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Reference-dependent preferences, time inconsistency, and unfunded pensions

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Keywords
Reference-Dependence, Crowding-out, Pensions, Dynamic Efficiency

Disciplines
Behavioral Economics | Macroeconomics | Social Welfare

Reference-Dependent Preferences, Time Inconsistency, and Unfunded Pensions*

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Abstract

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JEL Classification: H55, E6
Keywords: Reference-Dependence, Crowding-out; Pensions, Dynamic Efficiency

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1 Introduction

As an institution, social security has been around for more than a century, and is nearly ubiquitous. Over the years, this institution has been called on to play different roles, chiefly pension (or old-age support) and insurance (e.g., dependent survivor benefits and income redistribution). These roles are, necessarily, somewhat entwined. But if one asks, what makes social security *unique* as an institution, the answer would have to be its pension role. Why? The other role, insurance, strictly speaking, does not need to be played by social security: it may be played, for instance, by other institutions embodying *intra*generational means of income redistribution and social insurance. But social security in its old-age support function is unique, the biggest intermediary of intergenerational transfers, the largest institution offering life-cycle, consumption smoothing opportunities to all.[1,2] This paper focuses *solely* on social security's primal role of providing a public pension system, a role debated to this day among academic economists – see Blake (2006) for a detailed discussion. It informs this debate by showing that a non-paternalistic justification for a public pension system exists in a dynamically efficient economy.

The debate started when researchers asked, why introduce a public pension system into the mix if agents can use private capital markets to achieve desired levels of consumption smoothing? Especially when capital markets offer a higher, safe return? Aaron (1966) and Samuelson (1975) – henceforth, the Aaron-Samuelson result – showed that the introduction of a public pension system has *no* welfare rationale if the economy is initially dynamically *efficient* (i.e., if private capital markets offer a higher return that the public system does). More correctly, a PAYG system in such a setting generates lower

[1] As Barnett et al. (2018) put it "[...] its principal identity is (and has always been) intergenerational, its chief function, pension provision to the elderly. To reiterate, in its identity and function as the chief intermediator of intergenerational transfers, social security is unique. [....] Interestingly, the issue of identifying a clear rationale for the creation of a public pension was center stage in political deliberations surrounding U.S. President Franklin D. Roosevelt's (FDR's) New Deal. FDR's idea was to de-emphasize the social-insurance aspect of the social security program – contributions and benefits are intentionally not means-tested – and, instead, promote its public pension role. In 1935, the U.S. Congress created a national payroll tax to pay for old-age pensions so as to not rely on funding from general tax revenues. FDR and Social Security administrators took pains to distinguish the U.S. pension system from other welfare programs by portraying the payroll tax not as a form of taxation but instead as a contribution workers made to ensure their own security, an entitlement that people earned through hard work and their own contributions."

[2] How does social security deliver consumption smoothing? Most public pension programs have a major unfunded, pay-as-you-go (PAYG) component: the working, young are taxed and the proceeds finance a periodic payment, a transfer (pension), to the existing retired elderly (defined benefit).

long run welfare but offers a welfare gain to the inaugural, retired generation.[3] Abel et al. (1989) and Barbie et al. (2004) have argued that most developed economies are most likely dynamically *efficient*; by implication, a public PAYG pension system is not desirable even if they are popular in such countries.[4]

An approach to the why-do-public-pensions-exist question casts doubt on whether people, in reality, achieve as much life-cycle consumption smoothing as they wished they had. It turns out, not as much: many retirees experience a sharp drop in consumption (more so, in the absence of any public pensions) even though retirement is nearly perfectly anticipated.[5,6] Guided by this sort of evidence, researchers have suggested various "behavioral" modifications – present-bias (or more generally, time inconsistency or lack of self control) – to the textbook model in an attempt to explain the ineffective consumption smoothing. These modifications – see Findley and Caliendo (2008) and Chetty (2015) – employ a) the notion that individuals are comprised of multiple selves, possibly in conflict with one another, and b) the construct of a rift between a self's "true preferences" (experienced utility), that which she uses to determine how much she *should* save, versus her "choice" or "behavioral" preferences (decision utility),

[3]Interestingly the famous Beveridge report proposed a mandatory funded pension scheme, where contributions paid over the work-life were to be set on an actuarial basis to ensure the pension would be above some absolute poverty threshold. This scheme was not introduced since it would offer no pensions to those already old, something which could be achieved by the universal PAYG pension, see e.g. Bozio et al. (2010). Bismarck's initial idea was also to establish a funded scheme, but for the same reason as in the UK, the scheme was set-up as a PAYG scheme, see Scheubel (2013).

[4]In this paper, we are guided solely by the pension role played by a social security program, that too in a certain world. Dynamic inefficiency is, of course, *just one*, among a long list of reasons, justifying social security systems. Other rationales include risk sharing between or within generations (Smith, 1982; Enders and Lapan, 1982; Gordon and Varian, 1988; Sinn, 2004), income redistribution (Diamond, 1977), or fixing market failure in annuity markets (Diamond, 1977; Feldstein, 1990). The role played by unfunded pension systems in ameliorating idiosyncratic risks (such as, those involving mortality or labor income) in worlds with incomplete financial markets has been highlighted and surveyed in Krueger and Kubler (2006). Fuster, Imrohoroglu, and Imrohoroglu (2007) introduce bi-directional altruism along with mortality and earnings risks in a framework similar to one adopted by Conesa and Krueger (1999). Cooley and Soares (1999) explore a political-economy justification for PAYG pensions – see Galasso and Profeta (2004) for a survey.

[5]It has been documented that, among Americans 40-45 years of age, the median retirement account balance is just $14,500 — less than 4% of what the median-income worker will require in savings to meet his retirement needs. (Ghilarducci and James, 2018). Gillers et al (2018) cite a *Wall Street Journal* analysis that finds "more than 40 percent of households headed by people aged 55 through 70 lack sufficient resources to maintain their living standard in retirement."

[6]Aguiar and Hurst (2013) challenge the "retirement-consumption" puzzle and shows that the decline in consumption at retirement belies important heterogeneity across different types of consumption goods – a large part of the consumption decline at retirement is driven by declines in food and work-related expenses suggesting agents substitute away from market expenditures toward household production as the opportunity cost of time declines post retirement. See Olafsson and Pagel (2018) and Scott et al. (2020) for an updated take on this puzzle. Our analysis is not about this puzzle.

that which determines how much she *actually* saves. Intuitively, the idea here is that an agent may choose "overconsumption" in the current and postpone saving for retirement even though her "true" current self would not. This line of logic suggests a prima facie case for *paternalistic* government intervention in the form of a public pension.[7] What is not obvious, though, is whether paternalistic pension mandates actually prevent the consumption drop at retirement because present-biased individuals can *offset* the inherent forced saving by reducing, even eliminating, their own saving in response, leaving retirement wealth unchanged, possibly lower – see Feldstein (1985), Laibson et al. (1998), Feldstein and Liebman (2002), Imrohoroglu et al. (2003), Kaplow (2008), and Andersen and Bhattacharya (2011). These behavioral contrivances, at the very least, do not suggest a *non*-paternalistic role for government intervention.

Another approach to the why-do-public-pensions-exist question admits paternalism on the part of the government but prevents the aforediscussed offset by assuming agents face a borrowing constraint. In that case, present-biased individuals can, at most, eliminate any voluntary retirement saving but may not borrow against future pension payouts. In such a setting, as Andersen and Bhattacharya (2011) show, a paternalistic welfare rationale for public PAYG pensions in dynamically efficient economies exists if all private retirement saving by households is eliminated.[8]

The current paper is a generalization of the Aaron-Samuelson result and represents the latest argument in the aforementioned debate. It unifies many decades of work in this area under one umbrella by showing there may exist a *non-paternalistic* welfare rationale for public PAYG pensions in dynamically efficient economies and *not all retirement saving has to be socialized.* To that end, it studies another behavioral modification to textbook preferences – the reference-dependent or gain-loss utility setup of Kőszegi and Rabin (2006, 2007, 2009; KR, hereafter).[9] Here, in addition to consumption utility,

[7]Why saving through an unfunded public pension scheme and not a fully-funded one? Because the latter does not address the immediate problem faced by the inaugural, retired generation. Interestingly the famous Beveridge report proposed a mandatory funded pension scheme, where contributions paid over the work-life were to be set on an actuarial basis to ensure the pension would be above some absolute poverty threshold. This scheme was not introduced since it would offer no pensions to those already old, something which could be achieved by the universal PAYG pension, see e.g. Bozio et al. (2010). Bismarck's initial idea was also to establish a funded scheme, but for the same reason as in the UK, the scheme was set-up as a PAYG scheme, see Scheubel (2013).

[8]The analysis does not *literally* imply that all assets should be owned by pension funds. The paper focuses solely on pension/life-cycle saving for analytical clarity, and does not explicitly include other private motives such as buffer-stock or liquidity motives.

[9]Pagel (2017) incorporates the KR framework into a large-scale macro model in an effort to offer a uni-

the agent cares about "gain-loss utility" – whether she achieves a gain or loss in actual consumption relative to a benchmark, a reference level she had erstwhile set for herself. KR assume consumption gains please less than equivalent losses hurt. For our purposes, the selling point of the KR setup is its ability to endogenously generate "overconsumption" in pre-retirement years (a prelude to the consumption-drop that would ensue).[10] Borrowing jargon from the literature on time-inconsistent references and from Kramer (2016), we label agents as being sophisticated (naive) if they are aware their future self cares (does not care) about gain-loss utility.[11]

The Aaron-Samuelson result is really an impossibility result. Our flagship result, by contrast, shows that in a dynamically efficient, small-open KR economy populated by naive agents, there is a welfare case for PAYG pensions and private retirement saving is not fully crowded out.[12] Ours is a possibility result, a further generalization of the Aaron-Samuelson result to reference-dependent preferences. The bottom-line intuition is that agents face a tension: on the one hand, the PAYG scheme hurts because it is return dominated by competing, private saving instruments; on the other hand, the pension helps to bring down the planned reference levels – the same consumption yields a bigger gain than before, once the reference point is lower. In effect, the pension restricts the resources available to the naive young agent thereby decreasing the reference consumptions set by her and, in turn, improving her life-time utility. This offers a novel understanding of the role played by pensions beyond their well-understood part in preventing undersaving for retirement.

Another value-added to the pension literature is the following. As discussed before,

fied explanation for several empirical observations about lifecycle consumption: consumption responds to income shocks with a lag, and the hump-shaped consumption profile observed in micro data.

[10]Kőszegi and Rabin (2009) formulate a expectation-based, reference-dependent model and show that if the weight on prospective gain-loss is smaller than contemporaneous gain-loss, the agent might overconsume early relative to the optimal committed plan, increases immediate consumption following surprise wealth increases, and delay decreasing consumption following surprise losses. In contrast, by introducing borrowing constraints into the KR model, Kramer (2016) finds that reference dependence can increase optimal savings for people without access to credit because the expected future reference consumption will increase in anticipation of a liquidity constraint and liquidity constrained agents might want to concentrate loss in early periods to avoid future losses. Note though Kramer (2016) assumes that the weight on prospective gain-loss (γ) or loss aversion (λ) is large enough ($\gamma\lambda > 1$) so that overconsumption relative to pre-set reference points will not occur in his model.

[11]Also, this classification resonates with the literature on projection bias in predicting future utility (Loewenstein, O'Donoghue, Rabin, 2003; DellaVigna, 2009).

[12]If, instead, agents were fully sophisticated and correctly anticipated the gain-loss concerns of their future selves, then a welfare case for PAYG pensions exists only when all agents are borrowing constrained, in which case all saving is necessarily socialized.

behavioral biases generate differences between welfare from a policymaker's perspective, which depends on an agent's true utility (his actual well-being), and the agent's decision utility (the objective the agent maximizes when making choices). A paternalistic policymaker uses the agent's true utility as the yardstick and forces in a public PAYG pension scheme. While the construct of true and choice preferences is no doubt of great use, it leaves open the knotty issue of why governments would be paternalistic in the first place, and why they would use the agent's true and not her choice utility as the right welfare measure. (There is the additional complexity about whether the electorate votes with their true or their choice utility.) The current analysis stays away from this true-choice construct entirely so as to generate a *non*-paternalistic rationale for a public PAYG scheme.

What is critical to our result is the KR setup and two features therein. First, our result obtains only when the current self is fully naive, unaware that her future selves' preferences have gain-loss concerns.[13] Such an agent believes (naively) her future selves will honor the consumption plan she lays out for them. It is only in this case can the government help her out.[14] Second, the general formulation of the gain-loss value function has, in built, a *diminishing sensitivity* to the magnitude of gains and losses.[15] The most popular value function used in the literature (also in KR) is a two-part linear form, one that eschews diminishing sensitivity. We show diminishing sensitivity is central to our result.

The rest of the paper is organized as follows. Section 2 formulates our basic model within the KR framework while Section 3 derives the optimal saving decision and shows that agents generally undersave (overconsume). Section 4 solves for the optimal pension benefits. Section 5 concludes. Proofs of all results and other supplementary material is

[13] KR define a Preferred Personal Equilibrium (PPE) where each agent makes a reference consumption plan that is *consistent* with future action (O'Donoghue and Sprenger, 2018). The PPE imposes a strong restriction on the current decision maker's belief; he/she must correctly anticipate the overconsumption tendencies of her future selves and make a plan that will indeed be followed by these selves. In our language (and that of Kramer, 2016), KR, at least in the main body of their paper, only allow for fully sophisticated agents.

[14] A recent paper which studies the intergenerational redistribution effects of various social security schemes under the KR framework is Park (2018), where the naivete of the young agent is also implicitly assumed so that overconsumption occurs. Mostly, Park (2018) assumes a public PAYG pension system exists without questioning its raison d'être. Under logarithmic utility, he finds, just like we do, that a public PAYG pension is welfare dominated by partially or fully-funded pensions if and only if the economy is dynamically efficient. For more general consumption utility forms, however, we find that a PAYG pension could dominate the fully funded even if the economy is dynamically efficient.

[15] Diminishing sensitivity means a weakening reaction, for example, to marginal losses as losses get larger.

in the appendix.

2 The model

2.1 Primitives

Consider a three-period, stationary overlapping-generations economy wherein a unit mass of identical agents live through three phases: young, middle-aged and old; these phases are called "selves". Think of the young as a planning agent. Only in the middle period, does the agent have income, w. The environment is fully deterministic and in a steady state. Let c_m (c_o) denote the actual consumption of her middle-aged (old) self.

Following Kőszegi and Rabin (2006, 2007, 2009) and Kramer (2016), the young self makes consumption plans – reference points – for her middle-aged and old selves; let (c_m^r, c_o^r) denote any arbitrary reference point pair. When actual consumption by a self exceeds what had been planned for that self, the agent experiences a gain in utility. Similarly, the agents experiences a loss in utility if the opposite happens. Additionally, and most importantly, losses hurt more than gains please. Let felicity from consumption be given by the utility function, $u(.)$, satisfying all standard assumptions including twice differentiability, strictly increasing and strict concavity. In places below, we use the CES form, $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, $\sigma > 0$.

2.1.1 Gain-loss utility

Conditional on having formed reference points (c_m^r, c_o^r) when young, life-time utility for the middle-aged self is given by

$$\text{(1)} \qquad \Omega_m|_{c_m^r, c_o^r} = u(c_m) + \Phi\left(u(c_m) - u(c_m^r)\right) + \beta\left[\gamma\Phi\left(u(c_o) - u(c_o^r)\right) + u(c_o)\right];\ \beta \in (0,1]$$

where β is the subjective discount factor. Ω_m comprises of a standard part, utility gained from the very act of consumption, $u(c_m) + \beta u(c_o)$, and a gain-loss part captured by the function $\Phi(\cdot)$ defined over the difference in utility between actual and planned consumption, the utility gained or lost when one compares one's actual consumption with what had been planned.[16] Contemporaneous gain-loss is captured by $\Phi\left(u(c_m) - u(c_m^r)\right)$;

[16] In general, $\Phi(x)$ is assumed to have the following properties:

1. $\Phi(x)$ is continuous and strictly increasing for all x, twice differentiable for $x \neq 0$, and $\Phi(0) = 0$.

the other part, $\Phi\left(u\left(c_o\right)-u\left(c_o^r\right)\right)$, is prospective and is, hence, discounted by $\gamma \in (0,1)$.

Specifically, we work with gain-loss utility of the following form,

$$\text{(2)} \quad \Phi\left(x\right)=\begin{cases} G\left(x\right), \text{ if } x \geq 0, \\ -\lambda G\left(-x\right), \text{ if } x<0. \end{cases}$$

where $G_x\left(x\right) > 0$, $G_{xx}\left(x\right) < 0$, $G\left(0\right) = 0$, $G_{x+}\left(0\right) = G_{x-}\left(0\right) \equiv G_x\left(0\right)$. The parameter $\lambda > 1$ measures the degree of loss aversion. Then, for $x \geq 0$ we have $\Phi_x\left(x\right) = G_x\left(x\right) > 0$, $\Phi_{xx}\left(x\right) = G_{xx}\left(x\right) < 0$; for $x < 0$ we have $\Phi_x\left(x\right) = \lambda G_x\left(-x\right) > 0$, $\Phi_{xx}(x) = -\lambda G_{xx}(-x) > 0$. In words, this means when $x > 0$ (actual consumption exceeds what had been planned), an agent experiences a gain in utility, $G\left(x\right)$. Similarly, they experience a loss in utility $\left(-\lambda G\left(-x\right)\right)$ if the opposite happens. Additionally, since $\lambda > 1$, losses hurt more than gains please. Also, notice that agents are assumed to have a *diminishing sensitivity* to the magnitude of gains and losses: $\Phi_{xx}\left(x\right) < 0$ for $x > 0$ and $\Phi_{xx}(x) > 0$ for $x < 0$. For gains ($x > 0$), this property is analogous to the assumption of diminishing marginal utility. For losses ($x < 0$), to have diminishing sensitivity to marginal losses as losses get larger is equivalent to $\Phi\left(x\right)$ being convex in this domain.

In one special case, we will study $G\left(x\right) = \eta x$ $\left(x \geq 0\right)$ and thus the gain-loss function, $\Phi\left(x\right)$, is the two-part linear form, popular in the literature (O'Donoghue and Sprenger, 2018). Then,

$$\text{(3)} \quad \Phi\left(x\right)=\begin{cases} \eta x, \text{ if } x \geq 0, \\ -\lambda \eta\left(-x\right), \text{ if } x<0. \end{cases}$$

Note, the two-part linear form violates diminishing sensitivity. Later, we show why our flagship result requires the diminishing sensitivity property. (See Proposition 1).

Another special functional form for G of interest is the constant elasticity form, i.e., $-\frac{xG_{xx}(x)}{G_x(x)}$ is a constant. Consider, for example, an S-shaped gain-loss taking the following

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2. If $y > x \geq 0$, then $\Phi\left(y\right)+\Phi\left(-y\right) < \Phi\left(x\right)+\Phi\left(-x\right)$. (Loss Aversion For Large Stakes)

3. $\Phi''\left(x\right) \leq 0$ for $x > 0$ and $\Phi''\left(x\right) \geq 0$ for $x < 0$. Sometimes for simplicity we assume that $\Phi''\left(x\right) = 0$ for all $x \neq 0$, denoted as 3'. (Diminishing Sensitivity: Risk averse above reference point and risk loving below reference point)

4. $\frac{\Phi'_-(0)}{\Phi'_+(0)} \equiv \lambda > 1$, where $\Phi'_+\left(0\right) \equiv \lim_{x\to 0}\Phi'\left(|x|\right)$, $\Phi'_-\left(0\right) \equiv \lim_{x\to 0}\Phi'\left(-|x|\right)$. (Loss Aversion For Small Stakes)

form with $\alpha \in (0,1)$:

$$\Phi(x) = \begin{cases} Ax^{\alpha}, \text{ for } x \geq 0, \\ -\lambda A(-x)^{\alpha}, \text{ for } x < 0. \end{cases} \tag{4}$$

where $-xG_{xx}/G_x = (1-\alpha)$. This is used in Lemma 3 and in the numerics.

2.1.2 Budgets and pensions

The public sector offers a PAYG pension, b, to the old and finances it by a tax (τ) levied on the middle-aged. Let s denote voluntary middle-age saving. An agent's budget constraints when middle-aged and old are given by

$$c_m = (1-\tau)w - s, \text{ and} \tag{5}$$

$$c_o = Rs + b, \tag{6}$$

where $R > 1$ (dynamic efficiency) is the gross rate of return on saving, exogenously specified and time-invariant. These budget constraints may be combined into a lifetime budget constraint,

$$c_m + \frac{c_o}{R} = (1-\tau)w + \frac{b}{R} \equiv Y. \tag{7}$$

The budget constraint for the PAYG scheme is $b = \tau w$ (since there is no population growth). The government is benevolent and selects a b that maximizes lifetime utility of the middle-aged agent taking the decision rules of the agent as given.

For future use, a **borrowing constraint** is defined as $s \geq 0$. The borrowing constraint may or may not be binding. If b (think of this as forced saving) is sufficiently high, then it may be that voluntary saving, s, is driven to zero: the borrowing constraint binds. For even higher b, it is possible that $s < 0$ obtains: the agent in this case is borrowing against his future pension. When the borrowing constraint is imposed, this last possibility cannot arise.[17] In that case, voluntary saving is equal to zero, and any post-retirement consumption is financed by the public sector.[18] Henceforth, *assume the borrowing con-*

[17]Many countries with PAYG systems, such as the United Kingdom, Denmark, and the United States explictly disallow their citizens to borrow against their future pension. Practically, this makes sense since a PAYG pension payout is "legally not transferable; no lender can have any say in its ownership, the borrower's retirement date, and so on." (Andersen and Bhattacharya, 2011)

[18]We urge the reader not to take the "voluntary saving is equal to zero, and any saving for retirement is

straint is not imposed.

For future use, define the **double positive** result as one where in equilibrium, the optimal PAYG public pension is strictly positive and the associated private retirement saving, s, is also strictly positive. Recall, a borrowing constraint requires private saving to be weakly positive, $s \geq 0$. So, if the borrowing constraint binds, then by definition, a double positive result cannot obtain. For completeness sake, in Appendix A, we state and prove the Aaron-Samuelson result, where the double positive result does not obtain. It also does not obtain with myopic agents as shown in Proposition 2 in Andersen and Bhattacharya (2011).

2.2 Timing and equilibrium

The timeline of decision making is as follows. The young self makes consumption plans for her future selves, c_m^r and c_o^r respectively (and hence, by implication, plans for her middle aged self to save, s^r). These plans are chosen to maximize (1) subject to her own beliefs about her middle-aged self's decision problem. The middle-aged self chooses c_m and saving, s, so as to maximize (1) given c_m^r and c_o^r. The old self has no choice but to consume her gross interest income, Rs, where s is chosen by her past, middle-aged self, and a possible pension, b. If $c_m \neq c_m^r$, it will turn out that $c_o \neq c_o^r$. But c_o will exactly match the consumption plan made for the old self by her previous, middle-aged self.

It is time to define an equilibrium in this model. First, the young self chooses (c_m^r, c_o^r) based on her beliefs about the preferences of her future selves. The middle-aged self will choose c_m taking as given the young self's plans, (c_m^r, c_o^r). Thereafter, she updates her consumption plan from (c_m^r, c_o^r) to $(c_m, R(w - c_m))$ according to rational expectations. A plan is called an optimal consistent plan (OCP) if it satisfies the above description and the consumption plan made by the young self is implemented by the middle-aged self if the latter's preferences are the same as what the young imagined they would be.[19]

"socialized"" *too literally*. Obviously, this is an artifact of the simple nature of the model. Recall, our focus is solely on saving-for-retirement. If other saving motives (precautionary, demand for liquidity) are added *or* there is private and public physical capital, the statement will no longer be true.

[19] In the online appendix of Kőszegi and Rabin (2009), they point out that "since it is unlikely that a person would hold correct beliefs about all future contingencies from the moment of birth, and adjusting beliefs carries utility in our model, the question arises whether and in what situations she would actually arrive at PPE beliefs. In this appendix, we modify the model in the text to allow for unrestricted initial beliefs when the decisionmaker forms the first focused plans, and study the implications of rationality given our assumptions about preferences with minimal ancillary assumptions." This paper proceeds in a similar way by assuming the young agent is naive yet keeping the rationality part intact for the other selves.

A quick road map of where we are headed is in order. First, we compute the actual consumption choices of the middle-aged self where they take as given the reference points "sent up" from their past self. These issues are covered in Section 3.1. Further below, in Section 3.2, we ask, how do the young decide on which reference points to send up? This, as will turn out, will depend crucially on their beliefs about the preferences of their middle-aged self. Also, we would like to know if and when the middle-aged overconsumes relative to these plans. Next section, we will study if PAYG pensions help or hurt in this environment.

3 Consumption plans

3.1 Consumption plans of the middle-aged self

Using the specification of the gain-loss utility (2), life-time utility can be written as

(8)

$$\Omega_m|_{c_m^r, c_o^r} = \begin{cases} u(c_m) + G[u(c_m) - u(c_m^r)] - \beta\gamma\lambda G[u(c_o^r) - u(c_o)] + \beta u(c_o), & \text{for } c_m > c_m^r \\ u(c_m) - \lambda G[u(c_m^r) - u(c_m)] + \beta\gamma G[u(c_o) - u(c_o^r)] + \beta u(c_o), & \text{for } c_m \leq c_m^r \end{cases}.$$

The middle-aged takes (c_m^r, c_o^r) as given and chooses consumption c_m and saving s to maximize her life-time utility (1) subject to the budget constraints (5) and (6). Since our primary focus is on the possibility of "overconsumption" or undersaving, we look at $c_m > c_m^r$ ($\Leftrightarrow s < s^r$). Define

$$(9) \quad g_m \equiv u(c_m) - u(c_m^r),$$

$$(10) \quad l_o \equiv -[u(c_o) - u(c_o^r)].$$

The first order condition for saving reads[20]

$$(11) \quad \left.\frac{\partial \Omega_m}{\partial s}\right|_{c_m > c_m^r} = -[1 + G_x(g_m)]\, u_c(c_m^*) + \beta[1 + \gamma\lambda G_x(l_o)]\, R u_c(c_o^*) = 0,$$

where $G(g_m)$ is the utility gain and enters with a positive sign, while $G(l_o)$ enters with a negative sign signifying a utility loss since $c_m > c_m^r$ implies $c_o < c_o^r$. The first term

[20]Since we have defined $G(x)$ before, we use $G_x(\cdot)$ to denote its derivative. For ease of representation, let $G_x(g_m) \equiv G_x(x)|_{x=g_m}$, and $G_x(l_o) \equiv G_x(x)|_{x=l_o}$.

on the r.h.s. of eq. (11) is the marginal cost of saving more in terms of both forgone consumption as middle-aged and the lost gain, $G_x(g_m)$. The second term captures the marginal gain of having more consumption as old including the prospective gain. From (11), it follows

$$(12) \quad u_c(c_m^*) \equiv \kappa\beta R u_c(c_o^*),$$

where

$$(13) \quad \kappa(g_m, l_o) \equiv \frac{1+\gamma\lambda G_x(l_o)}{1+G_x(g_m)}.$$

An interior solution (c_m^*, c_o^*) is ensured by standard assumptions on $u(\cdot)$. For future use, the second order condition of the middle-aged self's problem is

$$(14) \quad \frac{d^2\Omega_m}{ds^2} = [1+G_x(g_m)]\, u_{cc}(c_m) + [1+\gamma\lambda G_x(l_o)]\, R^2 u_{cc}(c_o)$$
$$+ G_{xx}(g_m)[u_c(c_m)]^2 - \gamma\lambda G_{xx}(l_o) R^2 [u_c(c_o)]^2 < 0.$$

It bears emphasis here that κ is an endogenous variable, best thought of as a wedge arising from the gain-loss effect. Since κ and β enter side by side in (12), they play similar roles: higher β (and higher κ) is associated with higher saving (c_m^* decreases). A higher κ (since it does not enter the budget) releases only a substitution effect, while a change in R releases both a substitution and an income effect.

Having stated the middle-aged's choice problem, we need to characterize the optimal determination of (c_m^r, c_o^r) before proceeding further.

3.2 Consumption plans of the young self

The decision problem for the young self is given by

$$(15) \quad \max_{c_m^r, c_o^r} \Omega_y = u(c_m^*) + \Phi(u(c_m^*) - u(c_m^r)) + \beta[\gamma\Phi(u(c_o^*) - u(c_o^r)) + u(c_o^*)]$$

The question is, does the young self realize that actual choices made by her future selves, c_m^* and c_o^*, depend on the reference levels c_m^r and c_o^r she will set? If the young self is fully **naive,** she believes that $c_m^* = c_m^r$ and $c_o^* = c_o^r$ will hold. In effect, she believes her future selves will stick to her plan and choose $c_m^* = c_m^r$ and $c_o^* = c_o^r$ even when no commitment technology is used. She, in other words, (naively) believes she has **full commitment**. Another way to say this is, *a naive agent believes there is no gain-loss component to future utility*.

If, at the other extreme, she is fully **sophisticated**, she believes that the actual choices made by her future selves, c_m and c_o, will be given by $c_m^* = \Theta\left(c_m^r\right)$ and $c_o^* = \Theta\left(c_o^r\right)$ – she knows what $\Theta\left(\cdot\right)$ is and takes it into account when choosing c_m^r and c_o^r. Sophisticated agents are forward looking (rational expectations) and only form time-consistent plans. This is what Koszegi and Rabin (2009) define as a PPE-equilibrium.[21] *A sophisticated agent believes there will be a gain-loss unless he/she does something to prevent it.* Recall, there is never any gain-loss utility for the old.

Note that the optimal full-commitment (naive) consumption plan $\left(c_m^{r*}, c_o^{r*}\right)$ satisfies

$$(16) \quad u_c\left(c_m^{r*}\right) \equiv \beta R u_c\left(c_o^{r*}\right),$$

and the corresponding saving is $s^{r*} = w - b - c_m^{r*}$.

3.2.1 Consumption plans by the middle-aged when the young are naive: Overconsumption

How does welfare of the middle-aged vary with the levels of the reference points? Notice, that the optimal committed plans (c_m^{r*}, c_o^{r*}) must satisfy $\frac{\partial c_m^r}{\partial c_o^r} = -\frac{1}{R}$, cf. the budget constraint (7). Employing the optimal committed plan (16) and the first-order condition of the middle-aged (12), we have that

$$\begin{aligned}
&\frac{\partial \Omega_m}{\partial c_m^r} = \left[\left(1 + \gamma\lambda G_x\left(l_o\right)\right) u_c^*\left(c_o\right)\beta R - u_c\left(c_m^*\right)\left(1 + G_x\left(g_m\right)\right)\right]\frac{\partial s}{\partial c_m^r} \\
&+\beta R\gamma\lambda G_x\left(l_o\right) u_c\left(c_o^{r*}\right) - G_x\left(g_m\right) u_c\left(c_m^{r*}\right) \\
&= \left[\gamma\lambda G_x\left(l_o\right) - G_x\left(g_m\right)\right] u_c\left(c_m^{r*}\right).
\end{aligned}$$

[21]Our analysis also adds a bit more to a crucial question in prospect theory: how to discipline the determination of the reference points. KR define a Preferred Personal Equilibrium (PPE) where each agent makes a reference consumption plan that is *consistent* with future action (O'Donoghue and Sprenger, 2018). The PPE imposes a strong restriction on the current decision maker's belief; he/she must correctly anticipate the overconsumption tendencies of her future selves and make a plan that will indeed be followed by these selves. In our language (and that of Kramer, 2016), KR only allow for fully sophisticated agents. We broaden the discussion by allowing the current decision maker to be naive in the sense that she believes her future preferences are of the textbook kind with no gain-loss concerns. Such an agent believes (naively) she can commit to her consumption plan.

Since (c_m^{r*}, c_o^{r*}) maximizes $u(c_m) + u(c_o)$ it follows that

$$u(c_m^{r*}) + u(c_o^{r*}) > u(c_m) + u(c_o)$$

for any consumption pair $(c_m, c_o) \neq (c_m^{r*}, c_o^{r*})$. Therefore,

$$l_o = u(c_o^{r*}) - u(c_o) > u(c_m) - u(c_m^{r*}) = g_m$$

and since $G_{xx}(\cdot) < 0$, it follows

$$G_x(0) > G_x(g_m) > G_x(l_o).$$

Using $\kappa(g_m, l_o) \equiv \frac{1+\gamma\lambda G_x(l_o)}{1+G_x(g_m)}$, it follows that for $\gamma\lambda < 1$, $\kappa < 1$. It follows that

$$\text{(17)} \quad \text{sign}\left(\frac{\partial \Omega_m}{\partial c_m^r}\right) = \text{sign}\ [\gamma\lambda G_x(l_o) - G_x(g_m)] < 0 \text{ for } \gamma\lambda < 1$$

This means for $\gamma\lambda < 1$, higher reference consumption reduces life-time utility of the middle-aged if the plan set by the young is the optimal committed plan.

We wish to know if and when the middle-aged self will actually **overconsume** relative to the optimal committed consumption plan c_m^{r*}. Evaluating $\frac{\partial \Omega_m}{\partial s}\Big|_{s=s^{r*}}$ and using (12), we get

$$\left.\frac{\partial \Omega_m}{\partial s}\right|_{s=s^{r*}} = (\gamma\lambda - 1)\,\beta R u_c(c_o^r)\, G_x(0) \leq 0 \text{ iff } \gamma\lambda < 1.$$

Notice β plays no role in the decision to overconsume; henceforth, in many places, we set $\beta = 1$. What is interesting here is that overconsumption/undersaving, in the sense of consuming more (saving less) than previously planned, is possible in the KR gain-loss environment. Whether this is the case depends singularly on the gain-loss parameters (λ) and discounting (γ). Interestingly, it also responds to the pension scheme and the market rate of interest.

Lemma 1 *Given the consumption plan (c_m^{r*}, c_o^{r*}) set by the naive young self,*

$$\begin{array}{l} c_m^n = c_m^* > c_m^{r*} \text{ for } \gamma\lambda < 1 \\ c_m^n = c_m^* = c_m^{r*} \text{ for } \gamma\lambda \geq 1 \end{array},$$

This means, the middle-aged self consumes more than what her naive young self planned she would if $\gamma\lambda < 1$.

3.2.2 Consumption plans by the middle-aged when the young are sophisticated

Recall, any middle-aged agent uses (12)-(13) to determine her consumption choices, (c_m^*, c_o^*) . If the young are sophisticated, their optimal choice of reference plans, (c_m^{sr*}, c_o^{sr*}) , would satisfy (12) where[22]

$$\begin{aligned} c_m^* &= c_m^{sr*} \Leftrightarrow g_m = 0, \\ c_o^* &= c_o^{sr*} \Leftrightarrow l_o = 0, \end{aligned}$$

The following lemma is a restatement of the overconsumption result in KR.

Lemma 2 *[Köszegi and Rabin (2009)] If* $\gamma\lambda \geq 1$*, young sophisticated agents choose the same full commitment consumption plan* (c_m^{r*}, c_o^{r*}) *that is chosen by the naive young self. If* $\gamma\lambda < 1$*, sophisticated agents choose reference points* $(c_m^{sr}, c_o^{sr}) = (c_m^*, c_o^*)$ *where*

$$c_m^* > c_m^{r*}$$

holds, and (c_m^*, c_o^*) *satisfies equation (12) with*

$$\kappa(g_m, l_o) = \frac{1 + \gamma\lambda G_x(0)}{1 + G_x(0)} \equiv \kappa^s < 1.$$

This means if $\gamma\lambda \geq 1$, there is no incentive to overconsume; the naive and the sophisticated set the same plans. When $\gamma\lambda < 1$, sophisticated agents choose reference points (c_m^{sr}, c_o^{sr}) that do not coincide with the full commitment plan of the naive, (c_m^{r*}, c_o^{r*}) ; indeed, the actual consumption of the middle aged under the reference points (c_m^{sr}, c_o^{sr}) exceeds what the full commitment plan (c_m^{r*}, c_o^{r*}) deemed to be right.

In comparison, if the young is fully naive, she will *believe*

$$\begin{aligned} c_m^* &= c_m^{nr*} \\ c_o^* &= c_o^{nr*} \end{aligned}$$

[22] We use superscript "s" to denote "sophisticated", and let "sr" denote the reference if the agent is sophisticated. Analogously, we use "n" and "nr" to represent "naive".

and hence, she will believe $g_m = 0$ and $l_o = 0$. It will *turn out* that $c_m^* > c_m^{nr*} \Leftrightarrow g_m > 0$ and $c_o^* < c_o^{nr*} \Leftrightarrow l_o > 0$.

4 Optimal PAYG pensions

Inadequate retirement saving is usually understood to be the main reason for government intervention in the form of pension schemes. We have shown that KR-style gain-loss preferences may generate such undersaving. Now, we turn to questions such as, how does a public pension system affect private (under) saving? Is there a welfare rationale for government intervention via a pension system? Specifically, does there exist a positive optimal pension alongside positive voluntary saving, i.e., does the double positive result obtain?

Below, we study these questions for the fully naive agent. (Separately, in Appendix (C), we show that the *double-positive result does not obtain for fully sophisticated agents*, i.e., $\frac{d\Omega_m}{db} < 0$ for a fully sophisticated agent.[23]).

To set the scene, consider, first, the basic channel through which a PAYG pension may be successful in addressing the undersaving problem. The optimality condition of the naive, middle-aged reads: $u_c(c_m^n) \equiv \kappa\beta R u_c(c_o^n)$. Recall κ (< 1) is an endogenous variable, a wedge arising from the gain-loss effect that impels a naive agent to overconsume. The lower is κ, the more consumption is front-loaded, the more severe is the undersaving. If b could help raise κ, then the overconsumption problem would be somewhat mitigated. It turns out the effect of b on κ is, in general, too messy to be useful; but for the constant-elasticity gain-loss G function (4), a very precise result obtains.

Lemma 3 *For CES utility with $\sigma > 1$, the constant-elasticity gain-loss G function (4) and $\gamma\lambda < 1$, we have*

$$\frac{d\kappa}{db} > 0.$$

Therefore, the equilibrium actual consumption ratio $\frac{c_o^n}{c_m^n}$ is increasing in b, and the ratio of actual to reference consumption $\frac{c_m^n}{c_m^{nr*}}$ as middle-aged is declining in b.

[23] A welfare case for PAYG pensions for fully-sophisticated agents can be found in the presence of a binding borrowing contraint. In this case, the optimal public pension is positive but private retirement saving is zero. This is discussed in an Appendix (D).

In this case, an increase in b raises κ reducing the distortion from gain-loss concerns; the overconsumption problem is less severe, as is the discrepancy between actual and reference consumptions when middle-aged. While this offers some intuition for why a public pension may help the undersaving problem, by itself it does not prove there is a welfare case for such a pension since consumption *levels* are also affected. We now turn to this issue.

The effect of pensions b on life-time utility (using (12)) can be written as

(18)
$$\frac{d\Omega_m}{db} = \underbrace{\left[1 + G_x\left(g_m^n\right)\right] u_c\left(c_m^n\right)\left(\frac{1-R}{R}\right)}_{\text{return effect}} \underbrace{-G_x\left(g_m^n\right) u_c\left(c_m^{nr*}\right)\frac{\partial c_m^{nr*}}{\partial b} - \gamma\lambda G_x\left(l_o^n\right) u_c\left(c_o^{nr*}\right)\frac{\partial c_o^{nr*}}{\partial b}}_{\text{reference consumption effect}}.$$

The first term is the standard return effect underlying the classic Aaron-Samuelson result, and in a dynamically efficient economy, it is negative. Obviously, sans gain-loss effects ($G_x\left(g_m^n\right) = G_x\left(l_o^n\right) = 0$), the reference consumption effect is zero and $\frac{d\Omega_m}{db} \leq 0$: there is no welfare case for a PAYG pension when $R > 1$. The reference-consumption effect (the two last terms in (18)), is new and appears due to the gain-loss feature. It is immediate that a *necessary* condition for a welfare case for pensions ($\frac{d\Omega_m}{db} > 0$ for some b) requires the reference-consumption effect to be positive.

Lemma 4

(19) $$\frac{\partial s^{nr*}(b)}{\partial b} = -\frac{u_{cc}\left(c_m^{nr*}\right) + Ru_{cc}\left(c_o^{nr*}\right)}{u_{cc}\left(c_m^{nr*}\right) + R^2 u_{cc}\left(c_o^{nr*}\right)} \in \left(-1, -\frac{1}{R}\right),$$

$$\begin{aligned}
&(20) \quad \frac{\partial c_m^{nr*}}{\partial b} &=& \quad -1 - s_b^{nr*}(b) = -\frac{R(R-1)\, u_{cc}\left(c_o^{nr*}\right)}{u_{cc}\left(c_o^{nr*}\right) + R^2 u_{cc}\left(c_o^{nr*}\right)} < 0,\\
&(21) \quad \frac{\partial c_o^{nr*}}{\partial b} &=& \quad 1 + Rs_b^{nr*}(b) = -\frac{(R-1)\, u_{cc}\left(c_m^{nr*}\right)}{u_{cc}\left(c_o^{nr*}\right) + R^2 u_{cc}\left(c_o^{nr*}\right)} < 0,
\end{aligned}$$

implying, the introduction of pensions will always decrease the reference consumptions chosen by the naive.

The lower the reference level, the less the "temptation" to overconsume, and therefore the reference-consumption effect in (18) is positive.

4.1 Double-positive result

The necessary condition is, thus, always fulfilled, leaving it possible that there is a welfare case for the pension. But does the reference-consumption effect dominate the return effect? To address this question, note from (18)

(22)
$$\frac{d\Omega_m}{db} = \left(\frac{1-R}{R}\right)\left\{\left[1+G_x\left(g_m^n\right)\right]u_c\left(c_m^n\right) - \epsilon G_x\left(g_m^n\right)u_c\left(c_m^{nr*}\right) - \left[1-\epsilon\right]\gamma\lambda G_x\left(l_o^n\right)u_c\left(c_m^{nr*}\right)\right\},$$

where

(23) $$\epsilon \equiv \frac{R^2 u_{cc}\left(c_o^{nr*}\right)}{u_{cc}\left(c_m^{nr*}\right) + R^2 u_{cc}\left(c_o^{nr*}\right)} \in [0,1].$$

The optimality condition of the government, $\frac{d\Omega_m}{db} = 0$, can be simplified to

(24) $$\frac{u_c\left(c_m^n\right)}{u_c\left(c_m^{nr*}\right)} = \frac{\epsilon G_x(g_m^n) + [1-\epsilon]\gamma\lambda G_x\left(l_o^n\right)}{1+G_x(g_m^n)}.$$

More analytical progress is possible with the CES form $u(c) = c^{1-\sigma}/(1-\sigma)$. In that case, we have $\epsilon = \frac{R^{1-1/\sigma}}{1+R^{1-1/\sigma}}$. Define

(25) $$F(\kappa) \equiv \frac{1+\kappa^{-1}R^{1-1/\sigma}}{1+R^{1-1/\sigma}} - \left(\frac{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}{1+R^{1-1/\sigma}}\right)^{\sigma};$$

then (24) implies an optimal b^* solves

(26) $$F(\kappa) = \frac{1}{1+\gamma\lambda G_x\left(l_o^n\right)}.$$

It is instructive to reformulate (26) and (13) as follows:

$$\begin{aligned}
&(27) \quad \kappa \equiv \frac{1+\gamma\lambda G_x\left(l_o^n\right)}{1+G_x\left(g_m^n\right)} = \frac{1+\gamma\lambda G_x\left(A(\kappa,R,\sigma)C\right)}{1+G_x\left(B(\kappa,R,\sigma)C\right)},\\
&(28) \quad F(\kappa) = \frac{1}{1+\gamma\lambda G_x\left(l_o^n\right)} = \frac{1}{1+\gamma\lambda G_x\left(A(\kappa,R,\sigma)C\right)},
\end{aligned}$$

where

$$
\begin{aligned}
A(\kappa, R, \sigma) &\equiv \frac{1}{\sigma-1}\left[\left(\frac{1}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}\right)^{1-\sigma} - \left(\frac{1}{1+R^{1-1/\sigma}}\right)^{1-\sigma}\right] R^{1-\sigma} \geq 0, \\
B(\kappa, R, \sigma) &\equiv \frac{1}{\sigma-1}\left[\left(\frac{R^{1-1/\sigma}}{1+R^{1-1/\sigma}}\right)^{1-\sigma} - \left(\frac{\kappa^{-1/\sigma}R^{1-1/\sigma}}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}\right)^{1-\sigma}\right] \geq 0.
\end{aligned}
$$

For future use, define (κ^*, C^*) to be a solution to (27)-(28).

Lemma 5 *Suppose $\gamma\lambda < 1$ and the gain-loss function, G, satisfies $G_x(0) > \underline{G}_x(0)$, where $\underline{G}_x(0)$ is defined by*

$$
F\left(\frac{1+\gamma\lambda\underline{G}_x(0)}{1+\underline{G}_x(0)}\right) \equiv \frac{1}{1+\gamma\lambda\underline{G}_x(0)}.
$$

Then, there exists a w-independent solution (κ^*, C^*) to (27)-(28).

The possibility of a $b^* > 0$, if indeed validated, would be interesting for two reasons. First, in contrast to models with myopia, it would not require a distinction to be made between true and choice preferences. Secondly, a welfare case could potentially arise for positive voluntary saving (i.e., unlike Andersen and Bhattacharya (2011) and others, would not require private retirement saving to be driven to the zero corner). We start by laying out an impossibility result.

Proposition 1 *a) If $\sigma \leq 1$, there is no welfare role for pensions.*
b) If the gain-loss utility takes the two-part linear form as in (3), the "double-positive" result does not obtain.

Finally, we are able to state our flagship, double-positive, possibility result.

Proposition 2 ***("Double positive result")*** *Define*

$$
\begin{aligned}
\underline{w} &= (C^*)^{\frac{1}{1-\sigma}}, \\
\overline{w} &\equiv \frac{\left[1+(\kappa^*)^{-1/\sigma}R^{-1/\sigma}\right]R\,(C^*)^{\frac{1}{1-\sigma}}}{1+(\kappa^*)^{-1/\sigma}R^{1-1/\sigma}}.
\end{aligned}
$$

The "double positive" result holds when $w \in [\underline{w}, \overline{w}]$, i.e., in this range of w, positive pensions and positive private retirement saving coexists; for $w < \underline{w}$, saving is positive but *there is no welfare role for pensions; for $w > \overline{w}$, there is a role for positive pensions but private saving is negative*[24].

Recall, the weight on prospective gain-loss utility should be small enough $(\gamma\lambda < 1)$ such that overconsumption occurs (Lemma 1). This is a necessary condition for government intervention. The rest are a set of sufficient assumptions ensuring the double positive result. Let us go over some bits of intuition for these conditions.

First, $\sigma > 1$: Lemma 3 argues this ensures κ is increasing in b, that is, the intervention reduces the overconsumption problem; the ratio of consumption as middle-aged relative to consumption as old declines.

Second, the gain-loss part should be significant ($G_x(0) > \underline{G}_x(0)$) implying that there is a strong incentive to front-load consumption, or put differently, the overconsumption problem should be sufficiently severe. For there to be a welfare effect for the PAYG pension, see (18), we know that the reference consumption effect should dominate the classical return effect known from the Aaron-Samuleson result. This can happen only when the gain-loss part is significant enough.

Finally, and crucial for the double positive result, income has opposite effects on pensions and savings under the optimal policy. The optimal pension is increasing in income since the incentive to frontload consumption is stronger, the higher the income. A positive pension, therefore, requires a sufficiently high income ($\underline{w}$). On the other hand, the pension crowds out saving, and this effect is clearly stronger, the larger the pension. Hence, for income levels $w < \overline{w}$, voluntary saving is positive, and for a high level ($w > \overline{w}$), it is negative. It follows that the double positive result only arises for intermediary level of income ($w \in [\underline{w}, \overline{w}]$). For even higher levels of income, pensions remain positive, but voluntary saving is not – the double positive result no longer obtains. Note, it is always the case that $\overline{w} > \underline{w}$, i.e., there is always a range of income levels producing the double positive result.

4.2 Numerics

Our flagship result, Proposition 2 outlines a set of necessary (and sufficient) conditions required to obtain the double positive result. To ensure that the restriction set is indeed

[24]Introducing a borrowing constraint could also deliver a welfare role for pensions when it is binding. This is studied in Appendix (E).

non-empty, and to illustrate our finding, we showcase a numerical example. To that end, we set the parameters as follows:

$$R = 1.02\hat{}25,\ \gamma = 0.1,\ \lambda = 2,\ w = 10,\ \sigma = 3.$$

This is just one set; numerous other sets generate qualitatively similar results. We assume the G function takes the constant elasticity form as in (4) with $A = 1.42$ and $\alpha = 0.5$. In this case,

$$\begin{aligned} b^n &= 2.30,\ s^n = 1.34,\ s^{nr*} = 2.40, \\ c_m^n &= 6.35,\ c_o^n = 4.51,\ c_m^{nr*} = 5.30,\ c_o^{nr*} = 6.25. \end{aligned}$$

Clearly, the optimal b and optimal private retirement saving are both strictly positive. The effect of b on lifetime utility Ω_m and κ can be seen in Figure 1, where Ω_m is concave and κ is increasing in b. Consequently, compared with the case without pensions, the consumption ratio $\frac{c_o^n}{c_m^n}$ increased by 0.8%.

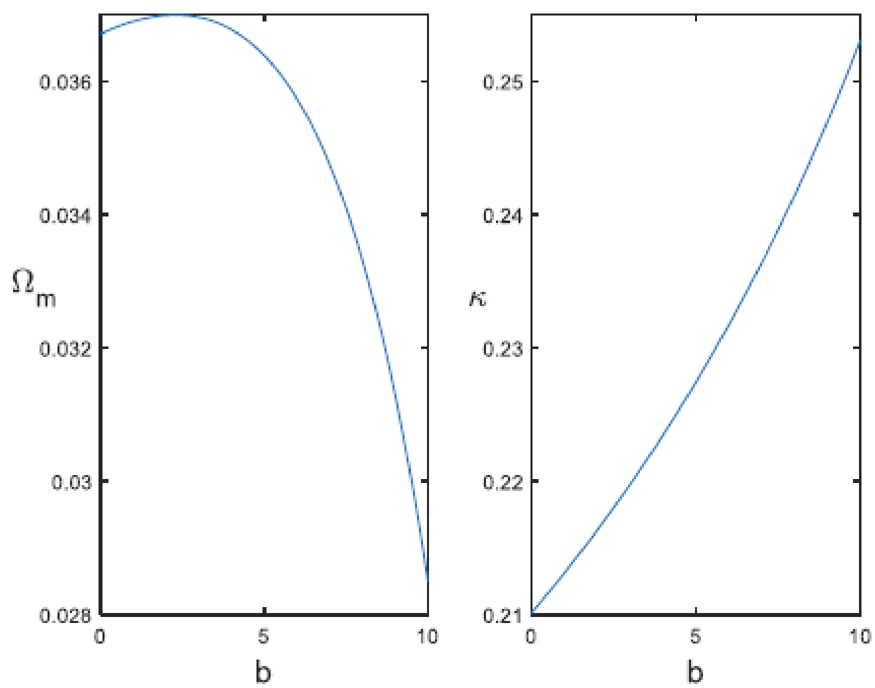

Figure 1: Ω_m **and** κ **against** b

Also, the size of pension depends on w, which can be seen in Figure 2. For low levels of income, no welfare role for pensions can be found and private saving is positive; for intermediate levels of income, the double-positive result obtains; for high levels of income, a role for positive pensions exists but private saving is negative.

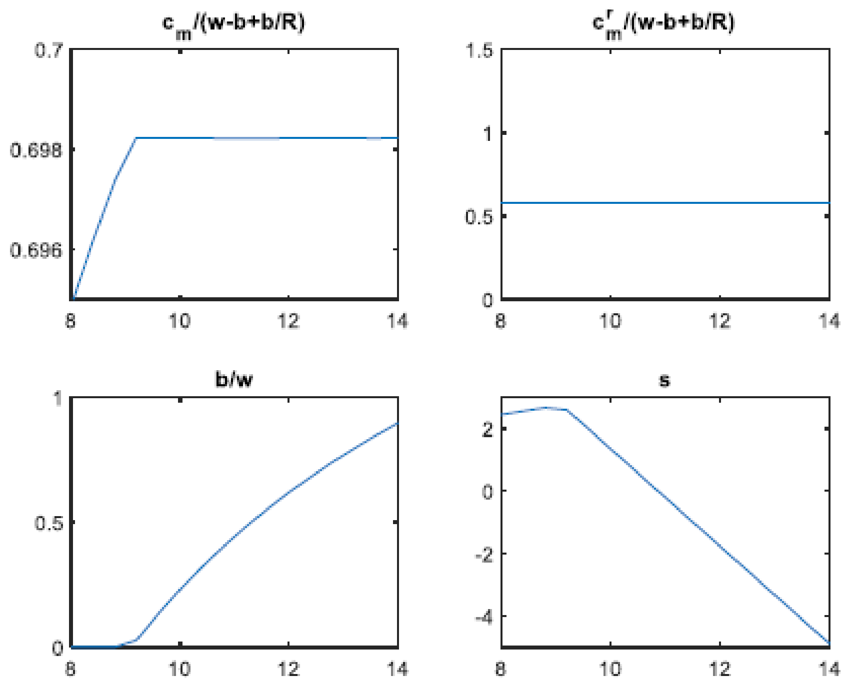

Figure 2: Various endogenous variables as a function of w

Finally, our result does not rely on the constant elasticity form. We can show, numerically, that if we adopt the expected S-shaped gain-loss of the following form

$$\Phi(x) = \begin{cases} A - Ae^{-\alpha x}, \text{ for } x \geq 0, \\ -\lambda(A - Ae^{\alpha x}), \text{ for } x < 0. \end{cases},$$

the double-positive result obtains. As before, the parameters are set as

$$R = 1.02\hat{\ }25,\ \gamma = 0.01,\ \lambda = 2,\ w = 10,\ \sigma = 3,\ A = 13.721,\ \alpha = 0.5,$$

The optimal solutions are

$$
\begin{aligned}
b^n &= 2.59,\ s^n = 0.89,\ s^{nr*} = 2.18, \\
c_m^n &= 6.53,\ c_o^n = 4.04,\ c_m^{nr*} = 5.23,\ c_o^{nr*} = 6.17.
\end{aligned}
$$

The effect of b on lifetime utility Ω_m and κ can be seen from the following Figure 3. Consequently, compared with the case without pensions, the consumption ratio $\frac{c_o^n}{c_m^n}$ increased by $3.3 * 10^{-5}$.

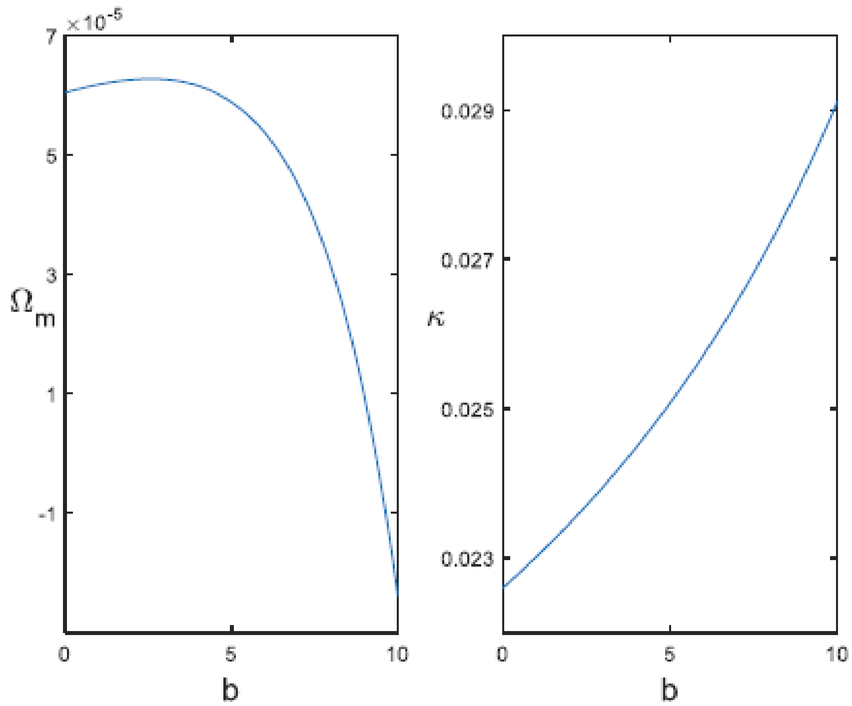

Figure 3: Ω_m and κ against b in general

For different levels of income w we have similar results as before (See Figure 4).

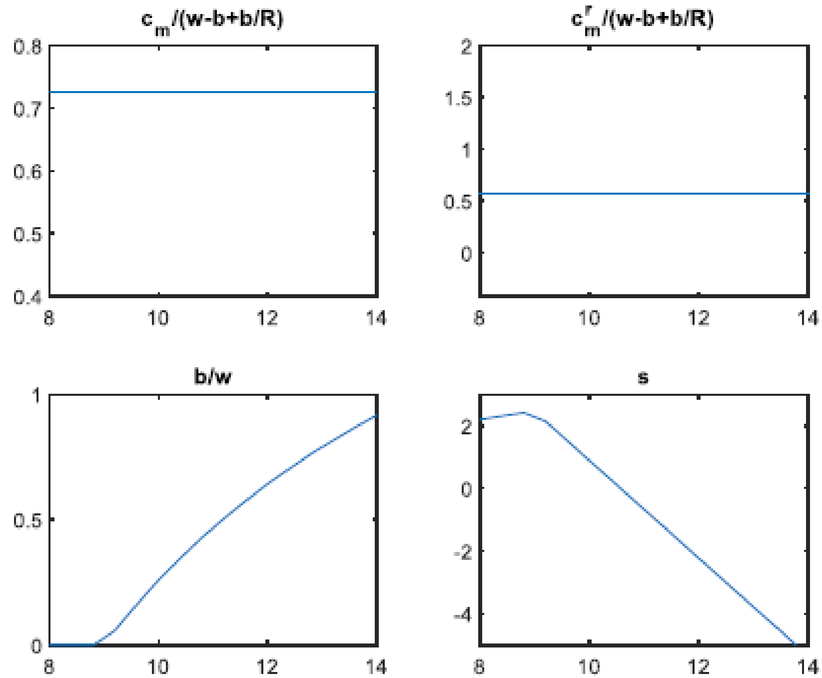

Figure 4: Endogeneous variables against w in general

5 Conclusion

The pension role of social security is paramount and unique to all the roles it has been called on to play. This paper contributes to a long line of research studying the long-run optimality of the pension role of PAYG social security in deterministic, dynamically efficient economies. Aaron (1966) and Samuelson (1975) had shown an impossibility result: the introduction of such a pension system can improve the stationary welfare of all two-period lived agents if and only if the economy is initially dynamically *in*efficient. Parenthetically, there is no welfare justification for introducing a PAYG pension scheme if the economy is initially dynamically efficient.

Numerous attempts, thereafter, have been made to generate a possibility result: after all, PAYG pensions are immensely popular in the real world and in economies that are dynamically efficient. Broadly speaking, most of these attempts rely either on the assumption of time-inconsistent preferences or, more generally, the construct of a chasm between true and choice utility. In such settings, it is indeed possible for a paternalistic government to usher in welfare-improving PAYG pensions but all private retirement saving has to be eliminated (and retirement consumption socialized). This paper represents another step in this agenda. It chases after the "holy grail" so to speak: generating a non-paternalistic rationale for public PAYG pensions in a dynamically efficient economy where private retirement saving is positive. It finds that in the KR world of reference-dependent preferences, if agents are naive (unaware of the impending gain-loss their future self is about to face), then, under some reasonable parametric restrictions, the so-called holy grail is reached.

The analysis has been silent on the issue of transition dynamics to a steady state with the optimal pension, b^*. Starting from laissez faire, how should such a pension system be initialized, knowing that the inaugural generation of retired individuals will receive a pension never having paid into the system? Further along, the sort of crowding-out issues we have discussed will appear as private agents cut private retirement saving in response to the public scheme. Their consumption benchmarks will also change along the transition. While these issues are of tremendous practical importance, we believe they are best left for future work. Some of these transition hurdles in different contexts are studied in Andersen and Bhattacharya (2017) and in the recent work by Bishnu et al. (2020).

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A Aaron-Samuelson result

In the general case, the agent's problem is to maximize $u(c_m, c_o)$ subject to (5)-(6). The first order conditions to the agent's problem reads $u_{c_m}(\cdot) = Ru_{c_o}(\cdot)$ for a solution in the interior, i.e., the optimal $s > 0$. The optimal c_m and c_o are $c_m^*(b, w)$ and $c_o^*(b, w)$ and long-run indirect utility of the agent is given by $u(c_m^*(b, w), c_o^*(b, w))$. Notice

$$\frac{\partial u(c_m^*(b,w), c_o^*(b,w))}{\partial b} = u_{c_m}(\cdot)\left[1 - \frac{\partial s}{\partial b}\right] + u_{c_o}(\cdot)\left[R\frac{\partial s}{\partial b} + 1\right]$$

which using $u_{c_m}(\cdot) = Ru_{c_o}(\cdot)$ yields

$$\frac{\partial u(c_m^*(b,w), c_o^*(b,w))}{\partial b} = \left(\frac{1-R}{R}\right) u_{c_m}(\cdot). \tag{29}$$

Since $u_{c_m}(\cdot) > 0$, it follows that for $R > 1$, $\frac{\partial u(c_m^*(b,w), c_o^*(b,w))}{\partial b} < 0 \Leftrightarrow b^* = 0$ implying an interior b^* cannot co-exist (ruling out negative pensions). In other words, there is no welfare case for an optimal positive PAYG pension when $R > 1$. This is the classic Aaron-Samuelson result. Under dynamic efficiency, and standard preferences, PAYG pensions are return-dominated by the market interest rate, and hence are never optimal in terms of long-run welfare.

In the rest of the paper, we will stay away from introducing the construct of true and choice preferences and borrowing constraints; after all, our goal is to look for a non-paternalistic rationale for public pensions consistent with positive voluntary savings. In this appendix, however, we bring it in only to outline what is known in the literature that allows for disagreement between choice and true utility and the role played by the borrowing constraint. To that end, let the utility be separable and let choice utility of the middle be given by $u(c_m, c_o) = u(c_m) + \beta u(c_o)$ and the true utility by $\Omega(c_m, c_o) = u(c_m) + \beta^* u(c_o)$. In that case, we say the agent is myopic if $\beta < \beta$, and there is undersaving in the sense that savings chosen under choice preferences fall short of the optimal level under true preferences (Andersen and Bhattacharya, 2011). Continue to assume the government is benevolent but, in addition, assume the government is paternalistic and uses $\Omega(c_m, c_o)$ to inform its decisions.

As before, the agent takes b as given and solves $u_c(\cdot) = R\beta u_c(\cdot)$ to compute optimal interior savings, $s^*(b)$ and consumptions, $c_m^*(b, w), c_o^*(b, w)$. The PAYG pension crowds out voluntary savings, specifically in equilibrium

$$\frac{\partial s}{\partial b} = -\frac{u_{cc}(w - s - b) + R\beta u_{cc}(Rs + b)}{u_{cc}(w - s - b) + R^2\beta u_{cc}(Rs + b)} < 0$$

and $-1 < \frac{\partial s}{\partial b} < -\frac{1}{R}$. The optimal pension, b^{*}, is chosen by setting

$$\frac{\partial \Omega \left(c_m^{*}\left(b,w\right), c_o^{*}\left(b,w\right)\right)}{\partial b} = \Omega_{c_m}\left(\cdot\right)\left[-1-\frac{\partial s^{*}\left(b\right)}{\partial b}\right] + \Omega_{c_o}\left(\cdot\right)\left[R\frac{\partial s^{*}\left(b\right)}{\partial b}+1\right] \leq 0 \text{ for } R > 1.$$

Introducing a behavioral explanation for undersaving does not as such overturn the Aaron-Samuelson result, there is still no welfare rationale for the PAYG pension under dynamic efficiency. The reason is that the pension is return dominated and crowds out voluntary savings. Due to the crowding out it follows, that there is a level of pensions so high that individuals would want to borrow. This happens if

$$u_c(w-b) > R\beta u_c(b)$$

Hence, for a pension $b \geq \underline{b}$ ($\underline{b} : u_c(w-\underline{b}) = R\beta u_c(\underline{b})$ voluntary savings is negative $s < 0$. Next impose a non-borrowing constraint ($s \geq 0$) it follows that true utility is $u\left(w-b\right)+\beta^{*}u\left(b\right)$ for $b \geq \underline{b}$, implying an optimal pension b^{*} satisfying

$$u_c(w-b^{*}) = \beta^{*}u_c(b^{*})$$

The welfare case for a positive pension in the presence of a borrowing constraint depends on the necessary condition $R\beta < \beta^{*}$, that is, the present bias should be sufficiently strong. In addition it is required that true utility for $b = b^{*}$ exceeds true utility for $b = 0$, see Andersen and Bhattacharya, 2011) for details. The bottomline is that behavioral "imperfections" give a rationale for PAYG pension only if the borrowing constraint is binding implying that positive PAYG pensions and voluntary savings do not coexist.

B Proof of Lemma 2

Consider the allocation (c_m^{r*}, c_o^{r*}) determined by (16), the optimal committed consumption plan. First, we show that for $\gamma\lambda \geq 1$, there is no incentive for the sophisticated to deviate from this plan. From (11) we have

$$\left.\frac{\partial \Omega_m}{\partial s}\right|_{c_m > c_m^r} = -\left[1+G_x\left(g_m\right)\right]u_c\left(c_m^{*}\right) + \beta\left[1+\gamma\lambda G_x\left(l_o\right)\right]Ru_c\left(c_o^{*}\right)$$

Employing the first-order conditions of the young and the middle-aged (16), we have that

$$\begin{aligned}\left.\frac{\partial \Omega_m}{\partial s}\right|_{c_m > c_m^r, c_m = c_m^{r*}} &= -\left[1+G_x\left(0\right)\right]u_c\left(c_m^{r*}\right) + \beta\left[1+\gamma\lambda G_x\left(0\right)\right]Ru_c\left(c_o^{r*}\right) \\ &= \left[\gamma\lambda - 1\right]G_x\left(0\right)u_c\left(c_m^{r*}\right)\end{aligned}$$

Hence, for $\gamma\lambda \geq 1$ there is no incentive to save less to accomplish a consumption level as middle-aged above c_m^{r*}. Second, for $\gamma\lambda < 1$, it follows that the allocation (c_m^{r*}, c_o^{r*}) cannot be implemented, since there is an incentive to decrease consumption to achieve higher consumption as middle-aged than c_m^{r*}. The sophisticated agent knows that for given reference consumption levels (c_m^r) optimal consumption plan is determined by

$$u_c\left(c_m^*\right) \equiv \kappa \beta R u_c\left(c_o^*\right)$$

where

$$\kappa\left(g_m, l_o\right) \equiv \frac{1+\gamma \lambda G_x\left(l_o\right)}{1+G_x\left(g_m\right)}$$

and the only PPE-equilibrium has $c_m = c_m^r$ which is determine by the above condition for $\kappa(0,0) \equiv \frac{1+\gamma\lambda G_x(0)}{1+G_x(0)}$.

C No double positive result for sophisticated agents

For sophisticated agents, the choice as middle-aged exactly matches the reference consumption set when young, that is, $c_m^s = c_m^{sr}, c_o^s = c_o^{sr}$. The first order condition reads

$$-u_c(w-s^s-b)\left[1+G_x(0)\right] + R\left[1+\gamma\lambda G_x(0)\right] u_c(Rs^s+b) = 0,$$

from which we get

$$\frac{ds^s}{db} = -\frac{\left[1+G_x(0)\right] u_{cc}(c_m^s) + R\left[1+\gamma\lambda G_x(0)\right] u_{cc}(c_o^s)}{\left[1+G_x(0)\right] u_{cc}(c_m^s) + R^2\left[1+\gamma\lambda G_x(0)\right] u_{cc}(c_o^s)} \in \left[-1, -\frac{1}{R}\right].$$

Also, the higher the pension benefits, the lower the voluntary saving, which in turn affects consumption as middle-aged and old,

$$\begin{aligned}
\frac{\partial c_m^s}{\partial b} &= -\left(1+\frac{ds^s}{db}\right) = \frac{R(1-R)\left[1+\gamma\lambda G_x(0)\right] u_{cc}(c_o^s)}{\left[1+G_x(0)\right] u_{cc}(c_m^s) + R^2\left[1+\gamma\lambda G_x(0)\right] u_{cc}(c_o^s)} < 0, \\
\frac{\partial c_o^s}{\partial b} &= R\frac{ds^s}{db} + 1 = \frac{(1-R)\left[1+G_x(0)\right] u_{cc}(c_m^s)}{\left[1+G_x(0)\right] u_{cc}(c_m^s) + R^2\left[1+\gamma\lambda G_x(0)\right] u_{cc}(c_o^s)} < 0.
\end{aligned}$$

The lifetime utility Ω_m^s is

$$\Omega_m^s = u(w-b-s^s) + G(0) - \gamma\lambda G(0) + u(Rs^s+b),$$

while the effect of pensions on life-time utility is

$$\frac{d\Omega_m^s}{db} = -u_c(c_m^s)\left(\frac{ds^s}{db}+1\right) + u_c(c_o^s)\left(R\frac{ds^s}{db}+1\right) < 0.$$

Therefore, there is no role for pensions if agents are sophisticated and there is interior savings.

In the Online Appendix we show that under a borrowing constraint there may be a welfare case for a positive PAYG pension. Voluntary savings is fully crowded out, and this does not deliver the "double" result of positive pensions and voluntary savings.

D Optimal pension for sophisticated agents at a corner

The preceding assumes that voluntary savings is determined by (12) with κ^s. However, due to the crowding out of voluntary savings, there is a pension level (b) so high that voluntary savings is non-positive. In the pensions literature it is customary to impose the condition that voluntary pension savings must be non-negative ($s \geq 0$), since borrowing with future public pensions (b) as collateral is not possible. This constraint is important for finding a welfare rationale for PAYG pensions in settings with myopic indviduals or self-control problems, see discussion in Andersen and Bhattacharya (2011). In the preset setting, the zero savings corner is reached if $b \geq \underline{b}$ defined by

$$u_c(w - \underline{b})\,[1 + G_x(0)] = R\,[1 + \gamma\lambda G_x(0)]\,u_c(\underline{b}).$$

Is it possible by choosing a b sufficiently high that there is a welfare case for the PAYG pension? If $s = 0$, we have

$$\frac{d\Omega_m}{db} = -u_c(c_m^s) + u_c(c_o^s)$$

and from the corner condition for savings, it follows that

$$\begin{aligned} 0 &= -u_c(c_m^s) + u_c(c_o^s) < \left[1 - R\frac{1 + \gamma\lambda G_x(0)}{1 + G_x(0)}\right] u_c(c_o^s) \\ &\Leftrightarrow \quad R < \frac{1 + G_x(0)}{1 + \gamma\lambda G_x(0)} = \frac{1}{\kappa^s}. \end{aligned}$$

Since $\kappa^s \equiv \frac{1+\gamma\lambda G_x(0)}{1+G_x(0)} > \gamma\lambda$, it is a necessary condition that $R < \frac{1}{\gamma\lambda}$ for $\frac{d\Omega_m}{db} > 0$ to be possible. Notice, that in models with myopia, it is also a requirement that R is not too large for pensions to be welfare improving. It follows straightforwardly, that the optimal pension b^s solves $-u_c(c_m^s) + u_c(c_o^s) = 0$ implying that $b^s = \frac{w}{2}$. To conclude whether a pension is welfare improving we also need to compare $\Omega_m|_{b=0}$ with $\Omega_m|_{b=b^s}$.

Notice that when the zero-savings constraint is binding, the optimal policy is effectively maximizing commitment utility. A high pension (making the zero savings corner binding) is thus a commitment device, and the analysis considers whether the sophisticated agents are better off under this commitment policy. The cost of the policy is that the implicit savings in the PAYG scheme has a lower return than the market return. In models with myopia, the distinction between true and choice utility is important and raises difficult issues. The preceding shows that there may be a welfare case for PAYG pension without this distinction, and it hinges on the commitment value being higher than the implicit cost in terms of a lower rate of return.

Suppose

$$u(c)=\frac{c^{1-\sigma}}{1-\sigma},G(x)=A-Ae^{-\alpha x}.$$

Under pensions $b=0$, the FOC (12) with κ^s gives

$$c_m^*=\frac{\kappa^{-1/\sigma}R^{1-1/\sigma}}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}w,\ c_o^*=\frac{R}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}w,$$

The life-time utility will be

$$\begin{aligned}\Omega_m|_{b=0} &= u(c_m^*)+G[u(c_m^*)-u(c_m^{sr})]-\gamma\lambda G[u(c_o^{sr})-u(c_o^*)]+u(c_o^*)\\ &= \frac{1}{1-\sigma}w^{1-\sigma}\left[\left(\frac{\kappa^{-1/\sigma}R^{1-1/\sigma}}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}\right)^{1-\sigma}+\left(\frac{R}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}\right)^{1-\sigma}\right].\end{aligned}$$

If $b=b^s$,

$$\begin{aligned}\Omega_m|_{b=b^s} &= u(c_m^s)+G(0)-\gamma\lambda G(0)+u(c_o^s)\\ &= 2\frac{(w/2)^{1-\sigma}}{1-\sigma}.\end{aligned}$$

A welfare role for pensions also requires $\Omega_m|_{b=b^s}>\Omega_m|_{b=0}$, which gives

$$2\frac{(w/2)^{1-\sigma}}{1-\sigma}>\ w^{1-\sigma}\Psi(R),$$

where

$$\Psi(R)=\frac{1}{1-\sigma}\left[\left(\frac{\kappa^{-1/\sigma}R^{1-1/\sigma}}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}\right)^{1-\sigma}+\left(\frac{R}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}\right)^{1-\sigma}\right].$$

If $\sigma=1$, the above equation cannot hold, which means that the log utility cannot deliver a role for pensions.

If $\sigma>1$, $\Psi(R)$ is an increasing function of R. Employing the above condition and letting $R\to\frac{1}{\kappa}$, we have

$$2^{\sigma-1}<(1+\kappa)^{\sigma-1}\Leftrightarrow\kappa>1,$$

which is impossible. Letting $R \to 1$, we have

$$\begin{aligned} 2^{\sigma} &< \left(\frac{\kappa^{-1/\sigma}}{1+\kappa^{-1/\sigma}}\right)^{1-\sigma} + \left(\frac{1}{1+\kappa^{-1/\sigma}}\right)^{1-\sigma} \\ &= \left(1+\kappa^{1/\sigma}\right)^{\sigma-1} + \left(1+\kappa^{-1/\sigma}\right)^{\sigma-1}, \end{aligned}$$

which might hold. This indicates that for $\sigma > 1$, we need the gross interest rate not to be very large $\left(R < \frac{1}{\kappa}\right)$. Also, κ should also be small, which implies that the weight on gain-loss $G_x(0)$ should be large and the weight on second-period gain-loss $\gamma\lambda$ should be small.

If $\sigma < 1$, letting $R \to \frac{1}{\kappa}$, we have

$$2^{\sigma-1} > (1+\kappa)^{\sigma-1} \Leftrightarrow \kappa < 1,$$

which directly holds. This indicates that it should be much easier to find a R such that a welfare role for pensions exists.

A numerical example might help us to see the result more clearly. Suppose

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}, G_x = \eta = 11.7775, w = 10, \sigma = .6, R = 1.6406, \gamma = .1, \lambda = 2.$$

The optimal pension is $b^s = 5$, which is marked by a blue circle in the following figure. The horizontal axis is the pension level b. The blue line is the optimal saving s without borrowing constraints. The green line is the life-time utility without borrowing constraints. The red line is the life-time utility with borrowing constraints.

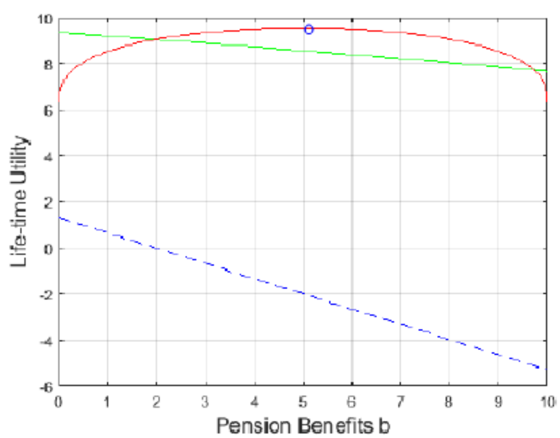

Figure A0: Pensions For Sophisticated Agents

E Optimal pension for naive agents at a corner

Also with naive agents, the PAYG pension crowds out voluntary savings. Hence, it is possible that the optimal pension is so high that the zero savings corner is binding. It is also possible, that there is a welfare case for a PAYG pension when taking into account the zero savings corner. The zero savings corner of the middle-aged is reached if $b^n \geq \underline{b}^n$ defined by

$$-u_c(w - \underline{b}^n)\left[1 + G_x(g_m^n)\right] + R\left[1 + \gamma\lambda G_x(g_o^n)\right]u_c(\underline{b}^n) = 0,$$

while the zero savings corner of the young is reached if $b^n \geq \bar{b}^n > \frac{w}{2}$ defined by

$$-u_c(w - \bar{b}^n) + Ru_c(\bar{b}^n) = 0.$$

Is it possible by choosing a b sufficiently high that there is a welfare case for the PAYG pension? If $s^n = 0$ and $s^{nr} = 0$ we have $g_m^n = g_o^n = 0$ and

$$\frac{d\Omega_m}{db} = -u_c(c_m^n) + u_c(c_o^n) = 0,$$

which gives $b^n = \frac{w}{2}$. But when $b^n = \frac{w}{2}$, the young agent's FOC satisfies

$$\frac{d\Omega_y}{ds}|_{s=0} = -u_c\left(\frac{w}{2}\right) + Ru_c\left(\frac{w}{2}\right) > 0,$$

which implies that the borrowing constraint is not binding for the young. Therefore, the optimal pension benefits in this scenario is that $b^n = \bar{b}^n$. Then the whole question can be reduced to the following problem, where the borrowing constraint is binding only for the middle-aged and $b^n \leq \bar{b}^n$.

Since $s^n = 0$ but $s^{nr} \neq 0$ in equilibrium, we have

$$\text{(30)} \quad \begin{aligned} \frac{d\Omega_m}{db} &= -\left[1 + G_x\left(g_m^n\right)\right]u_c(c_m^n) + \left[1 + \gamma\lambda G_x\left(l_o^n\right)\right]u_c(c_o^n) \\ &\quad -\frac{1-R}{R}\left[\epsilon G_x(g_m^n)u_c\left(c_m^{nr*}\right) + (1-\epsilon)\,\gamma\lambda G_x\left(l_o^n\right)u_c\left(c_m^{nr*}\right)\right] \end{aligned}$$

and from the corner condition for savings, it follows that

$$\frac{d\Omega_m}{db}\Big/\left(\frac{R-1}{R}\right) < -\left[1 + G_x\left(g_m^n\right)\right]u_c(c_m^n) + \left[\begin{array}{c} \epsilon G_x(g_m^n)u_c\left(c_m^{nr*}\right) \\ +(1-\epsilon)\,\gamma\lambda G_x\left(l_o^n\right)u_c\left(c_m^{nr*}\right) \end{array}\right].$$

A necessary condition would be

$$\left[1+G_x(g_m^n)\right]u_c\left(c_m^n\right)<\left[\epsilon G_x(g_m^n)+\left[1-\epsilon\right]\gamma\lambda G_x\left(l_o^n\right)\right]u_c\left(c_m^{nr*}\right)$$

or (using that $c_m^{nr*}=c_m^{r*}$)

$$\frac{u_c\left(c_m^n\right)}{u_c\left(c_m^{r*}\right)}<\frac{\epsilon G_x(g_m^n)+\left[1-\epsilon\right]\gamma\lambda G_x\left(l_o^n\right)}{1+G_x(g_m^n)}.$$

In order to find the optimal pension benefits, the agent needs to compare between $\Omega_m|_{b=0}$, $\Omega_m|_{b=\bar{b}^n}$, and $\Omega_m|_{s^n=0,s^{nr}\neq 0}$.

Assume constant marginal gain-loss $G_x(g_m^n)=G_x\left(l_o^n\right)=\eta$, the above condition can be written

$$\frac{u_c\left(c_m^n\right)}{u_c\left(c_m^{r*}\right)}<\left[\epsilon+\left(1-\epsilon\right)\gamma\lambda\right]\frac{\eta}{1+\eta}.$$

First, assuming constant absolute risk aversion, $a\equiv-\frac{u_{cc}(c_o^r)}{u_c(c_o^n)}$ - implies that

$$\epsilon=\frac{R^2u_{cc}\left(c_o^{r*}\right)}{u_{cc}\left(c_m^{r*}\right)+R^2u_{cc}\left(c_o^{r*}\right)}=\frac{\frac{R^2u_{cc}(c_o^{r*})}{Ru_c(c_o^{r*})}}{\frac{u_{cc}(c_m^{r*})}{u_c(c_m^{r*})}+\frac{R^2u_{cc}(c_o^{r*})}{Ru_c(c_o^{r*})}}=\frac{Ra}{a+Ra}=\frac{R}{1+R}.$$

With constant marginal gain-loss, the parameter κ becomes

$$\kappa=\frac{1+\gamma\lambda G_g\left(l_o^n\right)}{1+G_g\left(g_m^n\right)}=\frac{1+\gamma\lambda\eta}{1+\eta}.$$

The inequality can now be written

$$\frac{u_c\left(c_m^n\right)}{u_c\left(c_m^{r*}\right)}<\left[\frac{R}{1+R}+\frac{1}{1+R}\gamma\lambda\right]\frac{\eta}{1+\eta}$$

or

$$\frac{u_c\left(c_m^n\right)}{u_c\left(c_m^{r*}\right)}<\frac{R+\gamma\lambda}{1+R}\frac{\eta}{1+\eta}. \tag{31}$$

This inequality does not hold for $\eta=0$ (as should be expected, this is the standard case), since

$$\eta=0:\frac{u_c\left(c_m^n\right)}{u_c\left(c_m^{r*}\right)}=1>0=\frac{R+\gamma\lambda}{1+R}\frac{\eta}{1+\eta}$$

Returning to the inequality (31) we have that the LHS is decreasing in η, since κ is decreasing in η; the RHS is increasing in η. This suggests the possibility of a cut-off level of η ensuring a welfare case for PAYG pensions.

We know that (31) cannot hold for a low value of η - can it hold for a large value? Note the limit properties:

$$\kappa \to \gamma\lambda \text{ for } \eta \to \infty, \ \frac{\eta}{1+\eta} \to 1 \text{ for } \eta \to \infty$$

Hence, if it can be established that

$$\left.\frac{u_c\left(c_m^n\right)}{u_c\left(c_m^{r*}\right)}\right|_{\kappa=\gamma\lambda} < \frac{R+\gamma\lambda}{1+R} \tag{32}$$

there exists a $\overline{\eta}$ such that the inequality (31) holds for $\eta > \overline{\eta}$. Note that it is only required that this condition holds when LHS is evaluated for $b = 0$. Assume the following functional form for the utility function

$$u(c) = -e^{-ac}, \ a > 0$$

and recall the first-order conditions for consumption/savings

$$u_c\left(c_m^n\right) = \kappa R u_c\left(c_o^n\right), \ u_c\left(c_m^{r*}\right) = R u_c\left(c_o^{r*}\right).$$

Then it follows (determining consumption/savings for any pensions b, to be able to find the marginal utilities of consumption)

$$\begin{aligned} & e^{-a(w-s^n-b)} = R\kappa e^{-a(Rs^n+b)} \\ & \Rightarrow s^n = \frac{\ln\left(R\kappa\right) + a\left(w-2b\right)}{a(1+R)}. \end{aligned}$$

and therefore

$$\begin{aligned} c_m^n &= \frac{R}{1+R}\left(w-b+\frac{b}{R}\right) - \frac{\ln\left(R\kappa\right)}{a(1+R)}, \\ c_o^n &= \frac{R}{1+R}\left(w-b+\frac{b}{R}\right) - \frac{R\ln\left(R\kappa\right)}{a(1+R)}. \end{aligned}$$

Using the same procedure we find

$$\begin{aligned} c_m^{r*} &= \frac{R}{1+R}\left(w-b+\frac{b}{R}\right) - \frac{\ln\left(R\right)}{a(1+R)}, \\ c_o^{r*} &= \frac{R}{1+R}\left(w-b+\frac{b}{R}\right) - \frac{R\ln\left(R\right)}{a(1+R)}. \end{aligned}$$

Combining these expression, we get

$$\frac{u_c\left(c_m^n\right)}{u_c\left(c_m^{r*}\right)}=\frac{e^{-\alpha\left[\frac{aR\left(w-b+\frac{b}{R}\right)-\ln(R\kappa)}{a(1+R)}\right]}}{e^{-\alpha\left[\frac{aR\left(w-b+\frac{b}{R}\right)-\ln(R)}{a(1+R)}\right]}}=e^{\frac{\ln\kappa}{1+R}}=\kappa^{\frac{1}{1+R}}.$$

Condition (32) can now be written as

$$\kappa^{\frac{1}{1+R}}<\frac{R+\gamma\lambda}{1+R}.$$

or

$$\frac{1+\gamma\lambda\eta}{1+\eta}<\left(\frac{R+\gamma\lambda}{1+R}\right)^{1+R}.$$

This shows that $\frac{d\Omega_m}{db}\Big|_{any\, b}>0$ if $\eta>\overline{\eta}$ for some large $\overline{\eta}$. Therefore, without borrowing constraints we always have a corner pensions for some $\eta>\overline{\eta}$. Therefore, if $\eta>\overline{\eta}$, there is a welfare role for pensions and the borrowing constraints must be binding. Note that c_m^{r*},c_o^{r*} is the same as before while $c_m^n=w-b$, $c_o^n=b$. The FOC (30) becomes

$$\frac{d\Omega_m}{db}=-\left(1+\eta\right)ae^{-a(w-b)}+\left(1+\gamma\lambda\eta\right)ae^{-ab}+\frac{\eta a\left(R-1\right)}{R\left(1+R\right)}\left[Re^{-ac_m^{r*}}+\gamma\lambda e^{-ac_o^{r*}}\right].$$

If the second-order condition is satisfied, we could have an interior pensions $b^n\in\left(\underline{b}^n,\bar{b}^n\right)$.

F Proof of Lemma 3

The optimal actual and reference consumption can be written as

$$\begin{aligned}
s^n &= \frac{w-b-\kappa^{-1/\sigma}R^{-1/\sigma}b}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}},\ s^{nr*}=\frac{w-b-R^{-1/\sigma}b}{1+R^{1-1/\sigma}},\\
c_m^n &= \frac{\kappa^{-1/\sigma}R^{1-1/\sigma}}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}\left(w-b+\frac{b}{R}\right),\ c_m^{nr*}=\frac{R^{1-1/\sigma}}{1+R^{1-1/\sigma}}\left(w-b+\frac{b}{R}\right),\\
c_o^n &= \frac{1}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}R\left(w-b+\frac{b}{R}\right),\ c_o^{nr*}=\frac{1}{1+R^{1-1/\sigma}}R\left(w-b+\frac{b}{R}\right).
\end{aligned}$$

The ratio of old-age consumption to middle-age consumption can be written as

$$\frac{c_o^n}{c_m^n} = \frac{R}{R^{1-1/\sigma}}\kappa^{1/\sigma} = R^{1/\sigma}\kappa^{1/\sigma},$$

which is increasing in κ. Also,

$$\begin{aligned}\frac{c_m^n}{c_m^{nr*}} &= \frac{\frac{\kappa^{-1/\sigma}R^{1-1/\sigma}}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}}{\frac{R^{1-1/\sigma}}{1+R^{1-1/\sigma}}} = \frac{\kappa^{-1/\sigma}R^{1-1/\sigma}}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}\frac{1+R^{1-1/\sigma}}{R^{1-1/\sigma}} \\ &= \frac{\kappa^{-1/\sigma}\left(1+R^{1-1/\sigma}\right)}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}} = \frac{1+R^{1-1/\sigma}}{\kappa^{1/\sigma}+R^{1-1/\sigma}},\end{aligned}$$

which is decreasing in κ. The question boils down to how κ depends on b.

For later reference, the loss term is given as

$$\begin{aligned}l_o^n &= \frac{\left(c_o^{nr*}\right)^{1-\sigma} - \left(c_o^n\right)^{1-\sigma}}{1-\sigma} \\ &= \frac{1}{\sigma-1}\left[\left(\frac{R}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}\right)^{1-\sigma} - \left(\frac{R}{1+R^{1-1/\sigma}}\right)^{1-\sigma}\right]\left(w-b+\frac{b}{R}\right)^{1-\sigma},\end{aligned}$$

and hence

$$\begin{aligned}\frac{\partial l_o^n}{\partial b} &= \frac{l_o^n\left(\sigma-1\right)\left(1-\frac{1}{R}\right)}{w-b+\frac{b}{R}} = \frac{l_o^n\left(\sigma-1\right)\left(1-\frac{1}{R}\right)}{Y} \\ \frac{\partial l_o^n}{\partial \kappa} &= \left(-\frac{1}{\sigma}\right)\left(c_o^n\right)^{1-\sigma}\frac{\kappa^{-1-1/\sigma}R^{1-1/\sigma}}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}\end{aligned}$$

and similarly for the gain term

$$\begin{aligned}g_m^n &= \frac{\left(c_m^n\right)^{1-\sigma} - \left(c_m^{nr*}\right)^{1-\sigma}}{1-\sigma} \\ &= \frac{1}{\sigma-1}\left[\left(\frac{R^{1-1/\sigma}}{1+R^{1-1/\sigma}}\right)^{1-\sigma} - \left(\frac{\kappa^{-1/\sigma}R^{1-1/\sigma}}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}\right)^{1-\sigma}\right]\left(w-b+\frac{b}{R}\right)^{1-\sigma}.\end{aligned}$$

$$\begin{aligned}\frac{\partial g_m^n}{\partial b} &= \frac{g_m^n\left(\sigma-1\right)\left(1-\frac{1}{R}\right)}{Y} \\ \frac{\partial g_m^n}{\partial \kappa} &= \left(-\frac{1}{\sigma}\right)\left(c_m^n\right)^{1-\sigma}\frac{\kappa^{-1}}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}} < 0.\end{aligned}$$

Turning to the expression for κ, $\kappa \equiv \frac{1+\gamma\lambda G_x(l_o^n)}{1+G_x(g_m^n)}$, it follows that

$$\frac{d\kappa}{db} = \frac{\gamma\lambda G_{xx}\left(l_o^n\right)\left[\frac{\partial l_o^n}{\partial\kappa}\frac{d\kappa}{db} + \frac{\partial l_o^n}{\partial b}\right]\left[1+G_x\left(g_m^n\right)\right] - \left[1+\gamma\lambda G_x\left(l_o^n\right)\right]G_{xx}\left(g_m^n\right)\left[\frac{\partial g_m^n}{\partial\kappa}\frac{d\kappa}{db} + \frac{\partial g_m^n}{\partial b}\right]}{\left[1+G_x\left(g_m^n\right)\right]^2}$$

implying

$$\frac{d\kappa}{db} = \frac{-\kappa G_{xx}\left(g_m^n\right)\frac{\partial g_m^n}{\partial b} + \gamma\lambda G_{xx}\left(l_o^n\right)\frac{\partial l_o^n}{\partial b}}{\left[1+G_x\left(g_m^n\right)\right] - \gamma\lambda G_{xx}\left(l_o^n\right)\frac{\partial l_o^n}{\partial\kappa} + \kappa G_{xx}\left(g_m^n\right)\frac{\partial g_m^n}{\partial\kappa}} \tag{33}$$

Using the expression given above for $\frac{\partial g_m^n}{\partial b}$, $\frac{\partial l_o^n}{\partial b}$, $\frac{\partial l_o^n}{\partial\kappa}$, and $\frac{\partial g_m^n}{\partial\kappa}$ we get

$$\frac{d\kappa}{db} = \frac{\left(\sigma-1\right)\left(1-\frac{1}{R}\right)}{Y}\frac{-\kappa G_{xx}\left(g_m^n\right)g_m^n + \gamma\lambda G_{xx}\left(l_o^n\right)l_o^n}{\Phi}$$

where

$$\begin{aligned}\Phi \quad \equiv \quad & \left[1+G_x\left(g_m^n\right)\right] - \gamma\lambda G_{xx}\left(l_o^n\right)\left(-\frac{1}{\sigma}\right)\left(c_o^n\right)^{1-\sigma}\frac{\kappa^{-1-1/\sigma}R^{1-1/\sigma}}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}} \\ & +G_{xx}\left(g_m^n\right)\left(-\frac{1}{\sigma}\right)\left(c_m^n\right)^{1-\sigma}\frac{1}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}\end{aligned}$$

The second order condition to the savings problem (14), reads

$$\begin{aligned}\frac{d^2\Omega_m}{ds^2} \quad = \quad & \left[1+G_x\left(g_m^n\right)\right]u_{cc}\left(c_m\right) + \left[1+\gamma\lambda G_x\left(l_o^n\right)\right]R^2u_{cc}\left(c_o\right) \\ & + G_{xx}\left(g_m^n\right)\left[u_c\left(c_m\right)\right]^2 - \ \gamma\lambda G_{xx}\left(l_o^n\right)R^2\left[u_c\left(c_o\right)\right]^2 < 0.\end{aligned}$$

which implies

$$\begin{aligned}&\left[1+G_x\left(g_m^n\right)\right]\frac{\kappa u_{cc}\left(c_m\right)}{Ru_c\left(c_m\right)} + \left[1+\gamma\lambda G_x\left(l_o^n\right)\right]\frac{u_{cc}\left(c_o\right)}{u_c\left(c_o\right)} \\ &+u_c\left(c_o\right)\left[\ G_{xx}\left(g_m^n\right)\kappa^2 - \ \gamma\lambda G_{xx}\left(l_o^n\right)\right] < 0.\end{aligned}$$

and therefore

$$\Phi = \left[1+G_x\left(g_m^n\right)\right] - G_{xx}\left(g_m^n\right)\left(c_m\right)^{1-\sigma}\frac{1}{\sigma}\frac{1}{1+R^{1-\frac{1}{\sigma}}\kappa^{-1/\sigma}} + \gamma\lambda G_{xx}\left(l_o^n\right)\left(c_o\right)^{1-\sigma}\frac{1}{\sigma}\frac{R^{1-\frac{1}{\sigma}}\kappa^{-1-1/\sigma}}{1+R^{1-\frac{1}{\sigma}}\kappa^{-1/\sigma}} > 0,$$

which determines the sign of the denominator of $\frac{d\kappa}{db}$.Therefore,

$$\begin{aligned} sign\left(\frac{d\kappa}{db}\right) &= sign\left(-\kappa G_{xx}\left(g_m^n\right)\frac{\partial g_m^n}{\partial b} + \gamma\lambda G_{xx}\left(l_o^n\right)\frac{\partial l_o^n}{\partial b}\right) \\ &= sign\left[-\kappa G_{xx}\left(g_m^n\right)g_m^n + \gamma\lambda G_{xx}\left(l_o^n\right)l_o^n\right] \end{aligned}$$

Note that

$$\begin{aligned} -\kappa G_{xx}\left(g_m^n\right)g_m^n + \gamma\lambda G_{xx}\left(l_o^n\right)l_o^n &= \kappa G_x\left(g_m^n\right)\frac{G_{xx}\left(g_m^n\right)g_m^n}{G_x\left(g_m^n\right)} + \gamma\lambda G_x\left(l_o^n\right)\frac{G_{xx}\left(l_o^n\right)l_o^n}{G_x\left(l_o^n\right)} \\ &= (1-\alpha)\left[\kappa G_x\left(g_m^n\right) - \gamma\lambda G_x\left(l_o^n\right)\right] \end{aligned}$$

where it is assumed that $-\frac{xG_{xx}(x)}{G_x(x)} \equiv 1-\alpha$. Finally

$$\begin{aligned} \kappa G_x\left(g_m^n\right) - \gamma\lambda G_x\left(l_o^n\right) &= \frac{1+\gamma\lambda G_x\left(l_o^n\right)}{1+G_x\left(g_m^n\right)}G_x\left(g_m^n\right) - \gamma\lambda G_x\left(l_o^n\right) \\ &= 1-\kappa > 0 \text{ since } \kappa < 1. \end{aligned}$$

Therefore, if G is of the form in (4) and $\sigma > 1$, we have

$$\frac{d\kappa}{db} > 0.$$

Moreover,

$$\frac{d}{db}\left(\frac{c_o^n}{c_m^n}\right) > 0,\ \frac{d}{db}\left(\frac{c_m^n}{c_m^{nr*}}\right) < 0.$$

An increase in the pension b reduces the overconsumption problem; the agent allocates relatively more consumption into old-age consumption compared with the case without pensions. The tension between actual and reference consumption when middle-aged is reduced.

G Proof of Proposition 1

The optimal pension is determined by the expression (see main text),

$$F(\kappa) \equiv \frac{1+\kappa^{-1}R^{1-1/\sigma}}{1+R^{1-1/\sigma}} - \left(\frac{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}{1+R^{1-1/\sigma}}\right)^{\sigma}.$$

Observe that

$$\begin{aligned}
\frac{\partial F(\kappa)}{\partial\kappa} &= \frac{-\kappa^{-2}R^{1-1/\sigma}}{1+R^{1-1/\sigma}} - \sigma\left(\frac{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}{1+R^{1-1/\sigma}}\right)^{\sigma-1}\left(-\frac{1}{\sigma}\right)\frac{\kappa^{-\frac{1}{\sigma}-1}R^{1-1/\sigma}}{1+R^{1-1/\sigma}} \\
&= -\frac{\kappa^{-2}R^{1-1/\sigma}}{1+R^{1-1/\sigma}} + \left(\frac{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}{1+R^{1-1/\sigma}}\right)^{\sigma-1}\frac{\kappa^{-\frac{1}{\sigma}-1}R^{1-1/\sigma}}{1+R^{1-1/\sigma}} \\
&= \frac{\kappa^{-\frac{1}{\sigma}-1}R^{1-1/\sigma}}{1+R^{1-1/\sigma}}\left[-\kappa^{\frac{1}{\sigma}-1} + \left(\frac{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}{1+R^{1-1/\sigma}}\right)^{\sigma-1}\right] \\
&= \frac{\kappa^{-\frac{1}{\sigma}-1}R^{1-1/\sigma}}{1+R^{1-1/\sigma}}\left[-\left(\frac{\kappa^{-1/\sigma}+\kappa^{-1/\sigma}R^{1-1/\sigma}}{1+R^{1-1/\sigma}}\right)^{\sigma-1} + \left(\frac{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}{1+R^{1-1/\sigma}}\right)^{\sigma-1}\right] \\
&\geq 0 \text{ iff } \sigma \leq 1 \text{ since } \kappa < 1.
\end{aligned}$$

Therefore, for $\sigma \leq 1$, we have

$$F(\kappa) - \frac{1}{1+\gamma\lambda G_x(l_o^n)} \leq F(1) - \frac{1}{1+\gamma\lambda G_x(l_o^n)} < 0.$$

Hence, there is no welfare role for pensions if $\sigma \leq 1$.

Note that if the gain-loss utility is two-part linear, the first-order of the government becomes

$$F(\kappa) = \frac{1}{1+\gamma\lambda\eta},$$

with

$$\kappa = \frac{1+\gamma\lambda\eta}{1+\eta}.$$

Hence, there is no interior pensions in this case, and thus there is no "double-positive" result.

H Proof of Lemma 5

Existence of positive pension

The optimal pension is determined by the government's first-order condition

$$F(\kappa) = \frac{1}{1+\gamma\lambda G_x(l_o^n)}$$

Note that the loss as old is given as

$$
\begin{aligned}
l_o^n &= \frac{(c_o^{nr*})^{1-\sigma} - (c_o^n)^{1-\sigma}}{1-\sigma} \\
&= \frac{1}{\sigma-1}\left[\left(\frac{1}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}\right)^{1-\sigma} - \left(\frac{1}{1+R^{1-1/\sigma}}\right)^{1-\sigma}\right] R^{1-\sigma}\left(w-b+\frac{b}{R}\right)^{1-\sigma} \\
&= A(\kappa, R, \sigma)Y^{1-\sigma},
\end{aligned}
$$

where

$$
A(\kappa, R, \sigma) \equiv \frac{1}{\sigma-1}\left[\left(\frac{1}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}\right)^{1-\sigma} - \left(\frac{1}{1+R^{1-1/\sigma}}\right)^{1-\sigma}\right] R^{1-\sigma} \geq 0
$$

and the gain as middle-aged as

$$
\begin{aligned}
g_m^n &= \frac{(c_m^n)^{1-\sigma} - (c_m^{nr*})^{1-\sigma}}{1-\sigma} \\
&= \frac{1}{\sigma-1}\left[\left(\frac{R^{1-1/\sigma}}{1+R^{1-1/\sigma}}\right)^{1-\sigma} - \left(\frac{\kappa^{-1/\sigma}R^{1-1/\sigma}}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}\right)^{1-\sigma}\right] Y^{1-\sigma} \\
g_m^n &= B(\kappa, R, \sigma)Y^{1-\sigma}
\end{aligned}
$$

where

$$
B(\kappa, R, \sigma) \equiv \frac{1}{\sigma-1}\left[\left(\frac{R^{1-1/\sigma}}{1+R^{1-1/\sigma}}\right)^{1-\sigma} - \left(\frac{\kappa^{-1/\sigma}R^{1-1/\sigma}}{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}\right)^{1-\sigma}\right] \geq 0,
$$

Hence, using that $\kappa < 1$ we have

$$
\begin{aligned}
\frac{\partial F(\kappa, b)}{\partial \kappa} &= \frac{-\kappa^{-2}R^{1-1/\sigma}}{1+R^{1-1/\sigma}} - \sigma\left(\frac{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}{1+R^{1-1/\sigma}}\right)^{\sigma-1}\left(-\frac{1}{\sigma}\right)\frac{\kappa^{-\frac{1}{\sigma}-1}R^{1-1/\sigma}}{1+R^{1-1/\sigma}} \\
&= \frac{\kappa^{-\frac{1}{\sigma}-1}R^{1-1/\sigma}}{1+R^{1-1/\sigma}}\left[-\left(\frac{\kappa^{-1/\sigma}+\kappa^{-1/\sigma}R^{1-1/\sigma}}{1+R^{1-1/\sigma}}\right)^{\sigma-1} + \left(\frac{1+\kappa^{-1/\sigma}R^{1-1/\sigma}}{1+R^{1-1/\sigma}}\right)^{\sigma-1}\right] \\
&< 0 \text{ if } \sigma > 1.
\end{aligned}
$$

Hence, for $\sigma > 1$, $F(\kappa)$ is decreasing in κ. The problem, hereon, is to find an interior solution to the two equations in the two unknowns κ and C (27) (28), where $C \equiv Y^{1-\sigma}$. Note that $A(\kappa, R, \sigma) \in (0, \infty)$ and $B(\kappa, R, \sigma) \in (0, \infty)$.

(I) Equation (27)

This equation gives κ as an implicit function of C, $\kappa = \phi(C)$. We have that

$$\kappa_0 \equiv \phi(0) = \frac{1 + \gamma\lambda G_x(0)}{1 + G_x(0)} < 1$$

Moreover, for any $\kappa \in (\kappa_0, 1)$, we have the following boundary conditions

$$\begin{aligned} \frac{1 + \gamma\lambda G_x(A(\kappa, R, \sigma)C)}{1 + G_x(B(\kappa, R, \sigma)C)} &\rightarrow \kappa_0 < \kappa \text{ as } C \rightarrow 0, \\ \frac{1 + \gamma\lambda G_x(A(\kappa, R, \sigma)C)}{1 + G_x(B(\kappa, R, \sigma)C)} &\rightarrow 1 > \kappa \text{ as } C \rightarrow \infty, \end{aligned}$$

Note that $\phi(\cdot)$ is not monotone in C.

(II) Equation (28).

Define

$$\begin{aligned} \underline{\kappa} &: F(\underline{\kappa}) = 1 \\ \overline{\kappa} &: F(\overline{\kappa}) = \frac{1}{1 + \gamma\lambda G_x(0)} < 1 \end{aligned}$$

It follows that

$$\begin{aligned} \kappa &\rightarrow \overline{\kappa} \text{ for } C \rightarrow 0 \\ \kappa &\rightarrow \underline{\kappa} \text{ for } C \rightarrow \infty \end{aligned}$$

Since $F(\cdot)$ is declining in κ, $\underline{\kappa} < \overline{\kappa}$.

The information can now be summarized in the following figure where the red curve plots (27) (for simplicity assumed monotone) and the curve plots (28).

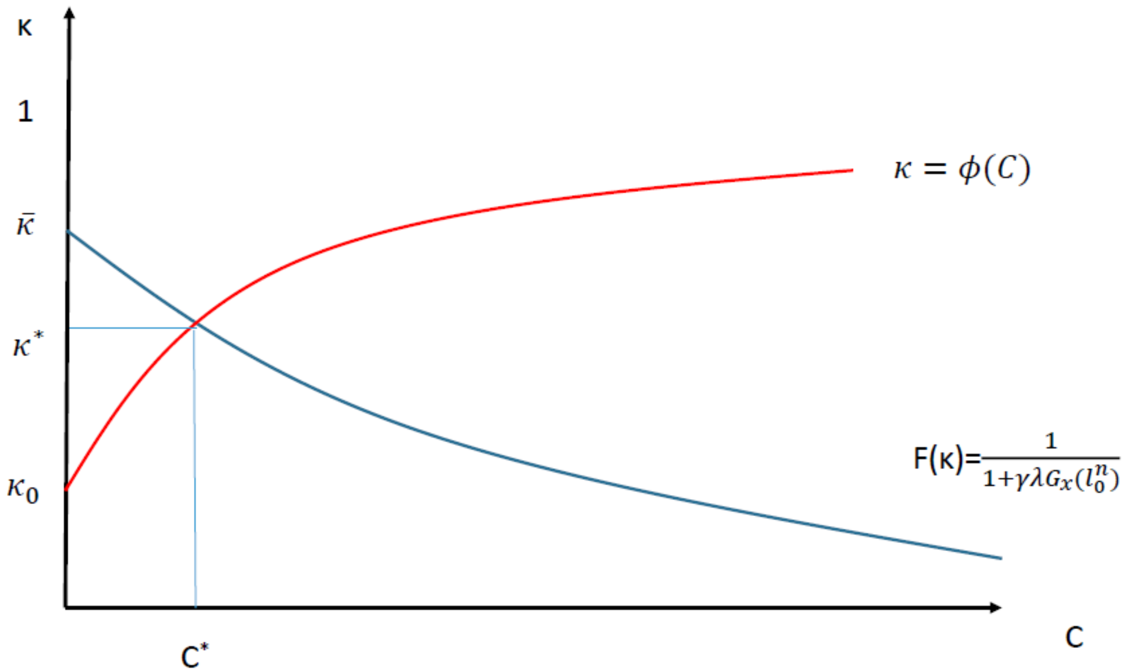

Figure A1: Determination of equilibrium κ and C

Existence of a solution requires that

$$\kappa_0 < \overline{\kappa}$$

Since $F(\kappa)$ is decreasing in κ this is equivalent to

$$F(\kappa_0) > F(\overline{\kappa})$$

or

$$(34) \quad F\left(\frac{1+\gamma\lambda G_x(0)}{1+G_x(0)}\right) > \frac{1}{1+\gamma\lambda G_x(0)},$$

Next we show that the above condition can be satisfied for large $G_x(0)$, a property of the gain-loss function. Note that if $G_x(0) \to \infty$ the RHS of (34) goes to 0, and the LHS goes to $F(\gamma\lambda) > F(1) = 0$, hence the inequality holds for a sufficiently large $G_x(0)$. For $G_x(0) \to 0$, we know that the RHS goes to 1 and the LHS to $F(1) = 0$, violating the inequality. Finally, using that the LHS of (34) is increasing in $G_x(0)$ while the RHS is decreasing in $G_x(0)$, there exists a $\underline{G}_x(0)$ such that (34) holds for any $G_x(0) > \underline{G}_x(0)$. Note this does not prove uniqueness, this requires $\phi_C(C) \geq 0$ for all C.

I Proof of Proposition 2

With $\kappa^* \in (\underline{\kappa}, \overline{\kappa})$ and C^* (they are independent of w) solved, we can solve sequentially the following.

$$
\begin{aligned}
Y^* &= (C^*)^{\frac{1}{1-\sigma}}, \ b^* = \frac{w - Y^*}{1 - \frac{1}{R}} = \frac{R}{R-1}(w - Y^*). \\
g_m^n &= B(\kappa^*, R, \sigma)C^*, \ l_o^n = A(\kappa^*, R, \sigma)C^*.
\end{aligned}
$$

$$
\begin{aligned}
s^n &= \frac{w - b^* - (\kappa^*)^{-1/\sigma} R^{-1/\sigma} b^*}{1 + (\kappa^*)^{-1/\sigma} R^{1-1/\sigma}}, \ s^{nr*} = \frac{w - b^* - R^{-1/\sigma} b^*}{1 + R^{1-1/\sigma}}, \\
c_m^n &= \frac{(\kappa^*)^{-1/\sigma} R^{1-1/\sigma}}{1 + (\kappa^*)^{-1/\sigma} R^{1-1/\sigma}} \left(w - b^* + \frac{b^*}{R} \right), \ c_m^{nr*} = \frac{R^{1-1/\sigma}}{1 + R^{1-1/\sigma}} \left(w - b^* + \frac{b^*}{R} \right), \\
c_o^n &= \frac{1}{1 + (\kappa^*)^{-1/\sigma} R^{1-1/\sigma}} R \left(w - b^* + \frac{b^*}{R} \right), \ c_o^{nr*} = \frac{1}{1 + R^{1-1/\sigma}} R \left(w - b^* + \frac{b^*}{R} \right).
\end{aligned}
$$

Wealth range supporting positive pensions and savings
Savings under the optimal pension is therefore

$$
\begin{aligned}
s^n &= \frac{w - \left[1 + (\kappa^*)^{-1/\sigma} R^{-1/\sigma} \right] b}{1 + (\kappa^*)^{-1/\sigma} R^{1-1/\sigma}} = \frac{w - \left[1 + (\kappa^*)^{-1/\sigma} R^{-1/\sigma} \right] \frac{R}{R-1}(w - Y^*)}{1 + (\kappa^*)^{-1/\sigma} R^{1-1/\sigma}} \\
&= \frac{\left[1 + (\kappa^*)^{-1/\sigma} R^{-1/\sigma} \right] \frac{R}{R-1} Y^* - \left[\frac{1}{R-1} + \frac{R}{R-1} (\kappa^*)^{-1/\sigma} R^{-1/\sigma} \right] w}{1 + (\kappa^*)^{-1/\sigma} R^{1-1/\sigma}}.
\end{aligned}
$$

Notice that $\left[1 + (\kappa^*)^{-1/\sigma} R^{-1/\sigma} \right] \frac{R}{R-1} Y^* > 0$ and

$$
\frac{\partial s^n}{\partial w} = -\frac{\frac{1 + \kappa^{-1/\sigma} R^{-1/\sigma} R}{R-1}}{1 + \kappa^{-1/\sigma} R^{1-1/\sigma}} < 0
$$

We thus have a situation as illustrated in the figure below (the saving function is, for simplicity, drawn as a straight line – but this is not generally the case, since κ is not a parameter). The straight positively slope line is showing $b = b_0 + \frac{R}{R-1} w$, where $b_0 = -\frac{R}{R-1} Y^*$.

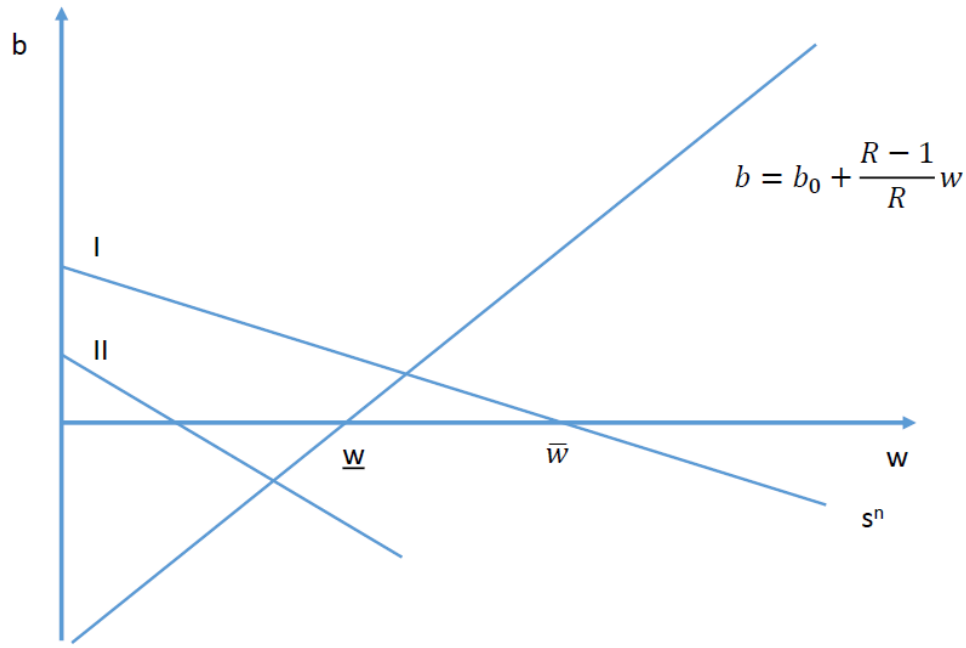

Figure A2: Income, savings and pensions

The interesting case is when the savings locus is as in case I, in which case we have positive pensions and savings for

(35) $w \in [\underline{w}, \overline{w}]$

where

$$\underline{w} : \frac{R}{R-1} (\underline{w} - Y^*) = 0 \Leftrightarrow \underline{w} = Y^* > 0$$

and

$$\bar{w} : s^n = 0 \Leftrightarrow \bar{w} = \frac{\left[1 + (\kappa^*)^{-1/\sigma} R^{-1/\sigma}\right] RY^*}{1 + (\kappa^*)^{-1/\sigma} R^{1-1/\sigma}}.$$

Note that there is no role for pensions if $w < \underline{w}$ and for $w > \overline{w}$ the optimal saving would be negative. Having both positive pensions and voluntary savings requires $\overline{w} > \underline{w}$

or

$$\frac{\left[1+(\kappa^*)^{-1/\sigma}R^{-1/\sigma}\right]RY^*}{1+(\kappa^*)^{-1/\sigma}R^{1-1/\sigma}}>Y^*,$$

which is true directly. Hence, in a dynamically efficient economy the double coincidence of positive pensions and savings is generally possible, i.e. Case I prevails. The range of wealth w supporting this is

$$Z=\frac{\overline{w}}{\underline{w}}>1.$$