_i given α_i:

$$\alpha_i \sim \text{Gamma}(\nu_{\alpha_i}, \, \xi_{\alpha_i}), \qquad \tau_i \,|\, \alpha_i \sim \text{Uniform}[0, \, \alpha_i]. \tag{14}$$

(Note that our Gamma distribution has a parameterization such that the above specification for α_i has mean $\nu_{\alpha_i} \cdot \xi_{\alpha_i}$.)

We may have detailed information about the threshold τ_i that leads us to specify an informative prior for τ_i. This can be difficult to specify using the preceding approach. However, consider the following. We specify an informative prior for τ_i and then a conditional prior distribution for α_i given τ_i. An example using Gamma distribution is given as follows:

$$\tau_i \sim \text{Gamma}(\nu_{\tau_i}, \xi_{\tau_i}); \qquad (\alpha_i - \tau_i) \,|\, \tau_i \sim \text{Gamma}(\nu_{\alpha_i - \tau_i}, \xi_{\alpha_i - \tau_i}). \tag{15}$$

Specifying prior distributions for non-basic events requires additional thought. Recall that the reliability and lifetime of non-basic events are functions of the parameters of the basic events and the degradation model have constraints from lower level events. This implies for non-basic events, we need to specify prior distributions on functions of parameters. In addition, the prior distributions specified on the parameters of basic events *induce* prior distributions on the reliability and lifetime of non-basic events. Consequently, if we also have prior information about the reliability or lifetime of non-basic events, we need a way to combine the information.

We use the *Bayesian melding* approach proposed in Poole and Raftery (2000). Suppose that we have independent prior distributions on parameters $\boldsymbol{\theta}$ and $\boldsymbol{\phi} = M(\boldsymbol{\theta})$, $q_1(\boldsymbol{\theta})$ and $q_2(\boldsymbol{\phi})$. $M(\cdot)$ is a deterministic function. The prior on $\boldsymbol{\theta}$ induces a prior on $\boldsymbol{\phi} = M(\boldsymbol{\theta})$, denoted by $q_1^*(\boldsymbol{\phi})$.

Poole and Raftery (2000) proposes pooling $q_1^*(\boldsymbol{\phi})$ and $q_2(\boldsymbol{\phi})$ and then inverting the pooled prior back to $\boldsymbol{\theta}$. Denote the inverted prior on $\boldsymbol{\theta}$ by $\tilde{q}(\boldsymbol{\theta})$, with its formula given by

$$\tilde{q}(\boldsymbol{\theta}) \propto q_1(\boldsymbol{\theta}) \left(\frac{q_2\big(M(\boldsymbol{\theta})\big)}{q_1^*\big(M(\boldsymbol{\theta})\big)} \right)^{1-\alpha}, \tag{16}$$

where α is the pooling weight.

In our system reliability setting, the reliability on a non-basic event at some time is a function of parameters of lower level events. Using the above notation, we have initial priors of the basic events that is denoted by $q_1(\boldsymbol{\theta})$ and initial priors on some reliability function of non-basic events that is denoted by $q_2(\boldsymbol{\phi})$. Then $q_1^*(\boldsymbol{\phi})$ is the prior induced by $q_1(\boldsymbol{\theta})$ on the reliability function. And $\tilde{q}(\boldsymbol{\theta})$ is the final prior on the parameters of basic events after melding. As a result, if we elicit the prior information on non-basic events as prior on the reliability, we can use the Bayesian melding approach to combine prior information given at the basic and non-basic events.

3. THREE-COMPONENT SERIES SYSTEM SCENARIOS

In this section, we apply the proposed methodology to analyze the three component system pictured in Figure 1. The system is composed of three components, and the system works if and only if the three components work. We denote the system by C_0 and the three components by C_1, C_2, and C_3.

Using the notation from Section 2, we have

$$R_0(t \mid \Theta_0) = R_1(t \mid \Theta_1) \cdot R_2(t \mid \Theta_2) \cdot R_3(t \mid \Theta_3), \tag{17}$$

where Θ_0 includes $\Theta_1, \Theta_2, \Theta_3$ and any other parameters involved in modeling the system. Note that by assumption, all the reliability functions are differentiable with respect to t and their parameters.

We consider three scenarios for the system. Each scenario has the same information for the components: pass/fail data for C_1; lifetime data for C_2; and degradation data for C_3. The degradation data for C_3 are collected for 20 units that are measured one time each. The component data are given in Table 1.

Scenario 1 has pass/fail data collected over time at the system, Scenario 2 has lifetime data at the system, and Scenario 3 has degradation data at the system. The system data are given in Table 2. The degradation data for the system are collected for five systems that are measured eight times each across different ages.

We first analyze the three scenarios when there is no prior information for the system. We then introduce prior information for the system and use Bayesian melding to reanalyze Scenario 1.

Table 1: Pass/fail data for component 1 (number of pass out of total), lifetime time (years) data for component 2, and degradation data for component 3. In the parenthesis is the age (years) when it is tested or measured.

C_1	25/25 (0), 25/25 (4), 25/25 (8), 24/25 (12), 22/25 (16), 23/25 (20), 20/25 (24), 14/25 (28), 9/25 (32), 7/25 (36), 3/25 (40)
C_2	23.8, 45.49, 64.61, 38.77, 11.22, 58.25, 29.93, 51.56, 75.42, 43.85, 44.01, 26.47, 26.9, 45.03, 21.11, 72.81, 64.04, 86.37, 56.67, 51.86, 69.88, 26.49, 71.24, 52.7, 67.84
C_3	93.61 (2), 95.80 (4), 80.59 (6), 83.79 (8), 80.25 (10), 54.60 (12), 70.20 (14), 58.06 (16), 38.63 (18), 26.18 (20), 87.93 (2), 85.44 (4), 86.31 (6), 71.48 (8), 70.73 (10), 57.85 (12), 60.43 (14), 70.45 (16), 40.88 (18), 51.15 (20)

Table 2: System data for the three scenarios: Pass/fail data (number of pass out of total) with ages indicated in the parenthesis, lifetime time (years) data, and degradation data for five systems (every consecutive eight are the measurements for each system at different ages showed in the parenthesis).

Pass/Fail	20/20 (0), 20/20 (2), 20/20 (4), 20/20 (6), 20/20 (10), 18/20 (15), 16/20 (20), 4/20 (30)
Lifetime	30.2, 36.55, 25.11, 39.35, 27.57, 25.91, 31.5, 29.24, 18.39, 16.65, 21.85, 24.88, 31.61, 18.74, 19.63, 28.98, 11.1, 21.66, 22.41, 26.04, 25.07, 23.48, 28.21, 25.21, 25.12, 27.76, 23.47, 23.51, 24.39, 21.93, 37.63, 20.32, 28.17, 24.66, 30.13, 21.42, 17.21, 19.98, 33.09, 16.04, 17.96, 19.57, 22.91, 25.69, 23.47, 16.91, 27.2, 27.23
Degradation	168.96 (2), 183.06 (4), 143.02 (8), 136.58 (12), 100.32 (16), 74.63 (20), 72.38 (24), 33.29 (28) 203.23 (2), 177.13 (4), 159.21 (8), 125.13 (12), 93.56 (16), 106.83 (20), 66.76 (24), 37.06 (28) 190.68 (2), 178.63 (4), 174.95 (8), 142.19 (12), 125.78 (16), 85.48 (20), 86.96 (24), 65.61 (28) 201.76 (2), 184.75 (4), 144.21 (8), 154.4 (12), 123.1 (16), 100.9 (20), 97.86 (24), 67.54 (28) 179.3 (2), 168.64 (4), 168.86 (8), 134.18 (12), 136.34 (16), 98.92 (20), 66.5 (24), 48.96 (28)

3.1 Models

We use the Logit model (Section 2.1) for C_1, the Weibull lifetime distribution model (Section 2.2) for C_2, and the degradation model (Section 2.3) for C_3. For the system (C_0), the reliability function is determined from the specifications for the components, as discussed in Section 2.

The three reliability functions for C_1, C_2, and C_3 are given below.

$$R_1(t \,|\, \Theta_1) = \text{logit}^{-1}(\theta_1 + \eta_1 t), \quad \Theta_1 = (\theta_1, \eta_1), \tag{18}$$

$$R_2(t \,|\, \Theta_2) = \exp\left[-\left(\frac{t}{\lambda_2}\right)^{\delta_2}\right], \quad \Theta_2 = (\delta_2, \lambda_2), \tag{19}$$

$$R_3(t \,|\, \Theta_3) = 1 - \Phi\Big(\frac{\log t - \mu_3 - \log(\alpha_3 - \tau_3)}{\psi_3}\Big), \quad \Theta_3 = (\mu_3, \psi_3, \alpha_3, \tau_3, \sigma_3). \tag{20}$$

Using (17), the reliability function for the system is

$$R_0(t \,|\, \Theta_0) = \text{logit}^{-1}(\theta_1 + \eta_1 t) \cdot \exp\left[-\left(\frac{t}{\lambda_2}\right)^{\delta_2}\right] \cdot \left[1 - \Phi\Big(\frac{\log t - \mu_3 - \log(\alpha_3 - \tau_3)}{\psi_3}\Big)\right]. \tag{21}$$

Let $\mathbf{b_1}$ denote the data for C_1; $\mathbf{t_2}$ for C_2; $\mathbf{d_3}$ for C_3; $\mathbf{b_0}$, $\mathbf{t_0}$, and $\mathbf{d_0}$ for C_0. Additionally, $\boldsymbol{\beta_3} = \{\beta_{3j} : j = 1, \ldots, v_3\}$, $\boldsymbol{\beta_0} = \{\beta_{0j} : j = 1, \ldots, v_0\}$.

In Scenario 1, with pass/fail data at the system, (21) specifies the probability of observing a "pass" at time t. For example, we have observed 16 passes out of total 20 tests when $t = 20$. The likelihood term for these 20 tests is given by

$$\binom{20}{16}\left[R_0(20\,|\,\Theta_0)\right]^{16}\left[1-R_0(20\,|\,\Theta_0)\right]^{4},$$

where $R_0(20\,|\,\Theta_0)$ is given in (21) with $t=20$.

In Scenario 2, with lifetime data at the system, the probability density function for the system lifetime distribution is determined by (21). Following (2), we have

$$f_0(t\,|\,\Theta_0) = -\frac{\mathrm{d}\,R_0(t\,|\,\Theta_0)}{\mathrm{d}\,t}, \tag{22}$$

where $R_0(t\,|\,\Theta_0)$ is given in (21). Since the data in Table 2 are independent, we have

$$L_0(\mathbf{t_0}\,|\,\Theta_0) = \prod_{j=1}^{48} f_0(t_{0j}\,|\,\Theta_0),$$

where for example $t_{01}=30.2$ and $t_{02}=36.55$.

In Scenario 3, with degradation data at the system, assume that we are modeling the data using the degradation model from Section 2.3. Following (10), we have the likelihood function:

$$L_0(\mathbf{d_0}\,|\,\boldsymbol{\beta_0},\alpha_0,\sigma_0) = \prod_{j=1}^{v_0}\prod_{k=1}^{z_{0j}}\frac{1}{\sigma_0}\phi\left(\frac{d_{0jk}-\alpha_0+\beta_{0j}^{-1}q_{0jk}}{\sigma_0}\right),$$

where $v_0=5$; $z_{0j}=8$ for $j=1,\ldots,5$. We specify the distribution for β_{0j}

according to equation (13). That is,

$$g_0(\beta_0 \,|\, \Theta_0) = (\alpha_0 - \tau_0) f_0\big[\beta_0(\alpha_0 - \tau_0) \,|\, \Theta_0\big], \tag{23}$$

where $f_0(\cdot \,|\, \Theta_0)$ is given in (22) such that the reliability function for the system still satisfies (17).

3.2 Prior distributions

When specifying prior distributions, we have the parameters from the basic events: θ_1, η_1, δ_2, λ_2, α_3, τ_3, ψ_3, μ_3, σ_3. In Scenario 3, we also have α_0, τ_0, σ_0. In real applications, these parameters are elicited; for illustration, we use fairly diffuse priors for some parameters here. These priors are detailed in Table 3. The priors for α_3, τ_3, α_0, and τ_0 follows the discussion in Section 2.4.

Table 3: Prior distributions

Parameters	Prior distribution
θ_1	$\text{Normal}(0, 100^2)$
η_1	$\text{Normal}(0, 100^2)$
δ_2	Gamma(1, 1) with mean 1
λ_2	$\text{Log-normal}(0, 100^2)$
α_3	Gamma(4, 30) with mean 120
τ_3	$\tau_3 \mid \alpha_3 \sim \text{Uniform}[0,\ \alpha_3]$
μ_3	$\text{Normal}(0, 10^2)$
ψ_3	Gamma(4, 0.2) with mean 0.8
σ_3	Gamma(4, 2.5) with mean 10
σ_0	Gamma(4, 3) with mean 12
τ_0	Gamma(100, 0.5) with mean 50
α_0	$\big(\alpha_0 - \tau_0\big) \mid \tau_0 \sim \text{Gamma}(150,\ 1)$

3.3 Joint Posterior Distribution

Let $L_1(\mathbf{b_1} \mid \Theta_1)$ be the likelihood function for component 1 (from (6)); $L_2(\mathbf{t_2} \mid \Theta_2)$ be the likelihood function for component 2 (from (8)); and $L_3(\mathbf{d_3} \mid \Theta_3)$ be the likelihood function for component 3 (from (10)). The likelihood function for the system L_0 is given above. By Bayes' theorem, we obtain the following unnormalized probability density functions for the joint posterior distribution for the three scenarios.

The unnormalized joint posterior probability density function for Scenario 1 is given by

$$\begin{aligned}
&\pi(\Theta_0, \boldsymbol{\beta_3} \,|\, \mathbf{b_1}, \mathbf{t_2}, \mathbf{d_3}, \mathbf{b_0}) \\
&\quad \propto L_1(\mathbf{b_1} \,|\, \Theta_1) L_2(\mathbf{t_2} \,|\, \Theta_2) L_3(\mathbf{d_3} \,|\, \boldsymbol{\beta_3}, \Theta_3) L_0(\mathbf{b_0} \,|\, \Theta_0) \\
&\qquad \cdot \prod_{j=1}^{v_3} \beta_{3j}^{-1} \phi\big[(\log \beta_{3j} - \mu_3)/\psi_3\big] \\
&\qquad \cdot \phi(\theta_1/100) \cdot \phi(\eta_1/100) \cdot \exp(-\delta_2) \cdot \lambda_2^{-1} \phi(\log \lambda_2/100) \cdot \phi(\mu_3/10) \\
&\qquad \cdot \alpha_3^3 \exp\left(-\alpha_3/30\right) \cdot \mathrm{I}(\alpha_3 > \tau_3 \geq 0) \cdot \psi_3^3 \exp\left(-\psi_3/0.2\right) \cdot \sigma_3^3 \exp\left(-\sigma_3/2.5\right)
\end{aligned} \tag{24}$$

The unnormalized joint posterior probability density function for Scenario 2 is given by

$$\begin{aligned}
&\pi(\Theta_0, \boldsymbol{\beta_3} \,|\, \mathbf{b_1}, \mathbf{t_2}, \mathbf{d_3}, \mathbf{t_0}) \\
&\quad \propto L_1(\mathbf{b_1} \,|\, \Theta_1) L_2(\mathbf{t_2} \,|\, \Theta_2) L_3(\mathbf{d_3} \,|\, \boldsymbol{\beta_3}, \Theta_3) L_0(\mathbf{t_0} \,|\, \Theta_0) \\
&\qquad \cdot \prod_{j=1}^{v_3} \beta_{3j}^{-1} \phi\big[(\log \beta_{3j} - \mu_3)/\psi_3\big] \\
&\qquad \cdot \phi(\theta_1/100) \cdot \phi(\eta_1/100) \cdot \exp(-\delta_2) \cdot \lambda_2^{-1} \phi(\log \lambda_2/100) \cdot \phi(\mu_3/10) \\
&\qquad \cdot \alpha_3^3 \exp\left(-\alpha_3/30\right) \cdot \mathrm{I}(\alpha_3 > \tau_3 \geq 0) \cdot \psi_3^3 \exp\left(-\psi_3/0.2\right) \cdot \sigma_3^3 \exp\left(-\sigma_3/2.5\right)
\end{aligned} \tag{25}$$

The unnormalized joint posterior probability density function for Scenario

3 is given by

$$
\begin{aligned}
&\pi(\Theta_0, \boldsymbol{\beta_3}, \boldsymbol{\beta_0} \,|\, \mathbf{b_1}, \mathbf{t_2}, \mathbf{d_3}, \mathbf{d_0}) \\
&\quad \propto L_1(\mathbf{b_1} \,|\, \Theta_1) L_2(\mathbf{t_2} \,|\, \Theta_2) L_3(\mathbf{d_3} \,|\, \boldsymbol{\beta_3}, \Theta_3) L_0(\mathbf{d_0} \,|\, \boldsymbol{\beta_0}, \Theta_0) \\
&\qquad \cdot \prod_{j=1}^{v_3} \beta_{3j}^{-1} \phi\big[(\log \beta_{3j} - \mu_3)/\psi_3\big] \cdot \prod_{j=1}^{v_0} g_0(\beta_{0j} \,|\, \Theta_0) \\
&\qquad \cdot \phi(\theta_1/100) \cdot \phi(\eta_1/100) \cdot \exp(-\delta_2) \cdot \lambda_2^{-1} \phi(\log \lambda_2/100) \cdot \phi(\mu_3/10) \\
&\qquad \cdot \alpha_3^3 \exp\left(-\alpha_3/30\right) \cdot \mathrm{I}(\alpha_3 > \tau_3 \geq 0) \cdot \psi_3^3 \exp\left(-\psi_3/0.2\right) \cdot \sigma_3^3 \exp\left(-\sigma_3/2.5\right) \\
&\qquad \cdot \big(\alpha_0 - \tau_0\big)^{149} \exp\left(-(\alpha_0 - \tau_0)\right) \cdot \tau_0^{99} \exp\left(-\tau_0/0.5\right) \\
&\qquad \cdot \sigma_0^3 \exp\left(-\sigma_0/3\right).
\end{aligned}
\tag{26}
$$

3.4 Model estimation and estimated reliabilities

We can use MCMC to draw samples from the unnormalized joint posterior distributions. In particular, we used a one-variable-at-a-time random walk Metropolis algorithm to draw samples from the posterior distributions specified in (24), (25), and (26).

The marginal posterior distributions of the parameters are summarized in Table 4 (Scenario 1), Table 5 (Scenario 2), and Table 6 (Scenario 3).

Table 4: Empirical mean, median, 2.5% and 97.5% quantiles, and standard deviation for each variable for Scenario 1.

	Mean	50%	2.5%	97.5%	SD
θ_1	6.240	6.205	5.034	7.651	0.669
η_1	-0.208	-0.207	-0.256	-0.167	0.023
δ_2	2.68	2.66	2.01	3.51	0.38
λ_2	55.7	55.4	47.8	64.9	4.3
μ_3	-0.928	-0.943	-1.183	-0.589	0.151
α_3	97.8	97.9	90.7	104.4	3.4
τ_3	17.1	16.6	1.1	37.2	9.9
ψ_3	0.274	0.264	0.138	0.466	0.084
σ_3	5.55	5.33	2.66	9.71	1.81

Table 5: Empirical mean, median, 2.5% and 97.5% quantiles, and standard deviation for each variable for Scenario 2.

	Mean	50%	2.5%	97.5%	SD
θ_1	6.299	6.271	5.066	7.694	0.669
η_1	-0.210	-0.209	-0.258	-0.167	0.023
δ_2	2.79	2.77	2.09	3.61	0.38
λ_2	56.3	56.0	48.6	65.3	4.2
μ_3	-0.963	-0.971	-1.206	-0.674	0.135
α_3	98.4	98.4	91.9	104.9	3.3
τ_3	22.0	22.4	3.7	38.3	8.9
ψ_3	0.265	0.259	0.182	0.381	0.051
σ_3	5.43	5.21	2.66	9.48	1.75
$\frac{\tau_3}{\alpha_3}$	0.225	0.228	0.037	0.404	0.094

Table 6: Empirical mean, median, 2.5% and 97.5% quantiles, and standard deviation for each variable for Scenario 3.

	Mean	50%	2.5%	97.5%	SD
θ_1	6.168	6.135	4.806	7.722	0.744
η_1	-0.205	-0.204	-0.258	-0.159	0.025
δ_2	2.67	2.65	1.90	3.55	0.42
λ_2	55.7	55.6	47.4	65.2	4.4
μ_3	-0.932	-0.947	-1.182	-0.590	0.151
α_3	97.9	97.9	90.7	104.6	3.4
τ_3	14.2	12.9	0.7	35.6	9.6
ψ_3	0.244	0.233	0.107	0.448	0.087
σ_3	5.80	5.59	2.81	10.01	1.86
α_0	199.7	199.7	193.4	205.9	3.2
τ_0	50.9	50.8	42.3	60.1	4.6
σ_0	10.92	10.79	8.69	13.86	1.32

Perhaps more interesting, we can obtain the posterior distributions of the reliability functions for both the components and the system from the samples from posterior distributions. Plots for the functions with respect to time t along with a credible interval band are presented in Figure 3 (Scenario 1), Figure 4 (Scenario 2), and Figure 5 (Scenario 3). Note that the estimation of the reliability function of C_3 is not as accurate as those for C_1 and C_2. The main reason is that we do not have much information about α_3

and τ_3 in the degradation model for C_3. Recall that a failure occurs when the degradation quantity passes the threshold. Here we have noninformative prior for the threshold τ_3, so the degradation data do not give much information about the reliability. Since we perform inference on the system as a whole, the information from the system contributes to the estimation of C_3; otherwise, we would not have information about the component reliability. Our methodology takes advantage of information at all levels to estimate the system reliability, but also it helps to estimate component reliability using data from the whole system.

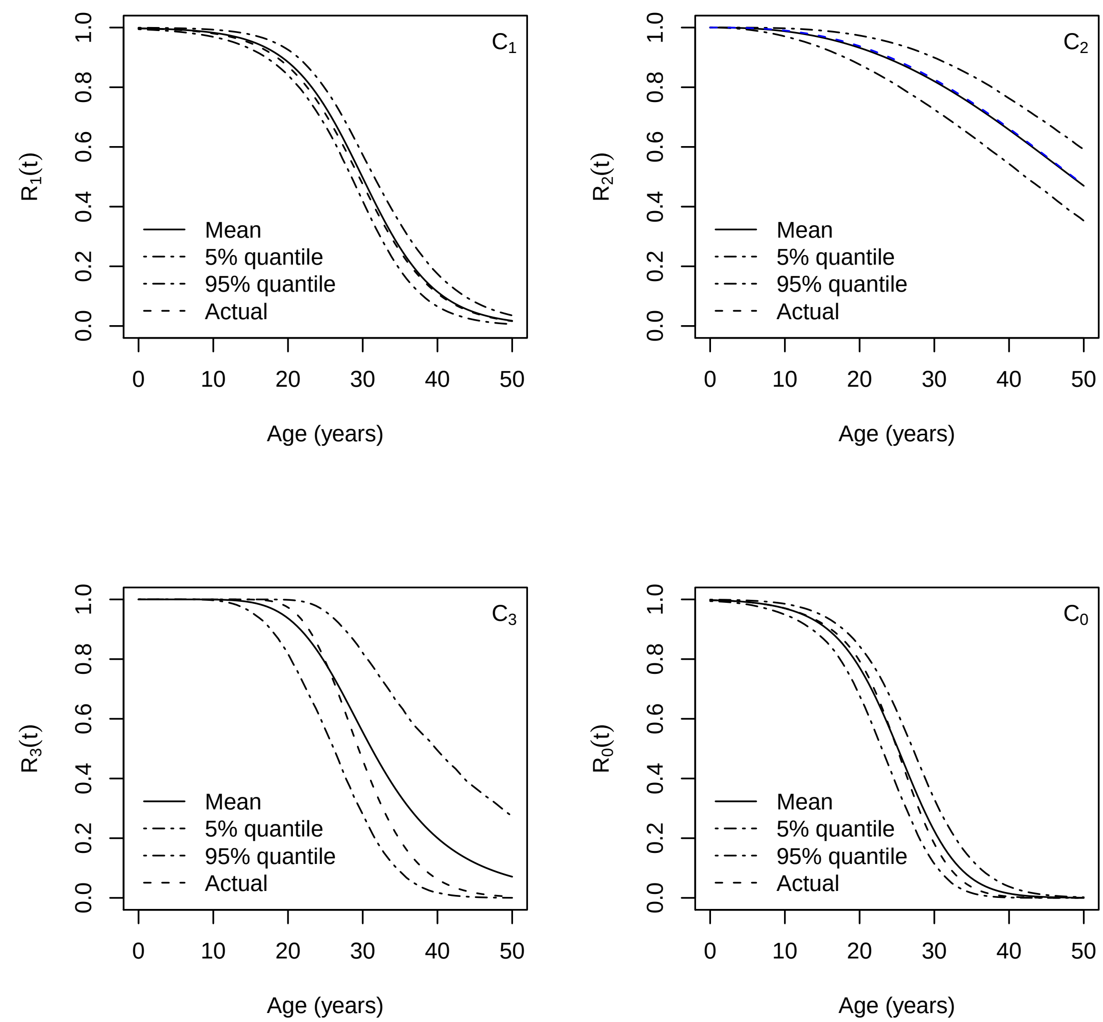

Figure 3: Reliability estimates and credible intervals with respect to the age. Upper left: Component 1, with pass/fail data using Logit model. Upper right: Component 2, with life time data assumed to have Weibull distribution. Lower left: Component 3, with degradation data. Lower right: The full system, for which pass/fail data are collected.

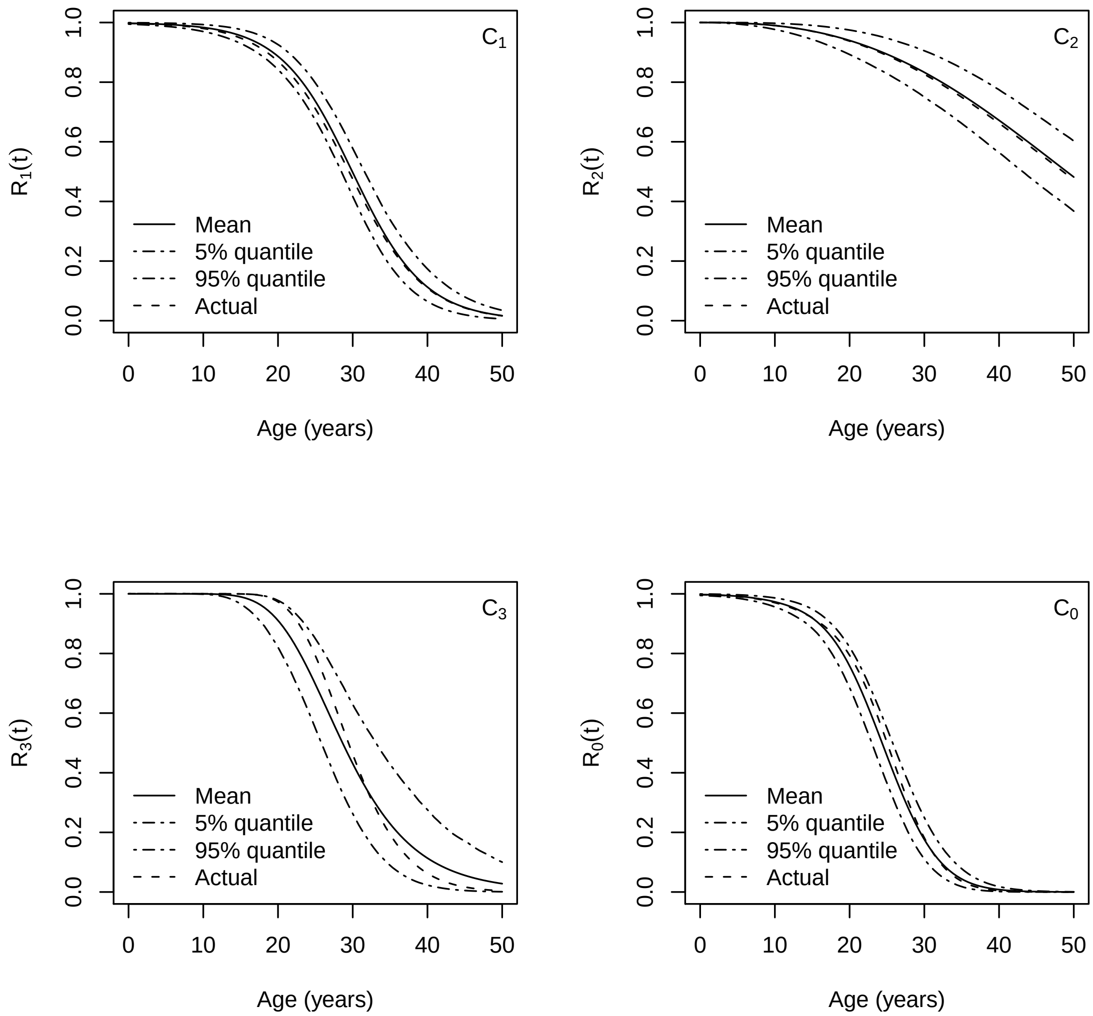

Figure 4: Reliability estimates and credible intervals with respect to the age. Upper left: Component 1, with pass/fail data using Logit model. Upper right: Component 2, with life time data assumed to have Weibull distribution. Lower left: Component 3, with degradation data. Lower right: The full system, for which lifetime data are collected.

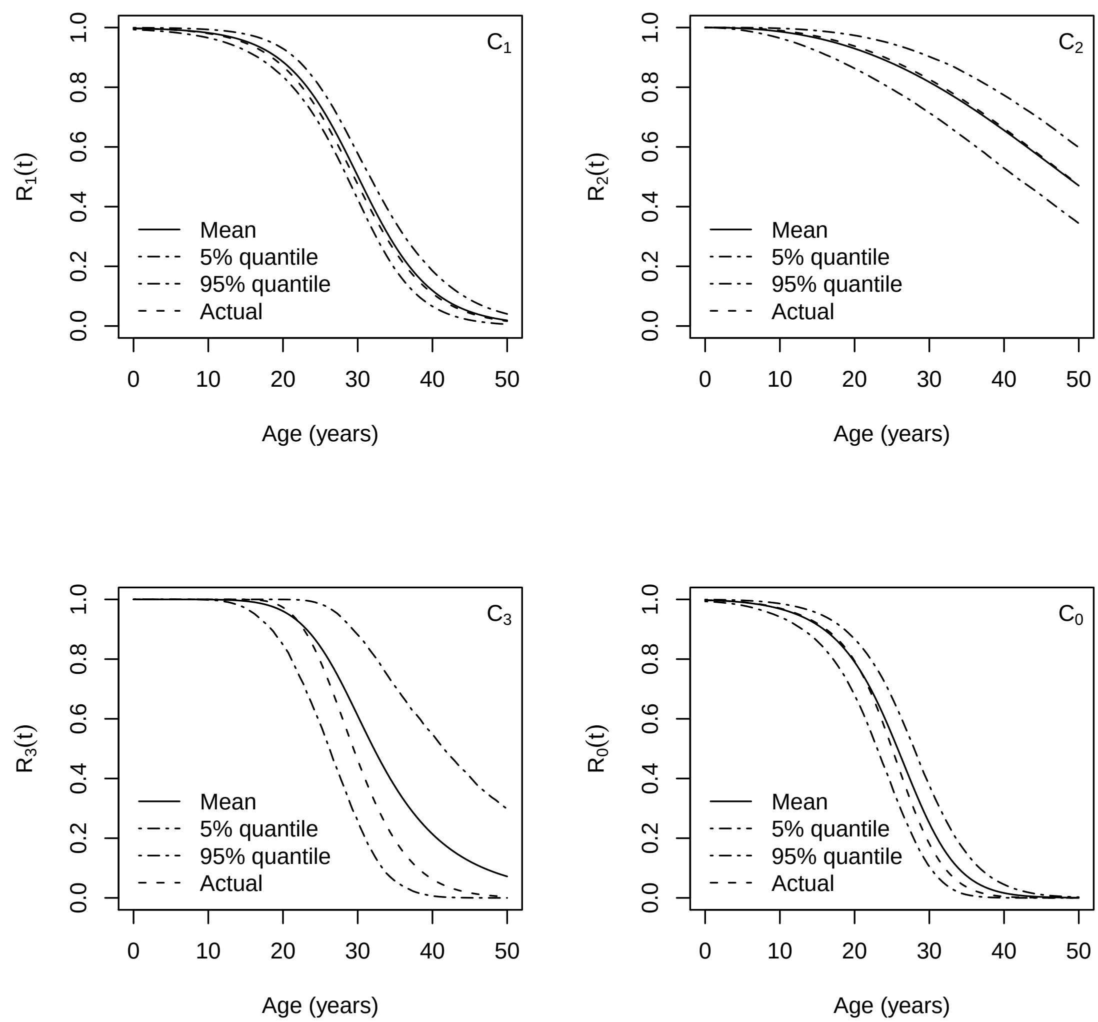

Figure 5: Reliability estimates and credible intervals with respect to the age. Upper left: Component 1, with pass/fail data using Logit model. Upper right: Component 2, with life time data assumed to have Weibull distribution. Lower left: Component 3, with degradation data. Lower right: The full system, for which degradation data are collected.

3.5 Incorporating prior information about the system

In the above analyses, we have not incorporated any additional prior information about the system. Suppose we have additional independent prior information for the system, and we believe the system reliability at age of 20 years, $R_0(t = 20 \,|\, \Theta_0)$, has a Beta$(4, 2)$ distribution. From (21), the system reliability $R_0(t = 20 \,|\, \Theta_0)$ is a deterministic function of parameters of the three components. Consequently the prior on Θ_0 induces a prior on $R_0(t = 20 \,|\, \Theta_0)$. Specifically, let $q_1(\boldsymbol{\theta})$ denote the prior in (24):

$$q_1(\boldsymbol{\theta}) \propto \phi(\theta_1/100) \cdot \phi(\eta_1/100) \cdot \exp(-\delta_2) \cdot \lambda_2^{-1}\phi(\log \lambda_2/100) \cdot \phi(\mu_3/10) \\ \cdot \alpha_3^3 \exp\left(-\alpha_3/30\right) \cdot \mathrm{I}(\alpha_3 > \tau_3 \geq 0) \cdot \psi_3^3 \exp\left(-\psi_3/0.2\right) \cdot \sigma_3^3 \exp\left(-\sigma_3/2.5\right). \tag{27}$$

$q_1^*(M(\boldsymbol{\theta}))$ is the prior distribution on $M(\boldsymbol{\theta})$ induced by the specification of (27). $q_2[M(\boldsymbol{\theta}) = R_0(t = 20 \,|\, \boldsymbol{\theta})]$ is the density function of the Beta$(4, 2)$ distribution.

In Figure 6, we plot the induced prior $q_1^*(M(\boldsymbol{\theta}))$; the initial prior on $M(\boldsymbol{\theta})$, $q_2(M(\boldsymbol{\theta}))$; and the pooling of $q_1^*(M(\boldsymbol{\theta}))$ and $q_2(M(\boldsymbol{\theta}))$. Inverting the pooled prior on $M(\boldsymbol{\theta})$ to prior on $\boldsymbol{\theta}$ gives the final Bayesian Melding prior. We use the melded prior as in (24) with pooling weight α being 0.5 instead to perform our posterior inference.

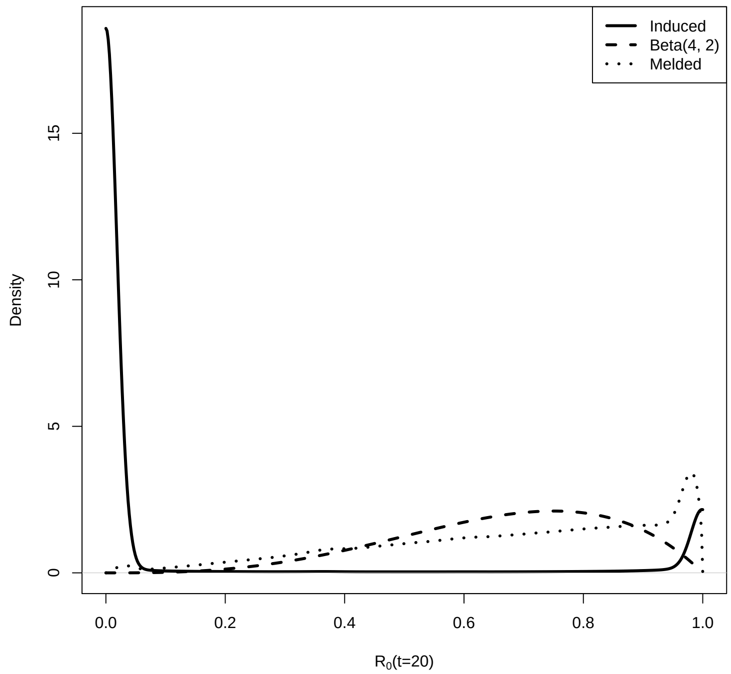

Figure 6: Probability density functions of priors on the system reliability when $t = 20$. The solid line represents the prior induced from the prior specified on the basic events; the dashed line $\text{Beta}(4, 2)$; the dotted line the pooled prior.

When executing the analyses, the induced prior often needs to be estimated numerically using, for example, kernel methods, since the deterministic function is complex. The MCMC can then be carried out with the updated posterior distribution. Notice that the induced prior is time-consuming to compute, and since its computation is required in every evaluation of the posterior distribution, the overall MCMC procedure can be quite slow.

We have employed two approximations to ease this computational burden.

First, we can approximate the induced prior distribution using a parametric form. For example, we can find a Beta distribution (or mixture of Beta distributions) to approximate the induced prior on system reliability. A second approach is to first evaluate the induced prior at multiple points (say 10^7 points). We can then use a "table lookup", which returns the density of the closest point to approximate the induced prior. This is the approach we used in our computations. The estimation results are presented in Table 7 and Figure 7.

Table 7: Empirical mean, median, 2.5% and 97.5% quantiles, and standard deviation for each variable.

	Mean	50%	2.5%	97.5%	SD
θ_1	6.214	6.178	5.018	7.618	0.666
η_1	-0.207	-0.206	-0.255	-0.165	0.022
δ_2	2.76	2.73	2.05	3.63	0.40
λ_2	55.6	55.4	47.9	64.5	4.2
μ_3	-0.928	-0.942	-1.184	-0.588	0.151
α_3	97.8	97.8	90.8	104.3	3.4
τ_3	17.6	17.2	1.1	37.7	10.0
ψ_3	0.278	0.268	0.141	0.468	0.083
σ_3	5.51	5.29	2.62	9.64	1.80

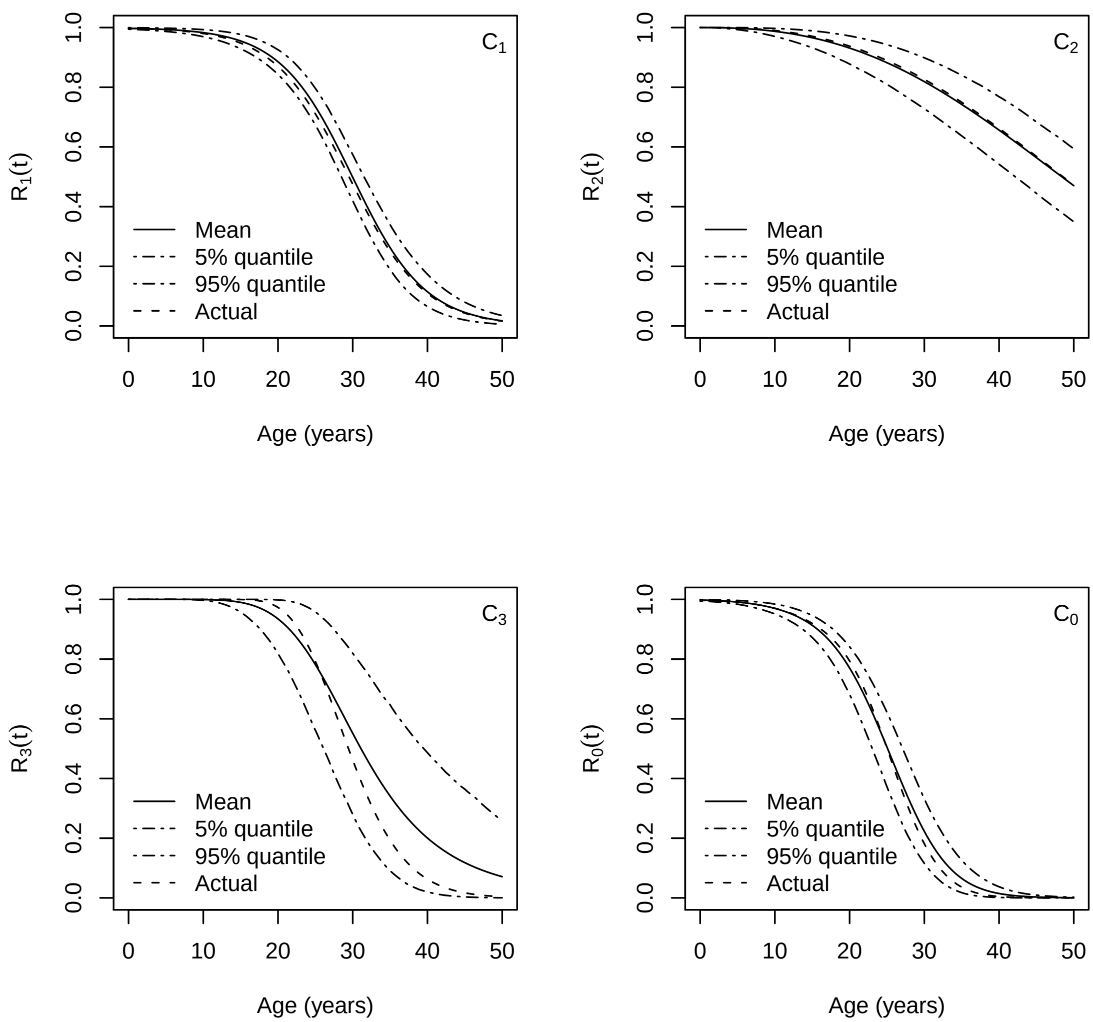

Figure 7: Reliability estimates and credible intervals with respect to the age. Upper left: Component 1, with Logit regression data. Upper right: Component 2, with life time data assumed to have Weibull distribution; Lower left: Component 3, with degradation data; Lower right: The full system, for which pass/fail data are collected.

4. EXTENSION AND DISCUSSION

In this paper, we propose a unified methodology to estimate system reliability for multi-components complex system with different types of information. This methodology uses the relationships among reliability functions between a system and its components to combine models at different levels into one model. The model for the system is developed in a consistent and compatible way so that it naturally eliminates the aggregation errors. As a result, all the data and information are used to assess the system and component reliabilities.

A real system might be much complex than the example system in Figure 1. Consider, for instance, the system analyzed by Hamada et al. (2004) and Reese et al. (2009). As the system complexity increases, finding the reliability function of a non-basic event in terms of basic events also is more complex. For systems represented by fault trees, techniques using *structure functions* and *path* or *cut sets* are helpful in finding the reliability functions. These algorithms are implemented in a variety of software packages; details of the methodology can be found in Rausand and Høyland (2004).

In addition, we may need more complex models for the data. For example, we might have dependence between basic events, which we could model using bivariate lifetime distributions, or different forms of degradation models.

The methodology can be easily extended to handle system with other features. For example, we can easily extend the approach to deal with censored lifetime data. In this case, we just need to replace $f_i(t_{ij} \mid \Theta_i)$ in the likelihood function of (8) by the corresponding forms given in Table 8.

Table 8: The likelihood contribution for a (censored) lifetime observation

Type of Observations	Failture Time	Contribution
Uncensored	$T_i = t_{ij}$	$f_i(t_{ij} \mid \Theta_i)$
Left censored	$T_i \leq t_{ij}$	$F_i(t_{ij} \mid \Theta_i)$
Interval censored	$t^*_{ij} \leq T_i \leq t^{**}_{ij}$	$F_i(t^{**}_{ij} \mid \Theta_i) - F_i(t^*_{ij} \mid \Theta_i)$
Right censored	$T_i > t_{ij}$	$1 - F_i(t_{ij} \mid \Theta_i)$

A second important extension is the application of the methodology to systems represented by generalizations of the fault tree. For example, consider the Bayesian networks in Figure 8. Using $C_i = 0$ (1) to denote that component i is working (not working), we could specify the relationships given in (28) to describe the dependence among the components.

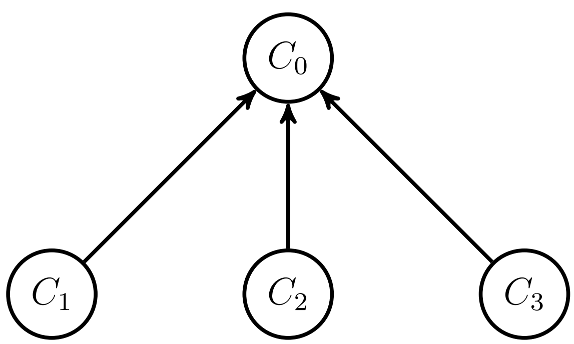

Figure 8: Bayesian network generalization of the example system.

$$
\begin{aligned}
&\Pr\left(C_0 = 1 \,|\, C_1 = 1, C_2 = 1, C_3 = 1\right) = 0.9,\\
&\Pr\left(C_0 = 1 \,|\, C_1 = 0, C_2 = 1, C_3 = 1\right) = 0.4,\\
&\Pr\left(C_0 = 1 \,|\, C_1 = 0, C_2 = 0, C_3 = 1\right) = 0.3,\\
&\Pr\left(C_0 = 1 \,|\, C_1 = 0, C_2 = 1, C_3 = 0\right) = 0.5,\\
&\Pr\left(C_0 = 1 \,|\, C_1 = 0, C_2 = 0, C_3 = 1\right) = 0.1,\\
&\Pr\left(C_0 = 1 \,|\, C_1 = 0, C_2 = 0, C_3 = 0\right) = 0.05,\\
&\Pr\left(C_0 = 0 \,|\, C_1 = 1, C_2 = 0, C_3 = 0\right) = 0.25,\\
&\Pr\left(C_0 = 1 \,|\, C_1 = 0, C_2 = 0, C_3 = 0\right) = 0.
\end{aligned}
\tag{28}
$$

For this generalized system, the relationships between reliability functions become more complicated. With the parameters suppressed for this BN, $R_0(t)$ is expressed as

$$
\begin{aligned}
R_0(t) = {}& 0.9R_1(t)R_2(t)R_3(t) + 0.4\big(1 - R_1(t)\big)R_2(t)R_3(t)\\
&+ 0.3R_1(t)\big(1 - R_2(t)\big)R_3(t) + 0.5R_1(t)R_2(t)\big(1 - R_3(t)\big)\\
&+ 0.1\big(1 - R_1(t)\big)\big(1 - R_2(t)\big)R_3(t) + 0.05R_1(t)\big(1 - R_2(t)\big)\big(1 - R_3(t)\big)\\
&+ 0.25\big(1 - R_1(t)\big)R_2(t)\big(1 - R_3(t)\big).
\end{aligned}
\tag{29}
$$

Then all the procedures for estimating the reliabilities for the system in fault tree qualification can be applied to this system with dependent components by updating the reliability function for the system.

In general, we can apply our methodology to any data structure as long as we can build up models for non-basic events from the relationships among the reliability functions of the basic events. As the systems become more complicated, it may be difficult to explicitly perform the differentiation required to determine the probability density function for, say, lifetime data. We can then employ numerical differentiation instead of writing down the explicit analytical form of the probability density function.

In summary, we have proposed a fully Bayesian methodology to estimate system reliability. The methodology provides a flexible and extendible approach to take advantage all available information arising from different levels. Further work concerns other specific models for different types of data; for example, other models for degradation data.

REFERENCES

Bergman, B. and Ringi, M. (1997a), "Bayesian System Reliability Prediction," *Scandinavian Journal of Statistics*, 24, 137–143.

— (1997b), "System Reliability Prediction Using Data From Non-identical Environments," *Reliability Engineering and System Safety*, 58, 185–190.

Chang, E. Y. and Thompson, W. E. (1976), "Bayes Analysis of Reliability for Complex Systems," *Operations Research*, 24, 156–168.

Hamada, M., Martz, H. F., Reese, C. S., Graves, T., Johnson, V., and Wilson, A. G. (2004), "A Fully Bayesian Approach for Combining Multilevel Failure Information in Fault Tree Quantification and Optimal Follow-on Resource Allocation," *Reliability Engineering & System Safety*, 86, 297–305.

Hamada, M. S., Wilson, A. G., Reese, C., and Martz, H. F. (2008), *Bayesian Reliability*, New York: Springer.

Hulting, F. L. and Robinson, J. A. (1994), "The Reliability of a Series System of Repairable Subsystems: A Bayesian Approach," *Naval Research Logistics*, 41, 483–506.

Johnson, V. E., Graves, T. L., Hamada, M. S., and Reese, C. S. (2003), "A Hierarchical Model for Estimating the Reliability of Complex Systems," in *Bayesian Statistics 7*, eds. Bernardo, J., Bayarri, M., Berger, J., David, A., Heckerman, D., Smith, A., and West, M., Oxford University Press, pp. 199–213.

Martz, H. F. and Waller, R. A. (1990), "Bayesian Reliability Analysis of Complex Series/Parallel Systems of Binomial Subsystems and Components," *Technometrics*, 32, 407–416.

Martz, H. F., Waller, R. A., and Fickas, E. T. (1988), "Bayesian Reliability Analysis of Series Systems of Binomial Subsystems and Components," *Technometrics*, 30, 143–154.

Mastran, D. V. (1976), "Incorporating Component and System Test Data into the Same Assessment: A Bayesian Approach," *Operations Research*, 24, 491–499.

Mastran, D. V. and Singpurwalla, N. D. (1978), "A Bayesian Estimation of the Reliability of Coherent Structures," *Operations Research*, 26, 663–672.

Poole, D. and Raftery, A. E. (2000), "Inference for Deterministic Simulation Models: The Bayesian Melding Approach," *Journal of the American Statistical Association*, 95, 1244–1255.

Rausand, M. and Høyland, A. (2004), *System Reliability Theory: Models, Statistical Methods and Applications*, Hoboken, New Jersey: Wiley, 2nd ed.

Reese, C. S., Johnson, V., Wilson, A. G., and Hamada, M. S. (2009), "A Hierarchical Model for Integrating Multiple Sources of Lifetime Information in System Reliability Assessments," Tech. Rep. LA-UR-09-06970, Los Alamos National Laboratory.

Robinson, D. and Dietrich, D. (1988), "A System-Level Reliability-Growth Model," in *Reliability and Maintainability Symposium, 1988. Proceedings., Annual*, pp. 243–247.

Sharma, K. and Bhutani, R. (1994), "Analysis of Posterior Availability Distributions of Series and Parallel Systems," *Microelectronics Reliability*, 34, 379–381.

Tang, J., Tang, K., and Moskowitz, H. (1997), "Exact Bayesian Estimation of System Reliability From Component Test Data," *Naval Research Logistics*, 44, 127–146.

Thompson, W. E. and Chang, E. Y. (1975), "Bayes Confidence Limits for Reliability of Redundant Systems," *Technometrics*, 17, 89–93.

Wilson, A. G., Graves, T. L., Hamada, M. S., and Reese, C. S. (2006), "Advances in Data Combination, Analysis and Collection for System Reliability Assessment," *Statistical Science*, 21, 514–531.

Wilson, A. G. and Huzurbazar, A. V. (2007), "Bayesian Networks for Multilevel System Reliability," *Reliability Engineering & System Safety*, 92, 1413–1420.

Winterbottom, A. (1984), "The Interval Estimation of System Reliability from Component Test Data," *Operations Research*, 32, 628–640.