1980

A Note on the Asymptotic Expansions of Eigenvalues

James Murdock
Iowa State University, jmurdock@iastate.edu

Clark Robinson
Northwestern University

SIAM J. MATH. ANAL.
Vol. 11, No. 3, May 1980

0036-1410/80/1103-0006 $01.00/0

A NOTE ON THE ASYMPTOTIC EXPANSION OF EIGENVALUES*

JAMES MURDOCK† AND CLARK ROBINSON‡

Abstract. Under certain conditions k-term asymptotic expansions of the eigenvalues of a matrix can be deduced from a k-term asymptotic expansion of the matrix.

Suppose $L_\varepsilon \sim L_0 + \varepsilon L_1 + \varepsilon^2 L_2 + \cdots$ is an asymptotic expansion of a matrix function of a small parameter ε, and it is desired to find a few terms of the expansion of the eigenvalues. Does it suffice to take k terms in the expansion of L_ε to obtain k terms in the expansion of the eigenvalues? The example

$$L_\varepsilon = \begin{bmatrix} \varepsilon^k & \varepsilon^{k-1} \\ \delta\varepsilon^{k+1} & -\varepsilon^k \end{bmatrix}$$

which has eigenvalues $\pm\varepsilon^k\sqrt{1+\delta}$, shows that this is not always the case. The following theorem gives a sufficient condition for this to be true.

THEOREM. *Let L_ε and N_ε be continuous real or complex matrix functions of ε defined for $\varepsilon \geqq 0$, and let $M_\varepsilon = L_\varepsilon + \varepsilon^{k+1}N_\varepsilon$. Suppose there exists a matrix C_ε defined in some interval $0 \leqq \varepsilon < \varepsilon_0$, continuous in ε and nonsingular, such that $C_\varepsilon^{-1}L_\varepsilon C_\varepsilon = D_\varepsilon =$ $\operatorname{diag}(\lambda_1(\varepsilon), \cdots, \lambda_n(\varepsilon))$. Suppose further that each pair of eigenvalues $\lambda_i(\varepsilon), \lambda_j(\varepsilon)$ satisfies either $\lambda_i(\varepsilon) = \lambda_j(\varepsilon) + O(\varepsilon^{k+1})$ or $|\lambda_i(\varepsilon) - \lambda_j(\varepsilon)| \geqq c\varepsilon^k$ for some $c > 0$ (this condition is satisfied automatically if each eigenvalue $\lambda_i(\varepsilon)$ is a C^{k+1} function of ε). Then M_ε has n eigenvalues of the form*

$$\nu_i(\varepsilon) = \lambda_i(\varepsilon) + \varepsilon^{k+1}\sigma_i(\varepsilon)$$

for $i = 1, \cdots, n$.

Remarks. The hypotheses are satisfied for $L_\varepsilon = L_0 + \varepsilon L_1 + \cdots + \varepsilon^k L_k$ if L_0 has distinct eigenvalues, or if $L_0 = I$ and L_1 has distinct eigenvalues. The referee has informed us that according to a theorem of Rellich, the hypotheses are also satisfied if $L_0, \cdots L_k$ are Hermitian; see [4, p. 376]. In the example preceding the theorem, C_ε exists for $\varepsilon > 0$ but either becomes unbounded or singular as $\varepsilon \to 0$. Thus it is necessary to insist on the continuity and nonsingularity of C_ε at $\varepsilon = 0$ even if L_0 is already diagonal.

The proof is based on a degree argument of Levinson [2], previously exploited by Coppel and Howe [1]. We first obtained this theorem in connection with our work on asymptotic expansions in dynamical systems ([3]). Although we eventually used a different argument there, we thought this result might have independent interest.

Proof. The eigenvalues of L_ε may be partitioned into equivalence classes, λ_i and λ_j being equivalent if $\lambda_i(\varepsilon) = \lambda_j(\varepsilon) + O(\varepsilon^{k+1})$. By re-numbering the eigenvalues and permuting the columns of C_ε, we may assume that $\lambda_1, \cdots, \lambda_p$ are equivalent and that none of these are equivalent to $\lambda_{p+1}, \cdots, \lambda_n$. We shall show the existence of p eigenvalues of the form $\nu_i(\varepsilon) = \lambda_i(\varepsilon) + \varepsilon^{k+1}\sigma_i(\varepsilon)$, $i = 1, \cdots, p$. The existence of n such eigenvalues follows by repeating the argument with different equivalence classes of eigenvalues placed first.

Let $\lambda(\varepsilon) = \lambda_1(\varepsilon)$ and observe that for $i = 1, \cdots, p$ we have $\lambda_i(\varepsilon) =$ $\lambda(\varepsilon) + \varepsilon^{k+1}\phi_i(\varepsilon)$, with $\phi_i(\varepsilon)$ continuous, hence $\lambda_i(\varepsilon) = \lambda(\varepsilon) + \varepsilon^{k+1}\phi_i(0) + o(\varepsilon^{k+1})$. Let

* Received by the editors February 28, 1979, and in revised form August 13, 1979.

† Department of Mathematics, Iowa State University, Ames, Iowa 50011.
‡ Department of Mathematics, Northwestern University, Evanston, Illinois 60201.

$\Lambda(\varepsilon)=\lambda(\varepsilon)I_p$, where I_p is the $p\times p$ identity matrix, and let $\Phi=\operatorname{diag}(\phi_1(0),\cdots,\phi_p(0))$. Then

$$D_\varepsilon=\left[\begin{array}{c|c}\Lambda_\varepsilon+\varepsilon^{k+1}\Phi+o(\varepsilon^{k+1}) & 0\\ \hline 0 & \hat{D}_\varepsilon\end{array}\right]$$

where $\hat{D}_\varepsilon=\operatorname{diag}(\lambda_{p+1}(\varepsilon),\cdots,\lambda_n(\varepsilon))$. Now

$$C_\varepsilon^{-1}M_\varepsilon C_\varepsilon=D_\varepsilon+\varepsilon^{k+1}C_0^{-1}N_0C_0+o(\varepsilon^{k+1}),$$

and we write

$$C_0^{-1}N_0C_0=B=\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}$$

where B_{11} is a $p\times p$ block. Now ν is an eigenvalue of M_ε if and only if it is an eigenvalue of $C_\varepsilon^{-1}M_\varepsilon C_\varepsilon$, hence if and only if

$$\det\left[\begin{array}{c|c}\Lambda_\varepsilon+\varepsilon^{k+1}(\Phi+B_{11})+o(\varepsilon^{k+1})-\nu I_p & \varepsilon^{k+1}B_{12}+o(\varepsilon^{k+1})\\ \hline \varepsilon^{k+1}B_{21}+o(\varepsilon^{k+1}) & \hat{D}_\varepsilon+\varepsilon^{k+1}B_{22}+o(\varepsilon^{k+1})-\nu I_{n-p}\end{array}\right]=0.$$

This equation has the form $f(\varepsilon,\nu)=0$, to be solved for $\nu=\nu(\varepsilon)$. Make the ε-dependent change of variables $\nu\leftrightarrow\sigma$ defined by $\nu=\lambda(\varepsilon)+\varepsilon^{k+1}\sigma$; this will yield an equation $g(\varepsilon,\sigma)=0$ which we now determine. First note that $\Lambda_\varepsilon-\nu I_p=-\varepsilon^{k+1}\sigma I_p$. From the manner of partitioning the λ_i we see that there exist constants $c>0$, $\varepsilon_0>0$ such that for each $i>p$,

$$\frac{|\lambda_i(\varepsilon)-\lambda(\varepsilon)|}{\varepsilon^k}\geqq c\quad\text{for } 0<\varepsilon<\varepsilon_0.$$

Hence $\hat{D}_\varepsilon-\nu I_{n-p}=\varepsilon^kT_\varepsilon-\varepsilon^{k+1}\sigma I_{n-p}$ where T_ε is diagonal with each diagonal element bounded away from zero as $\varepsilon\to 0$. Inserting these relations in our determinant and canceling ε^{k+1} from the top p rows and ε^k from the remainder we find

$$\begin{aligned}0&=\det\left[\begin{array}{c|c}(\Phi+B_{11})-\sigma I_p & B_{12}\\ \hline 0 & T_\varepsilon\end{array}\right]+o(1)\\ &=\det[(\Phi+B_{11})-\sigma I_p]\det T_\varepsilon+o(1).\end{aligned}$$

Since $\det T_\varepsilon$ is bounded away from zero this reduces to

$$\det[(\Phi+B_{11})-\sigma I_p]+o(1)=0.$$

When $\varepsilon=0$ there exist p roots for σ by the fundamental theorem of algebra; these persist for small ε by Rouché's theorem. Q.E.D.

REFERENCES

[1] W. A. Coppel and A. Howe, *Dynamical instability of linear canonical systems*, Austr. Math. Soc. J., 7 (1967), pp. 247–251.

[2] N. Levinson, *The stability of linear, real, periodic self-adjoint systems of differential equations*, J. Math. Anal. Appl., 6 (1963), pp. 473–482.

[3] J. A. Murdock and C. Robinson, *Qualitative dynamics from asymptotic expansions: local theory*, J. Differential Equations, submitted.

[4] F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Frederick Ungar, New York, 1955.