2-2013

# Computing links and accessing arcs

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# COMPUTING LINKS AND ACCESSING ARCS

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Abstract. Sufficient conditions are given for the computation of an arc that accesses a point on the boundary of an open subset of the plane from a point within the set. The existence of a not-computably-accessible but computable point on a computably compact arc is also demonstrated.

## 1. Introduction

Let $\mathbb{C}$ denote the complex plane. We consider the following situation: we are given an arc $A \subseteq \mathbb{C}$, a point $\zeta_1$ on $A$, and a point $\zeta_0$ that does not lie on $A$. By the term *arc* we mean a continuous embedding of $[0, 1]$ into $\mathbb{C}$. Such an embedding will then be referred to as a *parameterization* of the arc. We suppose that we wish to compute a parameterization of an arc $B$ from $\zeta_0$ to $\zeta_1$ that contains no point of $A$ other than $\zeta_1$. However, we also assume $B$ must be confined to some open set. The gist of our results is that covering information about $A$ *i.e.* the ability to plot $A$ on a computer screen with arbitrarily good resolution) is not sufficient for the computation of such an arc $B$, but that covering information combined with local connectivity information is.

Such an arc $B$ is called an *accessing arc*. More generally, when $\zeta_0$ and $\zeta_1$ are points in the plane, and when $X$ is a subset of the plane, we say that an arc $A$ from $\zeta_0$ to $\zeta_1$ *links* $\zeta_0$ *to* $\zeta_1$ *via* $X$ if all of its intermediate points belong to $X$. If $\zeta_0$ is a point in an open set $U \subseteq \mathbb{C}$ and if $\zeta_1$ is a point on the boundary of $U$, then we say that an arc $A$ *accesses* $\zeta_1$ *from* $\zeta_0$ *via* $U$ if it links $\zeta_0$ to $\zeta_1$ via $U$.

Our examination of accessing arcs is motivated in part by their relevance to boundary extensions of conformal maps as in [8], [14], and [11], and to the narrow escape problem in the theory of Brownian motion. The computation of links between points on the boundary of a domain is the first step in domain decomposition methods such as the Schwarz alternating method [7], [5]. In addition to these connections, the problem of computing accessing arcs seems to be an intrinsically interesting problem that admits many intriguing variations such as higher-dimensional versions, computable metric spaces, and rectifiable or computably rectifiable accessing arcs.

Our investigations first lead us to consider the situation in Figure 1 in which we have an open disk $D$, an arc $A$, a point $\zeta_1$ in $D \cap A$, and a point $\zeta_0$ in $D - A$. From our computability questions a purely topological question naturally arises. Namely, how close does $\zeta_1$ have to be to $\zeta_0$ in order for there to be an arc that accesses $\zeta_1$ from $\zeta_0$ via $D - A$? An answer is given in Theorem 5.3. Moreover, the bound in this theorem can be computed from sufficient information about $D$, $\zeta_0$, $\zeta_1$, and $A$. We then show that when such an accessing arc exists, one of its parameterizations

2010 *Mathematics Subject Classification.* 03F60, 30C20, 30C30, 30C85.
*Key words and phrases.* computable topology, effective local connectivity.

can be computed from sufficient information about $D$, $\zeta_0$, $\zeta_1$, and $A$. In particular, local connectivity information about $A$ is used.

Effective versions of local connectivity are considered in [1], [4] and [6]. In [1], local connectivity information arises naturally in the consideration of the computational relationships between a function and its graph. In [4], it is used in the computation of space-filling curves, and in [11] it is used in the computation of boundary extensions of Riemann maps.

In Theorem 4.1, we show that mere covering information about the arc $A$ is insufficient for the computation of accessing arcs.

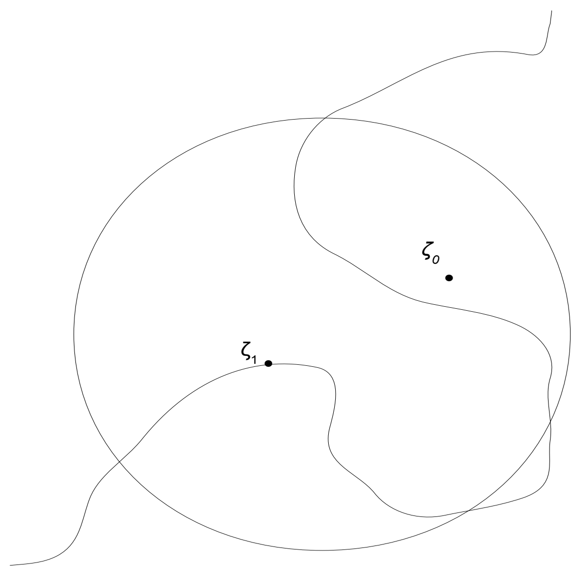


FIGURE 1.

The paper is organized as follows. Section 2 covers background and preliminaries from topology. Section 3 summarizes the prerequisites from computable analysis. Section 4 consists of the proof of Theorem 4.1. Section 5 presents the positive results on computing links.

## 2. Background from topology

When $X, Y \subseteq \mathbb{C}$, let

$$d(X,Y) = \inf\{|z - w| \ : \ z \in X \ \wedge \ w \in Y\}.$$

Let $d(p, X) = d(\{p\}, X)$ when $p \in \mathbb{C}$ and $X \subseteq \mathbb{C}$.

When $f, g : [0, 1] \to \mathbb{C}$ are bounded, let

$$\| f - g \|_\infty = \sup\{|f(t) - g(t)| \ : \ t \in [0, 1]\}.$$

$\| \|_\infty$ is called the *sup norm*.

Let $f :\subseteq A \to B$ denote that $f$ is a function whose domain is contained in $A$ and whose range is contained in $B$.

When $f :\subseteq \mathbb{C} \to \mathbb{C}$, a *modulus of continuity* for $f$ is a function $m : \mathbb{N} \to \mathbb{N}$ such that $|f(z) - f(w)| < 2^{-k}$ whenever $|z - w| \leq 2^{-m(k)}$ and $z, w \in \text{dom}(f)$. If a function has a modulus of continuity, then it follows that it has an increasing modulus of continuity. A function has a modulus of continuity if and only if it is uniformly continuous.

Let $D_r(z_0)$ denote the open disk whose radius is $r$ and whose center is $z_0$. Let $\mathbb{D} = D_1(0)$.

A *curve* is a set $C \subseteq \mathbb{C}$ for which there is a continuous function $f : [0, 1] \to \mathbb{C}$ whose range is $C$. The function $f$ is called a *parameterization* of the curve $C$. The term *parametrization* thus has two different though related uses. With respect to curves, it refers to a continuous surjection. But, with respect to arcs it always refers to a continuous bijection. We will follow the usual custom of identifying a curve and its parameterizations except when computability issues are of concern in which case the distinction is necessary by the results in [12].

With respect to a particular parameterization $f$ of a curve $C$, if $p = f(0)$ and $q = f(1)$, then the curve $C$ is said to be a curve *from $p$ to $q$*.

A *cut point* of a set $X \subseteq \mathbb{C}$ is a point $p \in X$ with the property that $X - \{p\}$ is disconnected. The following useful characterization of arcs is an immediate consequence of Theorem 2-27 of [10].

**Proposition 2.1.** *A set $A \subseteq \mathbb{C}$ is an arc if and only if it is compact, connected, and has just two non-cut points.*

It follows that if $f$ is a parameterization of an arc $A$, then $f(0)$ and $f(1)$ are the non-cut points of $A$.

Let $f : [0, 1] \to \mathbb{C}$ be a curve for which there exist numbers

$$0 = t_0 < t_1 < \ldots < t_k = 1$$

and points $v_0, v_1, \ldots, v_k \in \mathbb{C}$ such that

$$f(x) \quad = \quad \frac{x - t_j}{t_{j+1} - t_j}(v_{j+1} - v_j) + v_j \tag{2.1}$$

whenever $x \in [t_j, t_{j+1}]$. $f$ is called a *polygonal curve*. The points $v_0, \ldots, v_k$ are called the *vertexes* of $f$. We will call the points $v_1, \ldots, v_{k-1}$ the *intermediate vertexes* of $f$. A *rational polygonal curve* is a polygonal curve whose vertexes are all rational. We note that we may take $t_j$ to be $\frac{j}{k}$ in Equation 2.1.

The proof of the following is an easy modification of the proof of Theorem 3.5 of [10].

**Lemma 2.2.** *Suppose $U$ is a domain, and that $p, q$ are distinct points of $U$. Then, there is a polygonal arc $P$ from $p$ to $q$ that is contained in $U$ and whose intermediate vertexes are rational. Furthermore, if $\epsilon > 0$, then $P$ can be chosen so that the length of each of its line segments is smaller than $\epsilon$.*

A *Jordan curve* is a curve that has a parameterization $f$ that is injective except that $f(0) = f(1)$. When $J$ is a Jordan curve, let $\mathrm{Int}(J)$ denote its interior, and let $\mathrm{Ext}(J)$ denote its exterior.

The proof of the following is an easy exercise, but it is useful enough to warrant stating it as a proposition.

**Proposition 2.3.** *If $C \subseteq \mathbb{C}$ is connected, and if $X \subseteq \overline{C}$, then $C \cup X$ is connected.*

## 3. Preliminaries from computable analysis

Our work is based on the Type Two Effectivity foundation for computable analysis which is described in great detail in [15]. We give an informal summary here of the points pertinent to this paper. We begin with the naming systems we shall use. Intuitively, a name of an object is a list of approximations to that object that is sufficient to completely identify it.

A *name* of a point $z \in \mathbb{C}$ is a list of all the rational rectangles that contain $z$.

A *name* of a continuous function $f : [0, 1] \to \mathbb{C}$ is a list of rational polygonal curves $P_0, P_1, \ldots$ such that $\| P_t - P_s \|_\infty \leq 2^{-t}$ whenever $s \geq t$ and $f = \lim_{t\to\infty} P_t$. Here, the limit is taken with respect to the supremum norm. Such a sequence of curves is called a *strongly Cauchy sequence*.

A *plot* of a compact set $X \subseteq \mathbb{C}$ is a finite set of rational rectangles that each contain a point of $X$ and whose union contains $X$. A *name* of a compact $K \subseteq \mathbb{C}$ is a list of all plots of $K$. These names are called $\kappa_{mc}$-names in [15]. They provide precisely the right amount of information necessary to plot the set on a computer screen at any desired resolution.

However, whenever we speak of a name of an arc $A$, we always mean a name of a parameterization of $A$. And, whenever we speak of a name of a Jordan curve $\gamma$, we always mean a name of a parameterization of $\gamma$, $f$, with the property that $f(s) = f(t)$ only when $s = t$ or $s, t \in \{0, 1\}$.

Once we establish a naming system for a space, an object of that space is called *computable* if it has a computable name.

A sentence of the form

> "From a name of a $p_1 \in S_1$, a name of a $p_2 \in S_2$, ..., and a name of a $p_k \in S_k$, it is possible to uniformly compute a name of a $p_{k+1} \in S_{k+1}$ such that $R(p_1, \ldots, p_k, p_{k+1})$."

is shorthand for the following: there is a Turing machine $M$ with $k$ input tapes and a one-way output tape with the property that whenever a name of a $p_j \in S_j$ is written on the $j$-th input tape for each $j \in \{1, \ldots, k\}$ and $M$ is allowed to run indefinitely, a name of a $p_{k+1} \in S_{k+1}$ such that $R(p_1, \ldots, p_{k+1})$ holds is written on the output tape.

A *CIK* ("connected *im kleinen*") function for a set $X \subseteq \mathbb{C}$ is a function $f : \mathbb{N} \to \mathbb{N}$ with the property that whenever $k \in \mathbb{N}$ and $z_0 \in X$, there is a connected set $C \subseteq D_{2^{-k}}(z_0) \cap X$ that contains $D_{2^{-f(k)}}(z_0) \cap X$. Related notions are considered in [6], [2], [12], and [4].

A *ULAC* ("uniformly local arcwise connectivity") function for a set $X \subseteq \mathbb{C}$ is a function $f : \mathbb{N} \to \mathbb{N}$ with the property that whenever $k \in \mathbb{N}$ and $z_0, z_1$ are distinct points of $X$ such that $|z_0 - z_1| \leq 2^{-f(k)}$, there is an arc $A \subseteq X$ from $z_0$ to $z_1$ whose diameter is smaller than $2^{-k}$.

We will need the following two theorems which follow from the results in [6].

**Theorem 3.1.** *From a name of a compact and connected $C \subseteq \mathbb{C}$, a CIK function for $C$, and names of distinct $\zeta_0, \zeta_1 \in C$, it is possible to compute a name of an arc $A \subseteq C$ from $\zeta_0$ to $\zeta_1$.*

**Theorem 3.2.** ((1)) *From a name of an arc $A \subseteq \mathbb{C}$, it is possible to uniformly compute a name of $A$ as a compact set as well as a CIK function for $A$.*
((2)) *From a name of an arc $A \subseteq \mathbb{C}$ as a compact set and a CIK function for $A$, it is possible to uniformly compute a name of $A$.*

**Theorem 3.3.** ((1)) *Every ULAC function is a CIK function.*
((2)) *It is possible to uniformly compute, from a name of a compact set $X \subseteq C$ and a CIK function for $X$, a ULAC function for $X$.*

## 4. The insufficiency of plottability

**Theorem 4.1.** *The origin belongs to an arc $A$ from $-1$ to $1$ that is computable as a compact set and which has the property that $C \cap (A - \{0\}) \neq \emptyset$ whenever $C$ is a computable curve from $-i$ to $0$.*

*Proof.* We use a diagonalization argument. We build $A$ by stages $A_0, A_1, \ldots$. Each $A_t$ is a polygonal arc with all angles right that goes through 0.

Let $S_e = (-2^{-(e+1)}, 2^{-(e+1)})^2$.

Let $\{C_{e,t}\}_{e \in \mathbb{N}, t < k_e}$ be an effective enumeration of all possibly finite, computable, and strongly Cauchy sequences of rational polygonal curves. If $k_e = \omega$, then let $C_e = \lim_t C_{e,t}$. If $1 \leq k_e < \omega$, then let $C_e = C_{e,k_e - 1}$. Otherwise, let $C_e = \emptyset$.

For each $e$, let $R_e$ be the requirement

$$R_e : k_e = \omega \ \wedge \ C_e(1) = 0 \ \wedge \ C_e(0) \neq 0 \ \Rightarrow \ \exists t \ C_e(t) \in A - \{0\}.$$

**Stage 0:** Let $A_0 = [-1, 1] \times \{0\}$. No requirement acts at stage 0.

**Stage $\mathbf{t + 1}$:** Let us say that $R_e$ *requires attention at stage* $t + 1$ if after $t$ steps of computation it can be determined that there are rational numbers $0 < t_0 < t_1 < 1$ such that

- $C_e[0, t_0] \cap \overline{S_e} = \emptyset$,
- $C_e[t_0, t_1] \cap S_e \neq \emptyset$,
- $C_e[t_1, 1] \subseteq S_e$,
- $d(C_e[t_0, t_1], A_t) > 0$, and
- $R_e$ has not acted at any previous stage.

If no $R_e$ requires attention at stage $t + 1$, then go on to the next stage. Otherwise, let $e$ be the least number such that $R_e$ requires attention at stage $t+1$. We say that $R_e$ *acts at stage* $t + 1$. Compute $k \in \mathbb{N}$ such that $k \geq t$, and $2^{-k} < d(C_e[t_0, t_1], A_t)$. Compute $p_1, p_2 \in (A_t - \overline{S_e}) \cap \bigcap_{e' < e} S_{e'}$ such that 0 is between $p_1$ and $p_2$ on $A_t$ and the intersection of $S_e$ with the subarc of $A_t$ from $p_1$ to $p_2$ has exactly one connected component.

Let $q_j$ be a point on $A_t$ between $p_j$ and 0 such that the subarc of $A_t$ from $p_j$ to $q_j$ lies outside $\overline{S_e}$. Let $B$ denote the subarc of $A_t$ from $q_1$ to $q_2$. We create two parallel copies of $B$, $B_1$ and $B_2$, such that $B$ lies between them and

$$B_1 \cup B_2 \subseteq \{z \in \mathbb{C} \ : \ d(z, B) < 2^{-k}\}.$$

We also construct $B_1$ and $B_2$ so that they contain no point of $A_t$ and so that $B_j \cap S_e$ has only one component for $j = 1, 2$. Let $p_{i,j}$ be the endpoint of $B_j$ closest to $p_i$.

We form $A_{t+1}$ from $A_t$ as follows. We first remove the subarc of $A_t$ from $q_1$ to $p_1$. We then add a right angle polygonal arc from $p_1$ to $p_{1,2}$ and the arc $B_2$. We then remove the subarc from $q_2$ to $p_2$. We add a right angle polygonal arc from $p_{2,2}$ to $q_2$. We then add a right angle polygonal arc from $q_1$ to $p_{1,1}$ and the arc $B_1$. We then add a right angle polygonal arc from $p_{2,1}$ to $p_2$.

Thus, $S_e - A_{t+1}$ has two more connected components than $S_e - A_t$. One of these connected components is bounded by $B_1$, $B$, and the line segments along the sides of $S_e$ from $B_1$ to $B$. The other is bounded by $B_2$, $B$, and the line segments along the sides of $S_e$ from $B_2$ to $B$. Thus, 0 is a boundary point of each of these components. However, by the choice of $k$, if $k_e = \omega$, then $C_e$ can not enter either of these components without crossing either $B_1$ or $B_2$. If a requirement $R_{e'}$ with $e' < e$ acts at a later stage, its action will further split $B$, $B_1$, and $B_2$, but this will not make things any better for $C_e$. If a requirement $R_{e'}$ with $e' > e$ acts at a later stage, then $B$ will be further divided, but the situation for $C_e$ will remain the same. Thus, $R_e$ is satisfied if it ever acts. On the other hand, if $C_e$ is a curve from $-i$ to 0 that contains no point of $A$ but 0, then $R_e$ must eventually act. So, every requirement is satisfied.

It now follows that each requirement is satisfied and that $A =_{df} \lim_t A_t$, where the limit is taken with respect to the Hausdorff metric, is computable as a compact set. The only non-cut points of $A$ are $-1$ and 1. Thus, $A$ is an arc. □

In [9], an arc is constructed that is computable *as a curve* but not as an arc. That is, it has the property that it is the range of a computable function on $[0, 1]$, but is not the range of any computable injective function on $[0, 1]$. Thus, Theorem 4.1 is in fact stronger than the assertion that there is no accessing arc.

## 5. Computing links

We begin with two results which are purely topological but will drive our constructions later.

**Proposition 5.1.** *Suppose $\gamma$ is a Jordan curve and that $A \subseteq \overline{\mathrm{Int}(\gamma)}$ is an arc such that at most one endpoint of $A$ belongs to $\gamma$. Then, $\mathrm{Int}(\gamma) - A$ is connected.*

*Proof.* By the Carathéodory Theorem (see, *e.g.* Chapter I of [7]), we can assume $\gamma = \partial\mathbb{D}$. Let $p, q \in \mathbb{D} - A$. We show there is an arc from $p$ to $q$ in $\mathbb{D} - A$. By Theorem 4.5 of [13], $\mathbb{C} - A$ is connected. So, by Lemma 2.2, it is also arcwise connected. Let $B$ be an arc in $\mathbb{C} - A$ from $p$ to $q$. If $B \subseteq \mathbb{D}$, there is nothing left to prove. Suppose $B \not\subseteq \mathbb{D}$. There is a point $p_1 \in B \cap \partial\mathbb{D}$ such that the subarc of $B$ from $p$ to $p_1$ intersects $\partial\mathbb{D}$ only at $p_1$. There is a point $q_1 \in B \cap \partial\mathbb{D}$ such that the subarc of $B$ from $q$ to $q_1$ intersects $\partial\mathbb{D}$ only at $q_1$. Hence, $q_1$ is not between $p$ and $p_1$ on $B$. So, either $p_1 = q_1$ or $q_1$ is between $p_1$ and $q$ on $B$. Let $B_1$ denote the subarc of $B$ from $p$ to $p_1$. Let $B_2$ denote the subarc of $B$ from $q$ to $q_1$. Since $A$ is compact, it follows that there is a point $p_1' \in B_1$ and a point $q_1' \in B_2$ such that $|p_1'| = |q_1'|$ and such that one of the circular arcs from $p_1'$ to $q_1'$ that is concentric with $\mathbb{D}$ contains no point of $A$. For, otherwise, each subarc of $\partial\mathbb{D}$ from $p_1$ to $q_1$ contains a point of $A$. Since $p_1, q_1 \notin A$, these points would be distinct- a contradiction. It then follows that there is an arc from $p$ to $q$ in $\mathbb{D} - A$. □

**Proposition 5.2.** *Let $D$ be an open disk, and let $A$ be an arc. Let $C$ be a connected component of $D - A$. Let $p \in A \cap \partial C \cap D$, and suppose $q \in A \cap D - \partial C$. Then, the subarc of $A$ from $p$ to $q$ intersects the boundary of $D$.*

*Proof.* Let $B$ be the subarc of $A$ from $p$ to $q$. By way of contradiction, suppose $B$ contains no point of the boundary of $D$. Hence, since $p, q \in D$, $B \subseteq D$.

Since $D$ is open, there are points $p_1', q_1' \in A$ be such that the subarc of $A$ from $p_1'$ to $q_1'$ is contained in $D$, $p$ is between $p_1'$ and $q$ on $A$, and $q$ is between $p$ and $q_1'$ on $A$. By Theorem 3-18 of [10], there are points $p_1$ and $q_1$ on $A$ and points $p_2$, $q_2$ in $D - A$ such that $p_1$ is between $p_1'$ and $p$ on $A$, $q_1$ is between $q$ and $q_1'$ on $A$, $\overline{p_2p_1} \cap A = \{p_1\}$[1], and $\overline{q_2q_1} \cap A = \{q_1\}$. Let $B_1$ be the subarc of $A$ from $p_1$ to $q_1$. By Proposition 5.1, $D - B_1$ is connected.

By Lemma 2.2, there is a polygonal arc $P \subseteq D - B_1$ from $p_2$ to $q_2$. It follows that there is an arc $\sigma \subseteq P \cup \overline{p_2p_1} \cup \overline{q_2q_1}$ from $p_1$ to $q_1$. (Namely, follow $\overline{p_1p_2}$ until $P$ is first reached, then follow $P$ until $\overline{q_2q_1}$ is first reached after which $\overline{q_2q_1}$ is followed until $q_1$ is reached.) Hence, $\sigma \cap B_1 = \{p_1, q_1\}$. Thus, $J =_{df} B_1 \cup \sigma$ is a Jordan curve.

We first consider the case where there are points of $C \cap \mathrm{Ext}(J)$ arbitrarily close to $p$. Let $f$ be a conformal map of $D_1 =_{df} D - \overline{\mathrm{Int}(J)}$ onto an annulus $G =_{df} \{z \mid r_1 < |z| < r_2\}$. By Theorem 15.3.4 of [3], $f$ extends to a homeomorphism of $\overline{D_1}$ with $\overline{G}$; let $f$ denote this extension as well. We can assume $f$ maps $J$ onto the inner circle of $G$. It follows that $f(p), f(q) \in f[B_1] \subseteq \partial D_{r_1}(0)$. Let $f(p) = r_1 e^{i\theta_1}$, and let $f(q) = r_1 e^{i\theta_2}$. Without loss of generality, suppose $0 < \theta_1 < \theta_2 < 2\pi$. We claim there is an $R > r_1$ and an $\epsilon > 0$ such that

$$\{re^{i\theta} \mid \theta_1 - \epsilon < \theta < \theta_2 + \epsilon \ \wedge \ r_1 < r < R\} - f[A]$$

is connected. For, otherwise, there are points of $f[A - B_1]$ that are arbitrarily close to $f[B]$. This entails that $B \cap (\overline{A - B_1}) \neq \emptyset$ which violates the assumption that $A$ is an arc. Since $C \cap \mathrm{Ext}(J)$ contains points arbitrarily close to $p$, it now follows that $q$ is a boundary point of $C$.

If there are points of $C \cap \mathrm{Int}(J)$ arbitrarily close to $p$, then we proceed similarly except we first conformally map $\mathrm{Int}(J)$ onto $\mathbb{D}$.

Suppose by way of contradiction that neither of these cases holds. Then, there is a positive number $\epsilon$ such that $D_\epsilon(p)$ contains no point of $C \cap \mathrm{Ext}(J)$ nor any point of $C \cap \mathrm{Int}(J)$. Let $\epsilon_1$ be a positive number that is smaller than $\epsilon$ and that has the property that $D_{\epsilon_1}(p) \cap \sigma = \emptyset$. Let $w$ belong to $D_{\epsilon_1}(p) \cap C$. Thus, $w \in J$. Hence, $w \in B_1 \subseteq A$; this is a contradiction since $C \subseteq D - A$. □

The following answers the first question raised in the introduction.

**Theorem 5.3.** *Suppose $D$ is an open disk, $A$ is an arc with ULAC function $g$, and $\zeta_0 \in A \cap D$. Suppose $\zeta_1 \in D - A$ is such that $|\zeta_0 - \zeta_1| < 2^{-g(k)}$ where $k \in \mathbb{N}$ is such that $2^{-g(k)} + 2^{-k} \leq \max\{d(\zeta_0, \partial D), d(\zeta_1, \partial D)\}$. Then, $\zeta_0$ is a boundary point of the connected component of $\zeta_1$ in $D - A$.*

*Proof.* Let $l = \overline{\zeta_1\zeta_0}$. If $l \cap A = \{\zeta_0\}$, then there is nothing left to prove. So, suppose $l \cap A \neq \{\zeta_0\}$. Let $p$ be the point in $l \cap A$ that is closest to $\zeta_1$. Let $C$ be the connected component of $\zeta_1$ in $D - A$. Hence, $p \in \partial C$. Let $A_1$ be the subarc of $A$ from $p$ to $\zeta_0$. Since $|p - \zeta_0| < 2^{-g(k)}$, the diameter of $A_1$ is smaller than $2^{-k}$.

[1]Here, and elsewhere expressions of the form $\overline{zw}$ refer to the line segment from $z$ to $w$, not to the conjugate of $zw$.

We claim that $A_1 \subseteq D$. For, suppose otherwise, and let $q \in \partial D \cap A_1$. Hence, $|\zeta_0 - q| < 2^{-k}$. Thus, $d(\zeta_0, \partial D) < 2^{-k} < 2^{-g(k)} + 2^{-k}$. At the same time,

$$\begin{array}{rcl} |\zeta_1 - q| & \leq & |p - \zeta_1| + |p - q| \\ & < & 2^{-g(k)} + 2^{-k}. \end{array}$$

Hence, $d(\zeta_1, \partial D) < 2^{-g(k)} + 2^{-k}$. This is a contradiction since $2^{-g(k)} + 2^{-k} \leq \max\{d(\zeta_0, \partial D), d(\zeta_1, \partial D)\}$. Hence, $A_1 \subseteq D$.

It now follows from Proposition 5.2 that $\zeta_0$ is a boundary point of $C$. □

We now turn to the problem of computing accessing arcs.

**Theorem 5.4.** *From a name of an arc $A$, a point $z_0 \in \mathbb{D} - A$, and a name of a point $\zeta_0 \in A \cap \mathbb{D}$ that is a boundary point of the connected component of $z_0$ in $\mathbb{D} - A$, it is possible to uniformly compute a name of an arc $Q$ that links $z_0$ to $\zeta_0$ via $\mathbb{D} - A$.*

*Proof.* Compute an increasing ULAC function for $A$, $g$. Compute $s_0 \in \mathbb{N}$ such that $D_{2^{-s_0+2}}(\zeta_0) \subseteq \mathbb{D}$ and so that $2^{-s_0+2} < |z_0 - \zeta_0|$.

Since $\zeta_0$ is a boundary point of the connected component of $q_0$ in $\mathbb{D} - A$, there is a rational point $e_0$ in this component such that $|e_0 - \zeta_0| < 2^{-g(s_0)}$. It follows from Theorem 3-2 of [10] that this component is open. Hence, there is a polygonal arc $P_0$ from $z_0$ to $e_0$ contained in $\mathbb{D} - A$. It follows from Lemma 2.2 that such a point $e_0$ and such an arc $P_0$ can be discovered by a search procedure. Namely, we search for distinct rational points $q_1, \ldots, q_k \in \mathbb{D} - A$ that satisfy the following conditions.

((1)) $q_j \neq z_0$ when $j \in \{1, \ldots, k\}$.
((2)) $|q_k - \zeta_0| < 2^{-g(s_0)}$.
((3)) $\overline{z_0 q_1} \cap \overline{q_1 q_2} = \{q_1\}$.
((4)) $\overline{z_0 q_1} \cap \overline{q_j q_{j+1}} = \emptyset$ when $1 < j < k$.
((5)) $\overline{q_j q_{j+1}} \cap \overline{q_{j+1} q_{j+2}} = \{q_{j+1}\}$ when $1 \leq j < k - 1$.
((6)) $\overline{q_j q_{j+1}} \cap \overline{q_m q_{m+1}} = \emptyset$ when $m > j + 1$.

Condition (3) can be checked by checking that $\min\{d(z_0, \overline{q_1 q_2}), d(q_2, \overline{z_0 q_1})\} > 0$. By Lemma 2.2, we can also choose $q_1$ so that $|z_0 - q_1| < |z_0 - \zeta_0| - 2^{-s_0+2}$. Thus, $\overline{z_0 q_1}$ contains no point of the closed disk with center $\zeta_0$ and radius $2^{-s_0+2}$.

Now, by way of induction, suppose $|e_t - \zeta_0| < 2^{-g(s_t)}$, $s_t \geq t, s_0$. Let $\epsilon_t = 2^{-g(s_t)} + 2^{-s_t}$. We first note that

$$D_{\epsilon_t}(e_t) \subseteq D_{2^{-s_t+2}}(\zeta_0).$$

Since $s_t \geq s_0$, and since $D_{2^{-s_0+2}}(\zeta_0) \subseteq \mathbb{D}$, it follows that $D_{\epsilon_t}(e_t) \subseteq \mathbb{D}$.

Compute $s_{t+1} > \max\{s_t, t+1\}$ such that $d(\zeta_0, \bigcup_{s \leq t} P_s) > 2^{-s_{t+1}+2}$. It follows from Theorem 5.3 that $\zeta_0$ is a boundary point of the connected component of $e_t$ in $D_{\epsilon_t}(e_t) - A$. Hence, there is a rational point $e_{t+1}$ that belongs to this connected component such that $|e_{t+1} - \zeta_0| < 2^{-g(s_{t+1})}$. Since this component is open, there is a rational polygonal arc $P_{t+1}$ from $e_t$ to $e_{t+1}$ such that $P_{t+1} \subseteq D_{\epsilon_t}(e_t) - A$. It follows from Lemma 2.2 that such a point $e_{t+1}$ and such an arc $P_{t+1}$ can be discovered through a search procedure. Note that $P_{t+1} \subseteq D_{2^{-s_t+2}}(\zeta_0)$.

Note that by construction, $P_1$ contains no point of $\overline{z_0 q_1}$. Therefore, for each $j \in \mathbb{N}$ we can compute the least $t_j$ such that $P_j(t_j)$ belongs to $P_{j+1}$. Note that $t_j$ and $P_j(t_j)$ are rational. By construction, $z_0 \neq P_0(t_0)$ and $P_j(t_j) \neq P_{j+1}(t_{j+1})$. Let $Q_0$ be the subarc of $P_0$ from $z_0$ to $P_0(t_0)$. Let $Q_{j+1}$ be the sub arc of $P_{j+1}$ from

$P_j(t_j)$ to $P_{j+1}(t_{j+1})$. Define $Q(1)$ to be $\zeta_0$. When $\frac{j}{j+1} \leq t \leq \frac{j+1}{j+2}$, define $Q(t)$ to be $Q_j(s)$ where

$$s = \frac{t - \frac{j}{j+1}}{\frac{j+1}{j+2} - \frac{j}{j+1}}.$$

It follows that $Q$ can be uniformly computed from the given data. By construction, $Q \cap A = \{\zeta_0\}$. □

Finally, we turn to the problem of computing links between points on the boundary of a connected open set. We provide an answer when the boundary is locally arc-like. For example, when the boundary is a union of disjoint Jordan curves.

**Theorem 5.5.** *From a name of an open and connected $D \subseteq \mathbb{C}$, names of distinct points $\zeta_0, \zeta_1 \in \partial D$, arcs $B_0, B_1$, and a rational number $r > 0$ such that $D_r(\zeta_j) \cap \partial D \subseteq B_j$, it is possible to compute an arc $A$ that links $\zeta_0$ to $\zeta_1$ via $D$.*

*Proof.* Without loss of generality, we can assume $D_r(\zeta_0) \cap D_r(\zeta_1) = \emptyset$. Compute an increasing ULAC function for $B_j$, $g_j$. Compute a number $k \in \mathbb{N}$ such that $2^{-k+1} \leq r = d(\zeta_j, \partial D_r(\zeta_j))$. For each $j$, compute a rational point $\xi_j \in D - B_j$ such that $|\xi_j - \zeta_j| < 2^{-g_j(k)}$. By Theorem 5.3, $\zeta_j$ is a boundary point of the connected component of $\xi_j$ in $D_r(\zeta_j) - B_j$. Therefore, by Theorem 5.4, it is possible to uniformly compute from the given data an arc $A_j \subseteq D_r(\zeta_j)$ from $\xi_j$ to $\zeta_j$ such that $A_j \cap B_j = \{\zeta_j\}$. Hence, $A_j \cap \partial D = \{\zeta_j\}$. Furthermore, $A_1 \cap A_2 = \emptyset$. By Lemma 2.2, we can compute a rational polygonal arc $P \subseteq D$ from $\xi_1$ to $\xi_2$ from the given data. It may be that $P$ has one or more points in common with $B_1$ besides $\xi_1$, and it may have one or more points in common with $B_2$ besides $\xi_2$. However, by using the techniques in the proof of Theorem 5.4, we can cull an arc $A$ from $B_1 \cup P \cup B_2$ as required. □

## Acknowledgement

I thank the referee for his valuable comments. I also thank Jack Lutz for helpful and stimulating conversations on these topics. I also thank my wife Susan for support.

## References

1. V. Brattka, *Effective borel measurability and reducibility of functions*, Mathematical Logic Quarterly **51** (2005), no. 1, 19 – 44.
2. Vasco Brattka, *Plottable real number functions and the computable graph theorem*, SIAM J. Comput. **38** (2008), no. 1, 303–328.
3. J.B. Conway, *Functions of one complex variable II*, Graduate Texts in Mathematics, vol. 159, Springer-Verlag, 1995.
4. P.J. Couch, B.D. Daniel, and T.H. McNicholl, *Computing space-filling curves*, Theory of Computing Systems **50** (2012), no. 2, 370–386.
5. R. Courant and D. Hilbert, *Methods of mathematical physics. Vol. II*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1989, Partial differential equations, Reprint of the 1962 original, A Wiley-Interscience Publication.
6. D. Daniel and T.H. McNicholl, *Effective local connectivity properties*, Theory of Computing Systems **50** (2012), no. 4, 621 – 640.
7. J. B. Garnett and D. E. Marshall, *Harmonic measure*, New Mathematical Monographs, vol. 2, Cambridge University Press, Cambridge, 2005.
8. G. M. Golusin, *Geometric theory of functions of a complex variable*, American Mathematical Society, 1969.

9. X. Gu, J.H. Lutz, and E. Mayordomo, *Curves that must be retraced*, Information and Computation **209** (2011), 992 – 1006.
10. John G. Hocking and Gail S. Young, *Topology*, second ed., Dover Publications Inc., New York, 1988.
11. T.H. McNicholl, *An effective Carathéodory theorem*, Theory of Computing Systems **50** (2012), no. 4, 579 – 588.
12. J. Miller, *Effectiveness for embedded spheres and balls*, CCA 2002, Computability and Complexity in Analysis (V. Brattka, M. Schröder, and K. Weihrauch, eds.), vol. 66, Electronic Notes in Computer Science, no. 1, Elsevier, 2002, pp. 127 – 138.
13. Edwin E. Moise, *Geometric topology in dimensions* 2 *and* 3, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, Vol. 47.
14. Zeev Nehari, *Conformal mapping*, McGraw-Hill Book Co., Inc., New York, Toronto, London, 1952.
15. Klaus Weihrauch, *Computable analysis*, Texts in Theoretical Computer Science. An EATCS Series, Springer-Verlag, Berlin, 2000.

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