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Time domain inverse source problem and fluid-saturated porous media scattering problem

Zhiming Sun
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Time domain inverse source problem and fluid-saturated porous media scattering problem

Sun, Zhiming, Ph.D.
Iowa State University, 1991
Time domain inverse source problem and fluid-saturated porous media scattering problem

by

Zhiming Sun

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1991

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GENERAL INTRODUCTION
Wave Splitting and Invariant Imbedding

In this presentation, Corones and Krueger's time domain wave splitting and invariant imbedding techniques [1] [2] [3] are applied to solve (1) direct scattering and inverse source problems; (2) direct and inverse scattering problems for a fluid-saturated porous medium, which are studied in Part I and Part II, respectively. This chapter reviews the basic ideas of wave splitting and invariant imbedding techniques, and more details are cited on pages 51-52 and pages 123-126 of Parts I and II.

Wave Splitting

We begin with the following time domain wave equation

\[ E_{xx}(x,t) - c^{-2}(x)E_{tt}(x,t) = 0 \quad 0 < x < L \]  

(0.1)

which describes the propagation of a transversely polarized electromagnetic plane wave in a slab \((0, L)\) of inhomogeneous dielectric media. The wave speed, \(c(x)\), in equation (0.1) is assumed a \(C^1(0, L)\) function. The regions outside the slab are homogeneous with a constant wave speed \(c_0 = c(0-) = c(L+)\). Then we know that the wave equation outside the slab is

\[ E_{xx}(x,t) - c_0^{-2}E_{tt}(x,t) = 0 \quad x < 0 \text{ or } x > L. \]  

(0.2)

Also, we assume that a right going incident wave, \(E^{inc}(t - x/c_0)\), impinges on the
slab from left. A typical solution $E$ of (0.1) will be written as a sum $E(x,t) = E^+(x,t) + E^-(x,t)$ of so-called wave splitting components $E^+$ and $E^-$. These components will satisfy coupled first-order wave equations; with $E^+$ representing a part of $E(x,t)$ that travels to the right in the free-space regions, although the direction of propagation may be ill-defined in a nonvacuous medium; and with $E^-$ representing a part of $E(x,t)$ that travels to the left in the free-space regions. The key point of wave splitting is to find an appropriate dependent variable transform to obtain $E^+$ and $E^-$, the right and left going components of $E(x,t)$. We write (0.1) as

$$
\partial_x \begin{bmatrix} E \\ E_x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c^{-2}(x)\partial_t^2 & 0 \end{bmatrix} \begin{bmatrix} E \\ E_x \end{bmatrix} \equiv \bar{D} \cdot \begin{bmatrix} E \\ E_x \end{bmatrix}
$$

(0.3)

and define wave-splitting components $E^\pm$ (the wave splitting transform) by

$$
\begin{bmatrix} E^+(x,t) \\ E^-(x,t) \end{bmatrix} = \bar{T}^{-1} \cdot \begin{bmatrix} E(x,t) \\ E_x(x,t) \end{bmatrix}
$$

(0.4)

where

$$
\bar{T} = \begin{bmatrix} 1 & 1 \\ -c^{-1}(x)\partial_t & c^{-1}(x)\partial_t \end{bmatrix}
$$

(0.5)

and

$$
\bar{T}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -c(x)\partial_t^{-1} \\ 1 & c(x)\partial_t^{-1} \end{bmatrix}
$$

(0.6)

with

$$
\partial_t^{-1} g(t) = \int_0^t g(s) \, ds
$$

(0.7)
for any piecewise continuous function $g(t)$. We note that $\mathcal{T}$ diagonalizes $\mathcal{D}$ in the sense that $\mathcal{T}\mathcal{D}\mathcal{T}^{-1}$ is the unit matrix. We rewrite definition (0.4) as

$$E^+(x,t) = \frac{1}{2} \left[ E(x,t) + c(x) \partial_t^{-1} E_x(x,t) \right]$$

$$E^-(x,t) = \frac{1}{2} \left[ E(x,t) - c(x) \partial_t^{-1} E_x(x,t) \right]$$

(0.8)

and note that the total electric field is $E(x,t) = E^+(x,t) + E^-(x,t)$. From equations (0.8) and (0.3), we obtain a new split wave equation

$$\partial_x \begin{bmatrix} E^+(x,t) \\ E^-(x,t) \end{bmatrix} = \begin{bmatrix} \frac{\partial_x}{c} + \frac{c}{2c} & -\frac{c}{2c} \\ -\frac{c}{2c} & \frac{\partial_x}{c} + \frac{c}{2c} \end{bmatrix} \begin{bmatrix} E^+(x,t) \\ E^-(x,t) \end{bmatrix}.$$  

(0.9)

In regions $x < 0$ or $x > L$ the transform (0.8) of a general solution, $E(x,t) = f_1(t - x/c_0) + f_2(t + x/c_0)$, of (0.2), results in

$$E^+(x,t) = f_1(t - x/c_0)$$

$$E^-(x,t) = f_2(t + x/c_0).$$

(0.10)

Definition (0.8) then shows that the boundary conditions of the incident field $E^{inc}$ can be written conveniently as

$$E^+(0,t) = E^{inc}(t)$$

$$E^-(L,t) = 0.$$  

(0.11) (0.12)

This is the essential point of wave splitting, i.e., the components, $E^+$ and $E^-$, extends continuously across the boundaries of the slab to the exact physical right and left going waves both at the boundaries and in the regions outside the slab. Wave splitting makes the boundary conditions simple.
Invariant Imbedding

Invariant imbedding is that the total medium is thought of as containing an imbedded medium with associated scattering operators, which are independent of any incident waves. These scattering operators determine the reflection and transmission of fields incident on the boundaries of the imbedded medium. Variations in the boundaries of the imbedded medium will cause variations in the scattering operators such that a nonlinear differential equations can be derived [1] [2] [3]. Simply speaking, the special feature of invariant imbedding is that $\frac{\partial}{\partial x}$ represents a derivative with respect to the variable position $x$ of a slab's left boundary, while the right boundary
remains fixed at $L$ (See Figure 0.1). For the wave equation (0.9) the reflection and transmission operators, $\tilde{R}$ and $\tilde{T}$, are defined by

$$E^-(x,t) = \tilde{R}(x,t) \cdot E^{inc}(t)$$
$$E^+(L,t) = \tilde{T}(x,t) \cdot E^{inc}(t)$$

for the imbedded slab $(x, L)$. $\tilde{R}$ maps the incident wave $E^{inc}(t)$ to the reflected field $E^-(x, t)$ at $x$; $\tilde{T}$ maps the incident wave $E^{inc}(t)$ to the transmitted field $E^+(L, t)$ at $L$. Operators, $\tilde{R}$ and $\tilde{T}$, are integral convolution operators that can be derived from the Duhamel's principle [2]. Then the split wave equation (0.9) and the definition of (0.13) yield nonlinear integrodifferential equations for the integral kernels of $\tilde{R}$ and $\tilde{T}$. These nonlinear equations are used to study both the direct scattering problem and the inverse scattering problem.
This presentation applies wave splitting and invariant imbedding to two kinds of problems in the time domain. These problems are studied in Part I and Part II, respectively.

In Part I, invariant imbedding and wave splitting are extended to the case of a transient electric source $J(t)$ inside a dispersive or inhomogeneous dielectric slab. Representations of composite transmission operators are obtained. These operators are used to establish a delay Volterra type integral equation, which is used to infer the transient source $J(t)$ from the transmitted field. One analytical frequency-domain example and two numerical time-domain examples are presented. Also, Green's operators that map the source $J(t)$ to the field at an arbitrary observation point are defined and used to determine internal $E$ field. For the Green's operator kernels, we obtain linear integrodifferential equations with various initial, boundary and jump conditions.

In Part II, representations of reflection and transmission matrix operators are found, and integrodifferential equations for the operator kernels are derived from the Biot system of compressional wave equations for a finite slab of dispersive, dissipative, fluid-saturated porous medium. Some properties of these operator kernels, such as reciprocity relations and the multiple modes of propagation of discontinuities, are
discussed. A numerical scheme for solving the inverse problem is described, and specific numerical computations for a half-space direct and inverse scattering problem are presented.

**BASIC REFERENCES**


[3] Kristensson, G. and Krueger, R. J., *Direct and inverse scattering in the time domain for a dissipative wave equation*


PART I.

TIME DOMAIN DIRECT SCATTERING AND INVERSE SOURCE PROBLEMS
1. ABSTRACT

A time domain direct scattering and inverse source problem is studied for a transient electric source in a dispersive or nondispersive finite slab. The electric current source $J(t)$ is located at $z = 0$ with current flow along the $y$ axis. Operators that map the source current into transmitted waves are derived and used to establish Volterra type integral equations, which can be used to reconstruct the internal source function $J(t)$ from the transmitted $E$ field at one side of the medium. The direct problem for computing the transmitted fields at the two sides of the slab are solved by convolving the composite transmission operators with the source $J(t)$. Also, Green’s operators that map the source function to the field at an arbitrary observation point is defined and used to study the split wave equation to determine the internal $E$ field. For the Green’s operator kernels, linear integrodifferential equations with various initial, boundary and jump conditions are obtained.
Corones, Krueger and others [1] [2] [3] [5] [9] applied the invariant imbedding and wave splitting methods to do both direct and inverse scattering problems in the time domain for a variety of models. However, in all these problems the wave source has been assumed to be outside the scatterer (actually at infinity). The main purpose of this paper is to extend Corones, Krueger's these techniques to cases where the source is assumed to exist within the slab.

First, this paper defines the direct and inverse source problems for a finite slab where the electromagnetic excitation of the medium is due to a transient electric current within the medium. Then it presents a composite method to solve these direct scattering and inverse source problems. This composite method is a generalization of the Redheffer star product approach used in [6] and [12]. It treats the medium as being made of two adjoining layers with the current source at their interface. Left and right composite transmission operators ($T_l$ and $T_r$) of the full medium can be constructed from the reflection and transmission operators for the two layers. The direct scattering problem is to calculate the transmitted fields or the composite transmission kernels, given the material parameters of the medium and the source location and the source current function $J(t)$. The inverse source problem that will be treated here is to recover the source current function $J(t)$ from knowledge of the
transmitted fields, the source location and the material parameters of the medium. (Another type of inverse problem, that of reconstructing the material parameters from one-side knowledge of the transmitted fields, the source location and $J(t)$, will not be addressed in this paper.) For both the direct scattering and inverse source problems, the method yields Volterra integral equations for the unknown quantities. This part is based on the work of [13].

Second, this paper generalize Corones, Krueger's ideas [2] [4] [8] to obtain an inhomogeneous split wave equation (with source term at right hand side of the split wave equation) for a dispersive slab with a source existing at $z = 0$. The internal electric field at an arbitrary point $z$ for $b < z < a$ is related to the source excitation $J(t)$ through an operator that maps $J(t)$ to the electric field $E$ at an internal point $z$. This operator is defined as the Green's operator. To compute this operator will provide us an efficient way to obtain the internal field for different source functions by convolving them with the Green's kernel since the Green's operator is independent of source excitation (but depends on source location).

In what follows, a precise formulation of the electromagnetic wave propagation problem for a transient source in both dispersive and inhomogeneous medium is clearly described in Chapter 3. Chapter 4 will define and derive composite right and left transmission operators that map the source current into transmitted waves. In Chapter 5, the derived composite operators are used to set up Volterra type integral equations that are applied to solve the inverse problem for the internal source function $J(t)$. The direct problem is also solved by conducting a convolution computation. In Chapter 6, a frequency domain example is given for a dispersive model, which verifies the results obtained in Chapter 3. Two numerical computations for the dispersive
and nondispersive models are done to solve the direct and inverse problems in the time domain. In Chapter 7, the invariant imbedding and wave splitting method is applied to the inhomogeneous electromagnetic wave equation. Then a Green's operator's representation is given. Linear integrodifferential equations with various initial, boundary and jump conditions of this Green's operator kernel are derived.
3. PROBLEM FORMULATION

The scattering medium is a slab bounded on the left by the plane \( z = b \) (\( b < 0 \)) and on the right by the plane \( z = a \). Two kinds of models will be considered: homogeneous dispersive medium and inhomogeneous nondispersive medium as illustrated in Figure 3.1. For both models the medium is assumed to be isotropic, of constant free-space magnetic permeability \( \mu_0 \), nonconducting and devoid of free charge within the medium.

The electric source is located at \( z = 0 \). The electric source is fixed at \( z = 0 \), which is welded together with the neighborhood material, and the surface current flow is along the \( \hat{y} \) direction, i.e., we assume that the electric current density has the form: \( \vec{j}(z,t) = J(t)\delta(z - 0)\hat{y} \), where \( \hat{y} \) is the unit vector in the \( y \) direction, \( \delta(z) \) is the Dirac distribution function and \( J(t) \) is assumed a \( C^2 \) function for \( t > 0 \). Our problem is based on the following Maxwell's equations

\[
\nabla \times \vec{E}(\vec{x},t) = -\partial_t \vec{B}(\vec{x},t) \\
\nabla \cdot \vec{D}(\vec{x},t) = 0 \\
\nabla \times \vec{H}(\vec{x},t) = J(t)\delta(z - 0)\hat{y} + \partial_t \vec{D}(\vec{x},t) \\
\nabla \cdot \vec{B}(\vec{x},t) = 0
\]

(3.1)

where \( \vec{E} \) is the electric field vector, \( \vec{H} \) is the magnetic field vector, \( \vec{D} \) is the electric flux density, and \( \vec{B} \) is the magnetic flux density. According to the geometry of our
Figure 3.1: Schematic diagram of a 1-D inhomogeneous slab model

problem, the $\vec{E}$ and $\vec{H}$ fields will polarize in the $\hat{y}$ and $\hat{z}$ direction respectively. We let $\vec{E}(z, t) = E(z, t) \hat{y}$. From equation (3.1), we can see that Maxwell's equations give jump conditions on the magnetic field $\vec{H}$ at $z = 0$:

$$\hat{z} \times [\vec{H}(0+, t) - \vec{H}(0-, t)] = \lim_{\eta \to 0} \int_{-\eta}^{\eta} \vec{J}(z, t) \, dz = J(t) \hat{y}$$  \hspace{1cm} (3.2)

where $\hat{z}$ is the unit vector in the $z$ direction. From this we can obtain the jump condition for the $E$ field

$$\frac{\partial E(0+, t)}{\partial z} - \frac{\partial E(0-, t)}{\partial z} = \mu_0 J'(t)$$  \hspace{1cm} (3.3)

where $J(t)$ is the current per unit length.

For the dispersive case, the constitutive relation between the displacement vector
$D$ and the electric field $E$ inside the slab is

$$D(z, t) = \epsilon_0 \left[ E(z, t) + \int_0^\infty E(z, t - s)G(s) \, ds \right]$$

where $G(t)$ is assumed to be a $C^1$ function for $t > 0$. From Maxwell’s equations and the constitutive relation we obtain the following equation

$$E_{zz} - \left( \frac{1}{c_0^2} \right) \left[ E_{tt} + \frac{\partial^2}{\partial t^2} \int_0^\infty E(z, t - s)G(s) \, ds \right] = \mu_0 J'(t)\delta(z) \quad b < z < a \quad (3.4)$$

where $c_0 = (\epsilon_0 \mu_0)^{-1/2}$ is the speed of light in vacuum. The regions outside the slab are assumed to be free space with constant wave speed $c_0$.

For the nondispersive case as shown in Figure 3.1, the constitutive relation between $D$ and $E$ is

$$D(z, t) = \epsilon(z)E(z, t)$$

where $\epsilon(z)$ is assumed continuous in $(-\infty, +\infty)$ and is a $C^1$ function in $(b, 0) \cup (0, a)$. The wave equation of $E$ in this case is given by

$$E_{zz}(z, t) - c^{-2}(z)E_{tt}(z, t) = \mu_0 J'(t)\delta(z) \quad b < z < a \quad (3.5)$$

where $c^{-2}(z) = \epsilon(z)\mu_0$. The regions to the right and left of the slab are homogeneous with constant wave speed $c(a)$ and $c(b)$, respectively.

Observe that the jump condition in (3.3) on $\partial_z E(z, t)$ follows easily from integrating equations (3.4) or (3.5) in $z$ from $z = 0-$ to $z = 0+$ and using the fact that $\epsilon(z)$, $G(t)$ and $E(z, t)$ are continuous (by Maxwell’s equations, we know that the tangent $E$ field is continuous at any interface), and $\epsilon(z)$ and $G(t)$ are bounded functions.
4. DERIVATION OF COMPOSITE OPERATORS

In this chapter, we find a pair of operators, $\tilde{T}_R$ and $\tilde{T}_L$, that map the waves due to current $J(t)$ into the right and left going transmitted waves at $z = a$ and $z = b$, respectively. By using Corones, Krueger and others' direct scattering results [2] [4] [9], we can obtain the reflection operator $\tilde{R}_1$ and transmission operator $\tilde{T}_1$ for the right half slab ($0 < z < a$), and $\tilde{R}_2$ and $\tilde{T}_2$ for the left slab ($b < z < 0$). The composite idea is to find $\tilde{T}_R$ and $\tilde{T}_L$ in terms of $\tilde{R}_{1,2}$ and $\tilde{T}_{1,2}$. The idea used here, to divide the slab and deal with the pieces, is similar to that presented in [6] and [12]. However, in the present problem we divide the original slab into three pieces, not two as in [6] and [12]. This is necessary in order to take proper account of the current source, which is contained in the middle slab (See Figure 4.1).

Let $\delta$ be an arbitrary small number, $\delta > 0$. In Figure 4.1, where the nondispersive model is illustrated, we separate our original slab into three regions, $(b, -\delta), (-\delta, +\delta)$ and $(+\delta, a)$. In order to carefully analyze the wave propagation process around $z = 0$, we approximate the very thin slab, $(-\delta, \delta)$, by a homogeneous slab bounded by planes $z = -\delta$ and $z = \delta$ with a constant permittivity $\varepsilon(0)$. If we take $\delta \to 0$, the approximate problem will become our original problem.

A transient source $J(t)$ radiates electromagnetic waves into the slab. Waves within $(-\delta, \delta)$ are a superposition of the radiation wave and the scattered wave. In
Figure 4.1: Wave splitting in the neighborhood of $z = 0$

Figure 4.1, $E^-(z,t)$ and $E^+(z,t)$ are the left and right going radiation waves produced by the source $J(t)$, which can be obtained by imbedding $J(t)$ in an unbounded space with homogeneous permittivity. We can obtain the radiation waves $E^\pm_J$ by solving the following equation for $E^\pm_J(z,t)$:

\[
\begin{align*}
\frac{\partial^2 E^\pm_J(z,t)}{\partial z^2} - v_0^{-2} \frac{\partial^2 E^\pm_J(z,t)}{\partial t^2} &= \mu_0 J'(t)\delta(z) \quad -\infty < z < \infty, \quad t > 0 \\
\frac{\partial E^\pm_J(z,t)}{\partial t} &= 0 \quad E^\pm_J(z,t) = 0 \quad \text{for} \quad t \leq 0
\end{align*}
\]

(4.1)

where $v_0 = 1/\sqrt{\mu_0 \varepsilon}$, $\varepsilon$ is a constant. $\varepsilon$ and $v_0$ are identified with $\varepsilon_0$ and $c_0$, respectively, in the dispersive model and with $\varepsilon(0)$ and $c(0)$, respectively, in the nondispersive model.

The solution of equation (4.1) can be obtained by using integral transformation
methods. The resulting solution is

\[ E_J(z, t) = E_J(t - |z|/v_0) = -\frac{1}{2} \sqrt{\frac{\rho_0}{\varepsilon}} J(t - |z|/v_0) \]  

(4.2)

Let \( E_J(z, t) = E_J^+(z, t) + E_J^-(z, t) \), where \( E_J^+ \) and \( E_J^- \) are the right and left going radiation waves, which are obtained from equation (4.2) by

\[
E_J^+(z, t) = \begin{cases} 
-\frac{1}{2} \sqrt{\frac{\rho_0}{\varepsilon}} J(t - z/v_0) & z \geq 0 \\
0 & z < 0 
\end{cases}
\]

(4.3)

and

\[
E_J^-(z, t) = \begin{cases} 
0 & z > 0 \\
-\frac{1}{2} \sqrt{\frac{\rho_0}{\varepsilon}} J(t + z/v_0) & z \leq 0 
\end{cases}
\]

(4.4)

Observe from (4.3) and (4.4) that for \( z > 0 \) there exists only the right going radiation wave \( E_J^+ \) and for \( z < 0 \) there exists only the left going radiation wave \( E_J^- \). A derivation of this solution can be found in Appendix A. Actually we can verify (4.2) by directly inserting it into equation (4.1) by using the fact that \( |z| = z(2H(z) - 1) \), \( |z'| = 2H(z) \) and \( |z''| = 2\delta(z) \), where \( H(z) \) is the Heaviside function.

Let \( E(z, t) \) be the total electric field and \( E^\pm(z, t) \) be the right and left going fields at \( z \); then \( E(z, t) = E^+(z, t) + E^-(z, t) \). We define the scattered field \( E_s(z, t) \) by

\[
E_s(z, t) = \begin{cases} 
E(z, t) - E_J^-(z, t) & -\delta < z \leq 0, \ t > 0 \\
E(z, t) - E_J^+(z, t) & 0 < z < \delta, \ t > 0 
\end{cases}
\]

(4.5)

Then by equation (4.1) and the equation of \( E, \partial_z^2 E - v_0^{-2} \partial_t^2 E = \mu_0 J(t)\delta(z) \) for \( |z| < \delta \), which can be obtained by letting \( G(t) = 0 \) in (3.4) and \( c(z) = c(0) \) in (3.5), we see that the scattered field \( E_s(z, t) \) satisfies a homogeneous wave equation:
\[ \partial_z^2 E_s - v_0^{-2} \partial_t^2 E_s = 0. \]

We also can split \( E_s(z, t) \) into right and left going waves, \( E^+(z, t) \) and \( E^-(z, t) \), respectively. Then \( E_s(z, t) = E^+_s(z, t) + E^-_s(z, t) \). For the middle thin slab, we can set up an initial boundary value problem for \( E^\pm_s(z, t) \)

\[
\begin{cases}
\partial_z^2 E_s(z, t) - (1/v_0^2) \partial_t^2 E_s(z, t) = 0 & -\delta < z < \delta, \quad t > 0 \\
\partial_t E_s(z, t)|_{t=0} = 0 & E_s(z, t)|_{t=0} = 0 \\
E_s(z, t)|_{z=\delta} = E_s(\delta, t) & E_s(z, t)|_{z=-\delta} = E_s(-\delta, t).
\end{cases}
\] (4.6)

We know that the solution of equation (4.6) is of the form: \( E_s(z, t) = E^+_s(z, t) + E^-_s(z, t) = u(t - z/v_0) + w(t + z/v_0) \), where \( u, w \in C^2(-\delta, \delta) \). We can easily see that \( u \) and \( w \) correspond to the right-going and left-going scattered waves, \( E^+_s \) and \( E^-_s \), respectively. When we apply the boundary conditions in (4.6) to such solutions, we have

\[
\begin{align*}
u(t \mp \delta/v_0) &= E^+_s(\pm\delta, t) \\
w(t \pm \delta/v_0) &= E^-_s(\pm\delta, t).
\end{align*}
\] (4.7)

From above, we can find the solution of (4.6)

\[
\begin{bmatrix}
E^+_s(z, t) \\
E^-_s(z, t)
\end{bmatrix} =
\begin{bmatrix}
E^+_s(\delta, t - (z - \delta)/v_0) \\
E^-_s(\delta, t + (z - \delta)/v_0)
\end{bmatrix} =
\begin{bmatrix}
E^+_s(-\delta, t - (z + \delta)/v_0) \\
E^-_s(-\delta, t + (z + \delta)/v_0)
\end{bmatrix}.
\] (4.8)

It can be verified that \( E(z, t) = E_s(z, t) + E_J(z, t) \) satisfies the jump condition in (3.3). Then we can express the right-going and left-going waves at \( z = \delta \) and \( z = -\delta \) in terms of radiation and scattered waves:

\[
\begin{cases}
E^+(\delta, t) = E^+_J(\delta, t) + E^+_s(\delta, t) \\
E^-(\delta, t) = E^-_s(\delta, t)
\end{cases}
\] (4.9)
\[
\begin{aligned}
E^+(\delta, t) &= E^+_s(\delta, t) = E^+_s(\delta, t + 2\delta/v_0) \\
E^-(\delta, t) &= E^-_s(\delta, t) + E^-_s(\delta, t) = E^-_s(\delta, t) + E^-_s(\delta, t - 2\delta/v_0).
\end{aligned}
\] (4.10)

In (4.10) we have used the relation between \(E^+_s(\delta, t)\) and \(E^+_s(\delta, t)\) in equation (4.8).

On the other hand, if we just separately look at the right slab \((\delta < z < a)\), the input to the slab is a right going incident wave \(E^+(\delta, t)\), which comes from a medium with constant wave speed \(v_0\), impinging on the slab at \(z = \delta\). Now we can apply the discussion and results in [2] and [9] to our case. Let \(\tilde{R}^\delta_1\) and \(\tilde{T}^\delta_1\) be the reflection and transmission operators corresponding to the right slab which is bounded by planes \(z = \delta\) and \(z = a\). The region to the left of the slab is a homogeneous medium with constant wave speed \(v_0\). \(\tilde{R}^\delta_1\) maps the right going wave \(E^+(\delta, t)\) into reflection wave \(E^-\). We must note that there is a jump in wave speed at interface \(z = \delta\) for the inhomogeneous model. A similar discussion can be applied to the left slab \((b < z < -\delta)\). According to [2] [9], we then have

\[
\begin{aligned}
E^-(\delta, t) &= \tilde{R}^\delta_1 \cdot E^+(\delta, t) \\
&= \bar{r}_1 E^+(\delta, t) + \int_0^t \tilde{R}^\delta_1(\delta, a, s)E^+(\delta, t - s) ds \\
E^+(\delta, t) &= \tilde{R}^\delta_2 \cdot E^-(\delta, t) \\
&= \bar{r}_2 E^-(-\delta, t) + \int_0^t \tilde{R}^\delta_2(\delta, b, s)E^-(\delta, t - s) ds
\end{aligned}
\] (4.11)

where \(\bar{r}_1 = \bar{r}_2 = 0\) for our dispersive model, \(\bar{r}_1 = [c(\delta) - c(0)]/[c(\delta) + c(0)]\) and \(\bar{r}_2 = [c(-\delta) - c(0)]/[c(-\delta) + c(0)]\) for our nondispersive model. Then \(\tilde{R}^\delta_1\) and \(\tilde{T}^\delta_1\) are given by \(\tilde{R}^\delta_1 = \lim_{\delta \to 0} \tilde{R}^\delta_1\) and \(\tilde{T}^\delta_1 = \lim_{\delta \to 0} \tilde{T}^\delta_1\), respectively. Then for \(\tilde{T}^\delta_1\), we have
(See [2] [9])

\[ E^+(a,t) = \bar{T}_1^\delta \cdot E^+(\delta,t) \] (4.12)

\[ = \alpha_1 E^+(0,t-\tau_1) + \int_0^{t-\tau_1} T_1^\delta(\delta,a,t-s)E^+(\delta,s) \, ds \]

where, for the dispersive model \( \alpha_1 = \exp[-0.5(a-\delta)G(0+)/c_0] \), \( \tau_1 = (a-\delta)/c_0 \) [2]; and for the nondispersive model \( \alpha_1 = 2[c(a)c(\delta+)]^{1/2}/[c(0) + c(\delta+)] \), \( \tau_1 = f_0^{a-\delta} c^{-1}(z) \, dz \) [9]. Then from equations (4.9), (4.10) and (4.11) we have

\[ E_\delta^-(\delta,t) = \bar{R}_1^\delta \cdot \left[ E_J^+(\delta,t) + E_\delta^+(\delta,t) \right] \] (4.13)

\[ E_\delta^+(\delta,t+2\delta/v_0) = \bar{R}_2^\delta \cdot \left[ E_J^-(\delta,t) + E_\delta^-(\delta,t-2\delta/v_0) \right]. \]

Using (4.13a) to eliminate \( E_\delta^- \) in (4.13b), we obtain

\[ E_\delta^+(\delta,t+2\delta/v_0) = \bar{R}_2^\delta \cdot E_J^-(\delta,t) + \bar{R}_2^\delta \cdot \bar{R}_1^\delta \cdot E_J^+(\delta,t-2\delta/v_0) \] (4.14)

\[ + \bar{R}_2^\delta \cdot \bar{R}_1^\delta \cdot E_\delta^+(\delta,t-2\delta/v_0) \]

Let \( \delta \to 0 \) in (4.14) and notice that

\[ \lim_{\delta \to 0} \bar{R}_1^\delta \cdot f(t) = \lim_{\delta \to 0} \left[ \bar{r}_1 f(t) + \bar{R}_1 \ast f(t) \right] = \bar{R}_1 \ast f(t) \quad \text{for} \quad f \in C(0,\infty) \]

\[ \lim_{\delta \to 0} \bar{R}_2^\delta \cdot f(t) = \lim_{\delta \to 0} \left[ \bar{r}_2 f(t) + \bar{R}_2 \ast f(t) \right] = \bar{R}_2 \ast f(t) \quad \text{for} \quad f \in C(0,\infty) \]

\[ \lim_{\delta \to 0} E_J^-(\delta,t) = -\frac{1}{2} \sqrt{\mu_0/\epsilon} \lim_{\delta \to 0} J(t + \delta/v_0) = -\frac{1}{2} \sqrt{\mu_0/\epsilon} \, J(t) \]

\[ \lim_{\delta \to 0} E_J^+(\delta,t-2\delta/v_0) = -\frac{1}{2} \sqrt{\mu_0/\epsilon} \lim_{\delta \to 0} J(t-3\delta/v_0) = -\frac{1}{2} \sqrt{\mu_0/\epsilon} \, J(t) \]

\[ \lim_{\delta \to 0} E_\delta^+(\delta,t \pm 2\delta/v_0) = E_\delta^+(0,t) \] (4.15)

since \( \lim_{\delta \to 0} \bar{r}_1 = \lim_{\delta \to 0} \bar{r}_2 = 0, J \in C^2(0,\infty), E_\delta(z,t) \) is a continuous function in \( z \) and \( t \) and \( R_r, R_l \) are continuous in \( z \). The * in (4.15) is the usual convolution operator, i.e., \( f \ast g(t) = \int_0^t f(s)g(t-s) \, ds \) for two functions \( f \) and \( g \).
Then by applying equation (4.15) to equation (4.13) we obtain the equation to express $E_s^+(0,t)$ in terms of the source $J(t)$:

$$E_s^+(0,t) = -(1/2)\sqrt{\mu_0/\bar{\epsilon}} \tilde{R}_l J(t) \quad (4.16)$$

$$\tilde{R}_l = (1 - \tilde{R}_2 \tilde{R}_1)^{-1} \tilde{R}_2 (1 + \tilde{R}_1)$$

$$= \tilde{R}_2 + \tilde{R}_2 \tilde{R}_1 + \tilde{R}_2 \tilde{R}_1 \tilde{R}_2 + \tilde{R}_2 \tilde{R}_1 \tilde{R}_2 \tilde{R}_1 + \cdots \quad (4.17)$$

In the same way, we can find $E_s^-(0,t)$ from equation (4.14),

$$E_s^-(0,t) = -(1/2)\sqrt{\mu_0/\bar{\epsilon}} \tilde{R}_r J(t) \quad (4.18)$$

where

$$\tilde{R}_r = (1 - \tilde{R}_1 \tilde{R}_2)^{-1} \tilde{R}_1 (1 + \tilde{R}_2)$$

$$= \tilde{R}_1 + \tilde{R}_1 \tilde{R}_2 + \tilde{R}_1 \tilde{R}_2 \tilde{R}_1 + \tilde{R}_1 \tilde{R}_2 \tilde{R}_1 \tilde{R}_2 + \cdots \quad (4.19)$$

Now we use the same idea to find the composite transmission operators. For the right slab ($\delta < z < a$) and left slab ($b < z < -\delta$), by the definition of $\bar{T}_1^\delta$ and $\bar{T}_2^\delta$,

$$E^+(a,t) = \bar{T}_1^\delta \cdot E^+(\delta,t) = \bar{T}_1^\delta \cdot [E_s^+ (\delta,t) + E_0^+(\delta,t)]$$

$$E^-(b,t) = \bar{T}_2^\delta \cdot E^-(\delta,t) = \bar{T}_2^\delta \cdot [E_s^- (\delta,t) + E_0^-(\delta,t)] \quad (4.20)$$

Take $\delta \to 0$ in (4.20) and then insert (4.18) and (4.16) into the result. Thus we have,

$$E^+(a,t) = -(1/2)\sqrt{\mu_0/\bar{\epsilon}} \bar{T}_r J(t) \quad (4.21)$$

$$\bar{T}_r = \bar{T}_1 (1 - \tilde{R}_2 \tilde{R}_1)^{-1} (1 + \tilde{R}_2) = \bar{T}_1 (1 + \tilde{R}_l)$$

and

$$E^-(b,t) = -(1/2)\sqrt{\mu_0/\bar{\epsilon}} \bar{T}_l J(t) \quad (4.22)$$

$$\bar{T}_l = \bar{T}_2 (1 - \tilde{R}_1 \tilde{R}_2)^{-1} (1 + \tilde{R}_1) = \bar{T}_2 (1 + \tilde{R}_r)$$

$\bar{T}_r$ and $\bar{T}_l$ are the right and left composite transmission operators, which map the source function $J(t)$ into the transmitted waves at $z = a$ and $z = b$, respectively.
5. DIRECT SCATTERING AND INVERSE SOURCE PROBLEMS

5.1 Representation of Composite Operators

Now we can write down the integral representation of the transmission operator $\tilde{T}_1$ for the right slab by taking $\delta \to 0$ in (4.12)

$$E^+(a,t) = \tilde{T}_1 \cdot E^+(0,t)$$

$$= \alpha_1 E^+(0,t - \tau_1) + \int_0^{t-\tau_1} T_1(0,a,t - s) E^+(0,s) \, ds$$

where, for our dispersive and nondispersive models $\alpha_1$ and $\tau_1$ are given in [2] [9] by

$$\alpha_1 = \exp\left[-0.5aG(0+)/c_0\right], \quad \tau_1 = a/c_0 \quad \text{and} \quad \alpha_1 = \left[c(a)/c(0)\right]^{1/2}, \quad \tau_1 = \int_0^a c^{-1}(z) \, dz,$$

respectively.

From equations (4.19), (4.17) and (4.11) (by letting $\delta \to 0$), we see that $R_r$ and $R_l$ are also integral operators given by

$$R_r(a,b,t) = \int_0^t R_r(t,s) \, J(s) \, ds$$

where

$$R_r(a,b,t) = R_1 + R_1 \ast R_2 + R_1 \ast R_2 \ast R_1 + \cdots$$

$$R_l(a,b,t) = R_2 + R_2 \ast R_1 + R_2 \ast R_1 \ast R_2 + \cdots$$

We can prove that these two series in (5.3) are convergent series. For the $R_r$ series, for example, we define a new series $S_r(\omega) = r_1 + r_1 r_2 + r_1^2 r_2 + r_1^2 r_2 + r_1^3 r_2 + r_1 r_2 + \cdots$, where $r_i = r_i(\omega)$ is the Fourier transform of $R_i(x,t)$. From the law of energy
conservation, we know that $|r_1(\omega)| < 1$ for all $\omega$ and $i = 1, 2$. Then we know that $|r_1(\omega)r_2(\omega)|$ is also bounded. We assume that $|r_1(\omega)r_2(\omega)|$ attains a maximum at $\omega_0$, $|\omega_0| < \infty$, i.e., $|r_1(\omega_0)r_2(\omega)| \leq |r_1(\omega_0)r_2(\omega_0)|$ for all $\omega$. $S_r(\omega_0)$ is convergent. Then $S_r(\omega)$ is an absolutely uniformly convergent series and bounded by $S_r(\omega_0)$, and it converges to

$$S_r(\omega) = r_1 + r_1r_2 + r_1^2r_2 + r_1^3r_2^2 + r_1^4r_2^3 + \ldots = \frac{r_1(\omega)(1 + r_2(\omega))}{1 - r_1(\omega)r_2(\omega)}. \quad (5.4)$$

We then apply the inverse Fourier transform $F^{-1}$ to equation (5.4). We can apply $F^{-1}$ to the series $S_r(\omega)$ term by term since $S_r(\omega)$ is uniformly convergent. Then it shows that the series $R_r = R_1 + R_1 \cdot R_2 + R_1 \cdot R_2 \cdot R_1 + \ldots$ is convergent and it converges to $F^{-1}\{[r_1(\omega)(1 + r_2(\omega))]/[1 - r_1(\omega)r_2(\omega)]\}$. The same discussion can be applied to the $R_l$ series.

We apply the operator representation of $\bar{T}_1$ in (5.1) and the operator representation of $\bar{R}_r$ in (5.2) to equation (4.21), yielding the following equation:

$$E^+(a, t) = -\frac{\mu_0 v_0}{2} \left[ \alpha_1 J(t - \tau_1) \right. \right.$$

$$+ \int_0^{t - \tau_1} \left[ T_1(0, a, t - u) + \alpha_1 R_l(a, b, t - \tau_1 - u) \right. \right.$$

$$+ \int_0^{t - \tau_1 - u} T_1(0, a, t - u - u') R_l(a, b, u') du' \right] J(u) du \right.$$

$$\left. \equiv -\left(\mu_0 v_0/2\right) \left[ \alpha_1 J(t - \tau_1) + \int_0^{t - \tau_1} T_r(a, b, t - u) J(u) du \right] \right). \quad (5.5)$$
In the same way we can obtain the integral equation for the left transmission problem:

\[
E^-(b, t) = -\frac{\mu_0 v_0}{2} \left[ \alpha_2 J(t - \tau_2) + \int_{0}^{t-\tau_2} T_2(0, b, t - u) + \alpha_2 R_r(a, b, t - \tau_2 - u) + \int_{0}^{t-\tau_2-u} T_2(0, b, t - u - u') R_r(a, b, u') du' \right] J(u) du
\]

\[
\equiv -\left(\mu_0 v_0/2\right) \left[ \alpha_2 J(t - \tau_2) + \int_{0}^{t-\tau_2} T_1(a, b, t - u) J(u) du \right]
\]

where \(\alpha_2\) and \(\tau_2\) of the left slab are analogous to \(\alpha_1\) and \(\tau_1\). Equations (5.5) and (5.6) will be applied to the dispersive and nondispersive medium model by letting \(v_0 = c_0\) and \(v_0 = c(0)\), respectively. From (5.5) and (5.6) we obtain the left and right transmission kernels

\[
T_r(a, b, t) = T_1(0, a, t) + \alpha_1 R_l(a, b, t - \tau_1) + \int_{0}^{t-\tau_1} T_1(0, a, t - u) R_l(a, b, u) du
\]

\[
T_l(a, b, t) = T_2(0, b, t) + \alpha_2 R_r(a, b, t - \tau_2) + \int_{0}^{t-\tau_2} T_2(0, b, t - u) R_r(a, b, u) du
\]

5.2 Transmitted Fields and Reconstruction of Internal Source

Equations (5.5) and (5.6) provide the means of solution of both the direct scattering problem and the inverse source problem. For either problem, begin by calculating the kernels \(R_{1,2}\) and \(T_{1,2}\) as in [2] [3] [6] [8]. From (4.19) and (4.17), \(R_r(0, a, t)\) and \(R_l(0, b, t)\) are determined from the Volterra equations

\[
R_r - R_1 \ast R_2 \ast R_r - R_1 - R_1 \ast R_2 = 0
\]

\[
R_l - R_2 \ast R_1 \ast R_l - R_2 - R_2 \ast R_1 = 0
\]

or directly from \(R_r = R_1 + R_1 \ast R_2 + R_1 \ast R_2 \ast R_1 + \cdots\) and \(R_l = R_2 + R_2 \ast R_1 + R_2 \ast R_1 \ast R_2 + \cdots\). Use these and \(T_{1,2}\) in (5.7) to find \(T_r\) and \(T_l\). Finally, insert
$T_r$ and $T_l$ into (5.5) and (5.6), respectively. Then for the direct scattering problem, the transmitted field follow directly from evaluation of the right hand side of (5.5) and (5.6). For the inverse source problem, equation (5.5) and (5.6) are delay Volterra equation for $J(t)$, either of which may be used to determine the source function.
6. EXAMPLES

In the following, we will give an example of the dispersive medium model and obtain its composite transmission coefficient in the frequency domain. We will show that it agrees with the result we have in (4.21) if we take the Fourier transformation of (4.21). We will also give two numerical examples for dispersive and nondispersive models to solve the direct scattering problem and inverse source problem in the time domain by using equation (5.5) or (5.6).

6.1 A Frequency Domain Example

In Figure 4.1, let the medium be dispersive with the current source located in the middle, i.e., \( a = |b| \). By the space symmetry of this problem, we know that \( R_1(0, a, t) = R_2(0, |b|, t) \) and \( T_1(0, a, t) = T_2(0, |b|, t) \). Its frequency domain reflection and transmission coefficients are defined as usual:

\[
\begin{align*}
  r_0(\omega) &= F[\tilde{R}_1 \cdot J(t)] / F[J(t)] = R_1(\omega) \\
  t_0(\omega) &= F[\tilde{T}_1 \cdot J(t)] / F[J(t)] = 1 + T_1(\omega) \\
  t_r(\omega) &= F[\tilde{T}_r \cdot J(t)] / F[J(t)] = 1 + T_r(\omega)
\end{align*}
\]

(6.1)

where \( r_0(\omega) \) and \( t_0(\omega) \) are the frequency domain reflection and transmission coefficient for the right slab \( (0 < z < a) \) and \( t_r(\omega) \) is the right composite transmission
coefficient at \( z = a \). We have assumed that \( G(0) = 0 \). Then \( \alpha_1 = 1 \) in equations (5.1) and (5.5).

For simplicity we assume that the source has the form \( J(\omega) = e^{i\omega t} \), where \( \omega \) is the frequency. Then from the results in the previous section, the right and left going radiation waves are

\[
E^\pm_J = -(1/2)\sqrt{\mu_0/\varepsilon(\omega)}e^{i(\omega t \pm k z)}
\]

where \( k = \omega/\varepsilon(\omega)\). We will derive the right and left composite transmission coefficients by tracing the multiple scattering process inside the medium. From Figure 4.1 we see how the right going radiation wave \( E^+_J \) contributes to the right composite transmission coefficient. For the first scattering step, the reflection and transmission coefficients for \( E^+_J \) traveling from 0 to \( a \) and then back to 0 are

\[
\begin{align*}
    r_1^+ &= r_0 = \frac{\sqrt{\varepsilon(\omega)} - \sqrt{\varepsilon_0}}{\sqrt{\varepsilon(\omega)} + \sqrt{\varepsilon_0}} e^{i2ka} \\
    t_1^+ &= t_0 = \frac{2\sqrt{\varepsilon(\omega)}}{\sqrt{\varepsilon(\omega)} + \sqrt{\varepsilon_0}} e^{ika}
\end{align*}
\]

Then the wave continues to travel to the left. We can trace all subsequent reflections and transmissions at the two boundaries back and forth (multiple scattering processes) and obtain a series of transmission coefficients at the right edge for each process

\[
t_n^+ = r_0^{2(n-1)}t_0 \quad \text{for } n = 2, 3, 4, \ldots.
\]

We obtain the full transmission coefficient due to the right going radiation wave \( E^+_J \) by summing all these \( t_n^+ \) terms

\[
t^+_t = t_1^+ + t_2^+ + t_3^+ + t_4^+ + \cdots = \frac{t_0}{1 - r_0^2}.
\]
Note that we have used the fact that $|r_0| < 1$ and $|t_0| < 1$ which result from the law of energy conservation. In the same way, we trace the left going radiation wave $E_j^-$ to obtain the corresponding transmission coefficient at $z = a$

$$t_r^- = t_1^- + t_2^- + t_3^- + t_4^- + \cdots = \frac{r_0 t_0}{1 - r_0^2}. \quad (6.6)$$

From equations (6.5) and (6.6), we have the total right composite transmission coefficient

$$t_r = t_r^+ + t_r^- = \frac{t_0}{1 - r_0} \quad (6.7)$$

where $r_0$ and $t_0$ are given by (6.2), (6.3). On the other hand, from the time domain result in (4.21), we see that $\mathcal{T}_r = \mathcal{T}_1 (1 - \tilde{R}_2 \tilde{R}_1)^{-1} (1 + \tilde{R}_2)$, i.e.,

$$\mathcal{T}_r (1 - \tilde{R}_1) J(t) = \mathcal{T}_1 J(t) \quad (6.8)$$

since in this example $\tilde{R}_1 = \tilde{R}_2$. If we take the Fourier transformation of (6.8), and thus change the time domain equation into a frequency domain equation, we obtain

$$[1 + T_r(\omega)] [1 - R_1(\omega)] = 1 + T_1(\omega). \quad (6.9)$$

Here we have used the integral representations of the reflection operator $\tilde{R}_1$ in (4.11) (by taking $\delta \to 0$ in (4.11)) and transmission operators $\mathcal{T}_r$ and $\mathcal{T}_1$ in (5.5) and (5.1) and the Fourier transformation convolution formula. From (6.9) and (6.1), we then have

$$t_r = \frac{t_0}{1 - r_0} \quad (6.10)$$

which agrees with the direct frequency domain result in (6.7).
6.2 Two Time Domain Numerical Examples

The following are two numerical examples of direct scattering and inverse source problems for dispersive and nondispersive models. In both of the examples the electric current source is assumed located in the middle of the slab, i.e., $|b| = a$. The first numerical problem uses the dispersive model with $a = 0.5cm$. Figure 6.1 is the susceptibility kernel function for a two resonant frequency Lorentz dispersive medium. The second example uses the nondispersive inhomogeneous model with $a = 4.0m$. Figure 6.4 shows the given permittivity profile $\varepsilon(z)$. From Figure 6.4 we see that $\varepsilon(-z) = \varepsilon(z)$. In this case our right and left composite transmission kernels are symmetric, i.e., $T_r = T_l$. For the dispersive model, the transmission kernels also have this symmetry property since the source is located in the middle.

For these two models, Figures 6.2 and 6.5 shows the scaled calculated composite transmission kernel $T_r$ compared with $T_1$, the transmission kernel of a single right slab case. From Figure 6.2, we see that in the dispersive case the coupling effect of the two half slabs is very small, i.e., in this case the reflection of a single slab is weak compared with the transmission of the single slab, since in our example we have assumed that $G(0) = 0$, i.e., no hard wall exists. A single slab with higher reflectivity would give a composite transmission kernel differing more from $T_1$ (or $T_2$). Figure 6.5 shows that for the inhomogeneous slab the composite transmission differs markedly from $T_1$. We have seen that in Figure 6.5 $T_r$ has a jump at one round trip time of the single slab, since the reflection kernel $R_2$ contributes directly
to $T_r$. It can be seen from the following equation

$$T_r(a, b, t) = T_1(0, a, t) + \alpha_1 R_1(a, b, t - \tau_1) + \int_0^{t-\tau_1} T_1(0, a, t-u)R_l(a, b, u)\,du$$

$$= T_1(0, a, t) + \alpha_1 R_2(0, b, t - \tau_1) + \int_0^{t-\tau_1} T_1(0, a, t-u)R_l(a, b, u)\,du$$

$$+ \alpha_1 [R_2 * R_1 * + R_2 * R_1 * R_2 + \cdots] (a, b, t - \tau_1).$$

From the discussions following equation (5.2) we know that $R_2 * R_1 + R_2 * R_1 * R_2 + \cdots$ is convergent and continuous. From this and (6.11) we can see that $T_r(0, a, t)$ and $R_2(0, |b|, t)$ have the same discontinuity property since $T_1$ is continuous [9]. $R_2$ has a jump at one round trip time [9], so $T_r$ also has a jump at one and half round trip time as shown in Figure 6.5. Note that in Figure 6.2 and Figure 6.3 we have been shifted the graph to the left by $t = \tau_1$ units since $T_1 = T_r = 0$ for $t < \tau_1$, i.e., a time coordinate measured from the wave front.

The reconstruction of $J(t)$ is obtained by solving the Volterra equation (5.5) or (5.6) for $J(t)$. For input data $E^+(a, t)$ or $E^-(b, t)$ containing no noise, we get a perfect reconstruction of $J(t)$. Figures 6.3 and 6.6 show the reconstruction of the source current $J(t)$ when noise exists in the input data. The signal to noise ratio in Figures 6.3 and 6.6 is 3.3.
Figure 6.1: Susceptibility kernel of dispersive model

Figure 6.2: Single & composite transmission kernels for dispersive model
Figure 6.3: Reconstruction of source function for dispersive model

Figure 6.4: Permittivity profile of inhomogeneous model
Figure 6.5: Single & composite transmission kernels for inhomogeneous model

Figure 6.6: Reconstruction of source function for inhomogeneous model
7. INTERNAL ELECTRIC FIELD

7.1 Wave Splitting and Operator Definitions

In previous sections we have derived the expressions for the left and right composite operators $T_r$, $T_l$ and $R_r$, $R_l$. Composite transmission operators $T_r$, $T_l$ can be applied to calculate the transmitted fields at the end points $z = a, b$ (See equations (5.5), (5.6)). Composite operators $R_r$, $R_l$ can be used to calculate the internal field at $z = 0$ (See equations (4.16), (4.18)). Is there any way that we can determine the internal electric field $E$ at an arbitrary point $z$ for $z \in [b, a]$?

Recently Krueger and Ochs [11] applied a Green's function approach to the split wave equation of a finite slab which has been assumed that no internal source exits. This technique is very effective in determining the internal fields of an inhomogeneous slab. A similar question is raised, i.e., how to calculate the internal $E$ field when there exists a transient electric source at $z = 0$. We look for a Green's operator which maps the electric source current $J(t)$ at $z = 0$ into the electric field at an arbitrary internal observation point within the slab. This Green's function depends solely on the the properties of the media of the slab and the geometrical location of the current source. The physical model for the medium that we will study is our previously defined homogeneous, dispersive media model which is mathematically modelled by equation (3.4).
We start by rewrite the electromagnetic wave equation (3.4) as

\[
\frac{\partial}{\partial z} \begin{bmatrix} E(z, t) \\ E_z(z, t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ (\partial_t^2 + g \ast \partial_t^2) / c_0^2 & 0 \end{bmatrix} \begin{bmatrix} E(z, t) \\ E_z(z, t) \end{bmatrix} + \begin{bmatrix} 0 \\ \mu_0 J'(t) \delta(z) \end{bmatrix}
\]  

(7.1)

with initial conditions

\[
E(z, 0) = E_t(z, 0) = 0 \quad b < z < a .
\]  

(7.2)

Here \( \delta(z) \) is the Dirac \( \delta \)-function. Note that we have used the lower letter \( g \) to represent the susceptibility kernel to distinguish from the capital \( G \) which will be used for the Green's operator kernel.

The ideas of wave splitting and invariant imbedding have been applied to different wave scattering models [1] [2] [3] [9] [5]. The key point of the wave splitting technique is the wave splitting transform. We define the following dependent variable transform

\[
E^\pm(z, t) = \frac{1}{2} [E(z, t) \mp c_0 \partial_t^{-1} E_z(z, t)]
\]  

(7.3)
i.e.,

\[
\begin{bmatrix} E^+(z, t) \\ E^-(z, t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -c_0 \partial_t^{-1} \\ 1 & c_0 \partial_t^{-1} \end{bmatrix} \begin{bmatrix} E(z, t) \\ E_z(z, t) \end{bmatrix}
\]  

(7.4)

for \( z \in [b, a] \), where \( \partial_t^{-1} E_z(z, t) \) is defined by

\[
\partial_t^{-1} E_z(z, t) = \int_{-\infty}^{t} E_z(z, s) \, ds .
\]

\( E^+ \) and \( E^- \) are treated as the right and left going waves at the internal point \( z \), i.e., the total \( E \) field is decomposed into the right and left going waves

\[
E(z, t) = E^+(z, t) + E^-(z, t) .
\]
The definition of $E^{\pm}(z,t)$ in (7.4) is mathematically precise even though the physical meaning of $E^{\pm}$ is not quite clear for $E^{\pm}$ evaluated at an arbitrary internal point $z$. The advantage of the definition in (7.4) is that it will simplify our boundary condition at $z = a, b$, since at the boundaries and in the regions outside the slab $E^{+}$ and $E^{-}$ are exactly the physical right and left going waves. In regions, $z > a$ or $z < b$, which are free space, the electrical field $E$ satisfies the homogeneous wave equation, $E_{tt} - (1/c_0^2)E_{zz} = 0$, which has the following general form of solution:

$$E(z,t) = f_1(t - z/c_0) + f_2(t + z/c_0) \quad t > 0, \quad z > a \text{ or } z < b. \quad (7.5)$$

If we apply the wave splitting transform (7.4) to (7.5), we will see that

$$E^+(z,t) = f_1(t - z/c_0)$$
$$E^-(z,t) = f_2(t + z/c_0)$$

i.e., the transformation (7.4) extends the defined internal $E^+(z,t)$ and $E^-(z,t)$ to the physical right and left going waves of the external fields at the two boundaries $z = a, b$.

Then by applying the definition in (7.4) to (7.1) we obtain a first order P.D.E. system (the split wave equation) for the right and left components, $E^{\pm}$, of the internal $E$ field

$$\frac{\partial}{\partial z} \begin{bmatrix} E^+(z,t) \\ E^-(z,t) \end{bmatrix} - T D T^{-1} \begin{bmatrix} E^+(z,t) \\ E^-(z,t) \end{bmatrix} = T \begin{bmatrix} 0 \\ \mu_0 J'(t) \delta(z) \end{bmatrix}$$

$$E^-(a,t) = E^+(b,t) = 0 \quad t > 0$$
$$E^+(z,t) = E^-(z,t) = 0 \quad t \leq 0 \quad z \neq 0 \quad (7.6)$$
where
\[
T = \frac{1}{2} \begin{bmatrix} 1 & -c_0 \partial_t^{-1} \\ 1 & c_0 \partial_t^{-1} \end{bmatrix}. \quad (7.7)
\]
\[
D = \begin{bmatrix} 0 & 1 \\ (\partial_t^2 + g \cdot \partial_t^2)/c^2 & 0 \end{bmatrix}. \quad (7.8)
\]
and
\[
TD^{-1} = \begin{bmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{bmatrix} = \begin{bmatrix} -\frac{1}{c_0}(\partial_t + \frac{1}{2}g \cdot \partial_t) & -\frac{1}{2c_0} g \cdot \partial_t \\ \frac{1}{2c_0} g \cdot \partial_t & \frac{1}{c_0}(\partial_t + \frac{1}{2}g \cdot \partial_t) \end{bmatrix}. \quad (7.9)
\]

The boundary conditions in (7.6) actually mean that there are no external incident waves from the right and the left to the slab.

In what follows, we will define a Green's operator for equation (7.6) and then derive the equation satisfied by this Green's operator. Krueger [10] and Kristensson [8] have studied their Green's function problems when there is an incident wave from left to the slab but without an internal source. In our case, we have a simple boundary condition which is due to the fact that there are no external incident waves at \( z = a, b \), but we have a special term on the right hand side of the equation (7.6) which is contributed by the internal transient source \( J(t) \).

In a fashion analogous to our previous results (the application of Duhamel's principle [7]), we can prove that the \( \tilde{G}^{\pm} \) operators, which map the current source \( J(t) \) into the internal \( E^{\pm} \) fields, can be represented by
\[ E^+(z,t) = \mathcal{G}^+ J(t) = \begin{cases} 
\frac{-\mu_0 c_0}{2} \tau \left[ J(t - \frac{z}{c_0}) + \int_0^{t-z/c_0} G^+(z, t-s)J(s) \, ds \right] \\
\frac{-\mu_0 c_0}{2} \tau \int_0^{t-z/c_0} G^+(z, t-s)J(s) \, ds 
\end{cases} \quad \text{for } z > 0 \\
\frac{-\mu_0 c_0}{2} \tau \int_0^{t-z/c_0} G^+(z, t-s)J(s) \, ds 
\] for \( z < 0 \) \hspace{1cm} (7.10)

\[ E^-(z,t) = \mathcal{G}^- J(t) = \begin{cases} 
\frac{-\mu_0 c_0}{2} \tau^{-1} \int_0^{t-z/c_0} G^-(z, t-s)J(s) \, ds 
\end{cases} \quad \text{for } z > 0 \\
\frac{-\mu_0 c_0}{2} \tau^{-1} \left[ J(t + \frac{z}{c_0}) + \int_0^{t-z/c_0} G^-(z, t-s)J(s) \, ds \right] 
\] for \( z < 0 \) \hspace{1cm} (7.11)

\[ = -\frac{-\mu_0 c_0}{2} \tau^{-1} \left[ J(t + \frac{z}{c_0})H(-z) + \int_0^{t-z/c_0} G^-(z, t-s)J(s) \, ds \right] 
\] for \( z < 0 \) \hspace{1cm} (7.11)

where \( H(z) \) is the Heaviside step function and \( \tau \) is given by

\[ \tau(z) = e^{-(z/2c_0)}g(0) \hspace{1cm} (7.12) \]

From (7.10) and (7.11), we see that \( \mathcal{G}^+ \) maps the source \( J(t) \) into the right going wave at \( z \) and \( \mathcal{G}^- \) maps the source \( J(t) \) to the left going wave at \( z \). We call \( G^+ \) and \( G^- \) the right and left going Green's operator kernels, respectively.
7.2 Green's Operator Kernel Integrodifferential Equations

From equations (7.6), (7.10) and (7.11) we obtain the following equations that the operators $\tilde{G}^\pm$ have to satisfy

$$
\frac{\partial}{\partial z} \begin{bmatrix} \tilde{G}^+(z,t) \\ \tilde{G}^-(z,t) \end{bmatrix} - TDT^{-1} \begin{bmatrix} \tilde{G}^+(z,t) \\ \tilde{G}^-(z,t) \end{bmatrix} = \begin{bmatrix} -\mu_0 c_0 \delta(z) \\ \mu_0 c_0 \delta(z) \end{bmatrix}
$$

$$
\begin{align*}
\tilde{G}^+(b,t) &= \tilde{G}^-(a,t) = 0 & t > 0 \\
\tilde{G}^+(z,t) &= \tilde{G}^-(z,t) = 0 & z \neq 0, \quad t \leq 0 \\
\tilde{G}^+(0^+,t) - \tilde{G}^+(0^-,t) &= -c_0 \mu_0 / 2 \\
\tilde{G}^-(0^+,t) - \tilde{G}^-(0^-,t) &= c_0 \mu_0 / 2 
\end{align*}
$$

where the jump conditions of $\tilde{G}^\pm$ at $z = 0$ are derived from

$$
\begin{align*}
E^+(0^+,t) - E^+(0^-,t) &= -\frac{c_0 \mu_0}{2} J(t) & t > 0 \\
E^-(0^+,t) - E^-(0^-,t) &= \frac{c_0 \mu_0}{2} J(t)
\end{align*}
$$

which are obtained by integrating equation (7.6).

From the representations of the operators $\tilde{G}^\pm$ which are given by (7.10) and (7.11), we can derive the following equations for $G^\pm$ (called the Green’s function or Green’s operator kernel) by inserting (7.10) and (7.11) into (7.13) and performing a straightforward but lengthy calculation, which will be shown later. The equations for $G^+$ and $G^-$ are found to be
\[
G^+(z, t) + \frac{1}{c_0} G^+_t(z, t) = \frac{1}{2c_0} \int_0^{t-|z|/c_0} \left[ G^+(z, t-s) + \tau^{-2} G^-(z, t-s) \right] g'(s) \, ds \\
- \frac{1}{2c_0} g'(t-|z|/c_0) \left[ \tau^{-2} + (1 - \tau^{-2}) H(z) \right] - \frac{g(0)}{2c_0} \tau^{-2} G^-(z, t)
\]

\[
G^-(z, t) - \frac{1}{c_0} G^-_t(z, t) = -\frac{1}{2c_0} \int_0^{t-|z|/c_0} \left[ G^-(z, t-s) + \tau^2 G^+(z, t-s) \right] g'(s) \, ds \\
+ \frac{1}{2c_0} g'(t-|z|/c_0) \left[ \tau^2 + (1 - \tau^2) H(-z) \right] + \frac{g(0)}{2c_0} \tau^2 G^+(z, t)
\]

with the following initial and boundary conditions and discontinuity jumps along the characteristic curves of (7.15)

\[
G^+(z, -z/c_0+) = \frac{1}{4} \tau^{-2}(z)g(0) \quad z < 0
\]

\[
G^+(z, z/c_0+) = \frac{z}{2c_0} \left[ g^2(0)/4 - g'(0) \right] \quad z > 0
\]

\[
G^+(z, (z-2b)/c_0+) - G^+(z, (z-2b)/c_0-) = -\frac{1}{4} \tau^{-2}(-b)g(0) \quad b < z < a
\]

\[
G^-(z, z/c_0+) = -\frac{1}{4} \tau^2(z)g(0) \quad z > 0
\]

\[
G^-(z, -z/c_0+) = -\frac{z}{2c_0} \left[ g^2(0)/4 - g'(0) \right] \quad z < 0
\]

\[
G^-(z, (2a-z)/c_0+) - G^-(z, (2a-z)/c_0-) = \frac{1}{4} \tau^2(a)g(0) \quad b < z < a
\]

\[
G^+(b, t) = G^-(a, t) = 0 \quad t > 0
\]

\[
G^+(z, t) = G^-(z, t) = 0 \quad t \leq 0
\]

\[
G^+(0+, t) = G^+(0-, t) \quad t > 0
\]

\[
G^-(0+, t) = G^-(0-, t) \quad t > 0
\]

Figure 7.1 shows a complete picture of the characteristic curves along which \( G^\pm \) suffer discontinuity jumps. In Figure 7.1, for example, the notation \([G^\pm]\) means that
Figure 7.1: Jumps of $G^\pm$ along characteristic curves

$G^+$ and $G^-$ both have discontinuity jumps along $t = |z|/c_0$; $[G^+]$ is the discontinuity jump of $G^+$ along $t = (z - 2b)/c_0$ and $[G^-]$ is the discontinuity jump of $G^-$ along $t = (2a - z)/c_0$. To obtain the first equation in (7.15), we need to compute $\partial_z \tilde{G}^+ f(t)$, $\tilde{a} \tilde{G}^+ f(t)$ and $\tilde{b} \tilde{G}^+ f(t)$ for an arbitrary differentiable function $f(t)$, $f(t) = 0$ for $t < 0$. When performing this calculation, care must be taken to consider possible jump discontinuities of $G^+$ and $G^-$ along their characteristic curves. We assume that $G^+$ may have a jump discontinuity along $t = \phi(z)$ and $G^-$ may have a jump discontinuity along $t = \psi(z)$. The following is an example of the necessary computational steps to obtain the first equation of (7.15). (See Appendix B for details about the derivation.
of the second equation of (7.15)).

\[
\partial_z \tilde{G}^+(t) = -\frac{A\tau}{2c_0} g(0) \left[ f(t - z/c_0) H(z) + \int_0^{t-z/c_0} G^+(z, t - s) f(s) \, ds \right]
\]

\[
+ A\tau \left[ \delta(z) f(t - z/c_0) - \frac{1}{c_0} f'(t - z/c_0) H(z) \right]
\]

\[-[G^+(z, \phi(z)) - G^+(z, \phi(-))] \phi f(t - \phi) - \frac{2H(z) - 1}{c_0} G^+(z, |z|/c_0) f(t - |z|/c_0) + \int_0^{t-|z|/c_0} G^+(z, t - s) f(s) \, ds \right] \tag{7.17}
\]

\[
\tilde{G}^+ f(t) = -\frac{1}{c_0} (\partial_t + \frac{1}{2} g \ast \partial_t) \tilde{G}^+ f(t)
\]

\[
= -\frac{A\tau}{c_0} \left[ f'(t - \frac{z}{c_0}) H(z) + \left[ G^+(z, \phi(z)) - G^+(z, \phi(-)) \right] f(t - \phi) \right.
\]

\[+ \left. G^+(z, |z|/c_0) f(t - |z|/c_0) + \int_0^{t-|z|/c_0} G^+_t(z, t - s) f(s) \, ds \right] \tag{7.18}
\]

\[
\tilde{G}^- f(t) = -\frac{1}{2c_0} g \ast \partial_t \tilde{G}^- f(t)
\]

\[
= -\frac{A\tau^{-1}}{2c_0} g(0) \left[ f(t + z/c_0) H(-z) + \int_0^{t-z/c_0} G^-(z, t - s) f(s) \, ds \right] \tag{7.19}
\]

\[-\frac{A\tau^{-1}}{2c_0} \left[ \int_0^{t+z/c_0} g'(t - s + z/c_0) f(s) H(-z) \, ds \right.
\]

\[\left. - \int_0^{t-|z|/c_0} f(s) \, ds \int_0^{t-s-|z|/c_0} g'(s') G^-(t - s - s') \, ds' \right] \]
where $A = -\mu_0 c_0/2$. From equation (7.13)

$$G^+(z, \phi^+)_t = \tilde{\alpha}_f(t) + \tilde{\beta}f(t) - \frac{\mu_0 c_0}{2} \delta(z)f(t)$$  \hspace{1cm} (7.20)

and equations (7.17–7.19), we then are able to obtain a differential equation which is
given by the first equation of (7.15), the first condition of (7.16) and

$$[G^+(z, \phi^+) - G^+(z, \phi^-)](\phi' - 1/c_0) = 0. \hspace{1cm} (7.21)$$

Equation (7.21) tells us that $G^+$ along $t = z/c_0 + \text{const}$ may have a possible
jump. This jump is given by the third equation of (7.16), which is derived by ele­
mentary propagation of singularity arguments. In addition, the second condition in
equation (7.16) is obtained if we integrate the first equation of (7.15) with respect to
$z$ from 0 to $z$ along $t = z/c_0$ for $z > 0$.

If we let $z = a$ in (7.10), we know that $G^+(a, t)$ maps the source $J(t)$ into the
right going wave at $z = a$. This right going wave is just the transmitted wave at the
right boundary. Similar result can be found for $G^-(b, t)$. By comparison of equation
(7.10) and (7.11) with equation (5.5) and (5.6), we then have

$$G^+(a, t) = r^{-1}(a)T_1 = r^{-1}(a)T_1(a, t) + R_1(a, b, t - \tau_1) \hspace{1cm} \hspace{1cm} (7.22)$$

$$G^-(b, t) = r^{-1}(-b)T_1 = r^{-1}(-b)T_2(b, t) + R_1(a, b, t - \tau_2)$$

Also, from equations (4.18), (4.16), (4.9) and (5.2) we see that

$$E^+(0+, t) = -\frac{\mu_0 c_0}{2} \left[J(t) + R_1 * J(t)\right]$$

$$E^+(0-, t) = -\frac{\mu_0 c_0}{2} \left[R_1 * J(t)\right] \hspace{1cm} (7.23)$$
and

\[ E^-(0+, t) = -\frac{\mu_0 c_0}{2} [R_r * J(t)] \]
\[ E^-(0-, t) = -\frac{\mu_0 c_0}{2} [J(t) + R_r * J(t)] . \]

Then by comparison of (7.23) and (7.24) with (7.10) and (7.11), then we have

\[ G^+(0, t) = R_l(a, b, t) \]
\[ G^-(0, t) = R_r(a, b, t) . \]

Equation (7.15) gives us a very effective method to calculate the right and left transmission kernels. But for the composite method that we have discussed in previous sections we have to solve the nonlinear partial integrodifferential equations for \( R_1, R_2, T_1 \) and \( T_2 \) and then compute the composite kernels \( T_r \) and \( T_l \). This requires several steps. To solve the linear partial integrodifferential equation (7.15) is straightforward. It will give us the composite scattering kernels, \( R_r, R_l, T_r \) and \( T_l \) and also the ability to compute the internal fields.
8. CONCLUSION

Direct scattering and inverse source problems were studied in the time domain, using Corones and Krueger's wave splitting and invariant imbedding techniques. Composite transmission operators, $T_l$ and $T_r$, which map the source function $J(t)$ to the transmitted electric fields were derived. The direct scattering problem was solved by computing the composite transmission operator kernels $T_l$ and $T_r$ or by computing the transmitted fields $T_r \cdot J(t)$ and $T_l \cdot J(t)$ for any electric source function $J(t)$. The solution of the inverse problem for the internal source $J(t)$ was based on delay Volterra integral equations (5.5) and (5.6). A frequency domain example was given, which verified our time domain results. For our dispersive and nondispersive models, composite transmission operator kernel was computed and compared with a single slab transmission operator kernel. Numerical examples were also given for our inverse internal source problem, where the transmission field data were taken on only one side of the slab.

Also, a Green's operator was defined. This operator maps the electric current source $J(t)$ at $z = 0$ to the internal field at an arbitrary observation point inside a homogeneous dispersive slab. A system of linear partial integro-differential equations with various initial, boundary and jump discontinuities were derived for the Green's operator kernels. Relations between $G^\pm$ and $R_r$, $R_l$, $T_r$ and $T_l$ were also identified.
APPENDIX A

The wave equation for an electric surface current source $J(t)$ located at $z = 0$ ($J(t) = 0$ for $t > 0$) in an unbounded medium with constant permittivity $\varepsilon$ is

\[
\begin{align*}
E_{zz}(z, t) - \left(\frac{1}{v^2}\right)E_{tt}(z, t) &= \mu_0 J'(t)\delta(z) \quad -\infty < z < \infty \quad t > 0 \\
E_t(z, t) &= 0 \quad E(z, t) = 0 \quad \text{for} \quad t \leq 0
\end{align*}
\]

(8.1)

where $v = 1/\sqrt{\mu_0\varepsilon}$. The solution of (8.1) can be obtained by using the integral transform techniques. Taking the Fourier transform in $z$ and Laplace transform in $t$ on the above equation (8.1), then we have

\[
E(k, s) = -\frac{\mu_0v^2}{s^2 + k^2v^2}L[J'_t(t)].
\]

(8.2)

By taking the inverse Laplace transform of equation (8.2), we then obtain the following equation

\[
E(k, t) = -\mu_0v \int_0^t J'_t(y)\frac{\sin[vk(t - y)]}{k} dy.
\]

(8.3)

The solution of equation (8.1) are found by directly using the inverse Fourier transform to the above equation (8.3). The solution is

\[
E(z, t) = -\mu_0v^2 \int_0^t J'_t(y)F^{-1}\left[\frac{\sin(vk(t - y))}{vk}\right] dy
\]

\[
= -(1/2)\sqrt{\mu_0/\varepsilon} \int_0^t J'_t(y)H(v(t - y) - |z|) dy
\]

(8.4)

\[
= -(1/2)\sqrt{\mu_0/\varepsilon}J(t - |z|/v).
\]
APPENDIX B

The following computations are used to derive the second Green's operator kernel equation (7.15). We let $A = -\mu_0 c_0 / 2$, $f$ be an arbitrary function in $C^1(t > 0)$, $f(t) = 0$ for $t \leq 0$ and $t = \psi(z)$ be the possible curve along which $G^-$ may have a jump. The assumption of smoothness of the input transient source function $f(t)$ will not affect our result (7.15), since the Green's function given by solving (7.15) does not depend on the source function.

$$\partial_z G^- f(t) = \frac{A}{2c_0} \gamma^{-1} g(0) \left[ f(t + z/c_0) H(-z) + \int_0^{t-|z|/c_0} G^-(z, t-s) f(s) ds \right]$$

$$+ A \gamma^{-1} \left[ -\delta(z) f(t + z/c_0) + \frac{1}{c_0} \frac{f'(t + z/c_0)}{H(z)} \right]$$

$$- [G^-(z, \psi+) - G^-(z, \psi-)] \psi' f(t - \psi)$$

$$\frac{2H(z) - 1}{c_0} G^-(z, |z|/c_0) f(t - |z|/c_0) + \int_0^{t-|z|/c_0} G_x(z, t-s) f(s) ds$$

(8.5)
\[ \tilde{\gamma} \tilde{G}^+ f(t) = \frac{1}{2c_0} g * \partial_t \tilde{G}^+ f(t) \]
\[ = \frac{A \tau}{2c_0} g(0) \left[ f(t - z/c_0)H(z) + \int_0^{t-|z|/c_0} G^+(z, t - s)f(s) \, ds \right] \]
\[ + \frac{A \tau}{2c_0} \left[ \int_0^{t-|z|/c_0} g'(t - s - z/c_0)f(s)H(z) \, ds \right. \]
\[ - \left. \int_0^{t-|z|/c_0} f(s) \, ds \int_0^{t-s-|z|/c_0} g'(s')G^+(t - s - s') \, ds' \right] \tag{8.6} \]

\[ \tilde{\delta} \tilde{G}^- f(t) = \frac{1}{c_0}(\partial_t + \frac{1}{2} g * \partial_t)\tilde{G}^- f(t) \]
\[ = \frac{A \tau^{-1}}{c_0} \left[ f(t + z/c_0)H(-z) + \int_0^{t+|z|/c_0} G^-(z, \psi +) - G^-(z, \psi -)]f(t - \psi) \right. \]
\[ + G^-(z, |z|/c_0)f(t - |z|/c_0) + \int_0^{t-|z|/c_0} G^-_t(z, t - s)f(s) \, ds \right] \]
\[ + \frac{A \tau^{-1}}{2c_0} g(0) \left[ f(t + z/c_0)H(-z) + \int_0^{t+|z|/c_0} G^-(z, t - s)f(s) \, ds \right] \]
\[ + \frac{A \tau^{-1}}{2c_0} \left[ \int_0^{t+z/c_0} g'(t - s + z/c_0)f(s)H(-z) \, ds \right. \]
\[ - \left. \int_0^{t-|z|/c_0} f(s) \, ds \int_0^{t-s-|z|/c_0} g'(s')G^-(t - s - s') \, ds' \right] \tag{8.7} \]

where the right and left going Green's operators \( \tilde{G}^\pm \) are the integral operators which are defined in (7.10) and (7.11); operators \( \tilde{\gamma} \) and \( \tilde{\delta} \) are given by (7.9) and \( g(t) \) is the susceptibility kernel of the dispersive media.

The above equations (8.5), (8.6) and (8.7) are used to derive the second equation of (7.15) and the fourth equation in (7.16).
REFERENCES


[9] Kristensson, G. and Krueger, R. J., *Direct and inverse scattering in the time domain for a dissipative wave equation*


PART II.

TRANSIENT SCATTERING FOR POROUS MEDIA
1. ABSTRACT

Direct and inverse wave scattering problems for fast and slow compressional waves normally incident on a statistically homogeneous, dispersive, dissipative fluid-saturated porous slab are studied in the time domain. By applying wave splitting and invariant imbedding techniques to Biot system of compressional wave equations reflection and transmission scattering operator kernel integrodifferential equations are derived. Some properties of these operator kernel matrices, such as reciprocity relations and the multimodes of propagation of discontinuities, are discussed. A numerical scheme for solving the direct and inverse problems from $R$-equation is presented. Numerical computations for a half-space model direct and inverse problem are given.
2. INTRODUCTION

The theory of wave propagation in fluid-saturated porous media is widely applicable in geophysics and other fields of engineering. Biot was the first to develop a systematic theory for such a problem in his series of papers [5] [6] [7].

Holland and Brunson [20] reported their experiments performed in a shallow water area of the Mediterranean Sea in which waves interact with sediments. They inferred from their experimental data the coefficients of Biot's wave equation, then compared that equation's predictions with the compressional wave velocity and attenuation measured in situ and in the laboratory. Biot's theory agreed with their experiment. Within the last few years, Plona [25], Duta [17] and Berryman [4] observed experimentally the slow compressional wave (Biot-Plona wave) predicted by Biot. Chin, Berryman and Hedstrom [11], Geertsma [19] and Mainardi et al. [24] also have applied Biot's theory to study fluid-saturated porous media wave scattering and propagation problems.

The compressional wave equations in [5] comes with restrictions. For instance, if there is dispersion and dissipation, and if the wave frequency is not in the low frequency range, then the equation [5] will not be usable since at higher frequencies the Poiseuille type flow will not be valid because the Reynolds number of the relative flow exceeds a critical value. Biot introduced a so-called correction (or modified)
function $F(\kappa)$ [6] to make Biot's equations [6] valid in a wide range of frequencies. But such correction function is not well-known. To make Biot's equation valid in a wide range of frequencies or to apply it in the time domain, we must know the behavior of function $F(\kappa)$. Such a function is very important both in theory and in application. As stated in Berryman's paper [3]:

1. Biot's theory is a satisfactory model of wave speeds and attenuation in fluid-saturated porous media.

2. More experiments are needed to answer the questions raised by the few remaining anomalies. Further theoretical work is needed to find the most general form of the universal function $F(\kappa)$ and to establish more satisfactory estimates of the characteristic length $a$.

We can see from [3] [5] [6] Biot has postulated that there exists a universal correction function to fit a wide range of frequencies. Although Biot's theory has been widely studied in the frequency domain since its publication, we will study direct and inverse scattering problems of Biot's compressional wave equations in the time domain, using Corones and Krueger's wave splitting and invariant imbedding techniques. The inverse technique will provide a means to seek the modified function $F(\kappa)$ from experimental data and then further perfect Biot's equations and verify Biot's postulation. On the other hand, the inverse problem studied in this paper has potential of application in geophysics and oil-exploration, e.g., remotely to obtain the underground material properties, such as permeability of oil flow through a porous media, porosity and fluid viscosity of fluid.

In time domain a linear wave propagation in a dispersive/dissipative fluid-filled
porous medium is characterized by the fact that its amplitude, phase and group velocities are functions of the frequency. A transient wave will tend to spread, change the wave front and attenuate in that medium even if the porous medium is statistically homogeneous. The following chapters uses Biot's compressional wave equations to study direct and inverse scattering problems for a fluid-saturated porous media that are dispersive and dissipative. Our direct scattering problem is that of obtaining and studying reflection and transmission operator kernels. The inverse problem is to determine the dispersive/dissipative function $f(t)$, which contains lots of information about the porous medium.
3. BIOT’S WAVE EQUATION AND WAVE SPLITTING

3.1 Biot’s Compressional Wave Equations

A slab, that occupies the region \(0 \leq x \leq L\), of fluid-saturated porous media is described by the presence of a statistically isotropic, homogeneous elastic porous frame which is permeated by pores fully (no empty pockets) containing viscous fluid. The fluid is assumed to be compressible and may flow relative to the solid frame causing friction (dissipation) to arise. The dispersive and dissipative effect of the medium is characterized by the modified function \(F(\kappa)\) which was introduced by Biot [6]. The regions outside the slab \((x < 0 \text{ or } x > L)\) are also fluid-saturated porous media which are the same as that of the slab except that they are nondispersive and nondissipative, i.e., \(F(\kappa) = 0\) for \(x < 0 \text{ or } x > L\). We assume that the propagation wave in region \(x < 0\) is normally incident on the slab at the left boundary \((x = 0)\). We are particularly interested in the application to cases where the fluid is a liquid, typical model like underground oil saturated rocks or sands, so then we disregard the thermoelastic effect.

There are three kinds of waves that can propagate in the fluid-saturated porous medium, one shear wave and two compressional waves [3] [5] [6] [9]. In this paper, we only study the compressional waves. Our work is based on the following Biot system of compressional wave equations [5] [6] [9]
\( \partial^2_x (Pe + Qe) = \partial^2_x (\rho_{11} e + \rho_{12} \epsilon) + b \partial_x (e - \epsilon) \)

\( \partial^2_x (Qe + Re) = \partial^2_x (\rho_{12} e + \rho_{22} \epsilon) - b \partial_x (e - \epsilon) \)

\( 0 < x < L, \ t > 0 \)

where \( e = \nabla \cdot \bar{u} \) is the dilation of the solid frame, \( \epsilon = \nabla \cdot \bar{v} \) is the dilation of the fluid, and \( \bar{u}, \bar{v} \) are the displacement of the frame and the fluid, respectively. The coefficients \( Q, R, P, \rho_{ij} \) and \( b \) in (3.1) are constants. \( Q \) is of the nature of a coupling between the volume change of the frame and that of the fluid. \( R \) is a measure of the pressure required on the fluid to force the fluid into the aggregate while the total volume remains constant. \( P = A + 2N \), where \( A \) and \( N \) correspond to the familiar Lamé coefficients in elastic theory. \( \rho_{11} \) and \( \rho_{22} \) are the effective mass densities of the solid frame and the fluid, respectively. \( \rho_{12} \) is the coupling coefficient of the two mass densities, where \( \rho_{11}\rho_{22} - \rho_{12}^2 > 0 \) and \( \rho_{12} \) is negative [5]. \( b \) is given by \( b = \mu \beta^2 / k \), where \( \mu \) is the viscosity of the fluid, \( k \) is the coefficient of absolute permeability of the porous frame with unit volume (\( k = k / \mu \) is the hydraulic permeability), and \( \beta \) is the porosity of the porous material. Just as stated in [3] [5] and [6], equation (3.1) is based on the assumption that the fluid flow relative to the solid frame is of the Poiseuille type. That is, in the frequency domain equation (3.1) holds only for \( \omega \) below a certain value where Poiseuille flow will break down. In order to extend equation (3.1) to be valid for higher frequencies, Biot in [5] [6] and [7] introduced a frequency correction term \( F(\kappa) \) to make equation (3.1) valid in a full frequency range. Biot also calculated the \( F(\kappa) \) function in both low porosity and high porosity cases for different kinds of pore models. By introducing \( F(\kappa) \), then in the full frequency
range equation (3.1) is written as

\[ \frac{\partial^2}{\partial x^2}(P\bar{e} + Q\bar{e}) = -\omega^2(\rho_{11}\bar{e} + \rho_{12}\bar{e}) + ibF(\kappa)\omega(\bar{e} - \bar{e}) \]  
\[ \frac{\partial^2}{\partial x^2}(Q\bar{e} + R\bar{e}) = -\omega^2(\rho_{12}\bar{e} + \rho_{22}\bar{e}) - ibF(\kappa)\omega(\bar{e} - \bar{e}) \]  
\[ 0 < x < L, \quad t > 0 \]

where \(\bar{e}, \bar{e}\) are the Fourier transforms of \(e, e\) with respect to \(t\). \(\kappa = a(\omega/\nu)^{1/2}\). Here \(\nu = \mu/\rho_f\), where \(\rho_f\) is the fluid density and \(a\) is a parameter of the geometric configuration of the pore spaces (characteristic pore size), \(a \geq 1\). \(a\) is determined by experiment. Now we define a new function \(f(t)\). We assume that \(f(t)\) is a \(C^1(0, \infty)\) function and its Fourier transform is \(\bar{f}(\omega)\), which is given by

\[ \bar{f}(\omega) = \frac{bF(\kappa)}{i\omega}. \]  

By taking the inverse Fourier transform of equation (3.2), we obtain the following modified Biot compressional wave equations in the time domain

\[ \frac{\partial^2}{\partial x^2}(Pe + Qe) = \frac{\partial^2}{\partial t^2}(\rho_{11}e + \rho_{12}e) + f(t) \ast \frac{\partial^2}{\partial t^2}(e - \epsilon) \]  
\[ \frac{\partial^2}{\partial x^2}(Qe + Re) = \frac{\partial^2}{\partial t^2}(\rho_{12}e + \rho_{22}e) - f(t) \ast \frac{\partial^2}{\partial t^2}(e - \epsilon) \]  
\[ 0 < x < L, \quad t > 0 \]

where the symbol \(\ast\) represents the usual convolution operator, i.e.,

\[ f(t) \ast \frac{\partial^2}{\partial t^2}[e(x, t) - \epsilon(x, t)] = \int_0^t f(t - s)[e_{ss}(x, s) - \epsilon_{ss}(x, s)]ds. \]

We assume that \(e(x, t), \epsilon(x, t)\) are in \(C^2[x \times t : (-\infty, \infty) \times (0, \infty)]\). We note that the resulting \(R\)-equation derived in the next section will not be altered by such continuity requirements for \(e(x, t)\) and \(\epsilon(x, t)\) since our following reflection kernel
$R(x, t)$ is independent of $e(x, t)$ and $\epsilon(x, t)$. We assume that the moment that the incident wave first impinges on the slab from left at $x = 0$ is $t = 0$, i.e., $e(x, t) = \epsilon(x, t) = 0$ for $t \leq 0$, $x \in (0, L)$.

For the regions outside the slab, i.e., where $F(\kappa) = 0$, then in these regions Biot compressional wave equations are

$$\begin{align*}
\partial_x^2 (P e + Q \epsilon) &= \partial_t^2 (\rho_{11} e + \rho_{12} \epsilon) \quad x < 0 \text{ or } x > L, \ t > 0, \\
\partial_x^2 (Q e + R \epsilon) &= \partial_t^2 (\rho_{12} e + \rho_{22} \epsilon)
\end{align*} \tag{3.5}$$

### 3.2 Wave Splitting for Biot's Wave Equations

The concept of wave splitting was first presented by Corones and Krueger [12] [15] and has been successfully applied to solve direct and inverse wave scattering problems for different kinds of models [2] [14] [21] and [22]. In our fluid-saturated porous media case, we will use this wave splitting technique to decompose the internal fast and slow wave modes into a pair of right and left going wave components, respectively. In the next section, again we will apply Corones and Krueger's invariant imbedding idea to the split wave equation to obtain the scattering matrix operator equations.

We start by recasting equation (3.4) into the following PDE system

$$\begin{align*}
\frac{\partial}{\partial x} \begin{bmatrix}
e \\
\epsilon \\
e_x \\
\epsilon_x
\end{bmatrix} &= \begin{bmatrix}0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
a_{11} \partial_t^2 - l_1 f * \partial_t^2 & a_{12} \partial_t^2 + l_1 f * \partial_t^2 & 0 & 0 \\
a_{21} \partial_t^2 + l_2 f * \partial_t^2 & a_{22} \partial_t^2 - l_2 f * \partial_t^2 & 0 & 0
\end{bmatrix} \begin{bmatrix}
e \\
\epsilon \\
e_x \\
\epsilon_x
\end{bmatrix} \tag{3.6}
\end{align*}$$
with $a_{ij}$ and $l_i$ defined as

$$a_{11} = A(R_{11} - Q_{12})$$
$$a_{12} = A(R_{12} - Q_{22})$$
$$a_{21} = A(P_{12} - Q_{11})$$
$$a_{22} = A(P_{22} - Q_{12})$$
$$l_1 = -A(Q + R)$$
$$l_2 = -A(Q + P)$$

where $A = (PR - Q^2)^{-1}$ and $0 < A < \infty$ [5]. We rewrite equation (3.6) as

$$\begin{bmatrix}
\frac{\partial}{\partial x} e(x,t) \\
\epsilon(x,t) \\
e_x(x,t) \\
\epsilon_x(x,t)
\end{bmatrix} = (D_0 + \delta)
\begin{bmatrix}
e(x,t) \\
\epsilon(x,t) \\
e_x(x,t) \\
\epsilon_x(x,t)
\end{bmatrix}$$

(3.8)

where

$$D_0 =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
a_{11} \partial_t^2 & a_{12} \partial_t^2 & 0 & 0 \\
a_{21} \partial_t^2 & a_{22} \partial_t^2 & 0 & 0
\end{bmatrix}$$

(3.9)

$$\delta =
\begin{bmatrix}
0 & 0 \\
Y & 0
\end{bmatrix}_{4 \times 4}$$

(3.10)

and

$$Y =
\begin{bmatrix}
-l_1 & l_1 \\
l_2 & -l_2
\end{bmatrix} f(t) \ast \partial_t^2$$

(3.11)
With the choice \( \delta = 0 \), equation (3.6) is diagonalized and we may determine the solutions in the regions outside the slab \( (x < 0 \) or \( x > L ) \). Let \( \tilde{\Lambda} \) and \( E \) be the eigenvalue and eigenvector matrix of \( D_0 \), respectively. That is \( E^{-1} D_0 E = \tilde{\Lambda} \), where \( \tilde{\Lambda} \) and \( E \) are given by

\[
\tilde{\Lambda} = \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix}
\]

and

\[
E = \begin{bmatrix}
1 & 1 & -v_1^{-1} \partial_t & -v_2^{-1} \partial_t & v_1^{-1} \partial_t & v_2^{-1} \partial_t \\
-K_1 & K_1 & -K_2 & K_2 & v_1^{-1} \partial_t & -v_2^{-1} K_2 \partial_t \\
-v_1^{-1} K_1 \partial_t & -v_2^{-1} K_2 \partial_t & v_1^{-1} K_1 \partial_t & -v_2^{-1} K_2 \partial_t & v_1^{-1} \partial_t & v_2^{-1} \partial_t
\end{bmatrix}
\]

where \( N \) and \( M \) are \( 2 \times 2 \) matrices, and \( \Lambda, \nu_1, \nu_2, K_1 \) and \( K_2 \) are given by

\[
\Lambda = \begin{bmatrix} -v_1^{-1} & 0 \\ 0 & -v_2^{-1} \end{bmatrix}
\]

\[
\nu_1 = \sqrt{2} \left[ a_{11} + a_{22} + \left( (a_{22} - a_{11})^2 + 4a_{21}a_{12} \right)^{1/2} \right]^{-1/2}
\]

\[
\nu_2 = \sqrt{2} \left[ a_{11} + a_{22} - \left( (a_{22} - a_{11})^2 + 4a_{21}a_{12} \right)^{1/2} \right]^{-1/2}
\]
\[ K_1 = (1/2a_{12}) \left[ a_{22} - a_{11} + \left((a_{22} - a_{11})^2 + 4a_{21}a_{12}\right)^{\frac{1}{2}} \right] \]
\[ K_2 = (1/2a_{12}) \left[ a_{11} - a_{22} + \left((a_{22} - a_{11})^2 + 4a_{21}a_{12}\right)^{\frac{1}{2}} \right]. \]  

(3.16)

It can be easily shown that \( v_1 \) and \( v_2 \) are positive \([5]\) and \( v_2 > v_1 \). We will see that \( v_1 \) and \( v_2 \) are the wave speeds corresponding to the slow compressional wave and the fast compressional wave, respectively. It also can be verified that the expressions for \( v_1 \) and \( v_2 \) we found in equation (3.15) are the same as that given in \([5]\) (see Appendix C). In the regions \( x < 0 \) or \( x > L \), the solution of equation (3.8) (with \( \delta = 0 \)) can be obtained by using \( E^{-1}D_0E = \bar{\Lambda} \) to diagonalize the PDE system with \( (e, \epsilon, e_x, e_x)^t = E(\phi^+, \psi^+, \phi^-, \psi^-)^t \)

\[ \frac{\partial}{\partial x} \begin{bmatrix} \phi^+ \\ \psi^+ \\ \phi^- \\ \psi^- \end{bmatrix} = \begin{bmatrix} -v_1^{-1} \partial_t & 0 & 0 & 0 \\ 0 & -v_2^{-1} \partial_t & 0 & 0 \\ 0 & 0 & v_1^{-1} \partial_t & 0 \\ 0 & 0 & 0 & v_2^{-1} \partial_t \end{bmatrix} \begin{bmatrix} \phi^+ \\ \psi^+ \\ \phi^- \\ \psi^- \end{bmatrix}. \]  

(3.17)

Its solutions are

\[ \begin{align*}
\phi^+(x, t) &= \phi^+(x - v_1 t) \\
\phi^-(x, t) &= \phi^-(x + v_1 t) \\
\psi^+(x, t) &= \psi^+(x - v_2 t) \\
\psi^-(x, t) &= \psi^-(x + v_2 t)
\end{align*} \]  

(3.18)

i.e., in the regions outside the slab there are two kinds of compressional waves. One is the fast compressional wave with speed \( v_2 \). The other is the slow compressional wave with speed \( v_1 \). Our following use of the wave splitting method is based on the physics of the above \( \phi^\pm, \psi^\pm \). It proves effective to split the total fields in the
slab in such a way that the fields outside the slab \( \phi^\pm, \psi^\pm \) in (3.18) are continuous extensions of these fields inside the slab at boundary. This will make the boundary condition of incidence simple \([2] [12] [23]\). Motivated by this splitting idea, inside the slab we define \( \phi^\pm \) and \( \psi^\pm \) as a pair of right and left going waves corresponding to the slow and fast wave velocity \( v_1 \) and \( v_2 \) by the following

\[
\begin{bmatrix}
\phi^+(x,t) \\
\psi^+(x,t) \\
\phi^-(x,t) \\
\psi^-(x,t)
\end{bmatrix} = E^{-1}
\begin{bmatrix}
\epsilon(x,t) \\
\epsilon_x(x,t) \\
\epsilon(x,t) \\
\epsilon_x(x,t)
\end{bmatrix}
\]

i.e.,

\[
\phi^\pm(x,t) = \frac{a_{12}}{2B} \left[ (K_2 \epsilon + \epsilon) \pm \partial_t^{-1}(K_2 \epsilon_x + \epsilon_x) \right]
\]

\[
\psi^\pm(x,t) = \frac{a_{12}}{2B} \left[ (K_1 \epsilon - \epsilon) \pm \partial_t^{-1}(K_1 \epsilon_x - \epsilon_x) \right]
\]

where

\[
\partial_t^{-1} g(t) = \int_0^t g(s) \, ds \quad \text{for} \ g \in C(0, \infty)
\]

\[
E^{-1} = \frac{1}{2}
\begin{bmatrix}
\alpha & -\beta \\
\alpha & \beta
\end{bmatrix}
\]

Here

\[
\alpha = N^{-1} = (a_{12}/B) \begin{bmatrix}
K_2 & 1 \\
K_1 & -1
\end{bmatrix}
\]

\[
\beta = M^{-1} = (a_{12}/B) \begin{bmatrix}
v_1 K_2 & v_1 \\
v_2 K_1 & -v_2
\end{bmatrix} \partial_t^{-1}
\]
with $B = [(a_{22} - a_{11})^2 + 4a_{21}a_{12}]^{\frac{1}{2}}$. We see that in the regions outside the slab the definition in (3.19) or (3.20) with $e, \epsilon$ a solution of (3.5) will result in equation (3.18). Although the exact physical concept of such right and left going waves in the slab is not quite clear, the definition in (3.20) is mathematically precise. Actually, it introduces new dependent variables in equation (3.4). Note that $E^{-1}$ is equivalent to the wave splitting matrix $T$ in [15]. In equation (3.19), we see that the fields $e, \epsilon$ in the solid and in the fluid consist of $\phi^+, \phi^-, \psi^+$ and $\psi^-

\begin{align*}
e(x,t) &= [\phi^+(x,t) + \phi^-(x,t)] + [\psi^+(x,t) + \psi^-(x,t)] \\
\epsilon(x,t) &= K_1[\phi^+(x,t) + \phi^-(x,t)] - K_2[\psi^+(x,t) + \psi^-(x,t)]
\end{align*}

(3.24)

i.e., $e$ and $\epsilon$ are linear combinations of $(\phi^+ + \phi^-)$ and $(\psi^+ + \psi^-)$. Then from the concept of the right and left going waves in equation (3.20), the original wave equation (3.6) becomes

\begin{align*}
\frac{\partial}{\partial x} \begin{bmatrix} \Theta^+(x,t) \\ \Theta^-(x,t) \end{bmatrix} &= \begin{bmatrix} \Lambda \partial_t - \tilde{q} & -\tilde{q} \\ \tilde{q} & -\Lambda \partial_t + \tilde{q} \end{bmatrix} \begin{bmatrix} \Theta^+(x,t) \\ \Theta^-(x,t) \end{bmatrix} \\
\text{where} \\
\Theta^\pm(x,t) &= \begin{bmatrix} \phi^\pm(x,t) \\ \psi^\pm(x,t) \end{bmatrix} \\
\tilde{q} &= \frac{1}{2} \beta Y \alpha^{-1} = qf(t) * \partial_t \\
q &= \begin{bmatrix} m(K_1 - 1) & -m(K_2 + 1) \\ -n(K_1 - 1) & n(K_2 + 1) \end{bmatrix}
\end{align*}

(3.25)
where

\[ m = a_{12}v_1B^{-1}(l_1K_2 - l_2)/2 \]
\[ n = -a_{12}v_2B^{-1}(l_2 + l_1K_1)/2 \]  \hfill (3.29)

From (3.28) we see that \( q \) is a singular matrix. We set up the following initial and boundary value problems for the split wave equation

\[
\begin{align*}
\Theta^+(x,t) - \Lambda \Theta^+_t(x,t) &= -qf \ast \Theta^+(x,t) - qf \ast \Theta^-(x,t) \\
\Theta^-(x,t) + \Lambda \Theta^-_t(x,t) &= qf \ast \Theta^+(x,t) + qf \ast \Theta^-(x,t) \\
\Theta^+(0,t) &= \Theta^+_t(0) \quad t > 0 \\
\Theta^-(L,t) &= \Theta^-_t(L) \quad t \geq 0 \\
\Theta^+(x,0) &= \Theta^-(x,0) = 0 \\0 < x < L
\end{align*}
\]  \hfill (3.30)

i.e., a right going propagation wave vector \( \Theta^+_t \) in region \( x < 0 \), consisting of a slow wave \( \phi^+_t(t - x/v_1) \) and a fast wave \( \psi^+_t(t - x/v_2) \), impinges on the slab from the left at \( x = 0 \) at \( t = 0 \); a left going propagation wave vector \( \Theta^-_t \) in region \( x > L \) impinges on the slab from the right at \( x = L \) at \( t = 0 \). The split wave equation with the initial and boundary conditions in (3.30) is a well-posed problem \[16\]. It is clear physically that we can determine the internal field and the reflected, transmitted waves.

### 3.3 Invariant Imbedding and Scattering Operator Equations

For an imbedded slab, \((x, L)\), with left boundary plane at \( x \) and right boundary plane at \( L \), we let \( \Theta^-(x,t) \) and \( \Theta^+(L,t) \) be the left going reflected wave and right going transmitted wave at \( x \) and \( L \), respectively. Without loss of generality, we assume that at the left and right boundaries there are right going incident wave,
\( \Theta^+(x, t) \), and left going incident wave, \( \Theta^- (L, t) \), impinging on the slab from the left and right, respectively. The reflected and transmitted waves, \( \Theta^- (x, t) \) and \( \Theta^+ (L, t) \) are connected to these incident waves through a scattering matrix operator \( \tilde{S}(x, t) \), which is defined by [15]

\[
\begin{bmatrix}
\Theta^+(L, t) \\
\Theta^-(x, t)
\end{bmatrix} =
\begin{bmatrix}
\tilde{R}^+(x, t) & \tilde{R}^-(x, t) \\
\tilde{R}^+(x, t) & \tilde{R}^-(x, t)
\end{bmatrix}
\begin{bmatrix}
\Theta^+(x, t) \\
\Theta^-(L, t)
\end{bmatrix} = \tilde{S} \cdot \begin{bmatrix}
\Theta^+(x, t) \\
\Theta^-(L, t)
\end{bmatrix} .
\tag{3.31}
\]

Corones and Krueger's invariant imbedding idea [12] [15] will be applied to the split wave equation (3.30). Variations in the boundaries of the imbedded medium will give rise to variations in its scattering operators such that partial integrodifferential equations for the reflection kernel \( R \) and transmission kernel \( T \) can be derived which allow us to calculate the scattering operators for the imbedded medium. We rewrite equation (3.25) in the following form

\[
\frac{\partial}{\partial x} \begin{bmatrix}
\Theta^+(x, t) \\
\Theta^-(x, t)
\end{bmatrix} =
\begin{bmatrix}
\tilde{\alpha} & \tilde{\beta} \\
\tilde{\delta} & \tilde{\gamma}
\end{bmatrix}
\begin{bmatrix}
\Theta^+(x, t) \\
\Theta^-(x, t)
\end{bmatrix} .
\tag{3.32}
\]

where

\[
\begin{align*}
\tilde{\alpha}(t) &= \Lambda \partial_t - qf * \partial_t \\
\tilde{\beta}(t) &= -qf * \partial_t \\
\tilde{\gamma}(t) &= qf * \partial_t \\
\tilde{\delta}(t) &= -(\Lambda \partial_t - qf * \partial_t) .
\end{align*}
\tag{3.33}
\]

Using the invariant imbedding idea, i.e., \( \partial / \partial x \) represents a derivative with respect to the variable position \( x \) of a slab's left boundary, while the right boundary remains
fixed at $L$, then we differentiate (3.31) with respect to $x$

$$
\begin{bmatrix}
0 \\
\Theta_x^-(x,t)
\end{bmatrix} = \tilde{S}_x \begin{bmatrix}
\Theta^+(x,t) \\
\Theta^-(L,t)
\end{bmatrix} + \tilde{S} \begin{bmatrix}
\Theta_x^+(x,t) \\
0
\end{bmatrix} .
$$

(3.34)

We replace $\Theta^+(x,t)$ and $\Theta^-(x,t)$ in (3.34) by the followings

$$
\partial_x \Theta^+(x,t) = \tilde{\alpha} \cdot \Theta^+(x,t) + \tilde{\beta} \cdot \Theta^-(x,t)
$$

$$
= \tilde{\alpha} \cdot \Theta^+(x,t) + \tilde{\beta} \tilde{R}^+ \cdot \Theta^+(x,t) + \tilde{\beta} \tilde{T}^- \cdot \Theta^-(L,t)
$$

(3.35)

$$
\partial_x \Theta^-(x,t) = \tilde{\gamma} \cdot \Theta^+(x,t) + \tilde{\delta} \cdot \Theta^-(x,t)
$$

$$
= \tilde{\gamma} \cdot \Theta^+(x,t) + \tilde{\delta} \tilde{R}^+ \cdot \Theta^+(x,t) + \tilde{\delta} \tilde{T}^- \cdot \Theta^-(L,t)
$$

which are directly from equations (3.25) and (3.31). Using the above equation (3.35) to equation (3.34), we obtain

$$
\begin{bmatrix}
0 \\
\tilde{\gamma} + \tilde{\delta} \tilde{R}^+ \\
\tilde{T}^-
\end{bmatrix} \begin{bmatrix}
\Theta^+(x,t) \\
\Theta^-(L,t)
\end{bmatrix} = \tilde{S}_x \begin{bmatrix}
\Theta^+(x,t) \\
\Theta^-(L,t)
\end{bmatrix} + \tilde{S} \begin{bmatrix}
\tilde{\alpha} + \tilde{\beta} \tilde{R}^+ & \tilde{\beta} \tilde{T}^- \\
0 & 0
\end{bmatrix} \begin{bmatrix}
\Theta^+(x,t) \\
\Theta^-(L,t)
\end{bmatrix}
$$

(3.36)

But $\Theta^+(x,t)$ and $\Theta^-(L,t)$ are arbitrary functions, then we have the following scattering matrix operator equation [12] [15]

$$
\frac{\partial \tilde{S}(x,t)}{\partial x} = \begin{bmatrix}
\tilde{T}(x,t) & 0 \\
\tilde{R}(x,t) & \tilde{I}
\end{bmatrix} \begin{bmatrix}
\tilde{q} - \Lambda \tilde{\delta} t & \tilde{q} \\
\tilde{q} & \tilde{q} - \Lambda \tilde{\delta} t
\end{bmatrix} \begin{bmatrix}
\tilde{T}(x,t) & 0 \\
\tilde{R}(x,t) & \tilde{I}
\end{bmatrix} .
$$

(3.37)

Notice that we have used the space symmetry property of our problem in equation (3.36), i.e.,

$$
\tilde{T}^+(x,t) = \tilde{T}^-(x,t) = \tilde{T}(x,t)
$$

$$
\tilde{R}^+(x,t) = \tilde{R}^-(x,t) = \tilde{R}(x,t).
$$

(3.38)
In both direct and inverse scattering problems we have to use the operator equation (3.37) to obtain the reflection kernel and the transmission kernel integrodifferential equations for $R(x,t)$ and $T(x,t)$, where $R(x,t)$ and $T(x,t)$ are the reflection and transmission kernels that will be defined in the next two chapters.
4. DIRECT AND INVERSE PROBLEMS OF $R$-EQUATION

4.1 $R$-equation and Characteristic Curves

According to equation (3.37) we obtain the following reflection operator $\mathcal{R}(x, t)$ equation

$$\mathcal{R}x(x, t) = \mathcal{F}(t) + \mathcal{G}(t)\mathcal{R}(x, t) - \mathcal{R}(x, t)\mathcal{H}(t) - \mathcal{R}(x, t)\mathcal{I}(t)\mathcal{R}(x, t) \quad (4.1)$$

By using a variation of Duhamel's principle [16, pp. 512-515] to the split wave equation (3.25) of an imbedded slab, which is bounded on the left by $x$ and on the right by $L$, we can prove that the reflection operator $\mathcal{R}(x, t)$ is an integral operator, which is represented by (See Appendix A for detailed proof)

$$\mathcal{R}(x, t) = \int_0^t \mathcal{R}(x, t - s) \left[ \begin{array}{c} \phi^+(x, s) \\ \psi^+(x, s) \end{array} \right] ds \quad (4.2)$$

i.e.,

$$\Theta^-(x, t) = \mathcal{R}(x, t) \cdot \Theta^+(x, t) = R(x, t) \ast \Theta^+(x, t) \quad (4.3)$$

where $R(x, t)$ is the reflection matrix kernel

$$R(x, t) = \begin{bmatrix} R_{11}(x, t) & R_{12}(x, t) \\ R_{21}(x, t) & R_{22}(x, t) \end{bmatrix} \quad (4.4)$$
and Θ±(x,t) are the right and left going wave vectors, which are defined by

\[ \Theta^\pm(x,t) = \begin{bmatrix} \phi^\pm(x,t) \\ \psi^\pm(x,t) \end{bmatrix} \] (4.5)

When \( x = 0 \) we see that \( \tilde{R}(0,t) \) maps the physical incident wave \( \Theta^i(t) \) at plane \( x = 0 \) into the reflected wave, i.e.,

\[ \Theta^-(0,t) = \tilde{R}(0,t) * \Theta^i(t) \] (4.6)

where \( \Theta^i(t) \) and \( \Theta^-(0,t) \) are the incident and reflected compressional waves at \( x \). If let in incident waves have the form

\[ \Theta^i(t) = \begin{bmatrix} \phi^i(t) \\ \psi^i(t) \end{bmatrix} = \begin{bmatrix} \delta(t) \\ \delta(t) \end{bmatrix} \] (4.7)

where \( \delta(t) \) is the Dirac distribution function. Then it follows that \( \Theta^-(x,t) = R(x,t) \). So the physical meaning of the scattering kernel matrix \( R \) is that it is the impulse response of the imbedded slab, \( (x, L) \).

Our purpose is to derive the reflection operator kernel equation from the definition (4.2) by letting operator equation (4.1) acting on function \( g(t) \), where \( g(t) \) is defined as an arbitrary \( C^1(t > 0) \) function with \( g(t) = 0 \) for \( t \leq 0 \). We should be very careful in taking any partial derivative computations, since \( R \) may have possible jump discontinuities along certain curves (called the characteristic curves). Without loss of generality we assume that \( t = h_i(x) \), for \( i = 1, 2, \ldots, n \) and \( h_i \neq h_j \), for \( i \neq j \), are the \( n \) curves along which \( R \) will suffer jump discontinuities, where integer
$n$ will be determined later. We start to compute the $\tilde{R}$ operator acting on $g(t)$.

$$\partial_x \tilde{R} \cdot g(t) = \partial_x \left[ \int_0^{t-h_{1}^+} + \int_{t-h_1^-}^{t-h_2^+} + \cdots + \int_{t-h_{n-1}^-}^{t-h_n^+} R(x, t - s)g(s) \, ds \right]$$

$$= -\sum_{i=1}^{n} \left[ R(x, h_i^+) - R(x, h_i^-) \right] h_i'(x)g(t - h_i). \quad (4.8)$$

$$+ \int_{0}^{t-h_n} R_x(x, t - s)g(s) \, ds$$

where we have separated the integration in equation (4.2) by $\int_{0}^{t} = \int_{0}^{t-h_{1}^+} + \cdots + \int_{t-h_{n-1}^-}^{t-h_n^+}$ in order to take care of the characteristic curves. Also we should note that $R(x, t)$ is independent of $x$ for $t < h_n(x)$, i.e.,

$$\int_{0}^{t} R_n(x, t - s) \, ds = \int_{0}^{t-h_n} R_x(x, t - s) \, ds \quad (4.9)$$

We will see later that $h_n(x)$ is the one-round trip time of the fast wave front. The followings are the necessary computation steps to compute the right hand side of equation (4.1) acting on $g(t)$. They are

$$\tilde{\gamma} \cdot g(t) = qf \ast \partial_t g(t)$$

$$= qf(0)g(t) + q \int_{0}^{t} f'(t - s)g(s) \, ds \quad (4.10)$$

$$\delta \tilde{R} \cdot g(t) = (qf \ast \partial_t - \Lambda \partial_x) \tilde{R} \cdot g(t)$$

$$= qf(0)[R \ast g](t) + q[f' \ast R \ast g](t) - \Lambda R(x, 0)g(t)$$

$$- \Lambda [R_t \ast g](t) - \Lambda \sum_{i=1}^{n} [R(x, h_i^+) - R(x, h_i^-)]g(t - h_i) \quad (4.11)$$
\[
\vec{R}A \cdot g(t) = \vec{R}(\Lambda \partial_t - qf \ast \partial_t) \cdot g(t)
\]
\[
= R(x,0)\Lambda g(t) + [R_t \Lambda \ast g](t) - f(0)[R \ast q \ast g](t)
\]
\[-[f' \ast R \ast q \ast g](t) + \sum_{i=1}^{n} [R(x,h_i^+) - R(x,h_i^-)] \Lambda g(t - h_i)
\]
and
\[
\vec{R} \vec{R} \cdot g(t) = \vec{R}(-qf \ast \partial_t) \vec{R} \cdot g(t)
\]
\[-f(0)[R \ast q \ast R \ast g](t) - [f' \ast R \ast q \ast R \ast g](t).
\]

By inserting equations (4.8–4.13) to equation (4.1) we have
\[
[R_x \ast g](t) - \sum_{i=1}^{n} [R(x,h_i^+) - R(x,h_i^-)] h_i^{	ext{t}}(x)g(t - h_i)
\]
\[-qf(0)g(t) + q[f' \ast g](t) + qf(0)[R \ast g](t) + q[f' \ast R \ast R \ast g](t)
\]
\[-\Lambda[R_t \ast g](t) - \sum_{i=1}^{n} [R(x,h_i^+) - R(x,h_i^-)] g(t - h_i)
\]
\[-\Lambda R(x,0)g(t) - R(x,0)\Lambda g(t) - [R_t \Lambda \ast g](t) + f(0)[R \ast q \ast g](t)
\]
\[+ [f' \ast R \ast q \ast g](t) - \sum_{i=1}^{n} [R(x,h_i^+) - R(x,h_i^-)] \Lambda g(t - h_i)
\]
\[+ f(0)[R \ast q \ast R \ast g](t) + [f' \ast R \ast q \ast R \ast g](t).
\]

Then using the fact that \( g(t) \) is an arbitrary function, we then can obtain a partial integrodifferential equation (i.e., \( R \)-equation) with initial and boundary conditions
for the reflection kernel $R(x,t)$

$$R_x + (R_t \Lambda + \Lambda R_t) = qf'(t) + f(0)(qR + Rq + R* qR) +$$

$$f'(t) * (qR + Rq + R* qR) \quad 0 < x < L, \ t > 0$$

(4.15)

$$R(x,0^+)\Lambda + \Lambda R(x,0^+) = qf(0) \quad 0 < x < L$$

$$R(L,t) = 0 \quad 0 < t < \infty$$

where the boundary condition in (4.15) follows directly from the fact that the right and left sides of the slab are the same porous media. $\Lambda$ and $q$ are constant matrices which are given by equations (3.14) and (3.28). Both of these matrices depend only on the properties of the medium, such as the Lamé constants of the elastic frame, effective mass density of solid and fluid, viscosity and permeability.
The characteristic curves \( t = h_i(x) \) are determined from the following equation

\[
[R]_i \dot{h}'(x) = \Lambda [R]_i + [R]_i \Lambda, \quad i = 1, 2, 3, \ldots, n
\]  

(4.16)

where \([R]_i = [R(x, h_i+) - R(x, h_i-)]\). Equation (4.16) are solved by

\[
\begin{align*}
 h_1(x) &= \frac{2(L - x)}{v_1} \\
 h_2(x) &= \frac{(L - x)}{v_1} + \frac{(L - x)}{v_2} \\
 h_3(x) &= \frac{2(L - x)}{v_2}
\end{align*}
\]  

(4.17)

where \(n\) has been determined to be equal to 3 since \(h_i(x) \neq h_j(x)\) for \(i \neq j\). So, there are three characteristic curves along which \(R\) have jump discontinuities. The jump conditions of \([R]_i\) for \(i = 1, 2, 3\) along these three characteristic curves will be derived in two ways. One method to derive them are given in the next given by using usual theory of propagation of singularities. Another way to derive them are given in Chapter 5.

### 4.2 Jump Discontinuities and Reciprocity Relation of \(R\)

The characteristic curves of the P.D.E. system (4.15) can be obtained through solving the following equation [16, pp. 424-427]

\[
\begin{bmatrix}
2v_1^{-1} & 0 & 0 & 0 \\
0 & (v_1 + v_2)^{-1} & 0 & 0 \\
0 & 0 & (v_1 + v_2)^{-1} & 0 \\
0 & 0 & 0 & 2v_2^{-1}
\end{bmatrix}
\begin{bmatrix} dx + I_{4 \times 4} dt \end{bmatrix} = 0
\]  

(4.18)

where \(I_{4 \times 4}\) is a unit matrix. Actually the solutions of (4.18) are the same as that given by by equation (4.17). Along these three characteristic curves, we derive the fol-
lowing jump conditions for $R_{ij}(x, t)$ by using elementary propagation of singularities arguments [16]

$$R(x, h_1^-) - R(x, h_1^+) = \frac{v_1 f(0)}{2} \begin{bmatrix} \Delta_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

$$R(x, h_2^-) - R(x, h_2^+) = \frac{v_1 v_2 f(0)}{v_1 + v_2} \begin{bmatrix} 0 & \Delta_{12} \\ \Delta_{21} & 0 \end{bmatrix}$$ \hspace{1cm} (4.19)

$$R(x, h_3^-) - R(x, h_3^+) = \frac{v_2 f(0)}{2} \begin{bmatrix} 0 & 0 \\ 0 & \Delta_{22} \end{bmatrix}$$

for $0 < x < L$, where

$$\Delta_{11} = \frac{q_{11}}{2} \exp \left[ - f(0) q_{11} (L - x) \right]$$

$$\Delta_{12} = \frac{q_{12}}{2} \exp \left[ - f(0) (q_{11} + q_{22}) (L - x) \right]$$

$$\Delta_{21} = \frac{q_{21}}{2} \exp \left[ - f(0) (q_{11} + q_{22}) (L - x) \right]$$

$$\Delta_{22} = \frac{q_{22}}{2} \exp \left[ - 2 f(0) q_{22} (L - x) \right].$$ \hspace{1cm} (4.20)

since the initial jump discontinuities of $R$ is given by (4.15), i.e.,

$$\begin{bmatrix} R_{11}(x, 0+) - R_{11}(x, 0-) \\ R_{12}(x, 0+) - R_{12}(x, 0-) \\ R_{21}(x, 0+) - R_{21}(x, 0-) \\ R_{22}(x, 0+) - R_{22}(x, 0-) \end{bmatrix} = \begin{bmatrix} -\frac{v_1}{2} q_{11} \\ -\frac{v_1 v_2}{v_1 + v_2} q_{12} \\ -\frac{v_1 v_2}{v_1 + v_2} q_{12} \\ -\frac{v_2}{2} q_{22} \end{bmatrix} f(0)$$ \hspace{1cm} (4.21)
where \( R(x, 0-) = 0 \). In the next chapter, we will obtain the same result as in (4.19) by using another method.

Equation (4.19) shows that if \( f(0) \neq 0 \) then there are three jumps in the reflection kernel \( R \). First, at one round trip time for the fast wave, i.e., at time \( t = 2(L - x)/v_2 \), \( R_{22} \) has a discontinuity with a jump \( \Delta_{22} \). Second, at the one way travel time of the fast wave plus the one way travel time of the slow wave, i.e., at time \( t = (L - x)/v_1 + (L - x)/v_2 \), both \( R_{12} \) and \( R_{21} \) have discontinuities with jumps \( \Delta_{12} \) and \( \Delta_{21} \), respectively. Third, at the one round trip time for the slow wave, i.e., at time \( t = 2(L - x)/v_1 \), \( R_{11} \) has a discontinuity with a jump \( \Delta_{11} \).

From the definitions of the reflection operator in equation (4.2), we know that \( \tilde{R}_{11} \) maps the slow incident wave into the slow reflected wave, \( \tilde{R}_{12} \) maps the fast incident wave into the slow reflected wave, \( \tilde{R}_{21} \) maps the slow incident wave into the fast reflected wave, and \( \tilde{R}_{22} \) maps the fast incident wave into the fast reflected wave. Then from this discussion and equation (4.19) we see that all the jump discontinuities in \( R_{ij} \) come from the back wall \( (x = L) \). In the scattering problems of electromagnetic waves for a dispersive or inhomogeneous medium the jump discontinuity of \( R \) is also from the back wall [2] [22]. We call the three curves, \( t = h_1(x) \), \( t = h_2(x) \) and \( t = h_3(x) \), the slow wave characteristic, the coupling wave characteristic, and the fast wave characteristic, respectively. The jump conditions in (4.19) provide the connection between regions \( I, II, III \) and IV (See Figure 4.1). This is crucial for our further work on the inverse problems.

The PDE system in equation (4.15) and the jump condition (4.19) are very useful for our direct scattering scattering problem. If we are given all the properties of the fluid-saturated porous medium, i.e., \( f(t), q \) and \( \Lambda \) are given, then we can solve
equation (4.15) for the reflection kernel $R(x, t)$ and use (4.19) to pass our solutions to the different regions in Figure 4.1.

In equation (4.15), we should note that the four components of $R(x, t)$ are usually not independent. We will prove that the following reciprocity relation between the off-diagonal components of $R$

$$\frac{R_{12}(x, t)}{q_{12}} = \frac{R_{21}(x, t)}{q_{21}} \quad (x, t) \in (D \times (0, \infty))$$

(4.22)

where $D$ is the union of the open regions of $I$, $II$, $III$, $IV$ (See Figure 4.1).

We start to prove equation (4.22) by assuming that $R(x, t)$ be the solution of P.D.E. system (4.15) and defining that

$$a = -\left(\frac{1}{v_1} + \frac{1}{v_2}\right)$$

$$W(x, t) = \frac{R_{21}(x, t)}{q_{21}} - \frac{R_{12}(x, t)}{q_{12}}.$$  (4.23)

We decompose the $R$-equation (4.15) into four integrodifferential equations. From the two off-diagonal component integrodifferential equations and the definition of $W$ in (4.23), we then have

$$\begin{cases}
W_x(x, t) + aW_t(x, t) = (q_{11} + q_{22})f'(t) \ast W(x, t) + \\
\quad \quad f'(t) \ast [q_{11}R_{11}(x, t) + q_{22}R_{22}(x, t) + \\
\quad \quad q_{12}R_{21}(x, t) + q_{21}R_{12}(x, t)] \ast W(x, t)
\end{cases}$$

(4.24)

$$W(x, 0) = 0$$

$$W(L, t) = 0.$$  

The above equation has a unique solution $W(x, t) = 0$. This is true because if we
take the Laplace transformation $L$ of equation (4.24), then we have

$$\begin{align*}
W_x(x,s) + saW(x,s) &= Q(x,s)W(x,s) \\
W(x,0) &= 0 \\
W(L,s) &= 0
\end{align*}$$

(4.25)

with

$$W(x,s) = L[W(x,t)]$$

$$Q(x,s) = L[(q_{11} + q_{22})f'(t) + f'(t) \ast (q_{11}R_{11}(x,t) +$$

$$q_{22}R_{22}(x,t) + q_{12}R_{21}(x,t) + q_{21}R_{12}(x,t))]$$

where $Q$ is continuous in $x$ and $s$ since $R_{i,j} \in C^1(D \times (0, \infty))$, for $i, j = 1, 2$ and $f(t) \in C^1(t > 0)$. The solution of (4.25) is then $W(x,s) = C \exp(-asx + \int_0^T Q(v,s) dv)$ where $C$ is a constant. By applying the boundary condition to this solution, we obtain the unique solution $W(x,t) = 0$, for all $(x,t) \in (D \times (0, \infty))$.

Then we obtain the reciprocity relation, which is given by equation (4.22).

### 4.3 A High-Frequency Scattering Example

The following will give an example to solve equation (4.15) for the direct scattering problem by using the modified function derived by Biot in [3], [5]. Due to the coupling between modes, the following direct scattering problem will be solved in the high frequency domain [24]. Let's take $L = +\infty$, i.e., we consider a half-space problem. We know that the reflection operator for the half-space model is independent of $x$. Thus in (4.15), $R(x,t) = R(t)$ and $R_x = 0$. We integrate (4.15) with respect to $t$ and also take the Laplace transformation of the resulting equation. Then we have

$$R(s)\Lambda + \Lambda R(s) = qf(s) + f(s)[qR(s) + R(s)q + R(s)qR(s)]$$

(4.27)
where \( s = i\omega \) is the Laplace transform variable. By [3] [5], we apply the modified function derived by Biot,

\[
F(\kappa) = \frac{1}{4} \frac{\kappa T(\kappa)}{1 - \frac{2}{i\kappa} T(\kappa)}
\]

where, \( \kappa = \alpha(\rho_f/\mu)^{1/2} \), \( T(\kappa) = \frac{\text{ber}'(\kappa) + i\text{bei}'(\kappa)}{\text{ber}(\kappa) + i\text{bei}(\kappa)} \), where \( \text{ber} \) and \( \text{bei} \) are the Kelvin functions. We can derive that

\[
f(s) = \frac{pb}{4y} \frac{I_1(y)/I_0(y)}{1 - 2y^{-1}I_1(y)/I_0(y)} = \frac{pb}{4y} \frac{I_1(y)}{I_2(y)}
\]

where \( y = ps \), with \( p = \alpha(\rho_f/\mu)^{1/2} \) and \( I_j(x), j = 0, 1, 2 \) are the hyperbolic Bessel functions of order 0, 1, 2 respectively. At very high frequency we find the asymptotic form of (4.29) as

\[
f(s) = \frac{bp}{8} \left[ 2y^{-1} + 3y^{-2} + \frac{15}{4} y^{-3} + \frac{15}{4} y^{-4} + \ldots \right].
\]

We then asymptotically solve (4.27) by using (4.30) to obtain the high frequency solution as

\[
R(s) = \sum_{n=1}^{\infty} r n y^{-n}
\]
where the components, \((r_n)_{ij} = r_n(ij)\), \(n = 1, 2, 3, \ldots\), of \(r_n\) are given by

\[

ger_{1(ii)} = \frac{p b q_{ii}}{8 \Lambda_{ii}} \\

ger_{1(ij)} = \frac{p b q_{ij}}{4(\Lambda_{11} + \Lambda_{22})} \\

ger_{2(ii)} = \frac{p b}{16 \Lambda_{ii}} \left[ 3q_{ii} + \frac{p b q_{ii}^2}{2 \Lambda_{ii}} + \frac{p b q_{12} q_{21}}{\Lambda_{11} + \Lambda_{22}} \right] \\

ger_{2(ij)} = \frac{p b q_{ij}}{8(\Lambda_{11} + \Lambda_{22})} \left[ 3 + \frac{p b (q_{11} + q_{22})}{2(\Lambda_{11} + \Lambda_{22})} + \frac{p b}{4} \left( \frac{q_{11}}{\Lambda_{11}} + \frac{q_{22}}{\Lambda_{22}} \right) \right]
\]

\[
\text{where } ii = 11, 22; \quad ij = 12, 21.
\]

From the result of the asymptotic solution given in (4.32) we also have that

\[
\frac{r_{1(12)}}{r_{1(21)}} = \frac{r_{2(12)}}{r_{2(21)}} = \ldots = \frac{q_{12}}{q_{21}}
\]

which agrees with our previous result in equation (4.22) about the general reciprocity relation of \(R_{12}(x,t)\) and \(R_{21}(x,t)\).

### 4.4 Numerical Schemes for Direct and Inverse Scattering Problems

By the reciprocity relation between \(R_{12}(x,t)\) and \(R_{21}(x,t)\), which are given by equation (4.22), we know that only three of the four components of matrix \(R\) are independent. If we define a reflection kernel vector \(\bar{R}\) by

\[
\bar{R}(x,t) = \begin{bmatrix} R_{11}(x,t) \\ R_{12}(x,t) \\ R_{22}(x,t) \end{bmatrix}
\]

(4.33)
i.e., $R_1 = R_{11}$, $R_2 = R_{12}$ and $R_3 = R_{22}$ Then we can write down our $R$-equation (4.15) in the following vector form

$$\ddot{R}_x - A\dot{R}_t = \ddot{\theta} f'(t) + B \left[ f(0) \ddot{R} + f'(t) \ast \ddot{R} \right] + f(0) \ddot{M} + f'(t) \ast \dddot{M}$$

$$\ddot{R}(x, 0+) = \begin{bmatrix}
\frac{-v_1}{2} q_{11} \\
\frac{-v_1 v_2}{v_1 + v_2} q_{12} \\
\frac{-v_2}{2} q_{22}
\end{bmatrix} f(0)$$

$$\ddot{R}(L, t) = 0$$

where

$$A = \begin{bmatrix}
2v_1^{-1} & 0 & 0 \\
0 & v_1^{-1} + v_2^{-1} & 0 \\
0 & 0 & 2v_2^{-1}
\end{bmatrix}$$

$$B = \begin{bmatrix}
2q_{11} & 2q_{21} & 0 \\
q_{12} & q_{11} + q_{22} & q_{12} \\
0 & 2q_{21} & 2q_{22}
\end{bmatrix}$$

$$\ddot{\theta} = \begin{bmatrix}
q_{11} \\
q_{12} \\
q_{22}
\end{bmatrix}$$
Figure 4.2: Different region numerical grid structures

and

\[
\mathcal{M} (\overline{R} \ast \overline{R}) = \begin{bmatrix}
q_{11} R_1 \ast R_1 + 2q_{21} R_1 \ast R_2 + q_{22} q_{21} q_{12}^{-1} R_2 \ast R_2 \\
q_{11} R_1 \ast R_2 + q_{22} R_3 \ast R_2 + q_{21} R_2 \ast R_2 \\
q_{22} R_3 \ast R_3 + 2q_{21} R_3 \ast R_2 + q_{11} q_{21} q_{12}^{-1} R_2 \ast R_2
\end{bmatrix} \quad (4.38)
\]

Now in equation (4.34) we substitute \( t \) by \( \bar{t} \) and \( x \) by \( \bar{x} - \lambda^{-1} \bar{t} \). Then we can obtain the following

\[
\frac{d}{d\bar{t}} R_n(\bar{x} - \lambda^{-1} \bar{t}, \bar{t}) = -\lambda^{-1} \Omega_n(\bar{R}(\bar{x} - \lambda^{-1} \bar{t}, \bar{t})) \quad n = 1, 2, 3 \quad (4.39)
\]

where \( \Omega \) is given by

\[
\Omega_n(\overline{R}) = b_n f'(t) + \sum_{i=1}^{3} B_{ni} [f(0)R_i + f'(t) R_i] + f(0)M_n + f'(t) M_n \quad (4.40)
\]

Our problem is a multi-characteristic system, when we advance in time we will cross the three characteristic curves with possible discontinuities there. Care must
be taken to handle all the cases to avoid the wave front of $\tilde{R}$ getting deformed when we cross these characteristic curves. In Ayoubi's thesis [1] has the same type of problem. If we set up a mesh structure with uniform step size in both $t$ direction and $x$ direction for all regions in Figure 4.2, it is impossible to make all the three characteristic curves lie exactly on the grid points (corners of cells). What we have to do is to build different mesh structure in different regions to ensure that the characteristic curve in that region is exactly on the grid points (See Figure 4.2).

Let $\Delta t = h$ be the uniform mesh size in the $t$ direction in all regions as shown in Figure 4.2. We use a different $x$ direction mesh size for the regions $I$, $II$, $III$ such that in each region the characteristic curve is exactly on the grid point. Figure (4.2) clearly illustrates our ideas. We define an index $m$ such that $m$ is correspondent to regions $I$, $II$, $III$ (or $IV$) for $m = 1$, $2$, $3$, respectively. The $x$ direction grid step size is chosen as $\Delta x_m = A_{m,n}^{-1}h$ in region $m$. For a given time $T$, we look for the solution of (4.15) in the rectangular region $0 \leq x \leq L$ and $0 \leq t \leq T$. We should note that the $h$ is determined by requiring that

$$\frac{2L}{v_2N_1} = \left(\frac{L}{v_1} + \frac{L}{v_2}\right)\frac{1}{N_2} = \frac{2L}{v_1N_3} \tag{4.41}$$

where $N_1$, $N_2$ and $N_3$ are any integer numbers such that points $(0, 2Lv_1^{-1})$, $(0, L(v_1^{-1} + v_2^{-1}))$ and $(0, 2L/v_2)$ are on the mesh grid points. We can prove that equation (4.41) can be true for choosing some appropriate $N_1$, $N_2$ and $N_3$. On the other hand, we define the following notation for functions evaluated at $(x, t)$, where $(x, t)$ is on the
grid point \((i, j)\),

\[ f_j = f(t) \]

\[ R_n(i, j) = R_n(x, t) \]

\[ \overline{R}_n(i+1, j-1) = R_n(x + A_{nn}^{-1} h, t - h) \]

where point \((x + A_{nn}^{-1} h, t - h)\) is usually not on the grid point even though \((x, t)\) is. If we know \(R_n(i', j - 1)\) for all \(i' = 1, 2, 3, \cdots, i\), then \(\overline{R}_n(i+1, j-1)\) can be obtained by extrapolation method.

We integrate equation (4.39) with respect to \(\overline{t}\) from \(t - h\) to \(t\) and set \(x = \overline{x} - A_{nn}^{-1} t\). By doing this, the discretized \(\overline{R}\)-equation of (4.34) is obtained by the following

\[
\begin{align*}
\frac{2A_{nn}}{h} \left[ \overline{R}_n(i+1, j-1) - R_n(i, j) \right] \\
= b_n \left[ f'_j + f'_{j-1} \right] + f_0 \sum_{k=1}^{3} B_{nk} \left[ R_k(i, j) + \overline{R}_k(i+1, j-1) \right] \\
+ \sum_{k=1}^{3} B_{nk} \left[ \Gamma_k(i, j)(f_j, R_k(i, j)) \right. \\
\left. + \Gamma_k(i+1, j-1)(f_{j-1}, \overline{R}_k(i+1, j-1)) \right] \\
+ f_0 M_n \left( \Gamma_n(i, j)(R_n(i, j), R_n(i, j)) \right) \\
+ f_0 M_n \left( \Gamma_n(i+1, j-1)(\overline{R}_n(i+1, j-1), \overline{R}_n(i+1, j-1)) \right) \\
+ \Gamma_n(i, j) \left( f'_j, M_n(i, j) \right) + \Gamma_n(i+1, j-1) \left( f'_{j-1}, \overline{M}_n(i+1, j-1) \right) \\
n = 1, 2, 3, \ i = 1, 2, \cdots, I, \ j = 1, 2, \cdots, J
\end{align*}
\]
In equation (4.43) the discrete function \( \Gamma_n(i,j) \) and \( M_n(i,j) \) are defined by

\[
\Gamma_n(i,j)(g_n(i,j), h_n(i,j)) = \frac{h}{2} \left[ g_n(i,0)h_n(i,j) + g_n(i,j)h_n(i,0) \right] + \sum_{k=1}^{j-1} h g_n(i,j-k)h_n(i,k)
\]

\[
M_n(i,j) = M_n(h \Gamma_n(i,j)(R_n(i,j), R_n(i,j)))
\]

\[
\overline{M}_n(i+1,j-1) = M_n(h \Gamma_n(i+1,j-1)(\overline{R}_n(i+1,j-1), \overline{R}_n(i+1,j-1)))
\]

where \( g_n(i,j), h_n(i,j) \) are arbitrary functions evaluated at the grid point \((i, j)\).

In region \( I \), i.e., for \( t < 2(L - x)/v_2 \), \( \overline{R} \) is independent of \( x \), i.e., \( R_x = 0 \) in equation (4.34). Then we can integrate (4.34) in time and use the initial condition in (4.34). Thus we have the following equation of region \( I \)

\[
A\overline{R}(t) + \overline{\delta}f(t) + Bf(t) * \overline{R}(t) + f(t) * \overline{M} = 0 \quad (4.45)
\]

Then the same discretization method will be used as in (4.43), but we use notation \( R_n(i,j) \) to indicate that \( R_n(i,j) \) is independent of \( i \) and we have the following discretized equation of (4.45)

\[
A_{nn}R_n(i,j) + b_n f_j + \sum_k P_{nk} \Gamma_k(i,j)(f_j, R_k(i,j)) + \Gamma_n(i,j)(f_j, M_n(i,j)) = 0
\]

for \( n = 1, 2, 3, j = 1, 2, 3, \ldots, J_1 \) \quad (4.46)

1. Direct Scattering Problem:

The direct scattering problem is to compute \( \overline{R}(x, t) \) in equation (4.34) for a given modified function \( f(t) \). Thus the reflection kernel at \( x = 0 \) is obtained. At
time level \( j \) equation (4.46) is a linear equation system with three equations and three unknowns \( R_{n(j)} \), \( n = 1, 2, 3 \). Thus we can obtain \( R_{n(j)} \) for \( n = 1, 2, 3, j = 1, 2, \cdots, J_1 \) in region I from bottom to top since \( R_{n(0)} \), for \( n = 1, 2, 3 \), are known. Then \( R_{n(i,j)} \) in region I are given by

\[
R_{n(i,j)} = R_{n(j)} \quad i = 0, 1, 2, \cdots, J_1 - j \\
= 0, 1, 2, \cdots, J_1, \ n = 1, 2, 3
\]  

(4.47)

From these \( R_{n(i,j)} \) of region I, by using interpolation and extrapolation method we can obtain all new \( R_{n(i,j)} \) values where \( (i,j) \) is on the grid point with respect to the mesh structure of region II (See Figure 4.2). The jump condition of \( \tilde{R} \) on \( t = 2(L - x)/v_2 \), which is given by equation (4.19), is used to pass \( \tilde{R}(x, 2(L - x)/v_2 -) \) cross the fast characteristic curve to region II. Then we have to use equation (4.43) to compute \( R_{n(i,j)} \) in region II from right to left, from bottom to top. All \( R_{n(i,j)} \) are thus computed in region I and region II. In the same fashion, \( R_{n(i,j)} \) in region III, IV and V can be computed.

2. Inverse Scattering Problem:

The inverse problem is to compute the modified function \( f(t) \) from equation (4.34) when \( R(0,t) \) is given. In region I, we know that \( R_{n(i,j)} = R(0,jh) \) for \( j = 0, 1, 2, \cdots, J_1 \). Thus we can recover \( f(t) \) from equation (4.45) or equation (4.46) by solving a system of Volterra type integral equations for \( t < 2(L - x)/v_2 \). On the other hand, from equation (4.47), we have \( R_{n(i,j)} = R_{n(j)} = R(0,jh) \) for \( (i,j) \) in region I. We can make an easy adjustment to extend these \( R_{n(i,j)} \) to the new grid structure as in regions II, III and IV.
Then $f_j'$, for $j = 0, 1, 2, \cdots, J_1$ are obtained by usual difference method. The computed $f_j'$, for $j = 0, 1, 2, \cdots, J_1$, is used in equation (4.43) to solve $R_n(i,j)$, for $n = 1, 2, 3, j = 0, 1, 2, \cdots, J_1$ from left to right, form bottom to top in the region of II, III and IV with $t \leq 2L/v_2$. Thus all these $f_j'$ and $R_n(i,j)$ in the region with $t < 2L/v_2$, $0 < x < L$ are known. If we let $i = 0$ in equation (4.43), then we can obtain $f'_{(k+1)}$ from it, since $R_n(0,k+1)$ is known. From this computed $f'_{(k+1)}$, $R_n(i,k+1)$ can be determined by solving the linear system (4.43). We continue this way until obtaining all data of $f_j$ and $R_n(i,k+1)$ in region II. The jump condition in equation (4.19) is then used to pass $R_n(i,j)$ to region II. Then again we use the same idea in equation (4.43) to solve all $R_n(0,k+1)$ in regions III and IV. We keep this kind fashion all the way to determined $f_j'$ for any given time $J = T/h$. By the relation in equation (3.3) we then can reconstruct the correction function $F(\kappa)$ in Biot's equation by doing an inverse Fourier transform.

### 4.5 A Half-Space Direct and Inverse Scattering Computations

For a half-space slab case, where $L = \infty$, we are given the modified function $f(t)$, which is given in Figure 4.7 and other material parameters. They are adopted from [18]
(1) Material modulas:

\[ R = 6.964000E + 8 \text{ [kg/(m * sec^2)]} \]
\[ Q = 6.622764E + 8 \text{ [kg/(m * sec^2)]} \]
\[ A = 6.767000E + 8 \text{ [kg/(m * sec^2)]} \]
\[ N = 1.855000E + 8 \text{ [kg/(m * sec^2)]} \]

(2) Mass densities:

\[ \rho = 2.155E + 3 \text{ [kg/m}^3\text{]} \]
\[ \rho_f = 1.000E + 3 \text{ [kg/m}^3\text{]} \]
\[ \rho_s = 2.650E + 3 \text{ [kg/m}^3\text{]} \]

(3) Others

viscosity: \( \mu = 0.001 \text{ [kg * (sec * m)}^{-1}\text{]} \)

porosity: \( \beta = 0.3 \)

permeability: \( \kappa = 10^{-12} \text{ [m}^2\text{]} \)

where \( \rho = (1 - \beta)\rho_s + \beta\rho_f \).

\[ \rho_{11} = \rho + \rho_f \beta (a - 2) \]  \(\text{(4.48)}\)
\[ \rho_{12} = \rho_f (1 - a) \beta \]  \(\text{(4.49)}\)
\[ \rho_{22} = a \beta \rho_f \]

The two wave speeds we have found for this medium are

\[ v_1 = 339.19366 \text{ m/s} \]  \(\text{(4.50)}\)
\[ v_2 = 1389.1373 \text{ m/s} \]  \(\text{(4.51)}\)
The reflection operator kernel $R$ is computed by solving a system of Volterra type integral equations, which is given by equations (4.45) and (4.46). The computed reflection kernel $R_{11}$, $R_{12}$, $R_{21}$ and $R_{22}$ are shown in Figures 4.3–4.6. They have been scaled. The reconstructed modified function is given in Figure 4.7, which is obtained by solving the Volterra equation (4.45) to obtain $f(t)$ by assuming that $R(t)$ is known. We note that for the inverse problem the given data is over-specified. There are three Volterra equations for $f(t)$ in (4.45). We found that the inverse problem is very sensitive to the computation-errors even the double precision is used. For this half-space direct and inverse scattering problems, we should note that there are no jump discontinuities in $R(t)$. This can been seen by letting $L = \infty$ in (4.20), where $q_{11}$ and $q_{22}$ are computed to be greater than zero.
Figure 4.3: Reflection operator kernel $R_{11}$

Figure 4.4: Reflection operator kernel $R_{22}$
Figure 4.5: Reflection operator kernel $R_{12}$

Figure 4.6: Reflection operator kernel $R_{21}$
Figure 4.7: Reconstruction of modified function $f(t)$
5. TRANSIENT TRANSMITTED FIELD

5.1 Representation of Transmission Operator

For the fluid-saturated porous media model we considered in the previous chapters, we look for the transmission operators that map the incident compressional waves from one side (at \( x = 0 \)) of the slab into the other side of the slab (at \( x = L \)). Thus the direct scattering problem for obtaining the transient transmitted field can be solved if we know the transmission operator. This is done by convolving the transmission kernels with any given incident waves. We begin with the scattering matrix operator differential equation (3.37), which is obtained by using the split wave equation (3.25) and the invariant imbedding method for a subsection of the slab, \((x, L)\). From equation (3.37) we have the transmission operator equation, which is coupled with the reflection operator \( \bar{R}(x,t) \)

\[
\partial_x \bar{T}(x,t) = \bar{R}(x,t)(\bar{q} - \Lambda \partial_t) + \bar{T}(x,t)\bar{q} \bar{R}(x,t) \tag{5.1}
\]

\[
\partial_x \bar{T}(x,t) = (\bar{q} - \Lambda \partial_t)\bar{T}(x,t) + \bar{R}(x,t)\bar{q} \bar{T}(x,t) \tag{5.2}
\]

\[
\partial_x \bar{R}(x,t) = \bar{T}(x,t)\bar{q} \bar{T}(x,t) . \tag{5.3}
\]

By using a variation of Duhamel's principle [16, pp. 512-515], we can prove that
the transmission operators are represented by (See Appendix A for detailed proof)

\[
\phi^+(L,t) = \tilde{T}_{11} \cdot \phi^i(t) + \tilde{T}_{12} \cdot \psi^i(t)
\]

\[
= \sum_{i=1}^{n} d_{i(11)}(x)\phi^i(t - \tau_i(x)) + \int_0^{t-(L-x)/v_2} T_{11}(x,t-s)\phi^i(s) \, ds 
\]

\[
+ \sum_{i=1}^{n} d_{i(12)}(x)\psi^i(t - \tau_i(x)) + \int_0^{t-(L-x)/v_2} T_{12}(x,t-s)\psi^i(s) \, ds 
\]

\[
\psi^+(L,t) = \tilde{T}_{21} \cdot \phi^i(t) + \tilde{T}_{22} \cdot \psi^i(t)
\]

\[
= \sum_{i=1}^{n} d_{i(21)}(x)\phi^i(t - \tau_i(x)) + \int_0^{t-(L-x)/v_2} T_{21}(x,t-s)\phi^i(s) \, ds 
\]

\[
+ \sum_{i=1}^{n} d_{i(22)}(x)\psi^i(t - \tau_i(x)) + \int_0^{t-(L-x)/v_2} T_{22}(x,t-s)\psi^i(s) \, ds 
\]

where \( T_{ij}(x,t) \) are the transmission kernels and \( d_{ij}(x) \) are the attenuation factors.

We have assumed that transmission kernel \( T(x,t) \) has jump discontinuities along \( n \) curves (characteristic curves) \( t = \tau_i(x) \), where we assumed that \( \tau_i(x) \neq 0 \) for all \( i \) and \( \tau_i(x) \neq \tau_j(x) \) for \( i \neq j \). The positive integer \( n \) will be determined for our specific model later. \( \phi^i(t) \) is the slow wave component of the incident waves and \( \psi^i(t) \) is the fast wave component of the incident waves. They impinge on the slab at the left boundary \( x \) of the imbedded slab \( (x,L) \).

Let \( \Theta^i(t) = (\phi^i(t), \psi^i(t)) \) be the incident wave vector and \( \Theta^+(L,t) = (\phi^+(L,t), \psi^+(L,t)) \) be the transmission wave vector at the left boundary \( L \). Then we write equations (5.4) and (5.5) in the following matrix form

\[
\Theta^+(L,t) = \tilde{T} \cdot \Theta^i(t)
\]

\[
= \sum_{i=1}^{n} D_i(x)\Theta^i(t - \tau_i(x)) + \int_0^{t-(L-x)/v_2} T(x,t-s)\Theta^i(s) \, ds 
\]

where \( T(x,t) \) is the transmission kernel matrix of the slab and \( D_i(x), i = 1, 2, \ldots, n, \)
are the attenuation factor matrices. They are given in the following form

\[
T(x,t) = \begin{bmatrix}
T_{11}(x,t) & T_{12}(x,t) \\
T_{21}(x,t) & T_{22}(x,t)
\end{bmatrix}
\] (5.7)

and

\[
D_i(x) = \begin{bmatrix}
d_{i(11)}(x) & d_{i(12)}(x) \\
d_{i(21)}(x) & d_{i(22)}(x)
\end{bmatrix}.
\] (5.8)

5.2 Transmission Kernel Equations and Jumps Along Characteristics

Before we derive the following transmission kernel equations, we define \( g(t) \) as an arbitrary function in \( C^1(t > 0) \) and \( g(t) = 0 \) for \( t \leq 0 \). To derive the transmission kernel equations, we have to be careful in taking any derivatives involving \( T \), since \( T \) may have possible jump discontinuities. We let \( t = \tau_i(x) \), for \( i = 1, 2, \cdots, n \), along these \( n \) curves \( T \) may suffer jump discontinuities (for detailed discussions see Appendix A).

First we derive the transmission kernel equation from the operator equation (5.1). Let operator equation (5.1) act on \( g(t) \), then the left hand side is

\[
\partial_x \bar{T} \cdot g(t) = \partial_x \left[ \sum_{i=1}^{n} D_i(x)g(t) + \int_{0}^{t-\tau_n} T(x,t-s)g(s) \, ds \right]
\]

\[
= \sum_{i=1}^{n} D_i'(x)g(t-\tau_i) - \sum_{i=1}^{n} \tau_i'(x) D_i(x)g'(t-\tau_i)
\]

\[
- \sum_{i=1}^{n} \left[ T(x,\tau_i^+) - T(x,\tau_i^-) \right] \tau_i'(x) g(t-\tau_i)
\]

\[
+ \int_{0}^{t-\tau_n} T_X(x,t-s) g(s) \, ds
\]

where \( \int_{0}^{t-\tau_n} \) is decomposed into \( \int_{0}^{t-\tau_n} = \int_{0}^{t-\tau_1^+} + \int_{t-\tau_1^-}^{t-\tau_2^+} + \cdots + \int_{t-\tau_{n-1}^-}^{t-\tau_n^+} \) and \( \tau_n = (L - x)/v_2 \). In equation (5.9), we have used \( T(x,\tau_n^-) = 0 \), which is from
causality theorems [26], since for time \( t < \tau_2(x) \) the fast wave front have not arrived at the right boundary \( x = L \) yet. To obtain the right hand side of (5.1) acting on \( g(t) \), we need to conduct the following computations

\[
\widehat{T} \alpha \cdot g(t) = \widehat{T} (\Lambda \partial_t - qf * \partial_t) \cdot g(t)
\]

\[
= \sum_{i=1}^{n} D_i(x) \Lambda g'(t - \tau_i) + \sum_{i=1}^{n} \left[ T(x, \tau_i^+) - T(x, \tau_i^-) \right] \Lambda g(t - \tau_i)
\]

\[
+ \int_{0}^{t - \tau_n} T_t(x, t - s) \Lambda g(s) ds - f(0) \sum_{i=1}^{n} D_i(x) q g(t - \tau_i)
\]

\[
- f(0) \int_{0}^{t - \tau_n} T(x, t - s) q g(s) ds - \int_{0}^{t - \tau_n} T(x, t - s) q [f' * g](s) ds
\]

\[
- \sum_{i=1}^{n} D_i(x) q \int_{0}^{t - \tau_i} f'(t - \tau_i - s) g(s) ds
\]

and

\[
\widehat{T} \beta \cdot g(t) = -\widehat{T} (q f * \partial_t) \cdot g(t)
\]

\[
= -f(0) \sum_{i=1}^{n} D_i(x) q \int_{0}^{t - \tau_i} R(x, t - \tau_i - s) g(s) ds
\]

\[
- \sum_{i=1}^{n} D_i(x) q \int_{0}^{t - \tau_i} f'(t - \tau_i - s) [R * g](s) ds
\]

\[
- f(0) \int_{0}^{t - \tau_n} T(x, t - s) q [R * g](s) ds
\]

\[
- \int_{0}^{t - \tau_n} T(x, t - s) q [f' * R * g](s) ds
\]

By inserting equations (5.9-5.11) into equation (5.1), then we obtain the following equations

\[
D_i(x) \tau_i'(x) = D_i(x) \Lambda
\]

for \( i = 1, 2, ..., n \) (5.12)

\[
D_i'(x) - f(0) D_i(x) q = \left[ T(x, \tau_i^+) - T(x, \tau_i^-) \right] (\tau_i I_{2 \times 2} - \Lambda)
\]

for \( i = 1, 2, ..., n \) (5.13)
and

\begin{equation}
T_x(x,t) = -T(x,t) + \sum_{i=1}^{n} D_i(x) q f'(t - \tau_i(x)) + \sum_{i=1}^{n} D_i(x) q R(x,t - \tau_i(x)) + T(x,t) * q R(x,t)
\end{equation}

\begin{align}
&= \left( T(x,t) q + \sum_{i=1}^{n} D_i(x) q R(x,t - \tau_i(x)) + T(x,t) * q R(x,t) \right) \\
&= \left( T(x,t) q + \sum_{i=1}^{n} D_i(x) q R(x,t - \tau_i(x)) + T(x,t) * q R(x,t) \right)
\end{align}

\begin{equation}
0 < x < L, \quad t > 0.
\end{equation}

In order to completely determine the integrodifferential equation for \( T(x,t) \), we have to determine the function form of \( D_i \) and \( \tau_i(x) \), \( i = 1, 2, \ldots, n \). From equation (5.12), we note that the attenuation factor matrices \( D_i(x) \), \( i = 1, 2, \ldots, n \), must be singular matrices for all \( t \) and \( x \), since \( \tau_i(x) \neq \tau_j(x) \) for \( i \neq j \). From equations (5.12) and equation (5.6) we have

\begin{equation}
d_{i(kl)}(x) \tau_i(x) = -d_{i(kl)}(x) v_l
\end{equation}

\begin{equation}
\sum_{i=1}^{n} d_{i(kl)}(x) = \delta_{kl}
\end{equation}

\begin{equation}
\tau_i(L) = 0
\end{equation}

for \( k, l = 1, 2 \)

for \( i = 1, 2, \ldots, n \), where

\begin{equation}
\delta_{kl} = \begin{cases} 
0 & k \neq l \\
1 & k = l
\end{cases}
\end{equation}

The two equations (5.16) and (5.17) come from the fact that \( \Theta^+(L,t) = \Theta^i(t) \) when set \( x = L \) in equation (5.6), since the media of the two sides of the slab is the same.
By solving (5.15–5.17) we find that $n = 2$ and

$$
\tau_1(x) = (L - x)/v_1 \tag{5.19}
$$

$$
\tau_2(x) = (L - x)/v_2 \tag{5.20}
$$

and

$$
d_1(11) = \text{undetermined}
$$

$$
d_1(12) = d_1(21) = d_1(22) = 0 \tag{5.21}
$$

$$
d_2(22) = \text{undetermined}
$$

$$
d_2(11) = d_2(12) = d_2(21) = 0
$$

where we have used that the fact $v_1 \neq v_2$ and our assumption $\tau_i \neq \tau_j$ for $i \neq j$. The undetermined $d_1(11)(x)$ and $d_2(22)(x)$ in (5.21) will be determined later.

Curves, $t = \tau_1(x)$ and $t = \tau_2(x)$, actually are the characteristic curves of the P.D.E. system (5.14), which can be obtained by using the method in section 4.2. Also, we see that $\tau_1(x)$ and $\tau_2(x)$ are the one way travel time for the slow and the fast waves passing through the slab that extends from $x$ to $L$. It is physically clear that the possible jump discontinuities of $T(x, t)$ are from the fast and slow wave fronts, i.e., at time $\tau_1(x)$ and $\tau_2(x)$.

If we use equation (5.21) to equation (5.13), then we have the following equation

$$
\begin{bmatrix}
  d_1'(11) & 0 \\
  0 & 0 \\
\end{bmatrix} - f(0) \begin{bmatrix}
  d_1(11)q_{11} & d_1(11)q_{12} \\
  0 & 0 \\
\end{bmatrix} = \begin{bmatrix}
  0 & (v_2^{-1} - v_1^{-1})[T_{12}]_1 \\
  0 & (v_2^{-1} - v_1^{-1})[T_{22}]_1 \\
\end{bmatrix} \tag{5.22}
$$

and

$$
\begin{bmatrix}
  0 & 0 \\
  0 & d_2'(22) \\
\end{bmatrix} - f(0) \begin{bmatrix}
  0 & 0 \\
  d_2(22)q_{21} & d_2(22)q_{22} \\
\end{bmatrix} = \begin{bmatrix}
  (v_1^{-1} - v_2^{-1})[T_{11}]_2 & 0 \\
  (v_1^{-1} - v_2^{-1})[T_{21}]_2 & 0 \\
\end{bmatrix} \tag{5.23}
$$
where \([T_{ij}]_1 = T_{ij}(x, \tau_1+) - T_{ij}(x, \tau_1-)\) and \([T_{ij}]_2 = T_{ij}(x, \tau_2+) - T_{ij}(x, \tau_2-)\) for \(i, j = 1, 2\), the jump discontinuities of \(T(x, t)\). From equation (5.16), i.e., \(D_1(L) + D_2(L) = I_{2 \times 2}\), and equations (5.22), (5.23), we have

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
\{ d'_{i(ii)}(x) = f(0)d_{i(ii)} q_{i(i)} \\
d_{i(ii)}(L) = 1 
\end{array}
\right.
\end{aligned}
\]  

(5.24)

Then the attenuation factor matrices \(D_i(x)\) are determined from (5.24). They are

\[
D_1(x) = \begin{bmatrix}
e^{-f(0)q_{11}(L-x)} & 0 \\
0 & 0 
\end{bmatrix} 
\]  

(5.25)

\[
D_2(x) = \begin{bmatrix}
0 & 0 \\
0 & e^{-f(0)q_{22}(L-x)} 
\end{bmatrix} 
\]  

(5.26)

On the other hand, we can have the jump discontinuities of \(T(x, t)\) from equations (5.22) and (5.23)

\[
\begin{aligned}
T_{11}(x, \tau_1+) - T_{11}(x, \tau_1-) &= \text{undetermined} \\
T_{12}(x, \tau_1+) - T_{12}(x, \tau_1-) &= \frac{v_1 v_2}{v_1 - v_2} q_{12} e^{-f(0)q_{11}(L-x)} \\
T_{21}(x, \tau_1+) - T_{21}(x, \tau_1-) &= \text{undetermined} \\
T_{22}(x, \tau_1+) - T_{22}(x, \tau_1-) &= 0
\end{aligned}
\]  

(5.27)

and

\[
\begin{aligned}
T_{11}(x, \tau_2+) &= 0 \\
T_{12}(x, \tau_2+) &= \text{undetermined} \\
T_{21}(x, \tau_2+) &= \frac{v_1 v_2}{v_2 - v_1} q_{21} e^{-f(0)q_{22}(L-x)} \\
T_{22}(x, \tau_2+) &= \text{undetermined} .
\end{aligned}
\]  

(5.28)
By a reciprocity relation between $T_{12}$ and $T_{21}$, which is given by equation (5.31), the undetermined jumps of $T_{21}$ at $t = \tau_1$ and $T_{12}$ at $t = \tau_2$ in equations (5.27) and (5.28) are determined (Appendix B show another way to obtain $[T_{21}]_1$ and $[T_{12}]_2$). They are

$$T_{21}(x, \tau_1^+) - T_{21}(x, \tau_1^-) = \frac{v_1 v_2}{v_1 - v_2} q_{21} f(0) e^{-f(0)q_{11}(L-x)}$$

$$T_{12}(x, \tau_2^+) = \frac{v_1 v_2}{v_2 - v_1} q_{12} f(0) e^{-f(0)q_{22}(L-x)} .$$

(5.29)

Note that by causality [26] we know that $T(x, \tau_2(x)-) = 0$, since for time $t < \tau_2(x)$ the fast wave front have not arrived at the right boundary $x = L$ yet.

From the definitions of the transmission operators in (5.4) and (5.5), we know that $\tilde{T}_{11}$ maps the slow incident wave into the slow transmitted wave and $\tilde{T}_{22}$ maps the fast incident wave into the fast transmitted wave. Equations (5.27a) and (5.28d) tell us that $T_{11}$ and $T_{22}$ have possible discontinuities at $t = (L-x)/v_1$ and $t = (L-x)/v_2$, which correspond to the slow and fast wave fronts, respectively. This is what we have expected. Also, from the transmission operator definition (5.4) and (5.5) we see that $\tilde{T}_{12}$ maps the fast incident wave into the slow transmitted wave and $\tilde{T}_{21}$ maps the slow incident wave into the fast transmitted wave. Operators $\tilde{T}_{12}$ and $\tilde{T}_{21}$ represent the coupling relation between the fast and slow waves during wave transmission process. Equations (5.27b) and (5.28c) show that both $T_{12}$ and $T_{21}$ have possible discontinuities at $t = (L-x)/v_1$ and $t = (L-x)/v_2$, the slow and fast wave fronts. The first condition in (5.28) is understandable since by causality we know that at $t = (L-x)/v_2$ the slow wave has not reached the right boundary $x = L$ yet.
Similarly, from equation (5.2) we find that (for details see Appendix B)

\[
T_x(x, t) = -\Lambda T_t(x, t) + \sum_{i=1}^{2} q D_i(x) f'(t - \tau_i(x)) + \\
\left. f(0) \left[ q T(x, t) + \sum_{i=1}^{2} R(x, t - \tau_i(x)) q D_i(x) + R(x, t) * q T(x, t) \right] \right. \\
f'(t) \left[ q T(x, t) + \sum_{i=1}^{2} R(x, t - \tau_i(x)) q D_i(x) + R(x, t) * q T(x, t) \right] + \left. \right. \\
0 < x < L, \ t > 0 .
\]

From the compatibility of equations (5.14) and (5.30), we then have the reciprocity relation between the off-diagonal components of \( T \)

\[
\frac{T_{12}(x, t)}{q_{12}} = \frac{T_{21}(x, t)}{q_{21}}
\]

for \( x \in (0, L) \) and \( t > 0 \) but \( t \neq \tau_1(x), \tau_2(x) \). This relation is similar to the reciprocity relation between \( R_{12} \) and \( R_{21} \) in equation (4.22).

We see that by applying the above reciprocity relation in equations (5.14) and (5.30) the component \( T_{12} \) of transmission matrix \( T \) satisfies

\[
p \frac{\partial}{\partial t} T_{12}(x, t) = \left[ f(0) + f(t) * V \right] + \\
q_{12} \left[ d_{1(11)} f'(t - \tau_1) - d_{2(22)} f'(t - \tau_2) \right]
\]

for \( 0 < x < L, \ t > 0 \), where \( p \) and \( V \) are given by

\[
p = \frac{1}{v_2} - \frac{1}{v_1}
\]
\[ V(R,T) = q_{12} \left[ T_{11}(x,t) - T_{22}(x,t) + T_{12}(x,t)(q_{22} - q_{11}) \right] \\
+ d_{1(11)} [q_{11} R_{12}(x,t - \tau_1) + q_{12} R_{22}(x,t - \tau_1)] \\
- d_{2(22)} [q_{12} R_{11}(x,t - \tau_2) + q_{22} R_{12}(x,t - \tau_2)] \\
+ T_{11} * (q_{11} R_{12} + q_{12} R_{22}) - T_{22} * (q_{12} R_{11} + q_{22} R_{12}) \\
+ T_{12} * (q_{22} R_{22} - q_{11} R_{11}). \]  

(5.33)

A similar equation can be found for \( T_{21} \).

Next we derive the transmission equation from (5.3). Let operator equation (5.3) act on function \( g(t) \). We come across the lengthy derivations for computing \( \tilde{T} \tilde{\beta} \tilde{T} \cdot g(t) \). This is derived in the following equations.
\[-\tilde{T}(x,t) \tilde{\beta} \tilde{T}(x,t) \cdot g(t) = \tilde{T}(x,t) q f * \partial_t \tilde{T}(x,t) \cdot g(t)\]

\[= f(0)D_1 q \left[ D_1 g(t - 2\tau_1) + D_2 g(t - \tau_1 - \tau_2) + \int_{0}^{t-\tau_1-\tau_2} T(x,t-\tau_1-s)g(s)\,ds \right] \]

\[+ f(0)D_2 q \left[ D_1 g(t - \tau_1 - \tau_2) + D_2 g(t - 2\tau_2) + \int_{0}^{t-2\tau_2} T(x,t-\tau_2-s)g(s)\,ds \right] \]

\[+ D_1 q \int_{0}^{t-\tau_1} f'(t - \tau_1 - s) \left[ D_1 g(s - \tau_1) + D_2 g(s - \tau_2) \right] \,ds \]

\[+ D_1 q \int_{0}^{t-\tau_1} f'(t - \tau_1 - s) \int_{s}^{s-\tau_2} T(x,s-u)g(u)\,du \]

\[+ D_2 q \int_{0}^{t-\tau_2} f'(t - \tau_2 - s) \int_{s}^{s-\tau_2} T(x,s-u)g(u)\,du \]

\[+ f(0) \int_{0}^{t-\tau_2} T(x,t-s)g \left[ D_1 g(s - \tau_1) + D_2 g(s - \tau_2) \right] \,ds \]

\[+ f(0) \int_{0}^{t-\tau_2} T(x,t-s) \int_{s}^{s-\tau_2} T(x,s-u)g(u)\,du \]

\[+ \int_{0}^{t-\tau_2} T(x,t-s)g \left[ f'(s) \left( D_1 g(s - \tau_1) + D_2 g(s - \tau_2) \right) \right] \,ds \]

\[+ \int_{0}^{t-\tau_2} T(x,t-s)g \left[ f'(s) \int_{s}^{s-\tau_2} T(x,s-u)g(u)\,du \right] \,ds \]

\[= f(0)D_1 q \left( D_1 g(t - 2\tau_1) + D_2 g(t - \tau_1 - \tau_2) + [T * g](t - \tau_1) \right) \]

\[+ f(0)D_2 q \left( D_1 g(t - \tau_1 - \tau_2) + D_2 g(t - 2\tau_2) + [T * g](t - \tau_2) \right) \]

\[+ D_1 q \left( D_1 [f' * g](t - 2\tau_1) + D_2 [f' * g](t - \tau_1 - \tau_2) + [f' * T * g](t - \tau_1) \right) \]

\[+ D_2 q \left( D_1 [f' * g](t - \tau_1 - \tau_2) + D_2 [f' * g](t - 2\tau_2) + [f' * T * g](t - \tau_2) \right) \]

\[+ f(0) \left( [TqD_1 * g](t - \tau_1) + [TqD_2 * g](t - \tau_2) + [Tq * T](t) \right) \]

\[+ f' \left( [TqD_1 * g](t - \tau_1) + [TqD_2 * g](t - \tau_2) + [Tq * T](t) \right) . \]
By inserting equations (5.34) and (4.8) into equation (5.3), then we obtain

\[
R_x(x, t) = -\left[ D_1(x)qD_2(x) + D_2(x)qD_1(x) \right] f'(t - \tau_1 - \tau_2)
- D_1(x)qD_1(x)f'(t - 2\tau_1) - D_2(x)qD_2(x)f'(t - 2\tau_2)
- f(0)\left[ D_1(x)qT(x, t - \tau_1) + T(x, t - \tau_1)qD_1(x) \right]
+ T(x, t - \tau_2)qD_2(x) + D_2(x)qT(x, t - \tau_2) + T*gT(x, t)
- f'(t) * \left[ D_1(x)qT(x, t - \tau_1) + T(x, t - \tau_1)qD_1(x) \right]
+ T(x, t - \tau_2)qD_2(x) + D_2(x)qT(x, t - \tau_2) + T*gT(x, t)
\]

for 0 < x < L and t > 0, and

\[
- \sum_{i=1}^{3} \left[ R(x, h_i^+) - R(x, h_i^-) \right] h_i' g(t - h_i)
= f(0)D_1 q D_1 g(t - 2\tau_1) + f(0)D_1 q D_2 g(t - \tau_1 - \tau_2)
+ f(0)D_2 q D_1 g(t - \tau_1 - \tau_2) + f(0)D_2 q D_2 g(t - 2\tau_2).
\]

From section 4.1, we know that

\[
h_1(x) = 2\tau_1(x)
\]
\[
h_2(x) = \tau_1(x) + \tau_2(x)
\]
\[
h_3(x) = 2\tau_2(x)
\]

Then we have

\[
\left[ R(x, h_1^+) - R(x, h_1^-) \right] h_1'(x) = f(0)D_1 q D_1
\]
\[
\left[ R(x, h_2^+) - R(x, h_2^-) \right] h_2'(x) = f(0)D_1 q D_2 + f(0)D_2 q D_1
\]
\[
\left[ R(x, h_3^+) - R(x, h_3^-) \right] h_3'(x) = f(0)D_2 q D_2
\]
where $D_1$ and $D_2$ are given by equation (5.25) and (5.26). By solving (5.38), then we obtain the jump discontinuities of $R(x,t)$,

$$
\begin{bmatrix}
R(x, h_1^-) - R(x, h_1^+) \\
R(x, h_2^-) - R(x, h_2^+) \\
R(x, h_3^-) - R(x, h_3^+)
\end{bmatrix} = \frac{v_1 f(0)}{2} \begin{bmatrix}
\Delta_{11} & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
$$

(5.39)

$$
\begin{bmatrix}
0 & \Delta_{12} \\
\Delta_{21} & 0
\end{bmatrix}
$$

where $\Delta_{ij}$ are defined by (4.20). This shows that the jump conditions (5.39) of $R$ are the same as that given by (4.19), which is derived from the elementary propagation of singularities arguments (see section 4.2 for details).

We can prove that by applying the reciprocity relation for $R_{12}$ and $R_{21}$, which is given by (4.22), to equation (5.35), the reciprocity relation for $T_{12}$ and $T_{21}$ can be obtained directly from the relation of off-diagonal elements on both sides of equation (5.35).

Equations (5.14) (or (5.30)) and (4.15) (or (5.35)), with the reciprocity relation in (5.31), the jump conditions in (5.27), (5.28) and the boundary condition $T(L,t) = 0$, are used to solve the other part of our direct scattering problem, i.e., to find the transient transmission kernel $T(x,t)$. 
6. CONCLUSION

Corones and Krueger's wave splitting and invariant imbedding methods [12] [15] were applied to the time domain direct and inverse scattering problems for a finite slab of fluid-saturated porous media with dispersive and dissipative properties. The classical Biot frequency domain equations [5] [9] were extended to the time domain. Our studies are completely performed in the time domain other than in the traditional frequency domain to deal with scattering problems for fluid-saturated porous media [19]. The dependent-variable transform (3.20) is the key point of wave splitting technique. It split the total field $e(x, t)$, $e(x, t)$ into two pairs (the fast and slow wave modes) of right and left going components. Especially, it simplified the boundary conditions, because fields $\phi^\pm(x, t)$ and $\psi^\pm(x, t)$ outside the slab in equation (3.18) are continuous extensions of fields inside the slab. Reflection and transmission operators $\tilde{R}$ and $\tilde{T}$ were then defined for an imbedded slab $(x, L)$. Invariant imbedding techniques were used to obtain differential equations for those scattering operators. Representations of these scattering operators were derived in Appendix A.

Chapter 4 derived a system of $R$-equations, using invariant imbedding method. The reciprocity relation and the jump discontinuity conditions for $R$ were also discussed. A numerical scheme was given for reconstructing the modified function $f(t)$ from knowledge of the reflection kernel $R(0, t)$ and the parameter matrices $q$ and
A. Numerical computations were performed for a special case with $L = \infty$, i.e., a half-space model.

Chapter 5 derived a system of integrodifferential equations (5.14), (5.30) and (5.35) for the transmission matrix operator kernel. The discontinuities of the coupling operator kernel $T_{12}$ and $T_{21}$ along the fast and slow wave fronts were obtained. The reciprocity relation for $T$ also was discussed.
APPENDIX A: REPRESENTATION OF OPERATORS

We will apply Duhamel's principle [16, pp.512-515] to show that the operators $\tilde{R}$ and $\tilde{T}$ are integral operators that are represented by equations (4.2) and (5.4–5.5) or (4.3) and (5.6). We start with the split wave equation (3.25) and discuss the initial and boundary value problem of the following system of hyperbolic partial differential equations

\begin{align*}
\Theta_+^+(x,t) - \Lambda \Theta_+^+(x,t) &= -qf \ast \Theta^+(x,t) - qf \ast \Theta^-(x,t) \\
\Theta_-^-(x,t) + \Lambda \Theta_-^-(x,t) &= qf \ast \Theta^+(x,t) + qf \ast \Theta^-(x,t) \\
\Theta^+(0,t) &= \begin{bmatrix} H(t) \\ 0 \end{bmatrix} \quad t > 0 \\
\Theta^-(L,t) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad t \geq 0 \\
\Theta^\pm(x,0) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad 0 < x < L
\end{align*}

(6.1) (6.2) (6.3) (6.4)

where

\begin{align*}
\Theta^\pm(x,t) &= \begin{bmatrix} \phi^\pm(x,t) \\ \psi^\pm(x,t) \end{bmatrix}
\end{align*}

(6.5)
and $H(t)$ is the Heaviside step function. $\Lambda$ and $q$ are $2 \times 2$ material parameter matrices, which are given by equations (3.14) and (3.28). We assume that $f(t)$ is a $C^1(t > 0)$ function.

We let $U_{1}^{\pm}(x, t)$ be the solution of (6.1) with the initial and boundary conditions in (6.2–6.4). The initial and boundary conditions in equations (6.2–6.4) mean that at point $x = 0$ and time $t = 0$ a right going slow wave impulse which increase the value $\phi^+(0,0)$ instantaneously to the value 1.

If we replace the boundary condition in equation (6.2) by

$$\Theta^+(0, t) = \begin{bmatrix} c H(t - s) \\ 0 \end{bmatrix}$$

where $c$ is a constant. Let $U_{1}^{\pm}$ be the corresponding solution with this boundary condition. From the time translation invariance and linear properties of the P.D.E. system (6.1), we then have that

$$U_{1}^{+}(x, t) = c U_{1}^{+}(x, t - s)$$

$$U_{1}^{-}(x, t) = c U_{1}^{-}(x, t - s).$$

Now we consider further the following boundary condition, which consists of a series of impulses occurring at time $t = 0, t_1, t_2, \cdots, t_n$,

$$\Theta^+(0, t) = \begin{bmatrix} \alpha(0) + \sum_{i=1}^{n+1} [\alpha(t_i) - \alpha(t_{i-1})] H(t - t_i) \\ 0 \end{bmatrix}$$

where $t = t_{n+1}$. From the boundary condition in (6.6) and its corresponding solution in equation (6.7), we know that each pulse makes the value $U_1(0, t_i)$ jump by the increment $\alpha(t_i) - \alpha(t_{i-1})$. Then the solution $\Theta_1^{\pm}(x, t)$, corresponding to this series
of impulses that is given by the boundary condition (6.8), is obtained by additively in the form

\[ \Theta_1^+(x,t) = \sum_{i=1}^{n+1} \left[ \alpha(t_i) - \alpha(t_{i-1}) \right] U_1^+(x,t - t_i) + \alpha(0) U_1^+(x,t) \]

\[ \Theta_1^-(x,t) = \sum_{i=1}^{n+1} \left[ \alpha(t_i) - \alpha(t_{i-1}) \right] U_1^-(x,t - t_i) + \alpha(0) U_1^-(x,t) \]  

(6.9)

We define a norm

\[ \|\Delta t\| = \max_{0 \leq i \leq n} |t_{i+1} - t_i| . \]

We assume that the derivative of function \( \alpha(t) \) is piecewise continuous and \( \alpha(t) = 0 \) for \( t < 0 \). Then we have that

\[ \lim_{\|\Delta t\| \to 0} \left( \sum_{i=1}^{n+1} \left[ \alpha(t_i) - \alpha(t_{i-1}) \right] H(t - t_i) + \alpha(0) H(t) \right) = \int_0^t \alpha'(s) \, ds + \alpha(0) H(t) = \alpha(t) . \]  

(6.10)

and

\[ \lim_{\|\Delta t\| \to 0} \left( \sum_{i=1}^{n+1} \left[ \alpha(t_i) - \alpha(t_{i-1}) \right] U_1^\pm(x,t - t_i) + \alpha(0) U_1^\pm(x,t) \right) = \int_0^t U_1^\pm(x,t - s) \alpha'(s) \, ds + \alpha(0) U_1^\pm(x,t) \]  

(6.11)

Let \( \|\Delta t\| \to 0 \), then equation (6.9) becomes

\[ \Theta_1^+(x,t) = \alpha(0) U_1^+(x,t) + \int_0^t U^+(x,t - s) \alpha'(s) \, ds \]

\[ \Theta_1^-(x,t) = \alpha(0) U_1^-(x,t) + \int_0^t U^-(x,t - s) \alpha'(s) \, ds \]  

(6.12)

which is the solution corresponding to the following boundary condition

\[ \Theta(0,t) = \begin{bmatrix} \alpha(t) \\ 0 \end{bmatrix} \]  

(6.13)
In the same way, if we replace the boundary condition in equation (6.2) by the following boundary condition

\[ \Theta^+(0, t) = \begin{bmatrix} 0 \\ \beta(t) \end{bmatrix} \]  

(6.14)
i.e., a right going fast wave impinges on the slab at its left boundary. Then as did in obtaining solution (6.12) we have its corresponding solution

\[ \Theta^+_2(x, t) = \beta(0) U^+_2(x, t) + \int_0^t U^+_2(x, t-s) \beta'(s) \, ds \]  

(6.15)

\[ \Theta^-_2(x, t) = \beta(0) U_2^-(x, t) + \int_0^t U_2^-(x, t-s) \beta'(s) \, ds \]

where the derivative of \( \beta \) is piecewise continuous and \( \beta(t) = 0 \) for \( t < 0 \). \( U^\pm_2 \) is the solution with the following boundary condition

\[ \Theta^+(0, t) = \begin{bmatrix} 0 \\ H(t) \end{bmatrix} \]  

(6.16)

For a general case, we consider the initial and boundary problem for the following P.D.E. system

\[
\begin{align*}
\frac{\partial \Theta^+}{\partial x}(x, t) - \Lambda \frac{\partial \Theta^+}{\partial t}(x, t) &= -qf \ast \Theta^+(x, t) - qf \ast \Theta^-(x, t) \\
\frac{\partial \Theta^-}{\partial x}(x, t) + \Lambda \frac{\partial \Theta^-}{\partial t}(x, t) &= qf \ast \Theta^+(x, t) + qf \ast \Theta^-(x, t)
\end{align*}
\]  

(6.17)

\[
\begin{align*}
\Theta^+(0, t) &= \begin{bmatrix} \alpha(t) \\ \beta(t) \end{bmatrix} \\
\Theta^-(L, t) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\Theta^+(x, 0) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{align*}
\]
where $\Theta^+(0, t)$ is a general incident wave vector (consisting of slow and fast incident waves) that impinges on the slab at $x = 0$. We can obtain its solution form the linear property of the above system and the solutions given by equations (6.12) and (6.15). The solution is

$$\Theta^+(x, t) = \alpha(0) U^+_1(x, t) + \beta(0) U^+_2(x, t)$$

(6.18)

$$+ \int_0^t U^+_1(x, t-s) \alpha'(s) ds + \int_0^t U^+_2(x, t-s) \beta'(s) ds$$

$$\Theta^-(x, t) = \alpha(0) U^-_1(x, t) + \beta(0) U^-_2(x, t)$$

(6.19)

$$+ \int_0^t U^-_1(x, t-s) \alpha'(s) ds + \int_0^t U^-_2(x, t-s) \beta'(s) ds$$

Next we will study equation (6.18) and (6.19) in order to show that they can be represented by the reflection and transmission kernels as in equations (4.3) and (5.6). First, we set $z = 0$ in equation (6.19), then we have

$$\Theta^-(0, t) = \alpha(0) U^-_1(0, t) + \beta(0) U^-_2(0, t)$$

$$+ \int_0^t [U^-_1(0, t-s) \alpha'(s) ds + U^-_2(0, t-s) \beta'(s)] ds$$

From elementary propagation of singularities argument and the initial condition, $[U^-_1(x, 0+) - U^-_1(x, 0-)] = 0$ (for all $x$), we know that $U^-_1(x, t)$ has no jump discontinuities. Same discussion can be applied to $U^-_2(x, t)$. Then

$$\Theta^-(0, t) = \alpha(0) U^-_1(0, t) + \beta(0) U^-_2(0, t)$$

$$+ \left[ U^-_1(0, t-s) \alpha(s) + U^-_2(0, t-s) \beta(s) \right]_0^t$$

$$+ \int_0^t \left[ U^-_1(t, t-s) \alpha(s) ds + U^-_2(t, t-s) \beta(s) \right] ds$$

(6.20)

$$= \int_0^t \left[ U^-_1(t, t-s) \alpha(s) + U^-_2(t, t-s) \beta(s) \right] ds$$

$$= \int_0^t R(0, L, t-s) \Theta^i(s) ds$$
where

\[
R(0, L, t) = \begin{bmatrix}
\phi_1^+(0, t) & \phi_2^+(0, t) \\
\psi_1^-(0, t) & \psi_2^-(0, t)
\end{bmatrix}
\]

\[\Theta(t) = \begin{bmatrix}
\alpha(t) \\
\beta(t)
\end{bmatrix}\]  

(6.21)

Second, we set \(x = L\) in equation (6.18) then we have

\[
\Theta^+(L, t) = \alpha(0) U_1^+(L, t) + \beta(0) U_2^+(L, t) \\
+ \int_0^t \left[ U_1^+(L, t - s) \alpha'(s) + U_2^+(L, t - s) \beta'(s) \right] ds
\]

(6.22)

By causality [26] we know that \(U_{1,2}^+(x, t) = 0\) for \(t < (L - x)/v_2\), i.e., the medium, which the wave front has not reached yet, is not affected. Then we get

\[
\int_0^t \left[ U_1^+(L, t - s) \alpha'(s) + U_2^+(L, t - s) \beta'(s) \right] ds = \int_0^{t - L/v_2} \left[ U_1^+(L, t - s) \alpha'(s) + U_2^+(L, t - s) \beta'(s) \right] ds \\
+ \int_{t - L/v_2}^t \left[ U_1^+(L, t - s) \alpha'(s) + U_2^+(L, t - s) \beta'(s) \right] ds
\]

(6.23)

Since \(U_1^+(x, t)\) and \(U_2^+(x, t)\) have jump discontinuities at \(t = 0\) at the left boundary, then by applying elementary propagation of singularities argument we know that \(U_1^+(x, t)\) and \(U_2^+(x, t)\) have jump discontinuities along curtain curves (the characteristic curves of (6.1)). Then
\[
\int_{0}^{t-L/v_2} \left[ U_1^+(L, t-s) \alpha'(s) + U_2^+(L, t-s) \beta'(s) \right] ds
\]
\[
= \sum_{i=1}^{n} \int_{t-T_i}^{t-T_i+} \left[ U_1^+(L, t-s) \alpha'(s) + U_2^+(L, t-s) \beta'(s) \right] ds
\]
\[
= \sum_{i=1}^{n} \left[ U_1^+(L, \tau_i+) - U_1^+(L, \tau_i-) \right] \alpha(t - \tau_i) + \sum_{i=1}^{n} \left[ U_2^+(L, \tau_i+) - U_2^+(L, \tau_i-) \right] \beta(t - \tau_i)
\]
\[
- \left[ \alpha(0) U_1^+(L, t) - \beta(0) U_2^+(L, t) \right]
\]
\[
+ \int_{0}^{t-L/v_2} \left[ U_1^+(L, t-s) \alpha(s) + U_2^+(L, t-s) \beta(s) \right] ds
\]
(6.24)

where we define \( \tau_0 = t, \tau_n = L/v_2 \). \( U_{1,2}^\pm \) have jump discontinuities at \( n \) points, \( t = \tau_i, i = 1, 2, \ldots, n. \) From equations (6.22), (6.23) and (6.24), then

\[
\Theta^+(L, t) = \sum_{i=1}^{n} \left[ U_1^+(L, \tau_i+) - U_1^+(L, \tau_i-) \right] \alpha(t - \tau_i) + \sum_{i=1}^{n} \left[ U_2^+(L, \tau_i+) - U_2^+(L, \tau_i-) \right] \beta(t - \tau_i)
\]
\[
+ \int_{0}^{t-L/v_2} \left[ U_1^+(L, t-s) \alpha(s) + U_2^+(L, t-s) \beta(s) \right] ds
\]
(6.25)

\[
= \sum_{i=1}^{n} D_i(0, L) \Theta^i(t - \tau_i) + \int_{0}^{t-L/v_2} T(0, L, t-s) \Theta^i(s) ds
\]

where

\[
T(0, L, t) = \begin{bmatrix}
\phi_1^+(L, t) & \phi_2^+(L, t) \\
\psi_1^+(L, t) & \psi_2^+(L, t)
\end{bmatrix}
\]
(6.26)

and

\[
D_i(0, L, t) = \begin{bmatrix}
\phi_1^+(L, \tau_i+) - \phi_1^+(L, \tau_i-) & \phi_2^+(L, \tau_i+) - \phi_2^+(L, \tau_i-) \\
\psi_1^+(L, \tau_i+) - \psi_1^+(L, \tau_i-) & \psi_2^+(L, \tau_i+) - \psi_2^+(L, \tau_i-)
\end{bmatrix}
\]
(6.27)
From the invariant imbedding argument, we have to consider the reflection and transmission operators for the subsection of the medium, which extends from $x$ to $L$. (Here $x$ stands for the left boundary, not the interior point that used in equation (6.1).). Therefore replacing 0 by $x$, we then can obtain the representations of the reflection and transmission operators, $\tilde{R}$ and $\tilde{T}$, for a subsection of the slab. They are

$$
\Theta^+(x, t) = \tilde{R} \cdot \Theta^i(t) = \int_0^t R(x, t - s) \Theta^i(s) \, ds
$$

$$
\Theta^-(L, t) = \tilde{T} \cdot \Theta^i(t) = \sum_{i=1}^{n} D_i(x) \Theta^i(t - \tau_i) + \int_0^{t-(L-x)/\omega_2} T(x, t - s) \Theta^i(s) \, ds
$$

where we have omitted the $L$ variable in $R$, $T$ and $D$ since $L$ is fixed in our problem. Integer $n$ and attenuation factor matrix $D_i(x)$ have been determined in chapter 5. We also should note that $\tau_i$ is also a function of the left boundary, i.e., a function of $x$. 


APPENDIX B: TRANSMISSION EQUATION DERIVATION

In what follows, we will give derivations of the transmission kernel integrodifferential equation, which are given by equations (5.30). Let $g(t)$ as an arbitrary function in $C^1(t > 0)$ and $g(t) = 0$ for $t \leq 0$. From the scattering matrix operator differential equation (3.37), we obtain the transmission operator equations, which are given by the following equations To derive equation (5.30) from equation (5.2), we give the following main computation steps

$$\tilde{T} T \cdot g(t) = (qf \ast \partial_t - \Lambda \tilde{T}) \tilde{T} \cdot g(t)$$

$$= f(0) q \sum_{i=1}^{2} D_i(x) g(t - \tau_i) + f(0) \sum_{i=1}^{2} D_i(x) \int_{0}^{t} f'(t - s) g(s - \tau_i) ds$$

$$+ f(0) q \int_{0}^{t - \tau_n} T(x, t - s) g(s) ds + qf'(t) \int_{0}^{t - \tau_n} T(x, t - s) g(s) ds \quad (6.29)$$

$$- \Lambda \sum_{i=1}^{2} D_i(x) g'(t - \tau_i) - \Lambda \sum_{i=1}^{2} \left[ T(x, \tau_i +) - T(x, \tau_i -) \right] g(t - \tau_i)$$

$$- \Lambda \int_{0}^{t - \tau_n} T_t(x, t - s) g(s) ds$$
and
\[ R \beta \tilde{T} \cdot g(t) = -R(qf * \partial_t) \tilde{T} \cdot g(t) \]
\[ = -f(0)q \sum_{i=1}^{2} D_i(x) \int_{0}^{t} R(x, t - s) g(s - \tau_i) \, ds \]
\[ - q \sum_{i=1}^{2} D_i(x) \int_{0}^{t} R(x, t - s) \, ds \int_{0}^{s} f(s - u) g(u - \tau_i) \, du \]
\[ - f(0) \int_{0}^{t} R(x, t - s) \, ds \int_{0}^{s} g(u) \, du \]
\[ - R(x, t) * f'(t) * \left[ \int_{0}^{t - \tau_n} qT(x, t - s) g(s) \, ds \right] . \]

Insert equations (5.9) into equation (5.2) then we obtain equation (5.30) and we also have the following equations

\[ D_i(x) \tau'_i(x) = \Lambda D_i(x), \quad \text{for } i = 1, 2 \]  \hfill (6.31)

\[ D'_i(x) - f(0)qD_i(x) = (\tau'_i I_{2 \times 2} - \Lambda) [T(x, \tau_i^+) - T(x, \tau_i^-)], \quad \text{for } i = 1, 2 . \]  \hfill (6.32)

i.e.,
\[
\begin{bmatrix}
  d'_1(11) & 0 \\
  0 & 0
\end{bmatrix}
- f(0)
\begin{bmatrix}
  d_{1(11)} q_{11} & 0 \\
  0 & d_{1(11)} q_{21}
\end{bmatrix}
= 
\begin{bmatrix}
  0 & 0 \\
  (v^2_2 - v^2_1)[T_{21} 1] & (v^2_2 - v^2_1)[T_{22} 1]
\end{bmatrix}
\]  \hfill (6.33)

and
\[
\begin{bmatrix}
  0 & 0 \\
  d'_{2(22)} & 0
\end{bmatrix}
- f(0)
\begin{bmatrix}
  0 & 0 \\
  d_{2(22)} q_{21} & d_{2(22)} q_{22}
\end{bmatrix}
= 
\begin{bmatrix}
  (v^2_1 - v^2_2)[T_{11} 2] & (v^2_1 - v^2_2)[T_{12} 2] \\
  0 & 0
\end{bmatrix}
\]  \hfill (6.34)

Similarly, as did to obtain equations (5.19), (5.20), (5.27) and (5.28), the above
Equations (6.31) and (6.32) are solved by

\[ T_{11}(x, \tau_1^+) - T_{11}(x, \tau_1^-) = \text{undetermined} \]
\[ T_{12}(x, \tau_1^+) - T_{12}(x, \tau_1^-) = \text{undetermined} \]  
\[ T_{21}(x, \tau_1^+) - T_{21}(x, \tau_1^-) = \frac{v_1 v_2}{v_1 - v_2} q_{21} f(0) e^{-f(0) q_{11}(L-x)} \]
\[ T_{22}(x, \tau_1^+) - T_{22}(x, \tau_1^-) = 0 \]  

and

\[ T_{11}(x, \tau_2^+) = 0 \]
\[ T_{12}(x, \tau_2^+) = \frac{v_1 v_2}{v_2 - v_1} q_{12} f(0) e^{-f(0) q_{22}(L-x)} \]
\[ T_{21}(x, \tau_2^+) = \text{undetermined} \]
\[ T_{22}(x, \tau_2^+) = \text{undetermined} . \]

Equation (6.31) can be solved and its solutions are the same as given by equation (5.19), (5.20), (5.25), (5.26). Equations (6.35) and (6.36) actually give the same results as in equation (5.29), which is obtained by the reciprocity relation.
Here we will verify that the two fundamental wave speeds $v_1$ and $v_2$ given by equation (3.15) are the same as that obtained by [5], which are obtained in the frequency domain. As in equation (3.15), we know that

$$v = \sqrt{\frac{2}{\pm (a_{22}^2 - 2a_{11}a_{22} + 4a_{21}a_{12} + a_{11}^2)^{1/2} + a_{11} + a_{22}}}$$

(6.37)

where $a_{ij}'s$ are given by equation (3.7). Now we use the same definitions as in [5],

$$H = P + R + 2Q$$

(6.38)

$$\rho = \rho_{11} + 2\rho_{12} + \rho_{22}$$

(6.39)

$$\sigma_{11} = P/H$$

(6.40)

$$\sigma_{22} = R/H$$

$$\sigma_{12} = Q/H$$

and

$$\gamma_{11} = \rho_{11}/\rho$$

(6.41)

$$\gamma_{22} = \rho_{22}/\rho$$

$$\gamma_{12} = \rho_{12}/\rho$$

$$v_c = \sqrt{\frac{H}{\rho}}.$$

(6.42)
Then we find
\begin{align*}
a_{11} &= A(R\rho_{11} - Q\rho_{12}) = (Q\rho_{12} - R\rho_{11})/(Q^2 - PR) \\
   &= v_c^{-2}(\sigma_{12}\gamma_{12} - \sigma_{22}\gamma_{11})/(\sigma_{12}^2 - \sigma_{11}\sigma_{22}).
\end{align*}
(6.43)

In the same way,
\begin{align*}
a_{21} &= v_c^{-2}(\sigma_{12}\gamma_{11} - \sigma_{11}\gamma_{12})/(\sigma_{12}^2 - \sigma_{11}\sigma_{22}) \\
a_{12} &= v_c^{-2}(\sigma_{12}\gamma_{22} - \sigma_{22}\gamma_{12})/(\sigma_{12}^2 - \sigma_{11}\sigma_{22}) \\
a_{22} &= v_c^{-2}(\sigma_{12}\gamma_{12} - \sigma_{11}\gamma_{22})/(\sigma_{12}^2 - \sigma_{11}\sigma_{22}).
\end{align*}
(6.44)

As in [5], we define \( z \) as
\[
z = \frac{v_c^2}{v_s^2}.
\] (6.45)

Then from equation (6.37), we see that \( z \) satisfies
\[
z^2 - v_c^2(a_{11} + a_{22})z + v_c^4(a_{11}a_{22} - 4a_{12}a_{21}) = 0.
\] (6.46)

Then insert equation (6.43) and equation (6.44) into equation (6.46), we find
\[
(\sigma_{12}^2 - \sigma_{11}\sigma_{12})z^2-(2\sigma_{12}\gamma_{12} - \sigma_{22}\gamma_{11} - \sigma_{11}\gamma_{22})z+(\gamma_{12}^2 - \gamma_{11}\gamma_{22}) = 0.
\] (6.47)

The above equation (6.47) is the same as that given by [5]. Then we see that the fast and slow wave speeds we obtained agree with that in [5].
REFERENCES


SUMMARY

This dissertation has studied two direct and inverse scattering problems in the time domain, using Corones and Krueger's wave splitting and invariant imbedding techniques. The first part of the dissertation derived composite transmission operators, $\mathcal{T}_r$ and $\mathcal{T}_l$, which map the source function $J(t)$ to the transmitted electric field. The direct scattering problem was solved by computing the composite transmission operator kernels $T_l$ and $T_r$. The inverse problem for the internal source $J(t)$ was solved based on the solutions of delay Volterra integral equations (5.5) and (5.6). A frequency domain analytic example and two time domain direct scattering and inverse source numerical examples were given. Also, a Green's operator was defined. This operator maps the electric current source $J(t)$ at $z = 0$ to the internal field at an arbitrary observation point inside a homogeneous dispersive slab. A system of linear partial integrodifferential equations with various initial, boundary and jump discontinuities were derived for the Green's operator kernels. Relations between $G^\pm$ and $R_r$, $R_l$, $T_r$ and $T_l$ were also identified.

The second part of this dissertation applied Corones and Krueger's wave splitting and invariant imbedding methods to direct and inverse scattering problems for a finite slab of fluid-saturated porous media with dispersive and dissipative properties. The classical Biot frequency domain equations were extended to the time domain. Our
studies are completely performed in the time domain other than in the traditional frequency domain to deal with scattering problems for fluid-saturated porous media. The dependent-variable transform (3.20) is the key point of wave splitting technique. It split the total field \( e(x,t) \), \( e(x,t) \) into two pairs (the fast and slow wave modes) of right and left going components. Especially, it simplified the boundary conditions, because fields \( \phi_{\pm}(x,t) \) and \( \psi_{\pm}(x,t) \) outside the slab in equation (3.18) are continuous extensions of fields inside the slab. Reflection and transmission operators \( R \) and \( T \) were then defined for an imbedded slab \((x,L)\). Invariant imbedding techniques were used to obtain differential equations for those scattering operators. Representations of these scattering operators were derived in Appendix A. A system of reflection and transmission kernel integrodifferential equations were then derived. The reciprocity relations and the jump discontinuity conditions for \( R \) and \( T \) were also discussed. A numerical scheme was presented for reconstructing the modefied function \( f(t) \) from knowledge of the reflection kernel \( R(0,t) \) and the parameter matrices \( q \) and \( \Lambda \). Numerical computations were performed for a special case with \( L = \infty \), i.e., a half-space model.