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Direct Matching Mechanism Design for India with Comprehensive Affirmative Action

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Keywords

Market design, affirmative action, matching, cumulative offer

Disciplines

Economic Theory | Education Policy | Marketing | Social Policy

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JEL Codes*:* C78, D47

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1 Affirmative Action in India

1.1 Vertical Reservations

India has been implementing the most comprehensive affirmative action program in the world since 1950. The affirmative action program has been implemented via a **reservation system**. According to the reservation system, certain fractions of seats at government-sponsored jobs and publicly funded educational institutions are reserved for members of historically discriminated groups, namely *Schedule Castes* (SC), *Scheduled Tribes* (ST), and *Other Backward Classes* (OBC). SC, ST, and OBS are referred to as the *reserved categories*. People who do not belong to any of these categories are referred to as members of the *General Category* (GC). Currently, up to 50 percent of seats at publicly funded schools and government jobs are reserved for the members of SC, ST, and OBC. The remaining 50 percent of seats are considered *open category* seats.

The Hindu society is divided into four *varnas*: the Brahmans, the Kshatriyas, the Vaishyas, and the Shudras. The varnas are ranked in this order with respect to their purity, i.e., the Brahmans are considered to be the "purest" and the Shudras being the lowest. The people who do not belong to one of these varnas are referred as to *"dalits"* or *"untouchables'*. The dalits—now called the SC— are at the bottom of the hierarchy. Moreover, the Indian society is divided into *jatis* (or subcastes), where each jati is related with a specific varna. The jati is hereditary so that it is determined at birth (Dumont, 1970). Jatis are mostly geographically limited.

The SC and the ethnic minorities—now called the ST— had been forbidden from using common goods and limited to menial jobs. They suffered from discrimination because of their low statuses and were economically very disadvantaged. To level the playing field positive discrimination has been in effect for them. Affirmative action in India dates back to 1882 when the British established special schools for the dalits (Jaffrelot, 2011). Some of the major initiatives for SC and ST reservations started in the late 1940s. In 1954, the Ministry of Education suggested that 20 percent of positions should be reserved for the SC and ST. It was specified in 1982 that 15 percent and 7.5 percent of seats in government-sponsored jobs and publicly funded educational institutions should be reserved for SC and ST, respectively.

The reservation system was later extended to jatis that belong to the Shudras—now referred to as the OBC—which is the lowest of the four varnas. This extension was the most controversial component of the reservation policy, mostly due to its size. The OBC constitutes approximately 50% of the Indian population. In 1979, the Mandal Commission was established to evaluate the situation of the socially and educationally backward classes. The commission used the data from the 1931 census and estimated that 52 percent of the population qualified as OBC at the time. In 1980, the commission recommended that 27 percent reservation should apply to public sector entities. In 1993, the Union Government started to implement 27 percent OBC reservation for government-sponsored jobs. Since 2008 OBC reservations has been implemented in the admissions to publicly funded universities. The OBC population is very heterogenous as opposed to the SC/ST who have been very deprived almost without any exception. Among the OBC, some subcastes have been obtaining influential positions thanks to their landholding positions. Also, the OBC suffered less from discrimination compared to SC/ST. Many subcastes in OBC had the opportunity to improve their economic statuses and cannot be considered anymore as disadvantaged.

The SC, ST, and OBC reservations were referred to as **vertical reservations** by the Supreme Court of India (SCI) in its historic judgement in *Indra Sawhney and others vs Union of India*,[1] henceforth Indra Sawhney (1992). According to this judgement , not only are certain fractions of seats at publicly funded schools and government jobs secured for members of SC, ST, and OBC, but members of these reserved categories might also obtain open category positions on the basis of merit. In current practices, applicants from SC, ST, and OBC are first considered for open category positions. If they cannot secure an open category position, they are then considered for their respective reserved category seats. When a reserved category candidate obtains an open category seat, she is said to obtain it based solely on merit. Seats taken from open category by SC, ST, and OBC applicants based solely on merit are *not* counted against their vertical reservations.

In admissions to publicly funded educational institutions and allocations of government-sponsored jobs in India, reserved category applicants are asked to submit their preferences *only* over institutions. It is assumed that each reserve category applicant is indifferent between obtaining a position via a reserved category position and an open category position with a given institution. Even though this might be the case for some reserve category applicants, in the next section we provide evidence that some reserve category applicants differentiate reserved category seats and open category seats, and have preferences over them.

Preferences of Reserve Category Applicants[2]

Being vertical reserve eligible by itself is not sufficient to be considered for reserved category positions. Applicants must reveal their vertical category membership in application forms to be able to avail this concession. The revelation of vertical reserve eligibility is

[1]The judgement is available at https://indiankanoon.org/doc/1363234/ (last accessed 12/10/2019). Sönmez and Yenmez (2019a,b) relates the legal framework starting with the Indra Sawhney (1992) judgement to matching literature.

[2]We thank Alex Teytelboym for his suggestions and feedbacks on this section.

optional. Applicants from reserved categories are considered *only for open category positions* if they choose not to reveal their vertical category membership, i.e., not specifying their SC/ST/OBC membership in the application forms. When reserved category applicants reveal their vertical category membership, then they might be assigned to the same institutions in one of *two* ways: either through an open category position or a reserved category position. We argue in this section that *some* reserved category applicants care about the way through which they are admitted.

Allocation procedures must give the correct incentives to applicants to reveal their vertical category memberships voluntarily. In India, this is not the case because applicants can submit their preferences only over institutions. It is ubiquitous that some reserved category applicants do not avail their reservation facilities. By not revealing their vertical category membership they apply to positions as GC candidates. Evidence of this practice can be seen in many court cases, in online discussion forums, and in research articles. The following quote from a recent case, *Shilpa Sahebrao Kadam And Another vs The State of Maharastra And ...* (2019),[3] highlights this point:

> "*The petitioners contend that though they belong to reserved category, they have filled in their application forms as a general category candidates and have not claimed any benefit as a member belonging to reserved category.*"

Gille (2013) analyzes the determinants of reserved category members' applications for the reservation policy in education, and in particular on the role of social stigma[4] attached to the reservation policy. The author focuses on the OBC applicants and analyze the impact of the social positions of the individuals' reference group on their choice of applying for reservation.[5] She finds that, for a given wealth level, individuals who are from socially higher subcastes at the village level are less prone to apply for reservations in education than individuals with lower subcastes.

Pandey and Pandey (2018) argue that students from reserve categories are likely to face humiliation and harassment from their teachers and fellow students. The authors report, from a survey at an IIT campus, that 13 percent of students in SC/ST caste categories felt teacher attitudes toward them were hostile. Moreover, they report that 21 percent of

[3] https://indiankanoon.org/doc/89017459/ (last accessed on 02/12/2020).

[4] Moffitt (1983) defines stigma as the "disutility arising from the participation in a welfare program per se". Gille (2013) adopts this definition. She argues that the disutility comes from a psychological cost due to negative self images from the participation and from negative social attitudes towards the claimants.

[5] Gille (2013) argues that since the SC is at the bottom of the social hierarchy they may not care about the social stigma because they have nothing to lose in terms of social status. She discusses that the cost of social stigma is higher for people with higher social status. She further argues that, according to the literature on Indian caste system, status is even more important for the OBC.

students in SC/ST caste categories found the attitudes of fellow students hostile compared to zero percent in the GC. In an email exchange with Dr. Priyanka Pandey (Economist at the World Bank) and Dr. Sandeep Pandey (Mechanical Engineering Professor), Dr. Sandeep Pandey stated[6]

> *"Reserved category students would prefer to be selected on open seats to avoid the stigma."*

Gudavarthi (2012) explains the stigma as follows:

> "*The singularly debilitating limitation of the system of reservations in India has been to increasingly produce a large number of social groups that suffer forms of public humiliation, resentment and insult. The purpose of reservations to provide the disadvantaged social groups a head start in realizing their potential remains arrested and minimal, due to their inability to overcome the stigma that is attached to such policies.*"

In a correspondence by email, Dr. Ajay Gudavarthi (Political Science Professor in Jawaharlal Nehru University) stated[7]

> *"Of course if reserve category candidates can qualify in the general category, many of them would wish to do that for both demonstrating merit and also leave a vacancy for their caste-fellows. It is a different matter that many may not be able to qualify in the general category."*

Moreover, a cursory search on the internet reveals how pervasive it is that many reserved category candidates apply to positions without claiming the benefit of their vertical category. In Quora—one of the most popular online discussion forums—users exchanged ideas as a response to the following question: *What happens if an SC applicant fills an application form as a GC member in India?*[8] The following quote from a user (Anand Ganesaiyer) indicates the prevalence of the practice that many reserve category candidates do not claim the benefit of reservation policy:

> *"I know several persons from SC community who applied in general category and got admissions and government positions. There were several cases of SC candidates contesting and winning from general constituencies in Kerala... In short, availing a reservation facility is an option and not compulsory."*

[6]S. Pandey (personal communication, 05/13/2020).

[7]A. Gudavarthi (personal communication, 05/13/2020).

[8]This discussion is available at https://www.quora.com/What-happens-if-an-SC-fills-a-form-as-a-general-category-member-in-India (last accessed 05/05/2020).

It is important to note that reserved category candidates know as a matter of fact that claiming the benefit of reservation policy help them obtaining positions they want. The following quote from another user (Raksha Singh) in the same discussion manifests the dilemma reserve category applicants have been going through:

> *"Sir, reserved people bear the brunt of the heat from both side. If they don't use reservation, then you say they are blocking general category people. If they are using reservation then they are being blamed too. What do they do!!!"*

An anonymous user responded in the same discussion as follows:

> "*The thing is I'd have got a General seat even if I've filled up the form as an SC because I secured All India Rank 02. I've done that intentionally, it's not that I lost my caste certificate or anything which made me doing that. I've done that because I knew I could do it in one go; confidence you name it probably... I want to lead life of a general candidate professionally, I don't want any extra bucks/promotions. I want what I deserve as a general candidate."*

This user did not reveal his SC membership because his merit score was high enough to secure him an open category position in his top choice institution. Since candidates submit their preferences over institutions and vertical category membership after knowing their merit scores, they have a good understanding regarding their placements with or without revealing their vertical category membership.

We argue that when vertical category revelation is optional and some reserve category students differentiate open and reserve category positions, neglecting their preferences over position types leads to significant implementation issues, which we discuss next.

Implementation Issues with the Current Preference Domain

When reserve category applicants are not able to express their preferences over open and reserved category positions it might cause severe implementation issues. Currently, to allocate positions in publicly funded educational institutions and government sponsored jobs, each applicant is asked to report

1. a ranking of institutions alone, and

2. a vertical category membership for which the individual is considered for reserve category slots.

If an applicant does not declare a vertical category membership, then she will be considered only for open category seats. Individuals who report their vertical category membership in SC, ST, or OBC are first considered for open category seats. If they are not able to obtain an open category position, they will then be considered for reserved positions in their respective vertical reserve category. Optional vertical category revelation becomes critical in this setting as it might create a channel for reserved category students to strategically manipulate the system. To illustrate this point consider the following example. Suppose an ST candidate who cares about the vertical type she is admitted under reports that she prefers institution a over institution b. Suppose also that she reveals her ST membership. Then, she will be considered for positions in the following order: (1) a under open category, (2) a under ST, (3) b under open category, (4) b under ST. If she does not reveal her vertical category membership, then submitting the same preference over a and b she will be considered positions in the following order: (1) a under open category, (2) b under open category.

A similar issue arises in cadet-branch matching in the United States Military Academy (USMA) (Sönmez and Switzer, 2013). In cadet-branch matching, each cadet is asked to choose (1) a ranking of branches alone, and, (2) a number branches (possibly none) for which the cadet is asked to sign a branch-of-choice contract.[9] Then, using the branch preferences along with the set of signed branch-of-choice contracts, *a particular preference relation*[10] over branch-service time pairs is *constructed* to implement the DA algorithm. The DA mechanism with respect to these constructed preferences is an *indirect* mechanism because the strategy space available to cadets is not rich enough to express certain preferences over branch-service time pairs. Sönmez and Switzer (2013) show that there may not be a weakly dominant strategy that elicits branch preferences truthfully under this mechanism. The authors argue that one cannot assume submitted branch preferences are necessarily truthful under this mechanism. Moreover, they show that the difference between the strategy spaces results in the DA mechanism losing most of the desirable properties.

Similarly, one cannot assume that submitted preferences over institutions are necessarily truthful in admissions to publicly funded educational institutions and government-sponsored

[9]According to the Branch-of-Choice incentive program, a cadet may agree to serve three additional years in exchange for increased priority in branches of their choice.

[10]The USMA mechanism assumes that each cadet prefers base year contract (with five years of service requirement) over branch-of-choice contract (with eight years of service requirement) at branches she agrees to sign branch-of-choice contracts in exchange for higher priorities. That is, if a cadet reports that she prefers branch a over branch b and that she wants to sign branch-of-choice contracts with both branches, then the USMA mechanism constructs the following proxy preference relation for this cadet: (1) branch a with base year contract, (2) branch a with branch-of-choice contract, (3) branch b with base year contract, and (4) branch b with branch-of-choice contract.

jobs in India. The strategy space available to reserve category applicants is not large enough. When applicants care about the vertical category they are admitted under, the DA mechanism is not a direct mechanism if applicants are asked to choose (1) a ranking of institutions alone, and, (2) a vertical category membership for which they are considered for reserve category slots. The DA mechanism loses most of the desirable properties, if not all.

Expanding the strategy space of reserved category candidates to institution-vertical category pairs overcomes the implementation issues discussed above. Reserved category applicants who are indifferent between the open category and reserved category positions and care only about the institutions can express their preferences in the expanded strategy space. A valid concern regarding the expanded strategy space for reserved category candidates would be collecting more complex and longer preferences. This might not be an issue in many allocation problems in India for the following reasons:

1. In most applications, the number of institutions that applicants rank is small. For example, in the allocation of civil service positions (Thakur, 2018), candidates rank state cadres. In total, there are 24 cadres that applicants are asked to rank.

2. In all applications, applicants are asked to submit their preferences after they know their merit scores and ranks in related categories. The cutoff scores of each institution for each vertical category in previous years are public information. Thus, reserved category applicants have a very good understanding regarding their placements with and without availing the reservation benefits.

1.2 Horizontal Reservations

There are also special reservations that are referred to as **horizontal reservations**.[11] These reservations are provided for other disadvantaged groups, such as *women*, *disabled people*, *dependents of freedom fighters*, *children of deceased/disabled soldiers*, and *people from hill areas*. The implementation of horizontal reservations is quite different than that of vertical reservations. According to the judgement in Indra Sawhney (1992), when a horizontal reserve eligible individual obtains a seat without using this benefit, her seat is counted against the horizontal reservations. It is also emphasized in the same judgement that if the allocation is solely meritorious and satisfies the horizontal reservations, then no adjustment is made.

From the perspective of policy makers, there has been confusion regarding how to jointly implement vertical and horizontal reservations. The judgement in Indra Sawhney (1992)

[11] Horizontal reservations are enabled by Article 16(1) in the Indian Institution after the Supreme Court judgement in Indra Sawhney (1992). While vertical reservations were studied by Aygün and Turhan (2018), the joint implementation of vertical and horizontal reservations in the context of Indian resource allocation problems from market design perspective was first studied by Sönmez and Yenmez (2019a,b).

does not provide an exact procedure. This ambiguity led to widespread confusion and many court cases. In 1995, the SCI provided a procedure in its judgement in *Anil Kumar Gupta Etc. vs. State of Uttar Pradesh And Others*,[12] henceforth Anil Kumar Gupta (1995). The judgement in Anil Kumar Gupta (1995) has been the main reference with regard to the joint implementation of vertical and horizontal reservations. The judgement clarifies that horizontal reservations are to be implemented *separately within each vertical category,* including the open category. We follow this judgement and model horizontal reservations to be implemented separately within each vertical category.

Horizontal reservations introduce challenges in current practices in India due to the lack of well-defined procedures to implement horizontal reservations when integrated with vertical reservations. The judgement in Anil Kumar Gupta (1995) does not define a procedure when there are applicants who belong to multiple horizontal types. In the presence of applicants with multiple horizontal types, the order in which horizontal reservation types are filled is consequential. The use of different orders in different allocation processes has caused widespread confusion and many court cases.

From the theoretical point of view, there are two types of practical applications in India: with and without hierarchical horizontal reservation structure. We say that horizontal reservations are hierarchical if, for any pair of horizontal reservation types, having one type implies having the other, or else no applicant has both of them simultaneously. In the allocation of government positions by the Union Public Service Commission, the horizontal reservations are hierarchical (Thakur, 2018). On the other hand, in admissions to technical universities, horizontal reservations are not hierarchical (Baswana et al., 2018). When horizontal reservations are not hierarchical, then there might be complementarities between applicants, which might cause non-existence of stable allocation. In this work, we focus on applications with hierarchical horizontal reservation structures.

1.3 Sequential Admission Procedures in Practice

In admissions to publicly funded educational institutions and allocation of government sponsored jobs, positions are allocated in multiple rounds spaced out over several months.[13] There are two major reasons for sequential processes to be implemented in practice.

[12] The judgement is available at https://indiankanoon.org/doc/1055016/ (last accessed on 11/23/2019). This case is discussed in detail in Sönmez and Yenmez (2019a,b).

[13] The recently reformed admission process to technical universities is an exception. See Baswana et al. (2018) for details.

1. The first one is due to de-reservations of vacant OBC seats. Even though vacant SC and ST seats **cannot** be given to others, the situation is different for vacant OBC seats. In its historic judgement in Ashoka Kumar Thakur (2006), the SCI made the following decree that makes it possible to revert vacant OBC seats to GC applicants for admissions to publicly funded educational institutions:

> "*...Only non-creamy layer OBCs can avail of reservations in college admission, and once they graduate from college they should no longer be eligible for post-graduate reservation. 27% is the upper limit for OBC reservation. The government need not always provide the maximum limit. Reasonable cut off marks should be so that standards of excellence are not greatly affected. The unfilled seats should revert to the general category... Under such a scheme, whenever non-creamy layer OBCs fail to fill the 27% reservation, the remaining seats would revert to general category students.*"

Even though SCI's decision above suggests that vacant OBC seats would revert to GC students, it does not provide a well-defined procedure to implement this. In all applications where vacant OBC seats are transferred to GC students, a sequential procedure is used.

In admissions to publicly-funded educational institutions and allocation of government sponsored jobs, additional rounds for OBC de-reservations are implemented months later than the main round of admissions.[14] This means additional uncertainty and stress for applicants. Practically speaking, adding another round of admission is costly both for candidates and the government. Moreover, these sequential procedures may incentivize applicants to misreport their preferences. Both theoretically and practically, there are great benefits to allocating all available positions in a single round. Theoretically, it has been shown that sequential assignment procedures are incompatible with equity and efficiency. In our design, all available positions are allocated in a single round by carefully designing institutional choice rules that can also incorporate OBC de-reservations.

Baswana et al. (2018) designed a joint seat allocation process based on the deferred acceptance algorithm for admissions to technical universities in India. Since 2015, their heuristic have been implemented. Baswana et al. (2018) report how they solve the issue of OBC de-reservation as follows:

> "*Our approach was remarkably simple: Run the core algorithm with no de-reservations to completion. Move vacant seat capacity in each OBC virtual program to the*

[14]The new admission procedure for technical universities (adopted since 2015) is an exception. See Baswana et al. (2018) for details.

corresponding open virtual program. Rerun the core algorithm. Iterate until convergence."

In the solution of Baswana et al. (2018), positions are transferred from OBC to open category that precedes OBC in the choice procedures of institutions.[15] In our design, institutions' choice procedures first fill open category positions. Then, reserve category positions are filled in any order. If there is any vacancy in OBC category, then these vacancies are allocated at the very end. We introduce several OBC de-reservation schemes. In one, vacant OBC seats are provided to GC applicants only. In another one, vacant OBC seats are allocated on the basis of merit and open to all candidates, regardless of their vertical category membership.

2. There is also a commonly adapted sequential procedure in India that allocates positions in two stages due to high-performing reserve category candidates. In such procedures, open category positions are tentatively filled in the first stage, and SC/ST/OBC positions are filled in the second stage while reconsidering SC/ST/OBC candidates who received an open category position in the first stage, i.e., meritorious reserve candidates. The tentative allocation of open category positions causes problems.[16]

1.4 Overview of Model and Results

We model matching markets for publicly funded educational institutions and government jobs in India as follows: There are individuals and institutions to be matched. Each institution initially reserves a certain number of its seats for individuals from SC, ST, and OBC. Candidates from SC, ST, and OBC are able to obtain open category seats as well. The GC applicants and reserved eligible SC, ST, and OBC candidates who do not declare their vertical categories are considered only for open category seats. When there is low demand from SC and ST categories, their unfilled reserved seats remain vacant. With regard to vacancies in OBC category, depending on the application, there are two possibilities: (1) Unfilled OBC seats remains vacant in the case of low demand, or (2) Unfilled OBC seats are provided for

[15] We call this type capacity transfers as "*backward transfers*" in Aygün and Turhan (2020), where we discuss the unintended consequences of implementing backward capacity transfers in the context of admissions to technical universities in India. The backward transfer approach, for example, might cause the following issue: For a given application pool, there might be a set of SC/ST students who are assigned to open category positions due to OBC de-reservations but would be assigned SC/ST positions in the absence of OBC de-reservations. That is, when the number of open category positions increases due to OBC de-reservation and (some of) these additional positions are taken by SC/ST candidates, and this might cause vacancies in SC/ST categories. Since de-reservation of SC/ST vacancies are not allowed, it might lead to inefficiencies. In other words, high-scoring SC/ST applicants might get additional open category positions due to OBC de-reservation, which leads to vacancies in SC/ST reservations that cannot be utilized by GC applicants.

[16] Since Sönmez and Yenmez (2019b) address issues caused by such sequential procedures in detail, we do not provide additional details here. However, it is important to note that our design overcomes these issues.

the use of others. At each institution, open category seats are filled first. Then, SC, ST, and OBC positions are filled. In markets where vacant OBC seats are de-reserved, GC applicants are considered after filling the SC, ST, and OBC seats. Each individual has a preference over institution-category pairs.[17] Individuals care not only about which institution they are matched to but also about the category under which they are admitted.

We design a sub-choice function, C^{hier}, that integrates meritocracy and hierarchical horizontal reservations within a vertical category. Given a set of applicants who have the same vertical category membership, the choice rule C^{hier} selects applicants so that (i) horizontal reservations are satisfied, and (ii) the *"best"* set of individuals with respect to merit scores is selected among the set of individuals that satisfy the horizontal reservations. We define a comparison criteria , *merit-based domination*[18], to compare different choice rules on the basis of merit. Our comparison criteria is an *incomplete* binary relation that makes domination comparison only when it is unambiguous. When horizontal reservation structure is hierarchical, then choice rules that satisfy horizontal reservations can always be compared on the basis of merit-based domination criteria we define. Therefore, for the resource allocation problem we consider, our comparison criterion always make the necessary comparisons.

We, then, design institutional choice functions that take OBC de-reservation policies into account. We propose three institutional choice functions, where each vertical category chooses applicants according to its respective C^{hier}, that differ with regard to OBC de-reservation policy: (1) without de-reservations, (2) vacant OBC seats are provided to GC applicants, and (3) vacant OBC seats are provided to remaining applicants according to their

[17] There are three approaches to model reserve category applicants' preferences. The first one is to define reserve category applicants' preferences only over institutions. Currently, in admissions to publicly funded educational institutions and allocation of government-sponsored jobs in India, this approach is actually implemented in practice. Even though it is assumed that applicants are indifferent between open category seats and reserve category seats, they are always considered for open category positions before reserved category positions. This approach leads to incentive issues with regard to revealing caste membership and preference submission over institutions when reserved category candidates also care about the type of seat they receive. There might also be welfare consequences due to this specific tie-breaking rule in favor of open category positions. The second approach is to define preferences for institution and vertical category pairs. This is the approach we take in this paper. The downside of this approach is that it might impose additional constraints with respect to stability. This is because it requires every applicant to submit strict preferences over institution-vertical category pairs while some reserve category applicants may truly be indifferent between different categories of the same institution. However, we believe that imposing some additional constraints with regard to stability is less of a concern than the incentive issues the first approach causes. The first approach not only places a strategic manipulation burden on reserve category applicants, it might also cause issues with regard to fairness and welfare. The third approach, which is a more general modeling choice, is to model reserve category applicants' preferences as weak preferences so that they are allowed to report both strict preferences and indifferences. Seymour and Ertemel (2019) offer a model with weak preferences and design a particular choice correspondence for institutions.

[18] This criterion was first introduced in the second chapter Aygün (2014). The current paper supersedes this chapter.

merit scores.

We propose the COM with respect to the institutional choice functions we design as our allocation mechanism. We do so because our design objectives for the allocation mechanism are *stability, strategy-proofness,* and *respect for improvements.* Stability ensures that (1) no individual is matched with an unacceptable institution-category pair, (2) institutional choices are respected, and (3) no individual desires a seat at which she has a justified claim under a given merit ranking and horizontal reservation structure. Strategy-proofness guarantees that individuals can never game the allocation mechanism via preference manipulation. In our framework, it also relieves individuals of the strategic manipulation burden, which involves whether or not individuals declare their vertical and horizontal category memberships.[19] Respect for improvements[20] is an essential property in meritocratic systems. In allocation mechanisms that respect improvements, students have no incentive to lower their standings in schools' priority rankings. For the matching problems we consider, respect for improvement incentivizes individuals to report every horizontal reservation type they have.

We design matching mechanisms that:

- respect vertical reservations at each institution,
- respect horizontal reservations within each vertical category at each institution,
- respect merit scores to the extent possible,
- take OBC de-reservations into account,
- consider applicants' preferences over both institutions and the category through which they are admitted under.

We propose the COM with respect to institutional choice rules we design as the allocation rule. We show that the COM is stable with respect to institutions' choice rules, strategy-proof for applicants, and respects improvements. Stability, strategy-proofness, and respecting improvements have desirable normative implications for resource allocation problems in India with vertical and horizontal reservations. Stability implies a form of fairness in the sense that merit scores are respected to the extent possible under vertical and horizontal reservation constraints, and OBC de-reservations. Strategy-proofness ensures not only that applicants report their preferences over institution-category pairs truthfully but also that they report their vertical category truthfully. Respect for improvement makes it a weakly dominant strategy for applicants to report all of their horizontal types.[21]

[19] Strategy-proofness ensures that it is a weakly dominant strategy for each student to report their vertical category membership.

[20] See Kominers (2019) for detailed discussion of respect for improvements in matching markets.

[21] Respect for improvements in our setting also implies *privilege monotonicity* property of Aygün and Bó

1.5 Related Literature

This paper contributes to the literature on resource allocation problems in India with comprehensive affirmative action constraints from the mechanism design perspective. Baswana et al. (2018) designed and have been implementing a centralized joint seat allocation process for technical universities in India. Our model is different from theirs in that we consider individuals' preferences over vertical categories as well, while their analysis takes individuals' preferences only over institutions. In their solution, they re-run the deferred acceptance algorithm multiple times to adjust OBC de-reservation, while we use simple capacity transfer schemes to incorporate OBC de-reservation in a single run of the algorithm. Thakur (2018) studies the allocation of government positions by the Union Public Service Commission in India. Unlike our work, Thakur (2018) does not take horizontal reservations into account. The author considers applicants' preferences over institutions only as in Baswana et al. (2018).

Sönmez and Yenmez (2019a,b) formulate joint implementation of vertical and horizontal reservations while Aygün and Turhan (2018) formulate vertical reservations only. Sönmez and Yenmez (2019a) analyze the shortcomings of the choice procedure given in Anil Kumar Gupta (1995), and provide an alternative choice rule. Sönmez and Yenmez (2019b) criticize the widespread practice of tentative allocation of the open positions and offer an alternative design. Sönmez and Yenmez (2019a,b) take individuals' preferences only over institutions while we argue that some reserve category candidates, if not all, do care about the category through which they are admitted. The difference is crucial because reserve category students who prefer open category seats might be able to manipulate the allocation mechanism by not revealing their vertical category, when are asked to report their preferences only over institutions. To solve this important issue, we model individuals' preferences as strict preferences over institution-seat category pairs. Moreover, our design takes OBC de-reservations into account while Sönmez and Yenmez (2019a,b) do not consider the transfer of vacant OBC seats.

Other notable papers on affirmative action and diversity constraints include Abdulkadiroğlu and Sönmez (2003), Abdulkadiroğlu (2005), Hafalir et al. (2013), Ehlers et al. (2014), Westkamp (2013), Echenique and Yenmez (2015), Aygün and Bó (2016), Kominers and Sönmez (2016), Kurata et al. (2017), Fragiadakis and Troyan (2017), Nguyen and Vohra

(2016). Privilege monotonicity suggests that when an applicant applies to an institution under a certain vertical category, claiming an additional privilege (horizontal type in our setting) should not decrease her chance of being accepted. As a result, when an applicant applies to an institution under a certain vertical category that uses a sub-choice function with that property, it is always safe for her to claim all the horizontal types she can. As indicated by Aygün and Bó (2016), this creates strategic simplicity for those applicants when it comes to the decision of which horizontal types to claim. When sub-choice functions are not privilege monotonic there are circumstances in which, in order to be accepted, the applicant should not claim some horizontal types.

(2019), and Avataneo and Turhan (2020) among others.

The matching problems we consider in this paper are special cases of the matching with contracts model of Fleiner (2003) and Hatfield and Milgrom (2005). Important work on matching with contracts include Hatfield and Kojima (2010), Aygün and Sönmez (2013), Afacan (2017), Hatfield and Kominers (2019), and Hatfield et al. (2017, 2019), among others.

The present paper technically builds on our companion paper, Aygün and Turhan (2019), that develops a theory of matching with generalized lexicographic choice rules. This theory is built on the novel observability theory of Hatfield, Kominers, and Westkamp (2019). We model institutions' choice rules in the present paper as generalized lexicographic choice rules.

We model horizontal reservations as hierarchical constraints, which is similar to the laminar families that have been studied in market design.[22] Examples include Biro et al. (2010), Kamada and Kojima (2015, 2017, 2018), Goto et al. (2017), and Kojima et al. (2018). Moreover, Milgrom (2009) and Budish et al. (2013) study laminar families in auction and indivisible object allocation settings, respectively.

Our paper is related to many other papers that study policy relevant real-world allocation problems in different contexts. Some examples of such allocation problems include refugee resettlement (Delacrétaz, Kominers, and Teytelboym, 2016, Andersson, 2017, and Jones and Teytelboym, 2017), assignment of arrival slots (Schummer and Vohra, 2013, and Schummer and Abizada, 2017), course allocation (Sönmez and Ünver, 2010, Budish, 2011, and Budish and Cantillon, 2012), and organ allocation and exchange (Roth et al., 2014, Ergin et al. 2017, 2020), among many others.

2 The Model

There is a finite set of institutions $\mathcal{S}$, i.e., publicly funded educational institutions or government jobs, depending on the application, and a finite set of individuals $\mathcal{I}$, i.e., students or job candidates, depending on the application. Institution $s \in \mathcal{S}$ has $\overline{q}_s$ seats.

Vertical reservations. There are three designated communities whose members are eligible for vertical reservations, $\mathcal{V} = \{SC, ST, OBC\}$.[23] Individuals who do not belong to a

[22] A family of sets is said to be laminar family if, for any pair of sets in this family, either they are disjoint or one of them is a subset of the other. That is, the structure can be described as layers or a hierarchy.

[23] In some applications in India, vertical reservation groups are divided into sub-categories. OBC reservations, for example, are divided into sub-categories. For notational simplicity, we model reserve categories as SC, ST, and OBC only. The whole analysis can be straightforwardly extended to sub-categorization. In January 2019, the Union Cabinet in India approved a 10% reservation of government jobs and educational institutions for the Economically Weaker Section (EWS) in the GC. Our model and results can be

vertical reserve eligible category are considered to be the members of the GC. We denote the set of all vertical categories by $\mathcal{T} = \{SC, ST, OBC, GC\}$. The correspondence $t : \mathcal{I} \rightrightarrows \mathcal{T}$ denotes vertical reserve eligibility of individuals. That is, $t(i) \subseteq \mathcal{T}$ is the set of vertical categories that individual i is eligible to claim. For every individual $i \in \mathcal{I}$, $GC \in t(i)$. That is, every individual is a member of the GC. The correspondence t is such that for every pair $v, v^{'} \in \mathcal{V}$ such that $v \neq v^{'}$, we have $t^{-1}(v) \cap t^{-1}(v^{'}) = \emptyset$. That is, an individual can belong to at most one category in $\mathcal{V}$. Vertical reservations for ST, SC, and OBC categories are implemented by setting aside a number of seats for each one of these categories. Let $\overline{v}_s^{SC}$, $\overline{v}_s^{ST}$, and $\overline{v}_s^{OBC}$ denote the number of seats that are set aside for SC, ST, and OBC categories at institution $s \in \mathcal{S}$, respectively. The vector $\overline{\mathbf{v}}_s = (\overline{v}_s^{SC}, \overline{v}_s^{ST}, \overline{v}_s^{OBC})$ denotes the vector of vertical reservations at institution s. Then, $\overline{v}_s^{O} = \overline{q}_s - \left(\overline{v}_s^{SC} + \overline{v}_s^{ST} + \overline{v}_s^{OBC}\right)$ is the number of open category seats. According to Indra Sawhney (1992), when SC/ST/OBC applicants are assigned open category positions, these positions are not counted against their respective vertical category.

Horizontal reservations. In addition to a vertical category membership, each individual has a set of *horizontal types*. Let $\mathcal{H} = \{h_1, ..., h_{|\mathcal{H}|}\}$ be the set of all horizontal types, i.e., h_1 denotes being a woman, h_2 denotes being disabled, etc. The correspondence $\rho : \mathcal{I} \rightrightarrows \mathcal{H} \cup \{h_0\}$ represents individuals' horizontal type eligibility. That is, $\rho(i) \subseteq \mathcal{H} \cup \{h_0\}$ denotes the set of horizontal types that individual i can claim. If $\rho(i) = h_0$, then individual i does not have any horizontal type. Following the judgement in Anil Kumar Gupta (1995), we model horizontal reservations so that they cut across vertical reservations. That is, horizontal reservations are implemented at each vertical category, including the open category. At each vertical category a certain minimum number of individuals for each horizontal type must be selected, whenever possible. We denote by $\overline{h}_{(v,s)}^{j}$ the number of reserved seats for horizontal type $h_j \in \mathcal{H}$ at vertical category $v \in \mathcal{T}$ in institution $s \in \mathcal{S}$. We denote the vector of horizontal reservations for institution s in vertical category $v \in \mathcal{T}$ by $\overline{\mathbf{h}}_{(v,s)} \equiv (\overline{h}_{(v,s)}^{1}, ..., \overline{h}_{(v,s)}^{|\mathcal{H}|})$. Let $\overline{\mathbf{h}}_s \equiv \left\{\overline{\mathbf{h}}_{(v,s)}\right\}_{v \in \mathcal{T}}$ denote the horizontal reservations for institution s.

We say that horizontal reservations, $\overline{\mathbf{h}}_{(v,s)}$, are *applicable* for vertical category v in institution s if, for every set of individuals $I \subseteq \mathcal{I}$, there exists a subset $J \subseteq I$ such that

1. $\mid J \mid \leq \overline{v}_s^{v}$,
2. if $J \cap \rho^{-1}(h_j) < \overline{h}_{(v,s)}^{j}$ implies $(I \setminus J) \cap \rho^{-1}(h_j) = \emptyset$, for every $h_j \in \mathcal{H}$.

That is, horizontal reservations are applicable for vertical category v if the total capacity of vertical category v and number of reserved positions in each horizontal type are respected.

straightforwardly extended for any number of vertical categories.

We assume that horizontal reservations are applicable for all vertical categories at each institution.

Merit scores. Each individual has a *merit score* at each institution. The function $\kappa : \mathcal{I} \times \mathcal{S} \rightarrow \mathbb{R}_+$ denotes individuals' merit scores at institutions. We denote by $\kappa(i,s)$ the merit score of individual i at institution s. We assume that no two individuals have the same score at a given institution.[24] That is, for all $i,j \in \mathcal{I}$ and $s \in \mathcal{S}$ such that $i \neq j$, we have $\kappa(i,s) \neq \kappa(j,s)$. The function κ induces strict meritorious ranking of individuals at each institution.

Contracts. We define $\mathcal{X} \equiv \underset{i\in\mathcal{I}}{\cup} i \times \mathcal{S} \times t(i)$ as the set of all contracts. Each contract $x \in \mathcal{X}$ is between an individual $\mathbf{i}(x)$ and an institution $\mathbf{s}(x)$ and specifies a vertical category $\mathbf{t}(x)$ under which individual $\mathbf{i}(x)$ is admitted. There may be multiple contracts for each individual-institution pair. We extend the notations $\mathbf{i}(\cdot)$, $\mathbf{s}(\cdot)$, and $\mathbf{t}(\cdot)$ to the set of contracts, for any $X \subseteq \mathcal{X}$, by setting $\mathbf{i}(X) \equiv \underset{x\in X}{\cup}\{\mathbf{i}(x)\}$, $\mathbf{s}(X) \equiv \underset{x\in X}{\cup}\{\mathbf{s}(x)\}$ and $\mathbf{t}(X) \equiv \underset{x\in X}{\cup}\{\mathbf{t}(x)\}$. For $X \subseteq \mathcal{X}$, we denote $X_i \equiv \{x \in X \mid \mathbf{i}(x) = i\}$; analogously, we denote $X_s \equiv \{x \in X \mid \mathbf{s}(x) = s\}$, and $X_t \equiv \{x \in X \mid \mathbf{t}(x) = t\}$.

Individuals' preferences. We define individuals' preferences over institutions and vertical category pairs. That is, each individual $i \in \mathcal{I}$ reports a strict preference relation P_i over $(\mathcal{S} \times t(i)) \cup \{\emptyset\}$, where $\emptyset$ denotes the outside option.[25] We write, for example,

$$(s,v)P_i(s^{'},v^{'})$$

to mean that admission to institution s through vertical category v is strictly preferred over admission to institution $s^{'}$ through vertical category $v^{'}$. The *at-least-as-well* relation R_i is obtained from P_i as follows: $(s,v)R_i(s^{'},v^{'})$ if and only if either $(s,v)P_i(s^{'},v^{'})$ or $(s,v) = (s^{'},v^{'})$. An institution and vertical category pair (s,v) is *acceptable* to individual i if it is at least as good as the null contract $\emptyset$, i.e., $(s,v)R_i\emptyset$ and is *unacceptable* to her if it is worse than the outside option $\emptyset$, i.e., $\emptyset P_i(s,v)$. We assume that for all $v \in \mathcal{V}$ such that $v \notin t(i)$, $\emptyset P_i(s,v)$ for all $s \in \mathcal{S}$. Individuals have *unit demand* in the sense that they choose at most one contract from a given set of contracts. For each individual $i \in \mathcal{I}$ and a set of pairs

[24]In current practices in India, when two or more applicants have the same score, ties are broken with respect to some exogenously given objective criteria.

[25]We have not found any evidence that individuals care about the horizontal types under which they are admitted. Therefore, we assume that individuals do not have preferences for horizontal types.

$X \subseteq \mathcal{S} \times t(i)$, the chosen set of individual i is defined as

$$C_i(X) \equiv \underset{P_i}{max} X \cup \{\emptyset\}.$$

Institutions' selection criteria. Institutions in India with *multi-unit demand* have complicated selection procedures when vertical and horizontal reservations are jointly implemented. In what follows, we describe institutions' selection criteria via (overall) choice rules that have been proven to be useful tools in matching markets with complex constraints. In our context, a **choice rule** of an institution $s \in \mathcal{S}$ is a function $C^s(\cdot \mid \overline{q}_s, \overline{\mathbf{v}}_s, \overline{\mathbf{h}}_s)$, henceforth $C^s(\cdot)$ for short, such that for any set of contracts $X \subseteq \mathcal{X}$, number of available seats $\overline{q}_s$, vector of vertical reservations $\overline{\mathbf{v}}_s$, and vector of horizontal reservations $\overline{\mathbf{h}}_s$,

$$C^s(X) \subseteq X \textit{ such that } \mid C^s(X) \mid \leq \overline{q}_s.$$

We incorporate vertical reservations, horizontal reservations within each vertical category, respecting meritocracy, and de-reservation policies into institutions' overall choice rules. Furthermore, to comply with the spirit of the affirmative action and meritocracy, institutions' overall choice rules need to satisfy the following natural **fairness** criterion.[26]

For a given set of contracts $X \subseteq \mathcal{X}_s$, the chosen set of contracts $C^s(X)$ is **fair** if $\mathbf{i}(x) \notin \mathbf{i}[C^s(X)]$, then for every $y \in C^s(X)$ at least one of the following hold:

1. $\kappa(\mathbf{i}(y), s) > \kappa(\mathbf{i}(x), s)$,
2. $\mathbf{t}(x) \neq \mathbf{t}(y)$,
3. $\mathbf{t}(x) = \mathbf{t}(y)$ and $\rho(\mathbf{i}(x)) \not\supseteq \rho(\mathbf{i}(y))$.

That is, if all contracts of an individual are rejected, then chosen contracts must be associated with individuals who either have higher merit scores or claim (set-wise) more affirmative action characteristics (i.e., a combination of vertical reservation categories and horizontal reservation types). We say that an overall choice rule C^s is **fair** if for any given set of contracts $X \subseteq \mathcal{X}$, $C^s(X)$ is fair for institution s.

Matching

A *matching* is a set of contracts $X \subseteq \mathcal{X}$. We restrict our attention to the matchings that are feasible.

[26] The "fairness of a choice rule" criterion under affirmative action constraints was first introduced in Aygün and Bó (2016) in the context of Brazilian college admissions where students have multidimensional privileges.

Feasibility. A matching X is *feasible* if

- there is at most one contract for each individual in X, i.e., $\mid X_i \mid \leq 1$ for each $i \in \mathcal{I}$, and

- there are at most $\overline{q}_s$ contracts for each institution s in X, i.e., $\mid X_s \mid \leq \overline{q}_s$ for each $s \in \mathcal{S}$.

We are interested in matchings that are **fair** in the following sense.[27]

Fairness. A feasible matching $X \subseteq \mathcal{X}$ is *fair* if, for any given pair contracts $x, y \in X$, $(\mathbf{s}(y), \mathbf{t}(y)) P_{\mathbf{i}(x)} (\mathbf{s}(x), \mathbf{t}(x))$, then at least one of the following hold:

1. $\kappa(\mathbf{i}(y), s(y)) > \kappa(\mathbf{i}(x), s(y))$,

2. $\rho(\mathbf{i}(x)) \not\supseteq \rho(\mathbf{i}(y))$.

That is, a matching is fair if an individual envies assignment of another individual, then either the former individual has a lower merit score at that institution or the latter individual has horizontal reservation type that the former does not. Note that if $(\mathbf{s}(y), \mathbf{t}(y)) P_{\mathbf{i}(x)} (\mathbf{s}(x), \mathbf{t}(x))$, then individual $\mathbf{i}(x)$ has vertical type $\mathbf{t}(y)$, as well.

In the context of admissions to publicly funded educational institutions and government-sponsored jobs in India, this fairness criterion—that can be interpreted as respecting merit scores to the extent possible—is crucial. It says that at each vertical category in every institution, an individual with a higher score is given higher priority than an individual with lower score, unless the latter individual has some horizontal reservation types that the former does not.

Stability. A feasible matching $X \subseteq \mathcal{X}$ is *stable* if it is

1. *Individually rational:* $C^s(X) = X_s$ for all $s \in \mathcal{S}$, and $X_i R_i \emptyset_i$ for all $i \in \mathcal{I}$.

2. *Unblocked:* There does not exist a nonempty $Z \subseteq (\mathcal{X} \setminus X)$, such that $Z_s \subseteq C^s(X \cup Z)$ for all $s \in \mathbf{s}(Z)$ and $Z P_i X$ for all $i \in \mathbf{i}(Z)$.

Individual rationality requires that each individual finds her assignment, i.e., an institution-vertical category pair, acceptable. Individual rationality also requires that institutions' selection procedures are respected in the sense that when an institution is offered its set of

[27] A similar notion of fairness is defined in Aygün and Bó (2016) in the context of Brazilian affirmative action, where students have multidimensional privileges (types).

contracts, i.e., a set of pairs of individuals and vertical categories, it selects all of them. Unblockedness states that individuals and institutions cannot benefit by re-contracting outside of the match.

Remark 1. Stability crucially depends on how institutions' selection procedures are defined. In admissions to publicly funded educational institutions and the allocation of government-sponsored jobs in India, institutions' selection criteria embody legal requirements such as respect for vertical reservations, implementation of horizontal reservations within each vertical category including the open category, respect for merit scores, and OBC de-reservations. Stability, therefore, is a natural, desirable characteristic of an allocation. If each individual applies to only one institution, stability requires that the rules and regulations encoded in institutions' choice rules determine which contracts (i.e., which individual-vertical category pairs) are selected. Laws in India do not specify an exact procedure to determine the allocation of individuals to institutions when individuals apply to multiple institutions. Stability presents a natural desiderata for an allocation: an individual will only be matched to a less desirable institution if, by following the selection criteria of those institutions, she would not be accepted given the individuals (with vertical and horizontal category membership) who have been matched to these institutions. Unstable allocations, therefore, might lead to lawsuits from dissatisfied applicants.

Stability with respect to fair choice rules does not imply fairness of a matching. Our first result explains the relationship between fairness of a matching and stability with respect to fair choice rules.[28]

Proposition 1. *Fairness and stability with respect to fair choice rules are independent.*

3 Designing Institutional Choices

To design choice rules for Indian institutions, we first note that the selection criteria must take into account (1) vertical reservations, (2) horizontal reservations within each vertical category, (3) respect for merit scores (in conjunction with satisfying horizontal reservations), and (4) de-reservation schemes from OBC to other categories. We begin our analysis by designing a sub-choice function to be implemented within each vertical category.

The sub-choice function of a vertical category must satisfy horizontal reservations and take merit scores into account. In the absence of horizontal reservations, sub-choice functions

[28]Romm, Roth, and Shorrer (2020) generalize the definition of justified envy in matching with contracts environments and show that stable allocations might have justified envy. Our fairness definition for choice rules is different than their definition of strong priority, but the two notions are related.

must be induced from merit scores. In the presence of horizontal reservations, we design a particular sub-choice function[29] that selects the *best* set of applicants with respect to a criterion, *merit-based domination* , which we define in Section 4.1. This sub-choice function minimally deviates from the meritorious outcome while satisfying horizontal reservations.

Before proceeding to the design of the aforementioned sub-choice function, we define a specific horizontal reservation structure, *hierarchy*, that will be necessary for the existence of stable outcomes. Even though in some applications in India horizontal reservations fail to satisfy this condition, in many others they do.

Hierarchical structure of horizontal reservations. Horizontal reservation types are **hierarchical**, if for any pair of horizontal types $h_j, h_k \in \mathcal{H}$ such that $\rho^{-1}(h_j) \cap \rho^{-1}(h_k) \neq \emptyset$, either $\rho^{-1}(h_j) \subset \rho^{-1}(h_k)$ or $\rho^{-1}(h_k) \subset \rho^{-1}(h_j)$. We say that a horizontal reservation type $h_j \in \mathcal{H}$ **contains** $h_k \in \mathcal{H}$ if, for every individual $i \in \mathcal{I}$, $h_k \in \rho(i)$ implies $h_j \in \rho(i)$. That is, $\rho^{-1}(h_k) \subset \rho^{-1}(h_j)$.

We assume from now on that horizontal reserve structure is hierarchical.

Integrating Horizontal Reservations with Meritocracy

We denote by $C_v^s(\cdot \mid \overline{v}_s^v, \overline{\mathbf{h}}_{(v,s)})$ a sub-choice function of vertical category $v \in \mathcal{T}$ in institution $s \in \mathcal{S}$, where $\overline{v}_s^v$ is number of positions vertically reserved for category $v \in \mathcal{T}$ and $\overline{\mathbf{h}}_{(v,s)}$ is the vector of horizontal reservations at vertical category v. We use C_v^s to denote the sub-choice function of vertical category v in institution s for short, if there is no confusion.

An attractive property of sub-choice functions is the ability to select as many alternatives as possible without exceeding the capacity, $\overline{v}_s^v$. This is a commonly used property called *q-acceptance,* where q denotes the capacity. Technically, a sub-choice rule C_v^s is *q-acceptant* if for any set of contracts $X \subseteq \mathcal{X}_v$,

$$\mid C_v^s(X) \mid \leq min\{\mid X \mid, \overline{v}_s^v\},$$

where $\overline{v}_s^v$ is the capacity of vertical category v. That is, if the number of contract offers is less or equal to the capacity of the vertical category v, C_v^s selects all of them. If the number of alternatives is greater than the capacity of vertical category v, then it chooses as many alternatives as capacity allows.

Horizontal reservations must be satisfied within each vertical category. Given a set of contracts $X \subseteq \mathcal{X}$, we say that $Y \subseteq X_v$ **satisfies the horizontal reservation for** h_j at

[29]This sub-choice function was introduced in the second chapter of Aygün (2014) and also in Aygün (2017). The current paper subsumes and supersedes both of them.

vertical category v, if $\mathbf{i}(Y) \cap \rho^{-1}(h_j) < \overline{h}^j_{(v,s)}$ implies $\mathbf{i}(X_s \setminus Y_s) \cap \rho^{-1}(h_j) = \emptyset$. That is, if the number of individuals who have horizontal type h_j in the set $\mathbf{i}(Y)$ is less than or equal to the number of positions reserved for horizontal type h_j at vertical category v, $\overline{h}^j_{(v,s)}$, then there must be no individual who have horizontal type h_j in the set $\mathbf{i}(X \setminus Y)$. We say that $Y \subseteq$ **satisfies horizontal reservations** if it satisfies horizontal reservations for all h_j, $j = 1, ..., \mid \mathcal{H} \mid$ at each vertical category $v \in \mathcal{T}$.

Before introducing our sub-choice function that integrates horizontal reservations and meritocracy, we first define a criterion upon which we compare different choice rules on the basis of meritocracy.

Merit-based domination. We say that a set of individuals I *merit-based dominates* a set of individuals J with $\mid I \mid = \mid J \mid = n$ at institution s if there exists a bijection $g : I \longrightarrow J$ such that

1. for all $i \in I$, $\kappa(i, s) \geq \kappa(g(i), s)$, and,
2. there exists $j \in I$ such that $\kappa(j, s) > \kappa(g(j), s)$.

We now extend the merit-based comparison criterion that compares sets to a merit-based comparison criterion to compare q-acceptant choice rules. Consider q-acceptant sub-choice rules $\widetilde{C}$ and C. We say that $\widetilde{C}$ *merit-based dominates* C if, for any set of contracts $X \subseteq \mathcal{X}$, the set of individuals $\mathbf{i}(\widetilde{C}(X))$ merit-based dominates the set of individuals $\mathbf{i}(C(X))$. That is, the score of the *highest* scoring applicant in $\mathbf{i}(\widetilde{C}(X))$ is greater than or equal to the score of the *highest* scoring applicant in $\mathbf{i}(C(X))$, the score of the *second highest* scoring applicant in $\mathbf{i}(\widetilde{C}(X))$ is greater than or equal to the score of the *second highest* scoring applicant in $\mathbf{i}(C(X))$, and so on and so forth. Moreover, there exists an integer l such that the score of the l^{th} *highest* scoring applicant in $\mathbf{i}(\widetilde{C}(X))$ is greater than or equal to the score of the l^{th} *highest* scoring applicant in $\mathbf{i}(C(X))$.

Our domination relation does not induce a complete binary relation. It might be the case that two choice rules are incomparable with regard to our domination criterion. To illustrate this, consider the following example: Let $q = 2$ and $I = \{i_1, i_2, i_3, i_4\}$. Suppose that individuals' test scores are such that $\kappa(i_1) > \kappa(i_2) > \kappa(i_3) > \kappa(i_4)$. Suppose also that q-acceptant choice functions $\widetilde{C}(\cdot)$ and $C(\cdot)$ choose following sets of individual from I: $\widetilde{C}(I) = \{i_1, i_4\}$ and $C(I) = \{i_2, i_3\}$. According to our domination criterion, the sets $\{i_1, i_4\}$ and $\{i_2, i_3\}$ do not dominate each other. Therefore, q-responsive choice functions $\widetilde{C}(\cdot)$ and $C(\cdot)$ are incomparable with regard to our comparison criterion.[30]

[30]Sönmez and Yenmez (2019a) introduce a different domination criterion to compare two sets of individuals

Given a set of contracts, there might be multiple subsets that satisfy horizontal reservations in a given vertical category. We use the merit-based domination criteria defined above to determine which subset of applicants is more meritorious.

Definition 1. A sub-choice rule C^s_v for vertical category v in institution s is *merit-based undominated* if, for every set of contracts $X \subseteq \mathcal{X}$, (1) $C^s_v(X)$ satisfies horizontal reservations $\overline{h}_{(v,s)}$, and (2) $C^s_v(X)$ is not merit-based dominated by any set $Y \subseteq X_v$ that satisfies horizontal reservations.

In what follows, we introduce the sub-choice rule C^{hier} that chooses the best set of candidates that satisfy horizontal reservations from any given set of contracts.[31]

Sub-choice rule C^{hier} and Its Properties

We describe the sub-choice rule C^{hier} for vertical category $v \in \mathcal{T}$ and horizontal reservations $\overline{\mathbf{h}}_{(v,s)}$. Suppose $X \subseteq \mathcal{X}$ is offered to institution s. Then, $X_v \subseteq X$ is the set of contracts associated with vertical category v. Let $\mathbf{i}(X_v)$ be the set of individuals who have contracts in X_v.

Step 1: If no individual in $\mathbf{i}(X_v)$ has any horizontal type, then choose contracts of individuals with the highest merit scores for all seats. Otherwise, for every horizontal type h_j that does not contain any other horizontal type, choose the contracts of highest-scoring individuals up to the capacity. Adjust the number of available seats and the number of horizontal reservations for any horizontal type that contains h_j by the number of chosen contracts. Adjust the number of remaining seats and the number of horizontal reservations of the horizontal types that contain h_j. Eliminate h_j from the set of horizontal types. If there are no individuals or seats left, end the process and return the chosen set of contracts.

on the basis of merit. They say that a set of individuals I *dominates* J if there exists an individual in $I \setminus J$ with a merit score that is strictly greater than the merit scores of all individuals in $J \setminus I$. Their domination criterion induces a strict partial order. According to their criterion, in this particular example, the set of individuals $\{i_1, i_4\}$ dominates $\{i_2, i_3\}$, since the former set contains the highest score student i_1 while the latter does not. However, we believe that the comparison between $\{i_1, i_4\}$ and $\{i_2, i_3\}$ is rather ambiguous. Our domination criteria is not able to compare sets on the basis of merit when there is such ambiguity. Also, the following is an important consideration: In India, when applicants have the same merit score, ties are broken with some pre-determined criteria. Suppose that candidates' test scores are as follows: $\kappa(i_1) = 100$, $\kappa(i_2) = 100$, $\kappa(i_3) = 99$, and $\kappa(i_4) = 20$. Suppose that tie between i_1 and i_2 is broken in a way that favors candidate i_1. Applying Sönmez and Yenmez's (2019a) criterion leads to the conclusion that $\{i_1, i_4\}$ is a better set of applicants than $\{i_2, i_3\}$. However, considering merit scores, $\{i_2, i_3\}$ is not dominated by $\{i_1, i_4\}$.

[31] We describe the sub-choice rule C^{hier} formally in the Appendix A.

Step n (n≥2): If there is no horizontal type left to be considered, then choose contracts of individuals following the merit score ranking for the remaining seats. Otherwise, for every horizontal type h_j that does not contain any other remaining horizontal type, choose the contracts of highest-scoring individuals up to the adjusted capacity. Adjust the number remaining seats and the number of horizontal reservations for any horizontal reservation that contains individuals with h_j. Eliminate h_j from the set of horizontal reservations to be considered. If there are no individuals or seats left, the process ends and returns the chosen set of individuals.

We are now ready to present our first main result:

Theorem 1. $\mathbf{C}^{hier}$ *is the unique choice rule that is merit-based undominated.*

In other words, a choice rule is merit-based undominated if and only if it is C^{hier}. Theorem 1 normatively justifies the use of the sub-choice rule C^{hier} at each vertical category.

Overall Choice Rules: Incorporating Vertical and Horizontal Reservations with OBC de-reservations

The reservation policy in India is implemented in a lot of different admissions and recruiting processes at the national level in publicly funded educational institutions and government job positions. In some admission processes, such as admissions to technical universities, vacant OBC seats are required to be provided for the use of GC candidates. In some others, vacant OBC seats remain vacant. In this section, we integrate the sub-choice function C^{hier} with three different OBC de-reservation policies to define three different institutional choice procedures.

1. Choice Rule without Reverting Surplus OBC Seats to GC

In applications where vacant OBC seats are not allowed to be utilized by other applicants, we design a choice rule by applying C^{hier} in both steps of a *two-step* procedure to allocate GC seats in the first step and reserved category seats in the second step. We refer to this choice rule $C^{VHw/oT}$, i.e., the choice rule under vertical and horizontal reservations without transfers.

Choice Rule $C_s^{VHw/oT}$

Step 1: In this step, consider only contracts associated with GC. Use $C^{hier}(\cdot \mid \overline{v}_s^{GC}, \overline{h}_{(GC,s)})$ to select contracts. Remaining contracts of selected individuals are removed for the rest of the process.

Step 2: For each reserve eligible category $v \in \mathcal{V}$, consider only the contracts with term v. Then, for each category v, apply $C^{hier}(\cdot \mid \overline{v}_s^v, \overline{h}_{(v,s)})$ to the contracts with term v.

2. Choice Rule with Transfers from OBC to GC

We now introduce a choice rule in which all vacant OBC seats are reverted to GC. The first two steps of this new choice rule are as same as the choice rule. However, this new choice rule has an additional third step. In the third step, all of the seats reverted from OBC to GC are filled with candidates who offer contracts with the GC term. Contracts with SC and ST terms are not considered in the third step.

Choice Rule C_s^{VHwT}

Step 1: In this step, consider only contracts with the GC term. Use $C^{hier}(\cdot \mid \overline{v}_s^{GC}, \overline{h}_{(GC,s)})$ to select contracts. Remaining contracts of selected individuals are removed for the rest of the process.

Step 2: For each reserve eligible category $v \in \mathcal{V}$, consider only the contracts with term v. Then, for each category v, apply $C^{hier}(\cdot \mid \overline{v}_s^v, \overline{h}_{(v,s)})$ to the contracts with term v. Remaining contracts of selected individuals are removed for the rest of the process.

Step 3: Let $\overline{v}^{OBC \to GC}$ be the number of vacant OBC seats at the end of Step 2. Consider *only* contracts with the GC term. Select contracts following the test score ordering up to the capacity $v^{OBC \to GC}$.

3. Allocating the Surplus OBC Seats via Meritocracy

Many in India argue that vacant OBC seats should be provided to all candidates on the basis of merit rather than exclusively to GC candidates. Babu (2010), for example, criticizes OBC de-reservations only to GC applicants as follows:

> *"...And even if there is a case for converting those seats, how come only one category (the general category) has monopoly over these surplus seats? If there are more seats available, then everyone has an equal claim to those. And if the cut-off mark for the general category is reduced to admit more students to fill the converted seats, then fairness demands that the same principle be applied to all the other categories."*

We now introduce a choice rule $C^{VHwT-Merit}$ that differs from the choice rule C^{VHwT} in that it considers all remaining contracts in the third step. This choice rule allocates seats that are reverted from OBC in the third step solely based on test scores and ignores the vertical categories that contracts are associated with. However, if an individual has two contracts (one with the GC term and another with the reserve eligible type), then the contract with the GC term is chosen.

Choice Rule $C_s^{VHwT-Merit}$

Step 1: In this step, consider only contracts with the GC term. Use $C^{hier}(\cdot \mid \overline{v}_s^{GC}, \overline{h}_{(GC,s)})$ to select contracts. Remaining contracts of selected individuals are removed for the rest of the process.

Step 2: For each reserve eligible category $v \in \mathcal{V}$, consider only the contracts with term v. Then, for each category v, apply $C^{hier}(\cdot \mid \overline{v}_s^{v}, \overline{h}_{(v,s)})$ to the contracts with term v. Remaining contracts of selected individuals are removed for the rest of the process.

Step 3: Let $\overline{v}^{OBC\to Merit}$ be the number of vacant OBC seats at the end of Step 2. Consider all remaining contracts. Select contracts following the test score ordering up to the capacity $\overline{v}^{OBC\to Merit}$. If an individual has two contracts at the beginning of this step, one with the GC term and another one with a term $v \in \{SC, ST\}$, then her only GC contract is considered.

In admissions to publicly funded educational institutions and government-sponsored jobs, when institutions run their assignment procedures independently the match outcome is determined by the choice rule they implement. Therefore, it is crucial that they respect meritocracy to the extent possible, in conjunction with vertical and horizontal reservations and OBC de-reservations. Our next result shows that all of the three choice procedure we described are fair.

Theorem 2. *$C^{VHw/oT}$, C_s^{VHwT} and $C_s^{VHwT-Merit}$ are fair.*

In decentralized admissions to a single institution, the outcome is determined via a choice rule. In such applications, fairness of choice rules is critical. The failure of the fairness criterion might cause legal disputes. Theorem 2 justifies the use of $C^{VHw/oT}$, C_s^{VHwT} and $C_s^{VHwT-Merit}$ in the Indian resource allocation problems on the fairness ground.

4 Allocation via Centralized Clearinghouses

A *mechanism* $\mathcal{M}(\cdot; C)$ maps preference profiles $P = (P_i)_{i \in \mathcal{I}}$ to matchings, given a profile of institutional choice rules $C = (C^s)_{s \in \mathcal{S}}$. Unless otherwise stated, we assume that the choice rules of the institutions are fixed and write $\mathcal{M}(P)$ in place of $\mathcal{M}(P; C)$. A mechanism $\mathcal{M}$ is *stable (fair)* if $\mathcal{M}(P)$ is a stable (fair) matching for every preference profile P. A mechanism $\mathcal{M}$ is *strategy-proof* if for every preference profile P and for each individual $i \in \mathcal{I}$, there is no $\widetilde{P}_i$, such that $\mathcal{M}(\widetilde{P}_i, P_{-i}) P_i \mathcal{M}(P)$.

The cumulative offer algorithm is the central allocation mechanism used in the matching with contracts framework. Here, we provide an intuitive description of this algorithm. We give a more technical statement in Appendix B.

In the **cumulative offer process**, individuals propose contracts to institutions in a sequence of steps $l = 1, 2, ...$:

Step 1: Some individual $i^1 \in I$ proposes his most-preferred contract, $x^1 \in X_{i^1}$. Institution $s(x^1)$ holds x^1 if $x^1 \in C^{s(x^1)}(\{x^1\})$ and rejects x^1 otherwise. Set $A^2_{s(x^1)} = \{x^1\}$, and set $A^2_{s^{'}} = \emptyset$ for each $s^{'} \neq s(x^1)$; these are the sets of contracts available to institutions at the beginning of Step 2.

Step l: Some individual $i^l \in I$, for whom no contract is currently held by any institution, proposes his most-preferred contract that has not yet been rejected, $x^l \in X_{i^l}$. Institution $s(x^l)$ holds the contract in $C^{s(x^l)}(A^l_{s(x^l)} \cup \{x^l\})$ and rejects all other contracts in $A^l_{s(x^l)} \cup \{x^l\}$; institutions $s^{'} \neq s(x^l)$ continue to hold all contracts they held at the end of Step $l-1$. Set $A^{l+1}_{s(x^l)} = A^l_{s(x^l)} \cup \{x^l\}$ and set $A^{l+1}_{s^{'}} = A^l_{s^{'}}$ for each $s^{'} \neq s(x^l)$.

If at any time no individual is able to propose a new contract—that is, if all individuals for whom no contracts are on hold have proposed all contracts they find acceptable—then the algorithm terminates. The outcome of the cumulative offer process is the set of contracts held by institutions at the end of the last step before termination. In the cumulative offer process, individuals propose contracts sequentially. Institutions accumulate offers, choosing a set of contracts at each step (according to C^s) to hold from the set of all previous offers. The process terminates when no individual wishes to propose a contract.

We are now ready to present our first result in this section.

Theorem 3. *The COM with respect to the $C^{VHw/oT}$, C^{VHwT}, and $C^{VHwT-Merit}$ choice rules is fair.*

In centralized admissions to institutions, fairness of outcomes are crucial. In India, there are numerous court cases due to fairness violations both in admissions to publicly funded educational institutions and allocation of government-sponsored jobs. Theorem 3 justifies the use of COM as an allocation mechanism, under the choice rules $C^{VHw/oT}$, C^{VHwT}, and $C^{VHwT-Merit}$, with respect to fairness.

Our next result states that the COM gives the correct incentives to applicants and is also stable. It is a weakly dominant strategy for individuals to report their vertical category and preferences over institution-vertical category pairs truthfully.

Theorem 4. *The COM with respect to $C^{VHw/oT}$, C^{VHwT}, and $C^{VHwT-Merit}$ choice rules is stable and strategy-proof.*

5 Respecting Improvements

It is crucial in meritocratic assignment systems that an increase in an applicants' merit scores does not result in a penalty. This natural property is known as *respect for improvements* (Balinski and Sönmez, 1999), which is one of the main desiderata of allocation mechanisms, especially in meritocratic systems.

For each vertical category $v \in \mathcal{V}$, individuals are prioritized by removing individuals who do not qualify for vertical category v and using the merit-score function κ. In the context of Indian resource allocation problems, all institutions prioritize individuals following the merit-score function κ within each vertical category. In what follows, we define a notion of improvement with respect to merit-score functions.

Definition 2. We say that merit-score function $\widetilde{\kappa}$ is an *improvement over merit-score function κ* for individual $i \in \mathcal{I}$ if

1. $\widetilde{\kappa}(i,s) \geq \kappa(i,s)$ for all institutions $s \in \mathcal{S}$, and,
2. there exists an institution $s' \in \mathcal{S}$ such that $\widetilde{\kappa}(i,s') > \kappa(i,s')$, and,
3. $\widetilde{\kappa}(j,s) = \kappa(j,s)$ for all $j \neq i$ and $s \in \mathcal{S}$.

That is, $\widetilde{\kappa}$ is an improvement over κ for individual i if $\widetilde{\kappa}$ is obtained from κ by increasing i's merit scores at some institutions while leaving other individuals' merit scores at every institution unchanged. By slightly abusing notation, we let $\widetilde{C}(\cdot,\widetilde{\kappa})$ and $C(\cdot,\kappa)$ denote vectors of institutions' overall choice rules obtained from the merit-score functions $\widetilde{\kappa}$ and κ, respectively.

Definition 3. A mechanism φ **respects for improvements** for $i \in I$ if for any preference profile $P \in \times_{i\in\mathcal{I}}\mathcal{P}^i$

$$\varphi_i(P;\widetilde{C}(\cdot,\widetilde{\kappa}))R^i\varphi_i(P;C(\cdot,\kappa))$$

whenever $\widetilde{\kappa}$ is an improvement over κ for i. We say that φ respects improvement*s* if it respects for improvements for each individual $i \in I$.

Theorem 5. *The COM with respect to $C^{VHw/oT}$, C^{VHwT}, and $C^{VHwT-Merit}$ choice rules respects for improvements.*

One of the main desiderata for assignment procedures in India is the respect for meritocracy. Therefore, individuals should not have any incentive to try to lower their standings in institutions' priority orderings. Theorem 5 states that the COM under $C^{VHw/oT}$, C^{VHwT}, and $C^{VHwT-Merit}$ achieves this goal.

6 Conclusion

In this paper, we model matching markets for publicly funded educational institutions and government-sponsored jobs in India with comprehensive affirmative action programs. The laws do not provide a specific method to implement horizontal reservations when individuals qualify for multiple horizontal reservation types. We design a sub-choice function that integrates meritocracy and hierarchical horizontal reservations to be implemented in each vertical category. Our sub-choice function selects the best set of individuals that satisfy horizontal reservations with respect to a comparison criteria we define.

We design three different choice rules for institutions that incorporate different OBC de-reservation policies. By embedding these choice rules in the COM, we offer matching mechanisms that (1) respect vertical reservations at each institution, (2) respect horizontal reservations within each vertical category at each institution, (3) respect meritocracy to the extent possible, (4) take OBC de-reservations into account, and (5) consider applicants' preferences for both institutions and the category through which they are admitted under. We show that the COM is stable with respect to institutions' choice rules, strategy-proof for applicants, and respects improvements.

7 Appendices

A. Formal Description of C^{hier}

We describe the sub-choice rule C^{hier} for vertical category $v \in \mathcal{T}$. Let $X \subseteq \mathcal{X}$ be the set of contracts offered to institution s. Remove all contracts in $X \setminus X_v$. Let $\overline{h}_s = \overline{h}_s^1 = (\overline{h}_{(v,s)}^1)_{v \in \mathcal{T}}$ denote the horizontal reservation of institution s, where

$$\overline{h}_{(v,s)}^1 \equiv (\overline{h}_{(v,s)}^{(j,1)})_{h_j \in \mathcal{H}}$$

is the vector of horizontal reservations at vertical category $v \in \mathcal{T}$. In each step of C^{hier} the number of horizontal reservations is updated.

Let

$$\mathcal{H}^1 = \{h_j \in \mathcal{H} \mid \nexists h_k \in \mathcal{H} \text{ such that } \rho^{-1}(h_k) \subset \rho^{-1}(h_j)\}$$

be the set of horizontal reservation types that does not contain another horizontal reservation type.

Step 1: If no individual has any horizontal reservation, then choose the contracts of individuals with the highest merit scores for all seats. Otherwise, for every horizontal type $h_j \in \mathcal{H}^1$, if there are less than $\overline{h}^{(j,1)}_{(v,s)}$ individuals, choose contracts of all of them. Otherwise, choose the contracts of the $\overline{h}^{(j,1)}_{(v,s)}$ highest-scoring individuals with horizontal reservation h_j. Reduce the number of available seats and the number of horizontal reservations for any horizontal type that contains h_j by the number of chosen contracts. Eliminate h_j from the horizontal types to be considered and set

$$\mathcal{H}^2 \equiv \{h_j \in \mathcal{H} \setminus \mathcal{H}^1 \mid \nexists h_k \in \mathcal{H} \setminus \mathcal{H}^1 \text{ such that } \rho^{-1}(h_k) \subset \rho^{-1}(h_j)\}.$$

If there are no individual or seats left, end the process and return the chosen set of contracts.

Let $\overline{h}^2_s = (\overline{h}^2_{(v,s)})_{v \in \mathcal{T}}$ denote the updated numbers of horizontal reservations for Step 2, where

$$\overline{h}^2_{(v,s)} \equiv (\overline{h}^{(j,2)}_{(v,s)})_{h_j \in \mathcal{H} \setminus \mathcal{H}^1}$$

is the updated number of horizontal reservations for horizontal types that have not yet been considered.

Step n (n≥2): If there is no horizontal type left to be considered, then choose contracts of individuals following the merit score ranking for the remaining seats. Otherwise, for every horizontal type $h_j \in \mathcal{H}^n$, if there are less than $\overline{h}^{(j,n)}_{(v,s)}$ individuals in the remaining set, choose contracts of all of them. Otherwise, choose the contracts of the $\overline{h}^{(j,n)}_{(v,s)}$ highest-scoring individuals with horizontal reservation h_j. Reduce the number of available seats and the number of horizontal reservations for any horizontal type that contains h_j by the number of chosen contracts. Eliminate h_j from the set of horizontal reservations to be considered and set

$$\mathcal{H}^{n+1} \equiv \{h_j \in \mathcal{H} \setminus (\bigcup_{r=1}^{n} \mathcal{H}^r) \mid \nexists h_k \in \mathcal{H} \setminus (\bigcup_{r=1}^{n} \mathcal{H}^r) \text{ such that } \rho^{-1}(h_k) \subset \rho^{-1}(h_j)\}.$$

If there are no individuals or seats left, the process ends and returns the chosen set of individuals.

Let $\overline{h}_s^{n+1} = (\overline{h}_{(v,s)}^{n+1})_{v \in \mathcal{T}}$ denote the updated numbers of horizontal reservations for Step $(n+1)$, where

$$\overline{h}_{(v,s)}^{n+1} \equiv (\overline{h}_{(v,s)}^{(j,n+1)})_{h_j \in \mathcal{H} \setminus (\overset{n}{\underset{r=1}{\cup}} \mathcal{H}^r)}.$$

B. Formal Description of the Cumulative Offer Process

The cumulative offer process (COP) associated with proposal order Γ is defined by the following algorithm.

1. Let $l = 0$. For each $s \in S$, let $D_s^0 \equiv \emptyset$, and $A_s^1 \equiv \emptyset$.

2. For each $l = 1, 2, ...$
   Let i be the $\Gamma_l - maximal$ individual $i \in I$, such that $i \notin i(\underset{s \in S}{\cup} D_s^{l-1})$ and $\underset{P^i}{max}(X \setminus (\underset{s \in S}{\cup} A_s^l))_i \neq \emptyset_i$—that is, the first individual in the proposal order who wants to propose a new contract, if such an individual exists. (If no such individual exists, then proceed to Step 3 below.)
   (**a**) Let $x = \underset{P^i}{max}(X \setminus (\underset{s \in S}{\cup} A_s^l))_i$ be i's most preferred contract that has not been proposed.
   (**b**) Let $s = s(x)$. Set $D_s^l = C^s(A_s^l \cup \{x\})$ and set $A_s^{l+1} = A_s^l \cup \{x\}$. For each $s' \neq s$, set $D_{s'}^l = D_{s'}^{l-1}$ and $A_{s'}^{l+1} = A_{s'}^l$.

3. Return the outcome

$$Y \equiv (\underset{s \in S}{\cup} D_s^{l-1}) = (\underset{s \in S}{\cup} C^s(A_s^l)),$$

   which consists of contracts held by institutions at the point when no individuals want to propose additional contracts.

Here, the sets D_s^{l-1} and A_s^l denote the set of contracts **held by** and **available to** institution s at the beginning of the cumulative offer process step l. We say that a contract z is **rejected** during the cumulative offer process if $z \in A_{s(z)}^l$ but $z \notin D_{s(z)}^{l-1}$ for some l.

C. Monotone Capacity Transfers

Monotonicity of capacity transfer schemes requires that (1) whenever weakly more slots are left unfilled in *every* vertical category preceding the j^{th} vertical category, weakly more slots should be available for the j^{th} vertical category, and (2) an institution cannot decrease its total capacity in response to increased demand for some vertical categories.

The overall choice rule $C^{VHw/oT}$ trivially has monotonic capacity transfer scheme as no capacity is transferred across vertical categories. It is straightforward to see that the overall choice rule C^{VHwT} has monotonic capacity transfer. The processing order of vertical categories is Open-SC-ST-OBC-Open and the only capacity transfer occur between OBC and the last category Open. All of the remaining vacant OBC seats are transferred to Open category. This basic scheme satisfies both of the requirements of monotone capacity transfer given above.

The processing order of vertical categories for the overall choice rule $C^{VHwT-Merit}$ is Open-SC-ST-OBC-Merit. The last category selects contracts with respect to merit score and is vertical category free. If a candidate have two contracts, the contract with the GC is chosen. The only capacity transfer occur between OBC and the last category Merit. All of the remaining vacant OBC seats are transferred to Merit. This basic scheme satisfies both of the requirements of monotone capacity transfer given above, as well.

D. Proofs

Proof of Proposition 1. Consider the following problem with a single institution $\mathcal{S} = \{s\}$, and two individuals $\mathcal{I} = \{i, j\}$. There are two vertical categories, SC and GC. Institution s has two positions. One of them is reserved for SC applicants and the other one is an open category position. If open category position remains empty, its capacity is not transferred to SC. Suppose that both i and j have vertical category SC. Let $X = \{x_1, x_2, y_1, y_2\}$, where $\mathbf{i}(x_1) = \mathbf{i}(x_2) = i$, $\mathbf{i}(y_1) = \mathbf{i}(y_2) = j$, $\mathbf{t}(x_1) = \mathbf{t}(y_1) = GC$, and $\mathbf{t}(x_2) = \mathbf{t}(y_2) = SC$. Suppose that individuals have the following preferences: $x_2 P_i x_1$ and $y_2 P_j y_1$. Individual i has higher merit score than individual j at institution s, i.e., $\kappa(i, s) > \kappa(j, s)$. Let C^s be one of the three overall choice rule we described in Section 4, i.e., C^s is a fair choice rule. Note that C^s fills the open category position first and then fills the SC position.

The allocation $Y = \{x_1, y_1\}$ is fair but not stable with respect to C^s because Y is blocked via $\{y_2\}$. That is, $C^s\{x_1, y_1, y_2\} = \{x_1, y_2\}$. On the other hand, the allocation $Z = \{x_1, y_2\}$ is stable with respect to C^s but not fair because individual i envies j's assignment.

Proof of Theorem 1. We first show that C^{hier} is merit-based undominated. Since each individual can have at most one contract with a given vertical category, we consider a set of individuals rather than a set of contracts for a given vertical category.

Toward a contradiction, suppose that C^{hier} is merit-based dominated. Then, for some $X \subseteq \mathcal{X}$, there exists $Y \subseteq X$, such that $\mathbf{i}(Y)$ merit-based dominates $\mathbf{i}(C^{hier}(X))$. Let

$\widehat{I}_1 = \mathbf{i}(C^{hier}(X)) \setminus \mathbf{i}(Y)$ and $\widehat{J} = \mathbf{i}(Y) \setminus \mathbf{i}(C^{hier}(X))$. For each $i \in \widehat{I}_1$, let n_i be the step in C^{hier} at which i is chosen. Define $\widehat{n}_1 = \underset{i \in \widehat{I}_1}{min}\ n_i$. Let $\hat{i}_1$ be the highest-scoring individual among individuals $i \in \widehat{I}_1$ with $n_i = \widehat{n}_1$. Consider the horizontal type $\widehat{h}_1$ in which $\widehat{i}_1$ is chosen. Since $\mathbf{i}(Y)$ merit-based dominates $\mathbf{i}(C^{hier}(X))$, it must be the case that $\mathbf{i}(Y)$ satisfies horizontal reservations. Since the set of chosen individuals before Step $\widehat{n}_1$ is also in the set $\mathbf{i}(Y)$, and C^{hier} selects top-scoring individuals within each horizontal type, to fill the remaining positions, given that $\mathbf{i}(Y)$ does not contain $\widehat{i}_1$, there exists at least one individual in $\mathbf{i}(Y)$, who has the horizontal type $\widehat{h}_1$ and whose score is lower than that of $\widehat{i}_1$. Among those, let $\widehat{j}_1$ be the highest-scoring individual. The set $\mathbf{i}(Y) \cup \{\hat{i}_1\} \setminus \{\hat{j}_1\} = \widetilde{I}^1$ merit-based dominates $\mathbf{i}(Y)$.

Let $\widehat{I}_2 = \widehat{I}_1 \setminus \{\hat{i}_1\}$. For each $i \in \widehat{I}_2$, let n_i denote the step in C^{hier} at which i is chosen. Define $\widehat{n}_2 = \underset{i \in \widehat{I}_2}{min}\ n_i$. Let $\widehat{i}_2$ be the highest-scoring individual among individuals $i \in \widehat{I}_2$ with $n_i = \widehat{n}_2$. Consider the horizontal type $\widehat{h}_2$ in which $\widehat{i}_2$ is chosen. Since the set of chosen individuals before Step $\widehat{n}_2$ is also in the set $\widetilde{I}^1$, and C^{hier} selects top-scoring individuals within each horizontal type, to fill the remaining positions, given that $\widetilde{I}^1$ does not contain $\widehat{i}_2$, there exists at least one individual in $\widetilde{I}^1$, who has the horizontal type $\widehat{h}_2$ and whose score is lower than that of $\widehat{i}_2$. Among those, let $\widehat{j}_2$ be the highest-scoring individual. The set $\widetilde{I}^1 \cup \{\hat{i}_2\} \setminus \{\hat{j}_2\} = \widetilde{I}^2$ merit-based dominates $\widetilde{I}^1$.

We continue in the same fashion. The set $\mathbf{i}(Y) \cup \{\widehat{i}_1, ..., \widehat{i}_l\} \setminus \{\hat{j}_1, ..., \hat{j}_l\} = \widetilde{I}^l$ merit-based dominates the set $\mathbf{i}(Y) \cup \{\widehat{i}_1, ..., \widehat{i}_{l-1}\} \setminus \{\hat{j}_1, ..., \hat{j}_{l-1}\} = \widetilde{I}^{l-1}$. Since the set $\widehat{I}_1$ is finite, in finitely many steps, call it m, we reach

$$\mathbf{i}(Y) \cup \{\widehat{i}_1, ..., \widehat{i}_m\} \setminus \{\hat{j}_1, ..., \hat{j}_m\} = \widetilde{I}^m = \mathbf{i}(C^{hier}(X)).$$

Hence, $\mathbf{i}(C^{hier}(X))$ merit-based dominates $\mathbf{i}(Y)$. This contradicts our supposition. Thus, C^{hier} is merit-based undominated.

Next, we will show that C^{hier} is the unique merit-based undominated choice rule. Toward a contradiction, suppose that there is a merit-based undominated choice rule $C(\cdot)$, such that, for a given set of contracts $X \subseteq \mathcal{X}_s$, $C^{hier}(X) \neq C(X)$. Define $\widehat{I}_1 \equiv \mathbf{i}(C^{hier}(X)) \setminus \mathbf{i}(C(X))$. Let n_i be the step in C^{hier} at which individual i is chosen. Let $\widehat{n}_1 = \underset{i \in \widehat{I}_1}{min}\ n_i$. Among all individuals with $n_i = \widehat{n}_1$, call the individual with the highest merit score $\widehat{i}_1$. Consider the horizontal type $\widehat{h}_1$ in which $\widehat{i}_1$ is chosen. We know that $\mathbf{i}(C(X))$ satisfies horizontal reservations. Since the set of chosen individuals before Step $\widehat{n}_1$ is also in the set $\mathbf{i}(C(X))$, and C^{hier} selects top-scoring individuals within each horizontal type, to fill the remaining positions, given that $\mathbf{i}(C(X))$ does not contain $\widehat{i}_1$, there exists at least one individual in

$\mathbf{i}(C(X))$, who has the horizontal type $\widehat{h}_1$ and whose score is lower than that of $\widehat{i}_1$. Among those, let $\widehat{j}_1$ be the highest-scoring individual. The set $\mathbf{i}(C(X)) \cup \{\hat{i}_1\} \setminus \{\hat{j}_1\} = \widetilde{I}^1$ merit-based dominates $\mathbf{i}(C(X))$. This contradict with our supposition that $\mathbf{i}(CX))$ is merit-based undomianted. Thus, there is no other merit-based undominated choice rule than C^{hier}.

Proof of Theorem 2. We prove Theorem 2 for a given institution $s \in \mathcal{S}$.

(i) We first show that C^{hier} is fair for any given vertical type. Toward a contradiction, suppose that C^{hier} is not fair. Then, there exists a set of contracts X_s and a pair of contract $x, y \in X_s$ such that $x \notin C^{hier}(X_s)$, $y \in C^{hier}(X_s)$, $\kappa(\mathbf{i}(x), s) > \kappa(\mathbf{i}(y), s)$, and $\mathbf{t}(x) \times \rho(\mathbf{i}(x)) \supseteq \mathbf{t}(y) \times \rho(\mathbf{i}(y))$. Therefore, one can construct a set of contracts $Y \subset X_s$ such that

$$Y = (C^{hier}(X_s) \setminus \{y\}) \cup \{x\}.$$

It is easy to see that $\mathbf{i}(Y)$ merit-based dominates $\mathbf{i}(C^{hier}(X_s))$ which contradicts C^{hier} being merit-based undominated. Hence, C^{hier} is fair.

(ii) We now show that $C^{VHw/oT}$, C^{VHwT}, and $C^{VHwT-Merit}$ are fair. We start with the choice rule $C^{VHw/oT}$. Toward a contradiction, suppose that $C^{VHw/oT}$ is not fair. Then, there exists a set of contracts X_s and a pair of contract $x, y \in X_s$ such that $x \notin C^{VHw/oT}(X_s)$, $y \in C^{VHw/oT}(X_s)$, $\kappa(\mathbf{i}(x), s) > \kappa(\mathbf{i}(y), s)$, and $\mathbf{t}(x) = \mathbf{t}(y) = v$, and $\rho(\mathbf{i}(x)) \supseteq \rho(\mathbf{i}(y))$. Since x is not chosen by the overall choice rule $C^{VHw/oT}$, it must be rejected by sub-choice functions $C^{hier}(\cdot \mid \overline{v}_s^{v'}, \overline{h}_{(v',s)})$ for all $v' \in \mathcal{T}$, including vertical category v. Therefore, one can construct a set of contracts $Y \subset X$ such that

$$Y = (C^{hier}(X_s^{'} \mid \overline{v}_s^{v}, \overline{h}_{(v,s)}) \setminus \{y\}) \cup \{x\},$$

where $X_s^{'}$ is the set of contracts considered by $C^{hier}(\cdot \mid \overline{v}_s^{v}, \overline{h}_{(v,s)})$. Note that $\mathbf{i}(Y)$ merit-based dominates $\mathbf{i}(C^{hier}(X_s^{'} \mid \overline{v}_s^{v}, \overline{h}_{(v,s)})$. This contradicts C^{hier} being merit-based undominated.

We next show that C^{VHwT} is fair. Toward a contradiction, suppose that it is not fair. Then, there exists a set of contracts X_s and a pair of contracts $x, y \in X_s$, such that $x \notin C^{VHwT}(X_s)$, $y \in C^{VHwT}(X_s)$, $\kappa(\mathbf{i}(x), s) > \kappa(\mathbf{i}(y), s)$, $\mathbf{t}(x) = \mathbf{t}(y) = v$, and $\rho(\mathbf{i}(x)) \supseteq \rho(\mathbf{i}(y))$. Since x is not chosen by the overall choice rule C^{VHwT}, it must be rejected by sub-choice functions $C^{hier}(\cdot \mid \overline{v}_s^{v'}, \overline{h}_{(v',s)})$ for all $v' \in \mathcal{T}$, including vertical category v. Hence, there is a step of C^{VHwT} and a set of contracts X_s' such that $x, y \in X_s'$, $x \notin C^{hier}(X_s' \mid \overline{v}_s^{v}, \overline{h}_{(v,s)})$ and

$y \in C^{hier}(X'_s \mid \overline{v}^v_s, \overline{h}_{(v,s)})$. One can construct a set of contracts $Y \subset X_s$ such that

$$Y = (C^{hier}(X'_s \mid \overline{v}^v_s, \overline{h}_{(v,s)}) \setminus \{y\}) \cup \{x\}.$$

It is easy to see that $\mathbf{i}(Y)$ merit-based dominates $\mathbf{i}(C^{hier}(X'_s \mid \overline{v}^v_s, \overline{h}_{(v',s)}))$. This contradicts C^{hier} being merit-based undominated. Hence, C^{VHwT} is fair.

Finally, we show that $C^{VHwT-Merit}$ is fair. Toward a contradiction, suppose that it is not. Then, there exists a set of contracts X_s and a pair of contracts $x, y \in X_s$, such that $x \notin C^{VHwT-Merit}(X_s)$, $y \in C^{VHwT-Merit}(X_s)$, $\kappa(\mathbf{i}(x), s) > \kappa(\mathbf{i}(y), s)$, $\mathbf{t}(x) = \mathbf{t}(y) = v$, and $\rho(\mathbf{i}(x)) \supseteq \rho(\mathbf{i}(y))$. Since x is not chosen by the overall choice rule $C^{VHwT-Merit}$, it must be rejected by sub-choice functions $C^{hier}(\cdot \mid \overline{v}^{v'}_s, \overline{h}_{(v',s)})$ for all $v' \in \mathcal{T}$, including vertical category v. However, it cannot be the case that y is chosen in Step 3 of $C^{VHwT-Merit}$ while x is rejected in the same step because in Step 3 contracts are chosen on the basis of merit scores and $\kappa(\mathbf{i}(x), s) > \kappa(\mathbf{i}(y), s)$. So, either in Step 1 or Step 2 of $C^{VHwT-Merit}$ there is a set of contracts X'_s such that $x, y \in X'_s$, $x \notin C^{hier}(X'_s \mid \overline{v}^v_s, \overline{h}_{(v,s)})$ and $y \in C^{hier}(X'_s \mid \overline{v}^v_s, \overline{h}_{(v,s)})$. Then, one can construct a set of contracts $Y \subset X_s$ such that

$$Y = (C^{hier}(X'_s \mid \overline{v}^v_s, \overline{h}_{(v,s)}) \setminus \{y\}) \cup \{x\}.$$

Again, it is easy to see that $\mathbf{i}(Y)$ merit-based dominates $\mathbf{i}(C^{hier}(X'_s \mid \overline{v}^v_s, \overline{h}_{(v,s)}))$. This contradicts with C^{hier} being merit-based undominated. Hence, $C^{VHwT-Merit}_s$ is fair.

Proof of Theorem 3. First, note that choice rules $C^{VHw/oT}$, C^{VHwT}, and $C^{VHwT-Merit}$ are in the family of Generalized Lexicographic Choice Rules (GLCR), studied in Aygün and Turhan (2019). Moreover, as shown in Theorem 2, choice rules $C^{VHw/oT}$, C^{VHwT}, and $C^{VHwT-Merit}$ are fair. We need to show that the COM with respect to these choice rules are fair.

Toward a contradiction, suppose that it is not. Then, there must exist a preference profile P such that the outcome of the COP under preference profile P is not fair. Let X be the outcome of the COP under preference profile P. Since X is not fair, there must exist contracts $x, y \in X$ such that $(\mathbf{s}(y), \mathbf{t}(y)) P_{\mathbf{i}(x)} (\mathbf{s}(x), \mathbf{t}(x))$. Since $(\mathbf{s}(y), \mathbf{t}(y))$ is acceptable to individual $\mathbf{i}(x)$, she is eligible for vertical category $\mathbf{t}(y)$. There are two cases to consider.

Case 1: $\mathbf{s}(x) \neq \mathbf{s}(y)$. If $(\mathbf{s}(y), \mathbf{t}(y)) P_{\mathbf{i}(x)} (\mathbf{s}(x), \mathbf{t}(x))$, then individual $\mathbf{i}(x)$ must have offered the contract $(\mathbf{i}(x), \mathbf{s}(y), \mathbf{t}(y))$ before she offered contract x. The contract $(\mathbf{i}(x), \mathbf{s}(y), \mathbf{t}(y))$ must have rejected by institution $\mathbf{s}(y)$. Moreover, individual $\mathbf{i}(x)$'s all contracts that are associated with institution $\mathbf{s}(y)$ must be rejected. Since choice rules $C^{VHw/oT}$, C^{VHwT}, and

$C^{VHwT-Merit}$ are fair, we have either $\kappa(\mathbf{i}(y), s(y)) > \kappa(\mathbf{i}(x), s(y))$ or $\rho(\mathbf{i}(x)) \not\supseteq \rho(\mathbf{i}(y))$.

Case 2: $\mathbf{s}(x) = \mathbf{s}(y)$. Let $\mathbf{s}(x) = \mathbf{s}(y) = s$. There are two sub-cases to consider:

Sub-case 2.1. $\mathbf{t}(y)$ precedes $\mathbf{t}(x)$. Consider the last step of the COP. At this step, the contract $(\mathbf{i}(x), s, \mathbf{t}(y))$ is available to institution s, but is rejected. Then, it must be the case that either $\kappa(\mathbf{i}(y), s) > \kappa(\mathbf{i}(x), s)$ or $\rho(\mathbf{i}(x)) \not\supseteq \rho(\mathbf{i}(y))$ hold. To see why consider the case where $\kappa(\mathbf{i}(y), s) < \kappa(\mathbf{i}(x), s)$ and $\rho(\mathbf{i}(x)) \supseteq \rho(\mathbf{i}(y))$. Lex X be the outcome of the COP and X_s be the set of contracts assigned to institution s. If we replace the contract y with $(\mathbf{i}(x), s, \mathbf{t}(y))$, the resulting set of contracts $(X_s \setminus \{y\}) \cup \{(\mathbf{i}(x), s, \mathbf{t}(x)\}$ merit-based dominates X_s,[32] which is a contradiction because, for every vertical type in $C^{VHw/oT}$, C^{VHwT}, and $C^{VHwT-Merit}$, contracts are chosen according to C^{hier} that is merit-based undominated.

Sub-case 2.2. $\mathbf{t}(x)$ precedes $\mathbf{t}(y)$. We invoke Lemma 1 of Aygün and Turhan (2019). We first define some key notions. Let $X^M = \{x^1, ..., x^M\}$ be an observable offer process. We say $X^m = \{x^1, .., x^m\}$, i.e., X^m are the contracts proposed up to step m of the observable offer process X^M. We let $H_t(X^m)$ denote the set of contracts available to vertical category t in the computation of $C^s(X^m)$.[33] Let $F_t(X^m) = \underset{n \leq m}{\cup} H_t(X^n)$, i.e., $F_t(X^m)$ is the set of all contracts that were available to vertical category t at some point of offer process $X^m = \{x^1, ..., x^m\}$. (iii) of Aygün and Turhan's (2019) Lemma 1 states that the choices from the F and H sets are the same for each m at every vertical category, i.e., $C_t(H_t(X^m); q_t^m) = C_t(F_t(X^m); q_t^m)$.

Consider the last step of the COP. At this step, the contract $(\mathbf{i}(x), s, \mathbf{t}(y))$ is available to institution s, but is rejected. Since the contract $(\mathbf{i}(x), s, \mathbf{t}(y))$ was offered at some earlier step, we have

$$(\mathbf{i}(x), s, \mathbf{t}(y)) \in F_{\mathbf{t}(y)}(X^M).$$

Since $C_{\mathbf{t}(y)}(H_{\mathbf{t}(y)}(X^M); q^M_{\mathbf{t}(y)}) = C_{\mathbf{t}(y)}(F_{\mathbf{t}(y)}(X^M); q^M_{\mathbf{t}(y)})$, we have

$$(\mathbf{i}(x), s, \mathbf{t}(y)) \notin C_{\mathbf{t}(y)}(H_{\mathbf{t}(y)}(X^M), q^M_{\mathbf{t}(y)}).$$

If the COP outcome X is not fair, then it must be the case that

$$(\mathbf{i}(x), s, \mathbf{t}(y)) \in C^{hier}_{\mathbf{t}(y)}(H_{\mathbf{t}(y)}(X^M) \cup \{(\mathbf{i}(x), s, \mathbf{t}(y))\}.$$

[32]Note that $\rho(\mathbf{i}(x)) \supseteq \rho(\mathbf{i}(y))$ ensures that $(X_s \setminus \{y\}) \cup \{(\mathbf{i}(x), s, \mathbf{t}(x)\}$ satisfies horizontal reservations.

[33]For the overall choice rule $C^{VHw/oT}$, the processing order of the vertical categories are as follows: Open-C-ST-OBC. For the over all choice rule C^{VHwT}, the processing order of the vertical categories are as follows: Open-SC-ST-OBC-Open. For the over all choice rule $C^{VHwT-Merit}$, the processing order of the vertical categories are as follows: Open-SC-ST-OBC-Merit, where the vertical category "Merit" does not differentiate vertical categories.

Because, otherwise,

$$C^{hier}_{\mathbf{t}(y)}(H_{\mathbf{t}(y)}(X^M), q^M_{\mathbf{t}(y)}) \setminus \{y\} \cup \{(\mathbf{i}(x), s, \mathbf{t}(y))\}$$

merit-based dominates $C^{hier}_{\mathbf{t}(y)}(H_{\mathbf{t}(y)}(X^M), q^M_{\mathbf{t}(y)})$. But, we know that $C^{hier}_{\mathbf{t}(y)}$ is merit-based undominated.

We know that

$$C^{hier}_{\mathbf{t}(y)}(H_{\mathbf{t}(y)}(X^M); q^M_{\mathbf{t}(y)}) = C^M_{\mathbf{t}(y)}(F_{\mathbf{t}(y)}(X^M); q^M_{\mathbf{t}(y)}) \subseteq H_{\mathbf{t}(y)}(X^M) \subseteq F_{\mathbf{t}(y)}(X^M).$$

Therefore, by the IRC of $C^{hier}_{\mathbf{t}(y)}$, we must have

$$C^{hier}_{\mathbf{t}(y)}(H_{\mathbf{t}(y)}(X^M); q^M_{\mathbf{t}(y)}) = C^M_{\mathbf{t}(y)}(H_{\mathbf{t}(y)}(X^M); q^M_{\mathbf{t}(y)}) \cup \{\mathbf{i}(x), s, \mathbf{t}(y)).$$

This contradicts with

$$(\mathbf{i}(x), s, \mathbf{t}(y)) \in C^{hier}_{\mathbf{t}(y)}(H_{\mathbf{t}(y)}(X^M) \cup \{(\mathbf{i}(x), s, \mathbf{t}(y))\}.$$

Proofs of Theorem 4 and 5. We invoke Theorem 1 of Aygün and Turhan (2019) to prove our Theorems 5 and 6. We first show that the sub-choice rule C^{hier} satisfies substitutability, size monotonicity, and quota monotonicity. A choice function C is *substitutable* if for all $x, y \in X$, and $Y \subseteq X$,

$$x \notin C(Y \cup \{x\}) \implies x \notin C(Y \cup \{x, y\}).$$

A choice function C is *size monotonic* if $Y \subseteq X$ implies $\mid C(Y) \mid \leq \mid C(X) \mid$. A choice rule C is *quota monotonic* if for any $q, q^{'} \in \mathbb{Z}_+$ such that $q < q^{'}$, for all $Y \subseteq X$,

$$C(Y, q) \subseteq C(Y, q^{'}), \; and$$
$$\mid C(Y, q^{'}) \mid - \mid C(Y, q) \mid \leq q^{'} - q.$$

Lemma 1. *C^{hier} is substitutable, size monotonic, and quota monotonic.*

Proof of Lemma 1.

(i) We first show that C^{hier} is substitutable. Consider a vertical category v and a set of contracts $X \subseteq \mathcal{X}_v$. Note that each individual in $\mathbf{i}(X)$ has only one contract in X. Suppose that contracts $x, y \in \mathcal{X}_v \setminus X$ are such that $\mathbf{i}(x) = i$ and $\mathbf{i}(y) = j$. Suppose that

$x \in C_v^{hier}(X \cup \{x, y\})$. Let m be the step in C_v^{hier} at which x is chosen from the set $X \cup \{x, y\}$ when horizontal type h is considered. We must show that $x \in C_v^{hier}(X \cup \{x\})$.

Consider $y \notin C_v^{hier}(X \cup \{x, y\})$, i.e., if individual j's contract is not chosen. It must be the case that at each horizontal type that individual j has, individuals who have higher scores than j fill the capacity. Removing individual j does not change the set of chosen individuals and updated reservations. Then, contract x will be chosen at Step m in C_v^{hier} when horizontal type h is considered from the set $X \cup \{x\}$.

Now suppose that $y \in C_v^{hier}(X \cup \{x, y\})$. There are two cases to consider. In the first one, y is chosen when horizontal type h' is considered, where h does not contain h' and h' does not contain h. In this case, x will still be chosen at Step m in C_v^{hier} because removing individual j from the applicant pool does not change the set of chosen individuals by horizontal types that are contained by h and their updated capacities.

In the second case, y is chosen when horizontal type h' is considered, where either h' contains h or h contains h'. Suppose that y is considered and chosen at some Step l in C_v^{hier} where $l \geq m$. Then, x will still be chosen at Step m in C_v^{hier} from $X \cup \{x\}$, because considering and choosing y at the same or a later stage of C_v^{hier} from the set $X \cup \{x, y\}$ does not affect the chosen sets prior to Step m from the set $X \cup \{x\}$. Note that the updated capacities of the horizontal types will be unchanged. Also, the number of individuals who have higher score than i and considered at Step m does not increase.

Now consider the case where y is chosen from the set $X \cup [x, y\}$ when horizontal type h' is considered, where h contains h'. That is, y is considered and chosen from the set $X \cup \{x, y\}$ at some Step l in C_v^{hier} where $l < m$. When horizontal type h is considered at Step m in C_v^{hier} for the set $X \cup \{x\}$, the updated number of horizontal reserves at which individual i is considered is either the same or one more than the updated number of same horizontal types in the choice process beginning with the contract set $X \cup \{x, y\}$. Moreover, the number of individuals who are considered for the same horizontal types and have higher score than individual i does not increase. Hence, x will be chosen at Step m of C^{hier} from the set $X \cup \{x\}$.

(ii) We now show that C_v^{hier} is size monotonic. Note that the last step of C_v^{hier} guarantees that C_v^{hier} is a q-acceptant choice function. Because, in the last step of C_v^{hier}, if there is no horizontal type left to be considered, contracts are chosen following the merit score ranking for the remaining seats. It is well-known that q-acceptance implies size monotonicity. Therefore, C_v^{hier} is size monotonic.

(iii) Finally, we show that C_v^{hier} is quota monotonic. Consider a set of contracts $X \subseteq \mathcal{X}_v$. The capacity of vertical category v is $\overline{v}_s^v$. We need to show that $C_v^{hier}(X, \overline{v}_s^v) \subseteq C_v^{hier}(X, 1+\overline{v}_s^v)$ and $\mid C_v^{hier}(X, 1+\overline{v}_s^v) \mid - \mid C_v^{hier}(X, \overline{v}_s^v) \mid \leq 1$.

We first show that $C_v^{hier}(X, \overline{v}_s^v) \subseteq C_v^{hier}(X, 1+\overline{v}_s^v)$. In the computations of C_v^{hier} with total capacity $1+\overline{v}_s^v$, the updated capacities of horizontal types at each stage is not lower than the corresponding updated capacities of horizontal types at each stage in the computation of C_v^{hier} with total capacity $\overline{v}_s^v$. Moreover, the sets of individuals that compete for positions at each horizontal type at each stage do not expand when the total capacity is increased from $\overline{v}_s^v$ to $1+\overline{v}_s^v$. Therefore, if x is chosen in Step m of C_v^{hier} when the total capacity is $\overline{v}_s^v$, then it must be chosen in Stem m (or at an earlier step) of C_v^{hier} when the total capacity is $1+\overline{v}_s^v$.

We now show that $\mid C_v^{hier}(X, 1+\overline{v}_s^v) \mid - \mid C_v^{hier}(X, \overline{v}_s^v) \mid \leq 1$. First, note that C_v^{hier} is an q-acceptant choice function. There are two cases to consider. In the first case, suppose that all $X = C_v^{hier}(X, \overline{v}_s^v)$. Then, if the total capacity is increased from $\overline{v}_s^v$ to $1+\overline{v}_s^v$, by q-acceptance, we have $X = C_v^{hier}(X, 1+\overline{v}_s^v)$. Therefore, we have

$$\mid C_v^{hier}(X, 1+\overline{v}_s^v) \mid - \mid C_v^{hier}(X, \overline{v}_s^v) \mid = 0.$$

For the second case, suppose that $C_v^{hier}(X, \overline{v}_s^v) \subset X$, which implies $\mid C_v^{hier}(X, \overline{v}_s^v) \mid = \overline{v}_s^v$. Since all contracts in X are associated with vertical type v, when the total capacity is increase to $1+\overline{v}_s^v$, then we have $\mid C_v^{hier}(X, 1+\overline{v}_s^v) \mid = 1+\overline{v}_s^v$. Hence, we have

$$\mid C_v^{hier}(X, 1+\overline{v}_s^v) \mid - \mid C_v^{hier}(X, \overline{v}_s^v) \mid = 1.$$

Proof of Theorem 4. We showed in Appendix C that the capacity transfer scheme in each of $C^{VHw/oT}$, C^{VHwT}, and $C^{VHwT-Merit}$ is monotonic. Then, by Theorem 1 of Aygün and Turhan (2019), the COM with respect to $C^{VHw/oT}$, C^{VHwT}, and $C^{VHwT-Merit}$ is stable and strategy-proof.

Proof of Theorem 5. The proof follows from the Lemma 1 in conjunction with the monotonicity of capacity transfer schemes in each of $C^{VHw/oT}$, C^{VHwT}, and $C^{VHwT-Merit}$, and the Theorem 2 of Aygün and Turhan (2019).

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