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Minimum consumption requirements and cycles in an overlapping generations model of money [Resurrecting Equilibria Through Cycles]

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Abstract
This paper studies a pure-exchange monetary overlapping generations economy in which young and old agents face exogenous minimum consumption requirements, and money is the only asset. The presence of the minimum consumption requirement on the old is shown to produce multiperiodic monetary equilibria in which real balances cycle forever between "momentary" equilibrium points (those which generate monetary sequences that potentially violate equilibrium strictures asymptotically).

Keywords
overlapping generations models, cycles, monetary equilibria, minimum consumption requirements

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Resurrecting Equilibria Through Cycles

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Resurrecting Equilibria Through Cycles*

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Abstract

In an overlapping generations model, momentary equilibria are defined as points that lie on the intergenerational offer curve, i.e., they satisfy agents' optimality conditions and market clearing at any date. However, some dynamic sequences commencing from such points may not be considered valid equilibria because they asymptotically violate some economic restriction of the model. The literature has always ruled out such paths. This paper studies a pure-exchange monetary overlapping generations economy in which real balances cycle forever between momentary equilibrium points. The novelty is to show that segments of the offer curve that have been previously ignored, can in fact be used to produce asymptotically valid cyclical paths. Indeed, a cycle can bestow dynamic validity on momentary equilibrium points that had erstwhile been classified as dynamically invalid.

JEL Classification: E 31, E 42, E 63
Keywords: overlapping generations models, monetary equilibria, cycles, minimum consumption requirements

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The best thing about the future is that it only comes one day at a time.

Abraham Lincoln

1 Introduction

Almost a half century of research has gone into studying stationary and non-stationary equilibria generated by a standard pure-exchange two-period lived overlapping generations (OG) model. The textbook version of this model economy is populated by two-period lived agents endowed with the single consumption good when young and who receive utility from consumption over their lifetimes, as captured by a strictly concave and time-separable utility function. There may be an initial old generation endowed with fiat currency, and this asset becomes the sole means by which agents shift resources over time. As discussed in Azariadis (1993), the law of motion for real balances (intergenerational offer curve) for the standard "Samuelson case" has two steady states, one at zero and one at a positive level. The non-monetary (monetary) steady state is locally stable (unstable). Any sequence of real balances starting to the left of the monetary steady state eventually converges to the non-monetary steady state; paths to the right are ruled out because asymptotically, they require agents to hold real balances beyond what their endowment would allow.[1]

In this paper, we distinguish between two types of equilibria in the standard OG model with money described above. Momentary equilibria are points that lie on the intergenerational offer curve, i.e., they satisfy agents' optimality conditions and market clearing *at any date*. However, some dynamic sequences starting from such points may not be presumed valid because they would *eventually* violate some economic restriction of the model (such as, real balances have to stay bounded above by a young agent's endowment, and so on). If there exists an asymptotically valid sequence commencing at a momentary equilibrium point, that point will be called a "dynamically valid equilibrium" point, and the sequence, a "dynamically valid equilibrium sequence". Note that our notion of a momentary equilibrium is distinct from that of a temporary equilibrium (Grandmont, 2007). In a temporary equilibrium, markets clear at any date conditional on expectations of future prices which *need not* be market clearing ones. [2]

[1] More complicated dynamical patterns in monetary OG models seem to require substantial deviations from the standard paradigm. Researchers such as Grandmont (1985) have studied non-stationary monetary equilibria in OG models and established a strong set of conditions (such as backward bending offer curves) needed for two-period cycles to emerge endogenously. Higher order cycles (which have the potential to replicate realistic time series for nominal variables in stylized models) require conditions that are even more restrictive than those required to generate two period cycles. For further discussion, see Bullard and Duffy (1998), Bunzel (2006), and Koskela and Puhakka (2006).

[2] Momentary equilibria are the same as perfect foresight temporary equilibria.

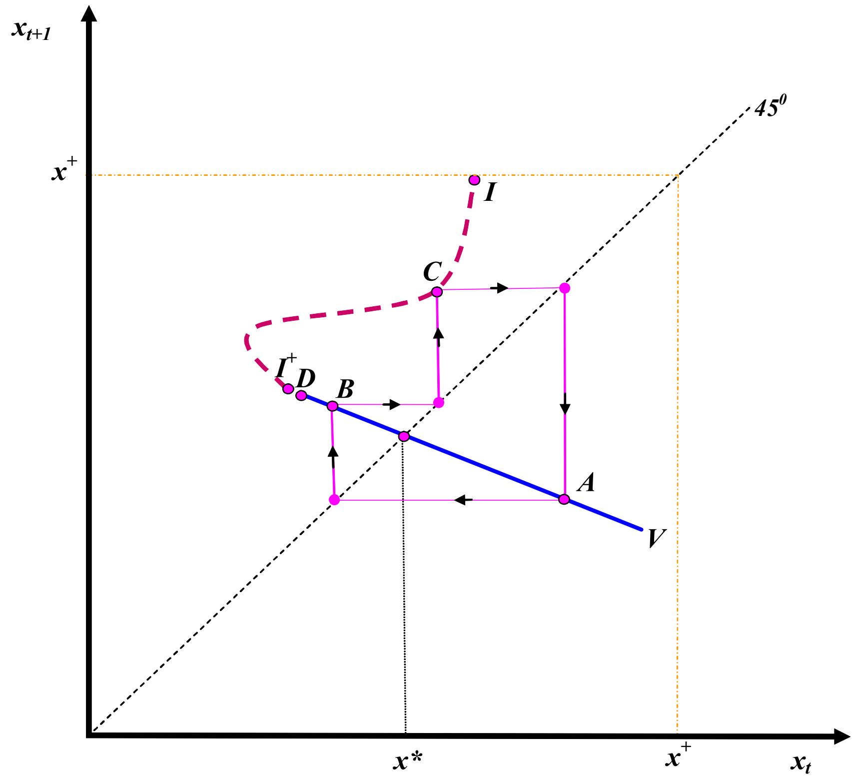

Figure 1: The time map $f(.)$

A picture would no doubt illuminate further. Consider a first-order discrete dynamical system $x_{t+1} = f(x_t)$ where x is an economic variable of interest in some model, t denotes time, and $f(x_t)$ summarizes all points that satisfy agents' optimality and market clearing conditions at any date. Assume that economic restrictions require x to stay below x^+; also assume that the unique steady state (x^*) is locally stable. In Figure 1, momentary equilibria are points on the entire curve IV. The literature has typically ruled out segments such as I^+I as being dynamically invalid (see the discussion surrounding Figure 5.10 in Blanchard and Fischer, 1989) deeming only the solid segment DV as dynamically valid. [3] After all, starting at any point such as C on the dashed curve I^+I and travelling exclusively along that path towards I will generate a dynamic sequence that eventually takes x_t beyond x^+.

However, as shown in the figure, an equilibrium can be constructed in which the economy cycles *forever* between points C, A, and B; this renders C *dynamically valid* because the cycle CAB always stays in the economically valid range! A major contribution of this paper is to show that segments of the intergenerational offer curve (such as I^+I) that may have been previously considered invalid, can in fact, be used to produce dynamically valid *cyclical* paths (such as CAB).[4]

[3] Note that at point D, the slope is -1. Since x^* is locally stable, sequences that start to the left of x^*(but right of D) and to the right of x^*, eventually converge to x^*. Such points are dynamically valid, because they stay within the range $(0, x^+)$. Points to the left of D get attracted towards I and beyond, taking them outside the range $(0, x^+)$.

[4] Points such as C become dynamically valid if agents believe, correctly, that at some date the economy will

We illustrate this fairly general idea using a simple OG model of money. Specifically, the paper introduces exogenous minimum consumption requirements on young and old agents into the textbook overlapping generations model with money as the only asset. Agents get utility from consumption only if it exceeds a certain minimum level (often identified with subsistence). We characterize the set of perfect-foresight monetary momentary and dynamically valid equilibria in the model economy. While analytical progress is hindered by the lack of a closed form solution for the equilibrium law of motion for real balances, it is still possible to show that there is a unique monetary steady state. We show that three – and higher – period cycles exist. As is well known, such multi-periodic equilibria may represent time paths for variables which closely resemble their real world counterparts.[5] The model economy generates high order cycles for a *wide* range of the parameter space. Also, if the minimum consumption requirement on the consumption of the old is removed, the economy studied would not produce any cyclical equilibria beyond periodicity two. In fact, our sole purpose of introducing the minimum consumption requirement is to generate a non-monotonic time map for real balances and exploit this non-monotonicity to generate cyclical paths.

The cycles in our model are interesting because they represent periodic behavior of real balances among momentary equilibrium points, some of which would have been labeled "dynamically invalid" and ruled out in the aforediscussed literature on cycles in OG models. The novelty here is to show that such previously-ignored momentary equilibria can get "resurrected" if, for example, they are part of a cycle. In other words, *a cycle can bestow dynamic validity on momentary equilibrium points (such as C above) that had erstwhile been classified as dynamically invalid.* And while the main premise of the paper is made within the context of an OG model with money, there is no reason to believe that in principle, a similar argument cannot be made for models without money.

The plan for the rest of the paper is as follows. Section 2 outlines the model and computes the law of motion for real balances. Section 3 shows uniqueness of the steady state while Section 4 studies the possibility of periodic behavior. Section 5 concludes.

return to the downward sloped portion of the offer curve, thereby preventing the sequence of money balances from extending beyond the feasible set. Incidentally, it is strictly not necessary that the path be a cycle. One can imagine paths that "bounce off" the upward sloping segment onto the segment DV and then converge to the steady state, x^*. Our thesis, however, is illustrated best with a cyclical path. Parenthetically, also note that it is not necessary that starting at B, the economy must go to C; it could, for example, go on to a point on the segment DV and eventually converge to x^*. All we are claiming is that it may be possible to construct an equilibrium cyclical sequence BCA which would resurrect the dynamic validity of C.

[5] Our model economy, for example, has the ability to produce time paths for inflation which have the property that in the short run, moderate inflation is followed by a sharp deflation and then a huge inflation.

2 The model

The economy consists of an infinite sequence of two-period lived overlapping generations of agents of unit mass. Let $t = 1, 2, ...$ index time. At each date t, a new generation appears. Each two-period-lived agent is endowed with $w > 0$ units of the good when young and nothing when old. Let c_t (x_t) denote the consumption of the final good by a representative young (old) agent born at t. All such agents have preferences representable by the utility function $u(c_t, x_t)$ where u is twice-continuously differentiable, strictly increasing, and strictly concave in its arguments and where

$$u(c_t, x_t) \equiv \frac{(c_t - \theta_t)^{1-\sigma}}{1-\sigma} + \frac{\beta (x_t - \delta_t)^{1-\sigma}}{1-\sigma}, \qquad \beta, \sigma > 0. \tag{1}$$

Here, θ_t and δ_t represent the exogenous minimum consumption requirement the agent faces when young and old respectively. For all that we do below, we will assume $\theta_t = \theta > 0$ and $\delta_t = \delta > 0$ $\forall t$. As will become evident below, θ plays a marginal role in the analysis presented below but is included here for completeness sake. The parameter β captures the relative importance of old-age utility relative to young-age utility.

It is important to point out that $1/\sigma$ does not have the standard interpretation of the elasticity of intertemporal substitution. In dynamic models of continuous time, the elasticity is often written as $\epsilon = -u'(c_t)/u''(c_t)\, c_t$. With a standard period utility with minimum consumption γ, $u(c-\gamma)$, $\epsilon = (c-\gamma)/\sigma c$. This term is increasing in consumption c and lies in the interval $(0, 1/\sigma)$. For finite-time horizons, where the time interval between periods is not small – as in the case of an OG model – the elasticity can be expressed as a weighted average of the first and second period consumptions, $(\zeta (c_t - \theta)/\sigma c_t + (1-\zeta)(c_{2t+1} - \delta)/\sigma c_{2t+1})$. Of course, when $\theta = \delta = 0$, ϵ is constant and equal to $1/\sigma$.[6]

Agents have access to a competitive market for fiat currency, the *sole* asset in this economy. The quantity of money in circulation at the end of period $t \geq 1$, per young agent, is fixed and denoted M. Let $0 < p_t < \infty$ denote the price level at date t and $\frac{p_{t+1}}{p_t} \equiv \pi_t$ denote the inflation rate between period t and $t+1$. Then the gross real rate of return on money (R_{t+1}) between period t and $t+1$ is given by $R_{t+1} \equiv p_t/p_{t+1} = \frac{1}{\pi_t}$. Also, let $m_t \equiv M/p_t$ denote real money balances at date t. Using the definitions for m and R, it is easily checked that $R_{t+1} = \frac{m_{t+1}}{m_t}$.

The agent's problem at date t is simply to choose positive first and second period consumption that maximize (1) subject to the budget constraints $c_t = w - m_t$, $x_t = R_{t+1} m_t$ and the constraints

[6] The intertemporal elasticity of substitution may not be unique in cases with non-homothetic preferences, as considered here. The problem stems from the fact that there may not be a one-to-one mapping between the ratio of the marginal utilities and the ratio of consumptions. Typically, the elasticity is simply reported as $(c-\gamma)/\sigma c$.

on minimum consumption, $c_t \geq \theta_t$ and $x_t \geq \delta_t$, taking R_{t+1} as given. The first order condition is given by

$$\frac{x_t - \delta_t}{c_t - \theta_t} = (\beta R_{t+1})^{\frac{1}{\sigma}} .$$

Incorporating the budget constraints, this can be written as

$$\frac{m_{t+1} - \delta}{w - m_t - \theta} = \beta^{\frac{1}{\sigma}} \left(\frac{m_{t+1}}{m_t} \right)^{\frac{1}{\sigma}} \quad \forall t > 1. \tag{2}$$

The non-negativity constraints on consumption place the restrictions $c_t \geq \theta$ and $x_t \geq \delta$ for all t. Using the agent's first and second period budget constraints, these amount to assuming

$$\delta \leq m_t \leq w - \theta, \tag{A.1}$$

for all t; (A.1) will be called the "valid range". [7]

3 Steady state equilibria

We restrict ourselves to cases in which $m_t \in \Re_{++} \forall t \geq 1$. All competitive equilibria satisfy (2). A *momentary equilibrium* at any date t represents a pair (m_t, m_{t+1}) that satisfies the difference equation (2) and (A.1). Any sequence of momentary equilibrium points $\{m_t\}_{t=1}^{\infty}$ with $m_t \in (\delta, w - \theta)$ *for all* $t \geq 1$ will be called a *dynamically valid equilibrium sequence;* each m_t in this sequence is a dynamically valid equilibrium point.

Setting $m_{t+1} = m_t = m^* > 0$ in (2), it is easy to check that the *unique* positive steady state is given by

$$m^* = \frac{\delta + (w - \theta)\, \beta^{\frac{1}{\sigma}}}{1 + \beta^{\frac{1}{\sigma}}} > 0. \tag{3}$$

It will evidently be useful to know if the difference equation implied by (2) is monotonic. Straightforward differentiation of (2) establishes

$$\frac{dm_{t+1}}{dm_t} = -\left(\frac{m_{t+1}}{m_t} \right)^{\frac{1}{\sigma}} \beta^{\frac{1}{\sigma}} \left[\frac{1 + \frac{1}{\sigma} \dfrac{(w - m_t - \theta)}{m_t}}{1 - \frac{1}{\sigma} \dfrac{(m_{t+1} - \delta)}{(m_{t+1})}} \right]. \tag{4}$$

From (4), it follows that the denominator of the term inside the square brackets can be positive or negative depending on whether

$$1 - \frac{1}{\sigma} \frac{(m_{t+1} - \delta)}{m_{t+1}} \lesseqgtr 0 \Leftrightarrow 1 \lesseqgtr \frac{1}{\sigma} \frac{(m_{t+1} - \delta)}{m_{t+1}}.$$

[7]The constraint $x_t \geq \delta$ imposes that $R_{t+1} m_t \geq \delta$. Since $R_{t+1} m_t = m_{t+1}$, the requirement $x_t \geq \delta$ translates into $m_{t+1} \geq \delta$. (A.1) follows from the fact that the minimum consumption constraint holds every period and specifically, that $x_{t-1} \geq \delta$.

Note that if $\delta = 0$, the offer curve is monotonic, the sign of its first derivative depending on σ. As pointed out, its sign is independent of the minimum consumption constraint on consumption when young, θ, for all values of m_t in the valid range.

Define $\check{m}$ as

$$\check{m} \equiv \frac{\delta}{1-\sigma} > 0 \text{ if } \sigma < 1.$$

The following result is then immediate.

Proposition 1 *When $\sigma < 1$, for $m_{t+1} > \check{m}$, $\frac{dm_{t+1}}{dm_t} > 0$ holds, and for $m_{t+1} < \check{m}$, $\frac{dm_{t+1}}{dm_t} < 0$ holds. At $m_{t+1} = \check{m}$, $\left|\frac{dm_{t+1}}{dm_t}\right| = \infty$. When $\sigma > 1$, $\frac{dm_{t+1}}{dm_t} < 0$ for all m_t, m_{t+1} in the valid range.*

In other words, when $\sigma < 1$, the possibility arises that the intergenerational offer curve given by (2) is non-monotonic. As is well known, a necessary (but not sufficient) condition for complex dynamics (cycles of order higher than two, etc.) is that the slope of the offer curve change sign. Henceforth, we restrict our attention to $\sigma < 1$.[8] In this case, the offer curve described implicitly by (2) is a correspondence in the forward dynamics, in much the same fashion as shown in Figure 1. In the standard model without minimum consumption requirements, as discussed in Grandmont (1985) and Boldrin and Woodford (1992), the offer curve bends backwards for very high returns to money (measured by the ratio m_{t+1}/m_t), due to the fact that the income effect of an increase in the return dominates the substitution effect. In the present case, the situation is *reversed;* the offer curve bends backwards for *low* returns on money.

Heuristically, when the interest rate is low, the agent must hold large cash balances in order to ensure that his consumption when old will exceed the minimum δ. Further reductions in the return on money forces him to hold even more cash to satisfy the minimum. However, holding more cash balances puts him closer to the young-age minimum consumption requirement, θ. The return can fall only so low – to $\delta/(w-\theta)$, to be exact – for at that return the agent holds $w-\theta$ in real balances and he just meets the young and old age minimum consumption requirements (θ and δ, respectively). This describes, intuitively, the downward sloped portion of the offer curve. Much the same thing happens initially along the upward sloped part of the curve. For low cash balances, the return m_{t+1}/m_t as measured along this portion of the curve may fall, despite the fact that m_{t+1} is rising. Facing falling returns, the agent compensates by holding more balances m_t in order to exceed the consumption minimum δ. At some point, however, further reductions in young-age consumption must be accompanied by a higher return; the offer curve then reflects

[8] When $\sigma > 1$, it is easily checked that (2) is downward sloping everywhere. In that case, it is possible to check that two (but no higher) period cycles around m^* are potentially possible.

the typical positive relationship between saving and rate of return.[9]

It will be readily apparent that more information on the various configurations of (2), and on the position of the steady state relative to the turning point ($\check{m}$) will be useful below. To that end, note that our focus is restricted to $m_{t+1} \geq \delta$ and $m_t \leq w - \theta$ (see (A.1) above). When $m_{t+1} = \delta$, (2) yields $m_t = w - \theta$ in the valid range for m_t; similarly, when $m_t = w - \theta$, (2) yields $m_{t+1} = \delta$. This gives us the endpoints of the offer curve.

If $m^* > \check{m}$ holds, the steady state m^* is on the upward sloping portion of the offer curve implying the slope at the steady state is positive and greater than 1. The condition $m^* > \check{m}$ can be expressed as

$$\frac{\delta + (w-\theta)\beta^{\frac{1}{\sigma}}}{1+\beta^{\frac{1}{\sigma}}} > \frac{\delta}{1-\sigma} \Longleftrightarrow \beta^{\frac{1}{\sigma}} > \frac{\delta\sigma}{(1-\sigma)(w-\theta)-\delta}. \tag{5}$$

The following lemma is then immediate.

Lemma 1 *Define $\tilde{m}$ as $m_{t+1}(\tilde{m}) = \frac{\delta}{1-\sigma}$. Then $\tilde{m} \geq \frac{\delta}{1-\sigma} \Leftrightarrow \frac{\sigma\delta}{((1-\sigma)(w-\theta)-\delta)} \leq \beta^{\frac{1}{\sigma}}$ or $\tilde{m} \geq \frac{\delta}{1-\sigma}$ holds under (5).*

Notice that $\tilde{m}$ is the m_t-coordinate of the turning point $m_{t+1} = \frac{\delta}{1-\sigma} \equiv \check{m}$. Also, $\tilde{m} \geq \frac{\delta}{1-\sigma}$ holds under exactly the same condition as for the steady state to be on the upward sloping part of the curve. Note also then, that in this case $m_t \in [\tilde{m}, w-\theta]$ and $\tilde{m}$ exceeds $\check{m} \equiv \frac{\delta}{1-\sigma}$. This is depicted in Figure 2a.

[9] As noted, as m_t approaches the maximum $w - \theta$, the agent must be compensated with increasingly higher returns in order to compensate for the decrease in young-age utility; this is reflected in the shape of the upper portion of the offer curve. This, however, does not occur along the downward portion of the curve. Here, the additional consumption the agent receives when old from holding an extra unit of cash goes a long way. The agent's consumption when old is close to the minimum δ, so the marginal gain in utility when old from holding an additional unit of cash is large and compensates for the large loss in utility when young as m_t approaches its upper bound (c_t approaches θ).

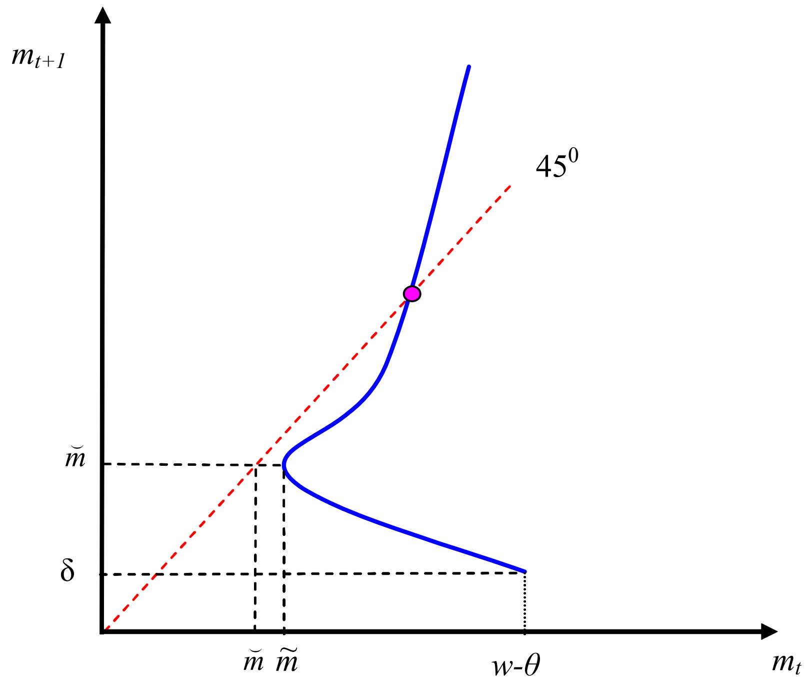

Figure 2a: Eq. (2) with the steady state on the upward sloping portion.

The steady state m^*, shown on the figure by a pink dot, is locally unstable. It follows that it is not possible to travel exclusively along the path described by the upward (or downward) sloping portion of the offer curve and stay in the economically valid region forever. Put differently, any of the momentary equilibrium points on the upward or downward sloping portion of the offer curve cannot by itself generate a dynamic sequence (along the offer curve) that stays asymptotically in $[\delta, w - \theta]$.

In Figure 2b, the steady state m^* is on the downward sloping portion of (2) and $\delta < \tilde{m} < \check{m}$. In other words, the turning point $(\tilde{m})$ occurs to the left of $\delta / (1 - \sigma)$ but still to the right of δ. This means that for $m_t \in [\tilde{m}, w - \theta]$, the corresponding m_{t+1} will lie in the valid range of $[\delta, \infty)$. As in Figure 1, it is not possible to travel exclusively along the path described by the upward sloping portion of the offer curve and stay in the economically valid region forever. Additionally, if the steady state is locally unstable (see Proposition 2), the same would be true of points on the downward sloping segment as well.

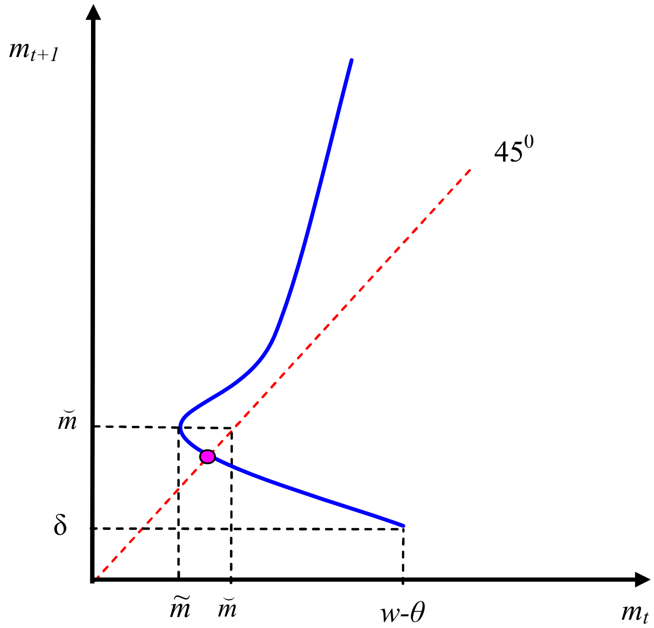

Figure 2b: Eq. (2) with the steady state on the downward sloping portion.

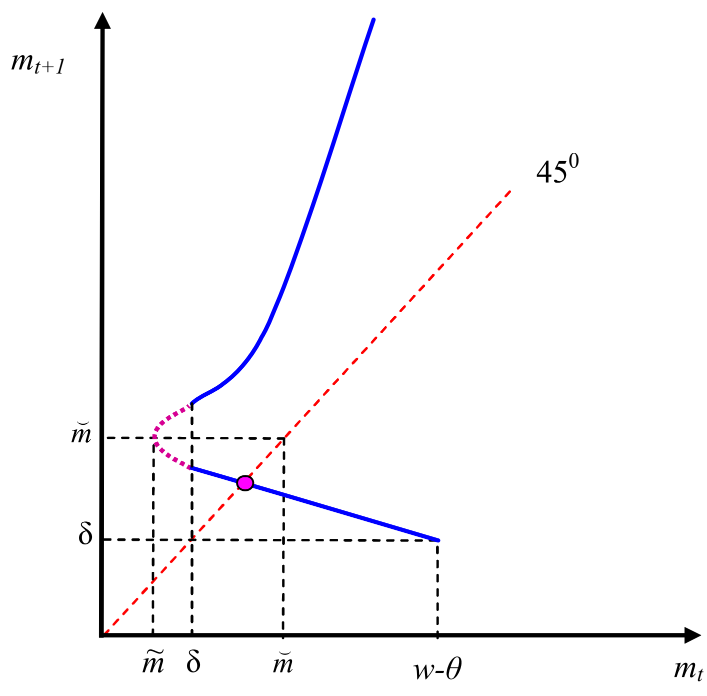

Figure 2c: Eq. (2) with the steady state on the downward sloping portion *and* $\tilde{m} < \delta$.

Finally, in Figure 2c, the unique steady state is again on the downward sloping segment of the law of motion; additionally $\tilde{m}$ is now to the left of δ. The implication is that for $m_t \in [\tilde{m}, w - \theta]$, the corresponding m_{t+1} will no longer lie in the valid range of $[\delta, \infty)$. Hence, there is a discontinuity at $m_t = \delta$ (marked on the figure by a dashed curve). This case in particular perhaps best illustrates our main point – the upper selection of the correspondence describing points that satisfy (2) is completely disconnected from its more familiar lower half, and, as in the previous cases, iterates of this mapping will not generate sequences that stay in the economically valid region forever.

Before closing this section, we consider the local stability properties of the steady state. Note that if the steady state is stable in the forward dynamics, it is unstable in the backward dynamics.

Proposition 2 *If*

A) $\delta \leq \frac{\beta^{\frac{1}{\sigma}}(1-\sigma)(w-\theta)}{\sigma+\beta^{\frac{1}{\sigma}}}$ *holds,* m^* *is unstable and lies on the upward sloping segment (as in Figure 2a).*

B) $\frac{\sigma(w-\theta)\left(\beta^{\frac{1}{\sigma}}\right)^2+(2-\sigma)(w-\theta)\beta^{1/\sigma}}{\sigma+(2-\sigma)\beta^{1/\sigma}} \leq \delta$*,holds,* m^* *is stable and lies on the downward sloping segment (as in Figure 2b or 2c).*

C) $\frac{\beta^{\frac{1}{\sigma}}(1-\sigma)(w-\theta)}{\sigma+\beta^{\frac{1}{\sigma}}} \leq \delta \leq \frac{\sigma(w-\theta)\left(\beta^{\frac{1}{\sigma}}\right)^2+(2-\sigma)(w-\theta)\beta^{1/\sigma}}{\sigma+(2-\sigma)\beta^{1/\sigma}}$*holds,* m^* *is unstable and lies on the downward sloping segment.*

As evident from the three cases A) - C) in Proposition 2, the level of δ (the minimum consumption requirement facing the old) matters crucially. When δ is low enough, the unique steady state is unstable on the upward sloping part of the offer curve. Beyond that, the steady state is on the downward sloping portion; in the intermediate range, it is locally unstable, and in the high range, it is locally stable (unstable in the backward dynamics).

Before proceeding, it is useful to summarize the salient features of our results. Thus far, we have shown that there is a unique steady state level of real balances. The dynamic law of motion for real balances is non-monotonic (it is a correspondence in the normal forward dynamics) opening up the possibility for complex dynamics. Three separate configurations are possible: a) the steady state is on the upward sloping segment of the law of motion, b) the steady state is on the downward sloping segment of the law of motion and the turning point is in the valid range of real balances, c) the steady state is again on the downward sloping segment but there is a discontinuity because the turning point is outside the aforementioned valid range. In each case it is not possible to travel exclusively along the path described by the upward sloping portion of the offer curve and stay in the economically valid region forever. Such momentary equilibrium points (such as those on the upward sloping portion of the offer curve) have traditionally been ruled out by standard textbook treatments and in the literature. In the next section, we study the possibility of cycles among such momentary equilibria and show that they may be indeed be dynamically valid as part of a periodic sequence.

4 Periodic equilibria

For most of this section, we focus on 3-period cycle equilibria. These are conveniently summarized by a triplet (m_a, m_b, m_c), $m_a \neq m_b \neq m_c$, with elements satisfying (A.1), and together, the system of equations:

$$\begin{aligned} \frac{m_b - \delta}{w - m_a - \theta} &= \beta^{\frac{1}{\sigma}} \left(\frac{m_b}{m_a}\right)^{\frac{1}{\sigma}} \\ \frac{m_c - \delta}{w - m_b - \theta} &= \beta^{\frac{1}{\sigma}} \left(\frac{m_c}{m_b}\right)^{\frac{1}{\sigma}} \\ \frac{m_a - \delta}{w - m_c - \theta} &= \beta^{\frac{1}{\sigma}} \left(\frac{m_a}{m_c}\right)^{\frac{1}{\sigma}} \end{aligned} \tag{6}$$

which represent the respective first-order conditions (2) for each date. Trivially, the steady state satisfies (6) with $m_a = m_b = m_c = m^*$.

More generally, when the offer curve is continuous, the existence of a homoclinic orbit around the steady-state ensures the existence of cycles of all orders.[10] To that end, in Figure 3, we illustrate such an orbit for our system.[11] Notice that the steady state is locally unstable in the backward dynamics. On the homoclinic orbit, a path that is repelled away from the steady state eventually gets attracted to it.

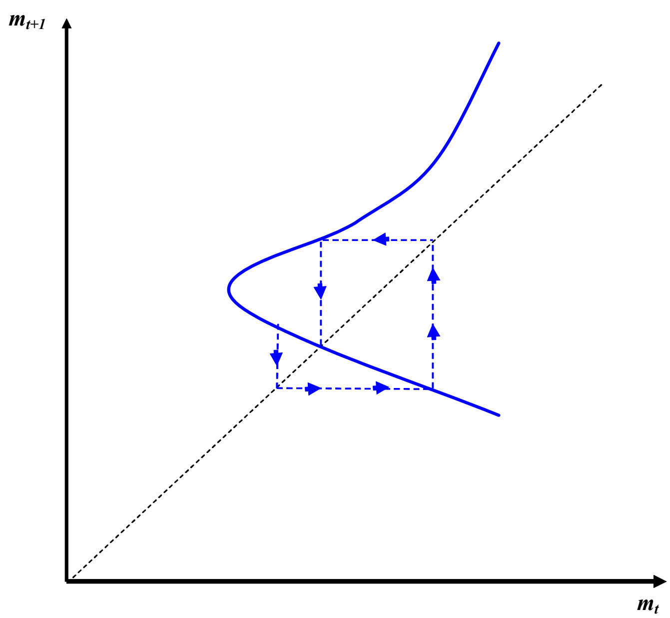

Figure 3: A homoclinic orbit.

[10] A homoclinic point lies in the neighborhood of a unstable steady state and converges to the steady state after n iterations. A homoclinic orbit is non-degenerate if its slope is bounded away from zero. Devaney (1989) establishes that if a map f admits a non-degenerate homoclinic point to the steady state, then in every neighborhood of the steady state f admits periodic orbits of periodicity 2^k, where $k = 1, 2, ...$ In our case, these periodic orbits must include points located on both the upward and the downward segments of the offer curve.

[11] For the parameter configuration used in Example 1, a homoclinic orbit is given by 0.141, 0.101, 0.264, and 0.115 (the steady state).

The following examples illustrate some of the salient features of our model. The first thing to note is that since the steady state on the upward sloping segment of the offer curve is locally unstable in Figure 2a, no cycles are possible under that configuration. We now illustrate some periodic sequences that are possible in the configurations described in Figures 2b and 2c. In each of these cases, the steady state is locally stable in the forward dynamics.

Example 1 *Suppose* $\delta = 0.1, w = 1, \theta = 0.1, \beta = 0.2528,$ *and* $\sigma = 0.35$. *Then, the economy cycles forever between real balance levels,* $0.10012, 0.14863,$ *and* 0.45002. *The steady state level of real balances is given by* $m^* = 0.11540$ *and* $\check{m} = 0.15385$. *The law of motion is continuous* $(\tilde{m} = .10001)$ *and is depicted (not to scale) in Figure 4.*

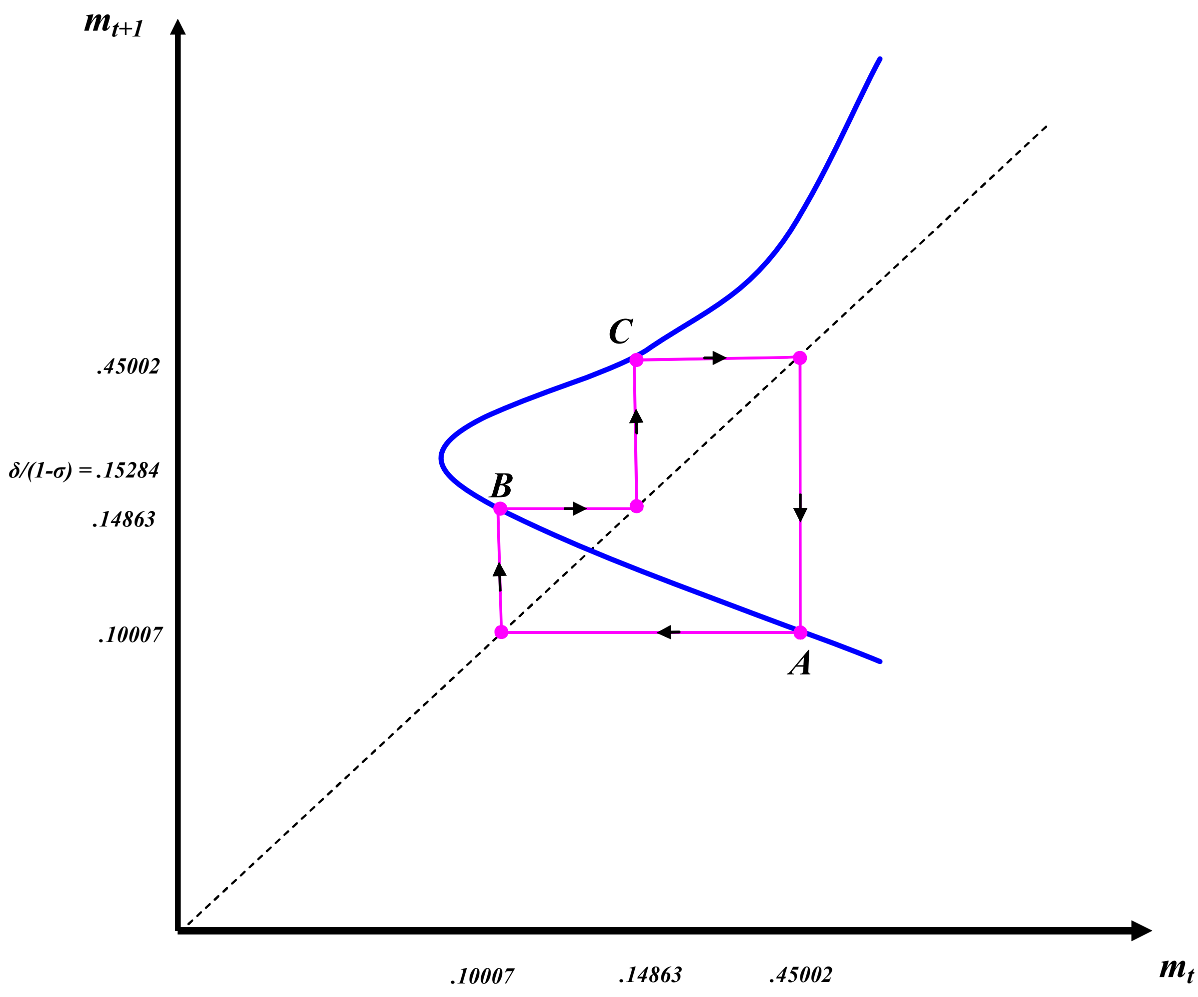

Figure 4: The three-period cycle in Example 1 (not to scale).

In the economy of Example 1, real balances cycle forever between CAB as depicted in Figure 3. The sequence CAB constitutes a dynamically valid equilibrium since points $C, A,$ and B satisfy

(A.1). The previous literature would have ruled out point C, since it lies on the dynamically invalid portion of the offer curve. Consequently, the possibility of CAB being a dynamically valid equilibrium would similarly have been ignored. Interestingly, it is the very existence of the cycle that resurrects the dynamic validity of point C. Therein lies the essence of this paper.[12]

It is fairly easy to generate cycles of periodicity higher than three in this environment.[13]

Example 2 *(5-period cycle) For the parametric specification outlined in Example 1, the economy cycles forever between real balance levels, 0.1039, 0.1290, 0.3387, 0.1003, and 0.1640.*

We close this section with an example of a three-period cycle for the case in which the law of motion is discontinuous.

Example 3 *Suppose* $\delta = 0.15, w = 1, \theta = 0, \beta = 0.42,$ *and* $\sigma = 0.2.$ *Then, the economy cycles between real balance levels,* $0.15006, 0.21087,$ *and* $0.39847.$ *The steady state level of real balances is given by* $m^* = 0.16096$ *and* $\check{m} = .18750.$ *The law of motion is discontinuous (as in Figure 2c) since* $\tilde{m} = .14710 < \delta = 0.15.$ *The example is depicted in Figure 5.*[14]

[12] We are not claiming the economy *must* transgress to a point on the upward sloped portion of the curve, such as point C in our example. When $m_t = .14863$, there is a perfectly valid momentary equilibrium value for $m_{t+1} < .14863$ on the downward sloped part of the offer curve. This sort of equilibrium sequence - one restricted to the downward sloped part of the curve - converges asymptotically to the steady state $m^* = 0.11540$.

[13] The celebrated Li-Yorke theorem does not apply here since the law of motion does not map a compact interval onto itself. Hence, establishing topological chaos via the "Li-Yorke route" is not possible.

[14] Incidentally, it is possible that an economy can support multiple 3-period cycles. For instance, a cycle with real balances 0.15040, 0.17125, and 0.28264 is an equilibrium for the economy described in Example 3. This result does not stem from the discontinuity in the offer curve: the 3-period cycle 0.10007, 0.15817, and 0.50723 obtains for the Example 2 economy.

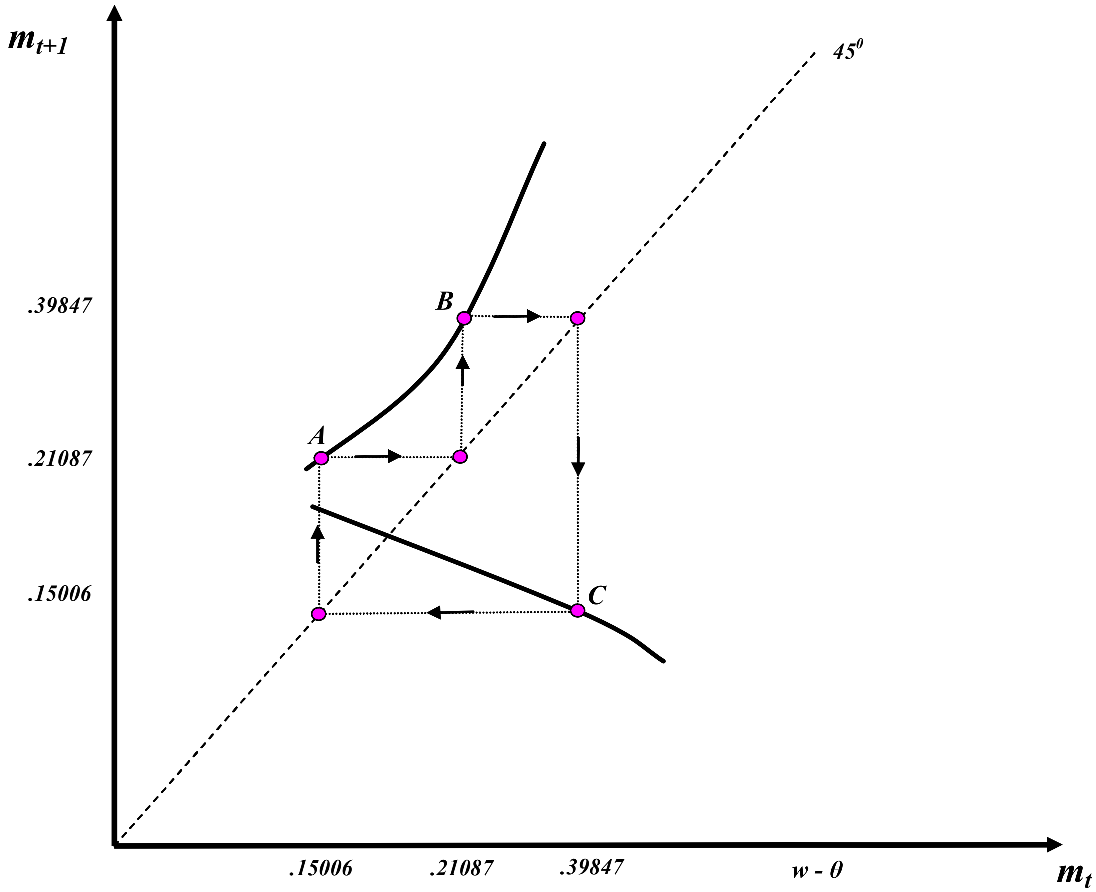

Figure 5: A three-period cycle for the economy in Example 3

Figure 4 illustrates the major contribution of the paper. It presents two separate segments of the intergenerational offer curve. On its own, the disjoint upward sloping segment would have been ruled out in the previous literature. Together with the downward sloping portion, however, they allow for the possibility of a dynamically valid cyclical sequence, such as BCA. Such a cycle bestows dynamical validity to points such as A and B.

5 Concluding remarks

A substantial amount of research has sought to understand the role of nonlinearity and periodic behavior in monetary macroeconomic models. Much of it has focussed on the possibility of generating periodic equilibria in reasonable-looking environments. To the best of our knowledge, all existing work in this area has restricted its attention on what Aguiar-Conraria and Shell (2006) call "long-run perfect foresight" equilibria. These are equilibrium sequences that stay economically valid asymptotically and expectations of agents on such paths are always fulfilled. A central point of Aguiar-Conraria and Shell (2006) is that this notion of notion of long-run perfect foresight equilibrium may be "too rigid" in an OG model where agents currently alive have to "predict the *market behavior* of all future generations". In their setup, there is a bubble that bursts eventually in that agents' expectations are ultimately unmet, but for a long while, expectations are fulfilled. In our present case, there may be cyclical paths on which the money bubble does not asymptotically burst, paths that would have been ignored by the previous literature. These paths contain momentary equilibrium points located along the upward sloped selection of the correspondence described implicitly by (2). Such points become dynamically valid precisely because agents believe that at some date, the economy will return to the downward sloped portion of the offer curve, thereby preventing the sequence of money balances from extending beyond the feasible set.

Though we haven't discussed it in this paper, cycles among momentary equilibria are more prevalent when a government that finances a deficit by seigniorage is introduced into the model. Indeed, it is possible to produce examples of economies where cycles *obtain only* when there is some positive level of seigniorage-financed government spending. Additionally, it appears a framework similar to ours should be capable of generating interesting chaotic and sunspot equilibria. Both seem potential avenues for further study.

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A Proof of Proposition 2

A) The proof of Proposition 2 A) follows from Lemma 1 along with the fact that the $\lim_{m_t \to w-\theta} \frac{dm_{t+1}}{dm_t} = \infty$. This latter point can be noted by rewriting (2) as

$$\frac{m_{t+1} - \delta}{m_{t+1}^{\frac{1}{\sigma}}} = \beta^{\frac{1}{\sigma}} \left(\frac{1}{m_t} \right)^{\frac{1}{\sigma}} (w - m_t - \theta)$$

The left-hand side is approximately $m_{t+1}^{1-\frac{1}{\sigma}}$ for large values of m_{t+1}. When $m_t = w - \theta$, the right-hand side equals 0, so it follows that $\lim_{m_t \to w-\theta} m_{t+1} = \infty$ when $\sigma < 1$.

B) and C) On the downward sloping part of the offer curve,

$$\beta^{\frac{1}{\sigma}} \leq \frac{\sigma\delta}{((1-\sigma)(w-\theta) - \delta)} \Leftrightarrow \frac{\beta^{\frac{1}{\sigma}} (1-\sigma)(w-\theta)}{\sigma + \beta^{\frac{1}{\sigma}}} \leq \delta \tag{7}$$

holds. The steady state is stable when

$$\frac{dm_{t+1}}{dm_t}|_{m^*} > -1,$$

or

$$\frac{dm_{t+1}}{dm_t}|_{m^*} + 1 > 0.$$

Using (4) and the expression for m^*,

$$w - m^* - \theta = \frac{w - \theta - \delta}{1 + \beta^{\frac{1}{\sigma}}}$$

the slope condition reduces to

$$\frac{dm_{t+1}}{dm_t}|_{m^*} + 1 = \frac{-\left[\frac{1}{\sigma}\beta^{1/\sigma} \left(\frac{w-\theta-\delta}{1+\beta^{\frac{1}{\sigma}}} \frac{1+\beta^{\frac{1}{\sigma}}}{\delta+(w-\theta)\beta^{\frac{1}{\sigma}}} \right) + \frac{1+\beta^{\frac{1}{\sigma}}}{w-\theta-\delta} \left(\frac{\delta+(w-\theta)\beta^{\frac{1}{\sigma}}}{1+\beta^{\frac{1}{\sigma}}} - \delta \right) \right]}{1 - \frac{\beta^{1/\sigma}}{\sigma} \left(\frac{w-\theta-\delta}{1+\beta^{\frac{1}{\sigma}}} \frac{1+\beta^{\frac{1}{\sigma}}}{\delta+(w-\theta)\beta^{\frac{1}{\sigma}}} \right)} + 1$$

which reduces further to

$$\frac{dm_{t+1}}{dm_t}|_{m^*} + 1 = \frac{1 - 2\frac{\beta^{1/\sigma}}{\sigma} \left(\frac{(w-\theta-\delta)}{\delta+(w-\theta)\beta^{\frac{1}{\sigma}}} \right) - \beta^{\frac{1}{\sigma}}}{1 - \frac{\beta^{1/\sigma}}{\sigma} \left(\frac{(w-\theta-\delta)}{\delta+(w-\theta)\beta^{\frac{1}{\sigma}}} \right)}.$$

Note that the denominator is positive, since

$$1 - \frac{\beta^{1/\sigma}}{\sigma} \left(\frac{(w-\theta-\delta)}{\delta + (w-\theta)\beta^{\frac{1}{\sigma}}} \right) > 0 \Leftrightarrow \beta^{1/\sigma} < \frac{\sigma\delta}{((1-\sigma)(w-\theta) - \delta)}$$

which holds on the downward sloping part of the offer curve under (7). What remains, then is to find conditions under which the numerator is positive. When this holds, the steady state will be stable. It is easy to check that

$$1-2\frac{\beta^{1/\sigma}}{\sigma}\left(\frac{(w-\theta-\delta)}{\delta+(w-\theta)\,\beta^{\frac{1}{\sigma}}}\right)-\beta^{\frac{1}{\sigma}}>0\Leftrightarrow\delta>\frac{\sigma\,(w-\theta)\left(\beta^{\frac{1}{\sigma}}\right)^{2}+(2-\sigma)\,(w-\theta)\,\beta^{1/\sigma}}{\sigma+(2-\sigma)\,\beta^{1/\sigma}}$$

and that (7) implies

$$\frac{\beta^{\frac{1}{\sigma}}\,(1-\sigma)\,(w-\theta)}{\sigma+\beta^{\frac{1}{\sigma}}}>\frac{\sigma\,(w-\theta)\left(\beta^{\frac{1}{\sigma}}\right)^{2}+(2-\sigma)\,(w-\theta)\,\beta^{1/\sigma}}{\sigma+(2-\sigma)\,\beta^{1/\sigma}}$$

never obtains.