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Keywords

Preference profiles, plurality voting, Borda Count, Condorcet cycles

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1 Introduction

A recent literature uses a vector decomposition approach to explain why several standard voting procedures produce different social choice outcomes on a given preference profile. A preference profile or profile for short, is defined here as a distribution of a fixed set of voters (or judges) across all possible rank orders of a set of candidates (or alternatives). With n-candidates and $n!$ different ways of ranking them, a profile is thus a $n!$ dimensional vector with each element denoting either a number of voters or a proportion of an electorate supporting a specific rank order of the candidates.[1] The approach decomposes a given profile into families of *component* profiles that lie in orthogonal subspaces. A family of component profiles lying in a specific subspace produce a *decisive* outcome - one in which not all the candidates in a field are tied - under some procedures but a complete tie under others. Thus, some standard procedures disagree on the social choice outcomes on a given profile, if the latter happens to be a sum of component profiles of significant weights but lying in different orthogonal subspaces.[2]

A component profile represents a *hypothetical* or virtual electorate (not an actual subset of voters) whose weight measures the extent to which the distribution of voters in the real electorate or given profile is explained by the distribution of the virtual voters. Besides explaining disagreements across certain procedures, the decomposition approach provides additional useful insight if the preference pattern of the virtual voters in a component profile captures some feature(s) of the collective psyche of a real electorate, in some socioeconomic context. In the past, the approach has inspired investigation into Condorcet cycles which are known to be responsible for disagreement between the Borda count and various types of procedures based on pairwise scores across candidate pairs (Zwicker (1991), Saari (1999, 2000 a,b)) and studies of how likely such cycles are in reality (Gehrlein (2001)).

The present paper uses the profile decomposition approach to characterize a family of component profiles responsible for conflicts in the induced social rank orders under the plurality and the Borda count methods. Both these procedures are widely used because of their many advantages and despite known weaknesses. Plurality is adopted because of its ability to reward merit but also often criticized for its potential to elect a deeply unpopular candidate.[3] The Borda count on the other hand is criticized for rewarding mediocrity but is also known to be a method with the fewest known anomalies compared to other methods.[4]

[1]Since only numbers and not individual identities of voters are included in this description, the vector is sometimes described as a *voting situation* as opposed to a *profile*. This distinction does not matter for us and hence the terms are here used interchangeably.

[2]The literature, often described as geometric voting theory, owes much of its development to the pioneering works of Saari and several co-authors. The papers that are most relevant for us are Saari (1999, 2000 a,b). The basics of the theory are also discussed in Hodge and Klima (2005), Balinski and Laraki (2010) and Nurmi (1996, 2002).

[3]Other useful properties of plurality that have been adduced, include, maximizing the number of voters who end up with their most preferred choice (Bossert and Suzumura, 2017), efficiency (Yeh, 2008,) etc.

[4]The Borda count is shown to be the method most likely to respect the Condorcet principle (Newenhizen (1992)), the

We build on an existing result which identifies a subspace of component profiles on which several standard sum-scoring procedures, plurality and the Borda count in particular, and some procedures based on pairwise comparison of candidates, such as the Copeland, Black and Kemeny methods, lead to the same outcome. Saari (1999, 2000a, 2000b) who is credited with this result describes these component profiles as *basic* profiles. These above-mentioned methods induce the same social rank order of the candidates on a given profile, if the latter happens to lie in the subspace spanned by basic profiles. When such is not the case and some component profiles of the given profile lie in subspaces orthogonal to basic profiles, these methods disagree. Furthermore, the rank order induced by the Borda count is always shown to follow the weights of the basic profile components only, in any given profile.

Our first main result characterizes a family of component profiles which span a subspace orthogonal to the span of the basic profiles and induced a rank order of the candidates under plurality which is decisive. In other words, not all the candidates are tied under plurality on these profiles. Under the Borda count on the other hand, these profiles produce a complete tie. Plurality and the Borda count are thus shown to disagree if these component profiles are present in a given profile and have larger weights relative to those of the basic profiles. The two methods concur if the weights of the basic profiles are relatively larger.

The above family of profiles is characterized by a most interesting preference structure. In each profile in this family, a specific set of rank orders of the candidates and the reverse set of these rank orders are supported by the same number of voters. In particular, each such profile is characterized by a specific candidate ranked first and last by an equal number of voters. In other words, the same candidate is "loved" (ranked first) and "hated" (ranked last) by an equal number of the virtual voters. We describe these component profiles as *reverse profiles* in light of this algebraic structure.[5] Plurality induces a rank order in which a specific candidate is strictly first ranked on each such profile (as it counts only the first choices). The Borda count, which has the *cancellation property* (Young, (1974)), produces a complete tie across all candidates on a reverse profile.

The (hypothetical) electorate of a reverse profile demonstrates an extreme like or dislike for a specific candidate or alternative. In other words, it is polarized around a specific candidate or alternative. Many recent studies provide evidence of such polarization around political candidates or on specific socioeconomic issues, in the US and elsewhere in the world. These include Baldassarri and Geiman, (2008); Boxell, Gentzkow and Shapiro (2017); Mason (2015); Abramowitz and Saunders (2008); Fiorina and Abrams

unique method to satisfy a modified version of the independence of irrelevant alternatives condition (Maskin (2020)) and is non-manipulable on a restricted domain (Barbie, Puppe and Tasnadi (2006). Other advantages are also discussed in Baharad and Nitzan (2002), Dellis (2009).

[5]Condorcet cycles and other symmetric profiles of rank orders similar to reverse profiles are well known in Engineering disciplines where a different terminology is used for them. See Zwicker (1991) and other works by Saari which are not cited here.

(2008); Fiorina, Abrams and Pope (2008); Krasa and Polborn (2014), among others.

The paper contributes to the substantial and varied literature on measures of divisiveness within a society, broadly defined as the presence of sizable groups who feel alienated from or hostile to each other. Established measures of polarization define groups on the basis of a *one-dimensional* population characteristic or attribute which may be a cardinal variable such as ”income” or an ordinal one such as a specific ”ethnicity”. Among other advantages, this uni-dimensionality allows some type of a distance function - either Euclidean or discrete - to measure alienation across groups (Esteban and Ray (1994), Duclos, Esteban and Ray (2004), Montalvo and Reynal-Querol (2008))[6].

The above criterion for defining groups fails if the population characteristic is not uni-dimensional or as in our case, the difference between one population member and another is a difference in a preference rank order. In political elections for high public offices for example, a voter usually assesses multiple candidates along multiple criteria, such as policies on the economy, trade, immigration, environment etc. The output of this assessment is usually a rank order of the candidates, either mental or sometimes explicitly stated on a ballot. These rank orders are also the information available for defining group-membership or constructing measures of inter-group alienation. With no generally accepted notion of a ”distance” between any two such orders, the established measures cannot be applied. Our paper proposes measures constructed from the weights of the reverse profiles as an alternative that is based on distribution of rank orders. Moreover, we show that these measures also indicate the extent to which a polarized electorate contributes to the selection of a winner of a race, when plurality or a plurality runoff is the procedure of choice.

When $n = 3$ and the profile space is 6-dimensional, two distinct basic profiles, two distinct reverse profiles and one Condorcet profile are sufficient to obtain a complete decomposition of any profile (see Section 3). When $n \geq 4$ however, the number of orthogonal basis vectors required to span the $n!$ dimensional profile space but not falling into either of these three classes, grows *factorially*. Not only is a complete profile decomposition computationally intensive and perhaps even impossible for arbitrary n, but previous attempts have shown that a majority of the component profiles outside of these three classes represent preference patterns that are unlikely in reality.[7]

A second result of the paper, Theorem 2, demonstrates that a complete decomposition of a profile is unnecessary in order to obtain the weights of the reverse profile components. It shows that the difference in

[6] See also Haimanko, Le Breton and Weber (2007) and Testa (2012) which use single peaked preferences on uni-dimensional policy spaces to study effects of polarization on secession, public policy etc.

[7] Saari (2000b) constructs a complete orthogonal basis for the case $n = 4$, to explain all possible voting paradoxes under standard procedures, including paradoxes that occur on subsets of candidates. The exercise involved constructing a set of 24 orthogonal basis vectors. The paper concludes that such an exercise although *possible* for $n = 5$ and $n = 6$, is ”not useful”. By way of comparison, the Democrat primaries of 2020 started with 10 official candidates.

the plurality tallies of a candidate pair can be decomposed into two parts - a part contributed by the weights of the basic profiles and another contributed by the weights of the reverse profiles only. By a previously published result (Chandra and Roy (2013)), the weights of the basic profiles can be directly obtained from the pairwise tallies across all candidate pairs, once effects due to Condorcet cycles are removed, using a computationally simple recursive procedure. This being the case, by Theorem 2, the weights of the reverse profiles can be directly obtained from the plurality tallies themselves.

The third and final Theorem 3 connects the differences in the weights of the basic and reverse profiles to disagreement between the plurality and Borda count rank orders. This provides a way to construct measures of polarization based on the relative weights of the basic and reverse profiles, some of which are discussed in Section 5.2.

The following 3-candidate examples illustrate some of our main ideas and their underlying intuition, with additional details provided in Section 3. The notation $i \succ j$ stands for "i is strictly preferred to j" and $i \succeq j$ denotes "i is weakly preferred to or at least as good as j". The notation $i \sim j$ implies i and j are preference equivalent.

EXAMPLE 1: Consider a profile consisting of 3 voters with preference $A \succ B \succ C$, another 3 voters with preference $C \succ B \succ A$ and 1 voter with preference $B \succ A \succ C$. The social choice outcome or winners under plurality are A and C and the induced social rank order of the candidates is $A \sim C \succ B$. Under the Borda count, the induced rank order is $B \succ A \succ C$ and B is the unique winner. A runoff in which the candidate with the lowest plurality tally is dropped, elects A who is the last choice of nearly 43% of the voters. The conflicts between the plurality and the Borda Count outcomes and the induced rank orders are due to the presence of two significant and *nearly* equal sized reverse profiles in the given one (see Section 3). Of the two reverse profiles, the one with a slightly smaller weight has the hypothetical voters evenly split across rank orders that place A in the first and last positions. The other reverse profile has hypothetical voters similarly split across rank orders with C in the first and last places. There are two basic profile components of unequal weights. The one with the larger weight favors B and accounts for the candidate's first position in the Borda rank order. The one with the smaller weight favors A and accounts for the candidate's second position in the Borda rank order and the win in the runoff.[8]

[8]Some evidence suggests that the scenario of Example 1 captures the last stage of the 2016 US Presidential Republican primaries with only three candidates standing - Donald Trump, Ted Cruz and John Kasich. See, http://www.fairvote.org, for article posted on Mar 04, 2016 by Andrew Douglas, Rob Richie, Elliott Louthen; http://www.newsweek.com for article posted on Oct 17, 2016 by Paul Raeburn; http://thefederalist.com. for article posted on April 5, 2016 by Kyle Sammin.

EXAMPLE 2: We next consider a different example in which preference polarization is visibly less stark and noticeable compared to the first. Consider a profile of 8 voters with 5 of them ranking the candidates $A \succ B \succ C$ and 3 of them ranking $B \succ C \succ A$. The plurality winner is A and the induced rank order under this method is $A \succ B \succ C$. Under the Borda count, candidate B is the winner and the induced rank order is $B \succ A \succ C$. Once again, the disagreement between the two methods can be explained by two reverse profile components with significant weights which are however difficult to envision without a formal profile decomposition. One of the reverse profiles has voters equally split across rank orders which place A in the first and last places. The other reverse profile has a *negative* weight, a novel feature of this example. The next section explains what a negative weight generally means in the context of profile decomposition. In a 3-candidate field specifically, a reverse profile with a negative weight reflects a strong support for a specific candidate in the second place but not in the first (and last) places. The candidate in question in this example turns out to be B. The given profile also has two basic profile components. The one with the larger weight favors B and the other favors A. The order of the basic profile weights explains the Borda outcome and rank order.

On a practical level, profile decomposition requires information of every voter's rank orders across all candidates. Currently, there are few elections across the world that require voters to rank all candidates on the ballot, although calls for such requirements are growing.[9] Many city councils in the US and most importantly and recently, the State of Maine, have adopted the procedure of ranked choice voting which require voters to rank candidates. Data availability nevertheless remains a critical limitation of this methodology.

The Cambridge City Council elections have a long history of using the ranked choice voting procedure. The last section of the paper applies our decomposition method and polarization measures on ballot data from these elections. This data has a major limitation in that, voters are required to rank at least one candidate but not *all*. Thus the rank orders on the ballot are not complete. The exercise and the results are nevertheless useful as a way of illustrating our technique and measures.

The structure of the rest of the paper is as follows. Section 2 lays out the definitions and the basic framework used. Section 3 illustrates the profile decomposition technique with three candidates, a scenario in which a complete profile decomposition is possible and easy. Sections 4 and 5 posits our main results. Finally, Section 6 reports the results of the analysis of the Cambridge City Council elections data.

[9]See Gehrlein, Lepelley and Plassmann (2016) and http://www.fairvote.org

2 Definitions and basic framework

We assume a field of n candidates or alternatives, indexed $i = 1 \dots n$, and a fixed electorate or set of voters/judges. Individual voters have strict, complete and transitive preferences over the alternatives. A *preference profile* is a distribution of voters across the $n!$ possible rank orders of the candidates and is denoted by $p = (p_1 \dots p_{n!}) \in \mathbf{R}_+^{n!}$, where p_k denotes either the number of voters or the proportion of the electorate with preferences given by the k-th rank order of the candidates.

A *voting procedure* or *social choice function* maps a profile p into a subset of the set of candidates, which we describe as the *social choice outcome* or the set of winners. A voting procedure may induce a weak rank order of the candidates which we describe as the *social rank order*.

A *sum-scoring* or *positional* voting procedure is a n-tuple, $(w_1, \dots w_n)$ where $w_m \geq 0$ equals the total points awarded by an individual voter to the candidate in the m-th position in his or her rank order. Only positional procedures with the property $w_m \geq w_{m+1}$ and a strict inequality for some $(m, m+1)$, are admitted. The *plurality* voting procedure is represented by the n-tuple, $(1, 0 \dots 0)$ and the *Borda count* by $(n-1, n-2 \dots 0)$. A sum-scoring method induces a social rank order of the candidates based on the sum total of the points awarded by all voters. The winner is (are) the first ranked alternative(s).

The *pairwise tally difference* for the (i, j) candidate pair, equals the number of voters who prefer i to j minus the number of voters who prefer j to i. The *normalized* pairwise tally difference is defined as the pairwise tally difference divided by the total number of voters. Denote by a_{ij}, the normalized tally difference for the (i, j) pair. Then

$$a_{ij} = \frac{\text{\# who prefer } i \text{ to } j \text{ - \# who prefer } j \text{ to } i}{\text{\# who prefer } i \text{ to } j + \text{\# who prefer } j \text{ to } i} = -a_{ji}$$

and $a_{ij} > 0$ implies, i beats j in a pairwise comparison across all voters. These scores by themselves are not guaranteed to induce a transitive social rank order of the candidates when Condorcet cycles are present in a profile. Methods based on pairwise comparisons of candidates are not the central object of our inquiry. It is important to note however that Copeland, Black and Kemeny methods are based on the a_{ij} scores across all candidate pairs and the next section discusses profiles on which these methods concur with sum-scoring methods.

The Borda score of candidate i has been shown to be an affine transformation of the sum of his/her a_{ij} scores across all rivals, namely, $\sum_{j \neq i} a_{ij}$. Thus, the social rank order of the candidates induced by the Borda count is the same as that induced by the set of such sums, $\{\sum_{j \neq i} a_{ij}\}_{i=1 \dots n}$ (Saari (2000a, 2000b), Zwicker (1991, 2016)). In particular, although the set of a_{ij} scores is not guaranteed to induce a transitive social rank order if Condorcet cycles are present, the set of sums, $\{\sum_{j \neq i} a_{ij}\}_{i=1 \dots n}$ is. (see Section 2.2 for more).

Let K^n be a $n!$ dimensional vector with each component equal to 1. Assume an electorate of a given and fixed size V, which is either the unit mass or a positive integer. A uniform distribution of voters within this electorate is denoted by the profile $\frac{V}{n!}K^n$.

A $\frac{V}{n!}K^n$ profile yields a tied outcome across all candidates under the admissible sum-scoring and some pairwise procedures. Thus, under these procedures, the induced social rank order on any given profile is not affected by adding or subtracting a scalar multiple of a K^n profile, although the candidate tallies change as a result.

The above observation provides us with a convenient way to represent profiles and component profiles in a decomposition. Express p as $p = \frac{V}{n!}K^n + p'$ where $p' \in \mathbf{R}^{n!}$. That is, p is *obtained from a uniform distribution of voters by moving some voters away from specific rank orders and adding them to others*. The negative components of p' mark which rank orders suffer depletion and by how many voters, whereas the positive components mark which ones experience gains and by how many. Stated alternatively, any given profile p is a result of enhanced support for some rank orders and reduced support for others compared with a uniform distribution of voters. Note that the components of the deviation profile p' or the increments and the decrements across all possible rank orders, must add up to zero. In other words, p' itself can be regarded as a deviation from a $0 \cdot K^n$ profile or electorate of size zero.

The deviation profile p' and the given profile p induce the same social rank order of the candidates (under admissible procedures), although candidate tallies on the two differ. In other words, the deviation profile p' ignores the neutral effects of the K^n component and captures the part of p which is decisive for the social outcome. By way of convenience, the component profiles of interest in this paper are expressed as deviations from a $0 \cdot K^n$ profile. To convert them into hypothetical electorates with non-negative voters, one simply needs to add appropriate K^n components.

In any profile decomposition exercise, component profiles can have, in general, positive or negative weights. A negative weight simply reverses the shifts across the rank orders in the component. Moreover, under sum-scoring rules, the induced rank order on p' and $-p'$ are also reversed. The following example illustrates the concepts presented so far.

EXAMPLE 3: Consider a field of three candidates and an electorate of 30 voters with profile $p = (7,6,2,8,0,7)$. The six possible rank orders are indexed as in Table 2. The profile can be expressed as $p = 5K^3 + p'$, where $p' = (2,1,-3,3,-5,2)$. That is, p is obtained from the uniform distribution of voters $5K^3$ by moving voters away from rank orders (3) and (5) in which A is middle ranked and adding them to the rank orders in which A is either first or last ranked.

The plurality tallies of the candidates on p are $(A:13, B:10, C:7)$ whereas on p', these are $(A:3, B:0, C:-3)$. The induced social rank order under plurality on both p and p' is $A \succ B \succ C$. The induced rank order under Borda count on both profiles is $B \succ A \sim C$.

The profile $-p'$ is given by $-p' = (-2,-1,3,-3,5,-2)$ which indicates a shift of voters away from rank orders (1), (2), (4) and (6), in which A is first or last ranked, to rank orders (3) and (5) in which A is ranked in the middle. To convert $-p'$ into an electorate of the same size as p, we add a $5K^3$ component to get $-p' + 5K^3 = (3,4,8,2,10,3)$. The plurality rank order on the latter is $C \succ B \succ A$, whereas the Borda rank order is $A \sim C \succ B$.

2.1 Basic profiles

We owe the concept and understanding of basic profiles to the writings of Saari (1997, 2000a, 2000b). Only the properties that are most relevant for our results are discussed here. The interested reader is referred to these papers for additional insights and results.

To obtain a basic profile, fix a candidate, i. Take a $0 \cdot K^n$ profile and shift one voter from each rank order which has i last ranked and add one voter to a rank order which has i first ranked. The component profile thus obtained has one voter for each ranking that has i top ranked, (-1) voter for each ranking that has i bottom ranked and 0 voter for each ranking that has i ranked somewhere in the middle. We denote this profile by B_i^n. If a neutral K^n profile is added to B_i^n, the profile $B_i^n + K^n$ has 2 voters for each rank order which has i first ranked, 1 voter for each rank order which has i ranked somewhere in the middle and 0 voter for each rank order which has i ranked last. Thus the virtual voters in a B_i^n profile demonstrate a ”strong to moderate” liking for candidate i and nobody ”dislikes” him/her.

It is straightforward to check that the set of all basic profiles, $\{B_i^n\}_{i=1}^n$, satisfy the constraint, $\sum_{i=1}^n B_i^n = 0$. That is, only $(n-1)$ of these profiles are linearly independent and without loss of generality, we assume that these are the ones indexed, $1 \ldots n-1$.

Under admissible sum-scoring procedures and Copeland, Black and Kemeny methods, the profile $a_i B_i^n$ where $a_i > 0$, has the i-th candidate top ranked and everyone else tied for the second place. In particular, the induced rank orders under plurality and the Borda count agree on a $a_i B_i^n$ profile. Moreover, a linear combination of basic profiles, $p = \sum_{i=1}^{n-1} a_i B_i^n$ where the a_i's are given constants (positive or negative), induces the same social rank order of the candidates under all these procedures.

Furthermore, it can be shown that the pairwise score for the candidate pair (i, j) on p, is given by $a_{ij} = a_i - a_j$, the difference in the coefficients of B_i^n and B_j^n in p. Moreover, this is true for differences in the positional tallies of the candidates i and j as well. Thus, under the above procedures, the relative rank of i and j in an induced social rank order on p is determined by the sign of $a_i - a_j$ alone and not by the coefficients or weights of the other basic profiles in p. If $a_i - a_j > 0$, i is ranked above j, if $a_i - a_j < 0$, i is ranked below j and if $a_i - a_j = 0$, i and j are tied in the social rank order. The social rank order on a linear combination of basic profiles is thus transitive and the same as the order of these weights.[10]

2.2 Condorcet profiles

To characterize a Condorcet profile, we first specify a *reference* rank order of the candidates, say $1 \succ 2 \succ 3 \ldots \succ n$, and index this rank order as (say), (1).

Consider two sets of cyclic rank orders generated by the reference rank order (1). We denote the first set by $c^n_{(1)}$ and the second set by $\rho(c_{(1)})^n$. Each rank order in the set $\rho(c_{(1)})^n$ is a reverse of the rank order in the set $c^n_{(1)}$. The two sets are shown in Table (1).

Table 1:

$c^n_{(1)}$	$\rho(c_{(1)})^n$
$1 \succ 2 \succ 3 \ldots \succ n$	$n \succ n-1 \succ n-2 \ldots \succ 1$
$2 \succ 3 \succ 4 \ldots \succ 1$	$n-1 \succ n-2 \succ n-3 \ldots \succ n$
$3 \succ 4 \succ 5 \ldots \succ 2$	$n-2 \succ n-3 \succ n-4 \ldots \succ n-2$
$\ldots$	$\ldots$
$n \succ 1 \succ 2 \ldots \succ n-1$	$1 \succ n-1 \succ n-2 \ldots \succ 2$

A *Condorcet profile* $C^n_{(1)}$ associated with the reference rank order (1), is a profile that has one voter for each rank order in the $c^n_{(1)}$ set, (-1) voter for each rank order in the $\rho(c^n_{(1)})$ set and 0 voter for each remaining rank order in the profile. The first or reference rank order of the set $c^n_{(1)}$, uniquely characterizes the Condorcet profile. There are $\frac{(n-1)!}{2}$ distinct ways of constructing the reference rank order for a field of n candidates. There are thus $\frac{(n-1)!}{2}$ *distinct* Condorcet profiles in such a field. Table (7) lists the three distinct Condorcet profiles for a 4-candidate field.

The $C^n_{(1)}$ profile is obtained from a $0 \cdot K^n$ profile by moving one voter away from each rank order in $\rho(c^n_{(1)})$ and adding one voter to a rank order in $c^n_{(1)}$. If a neutral K^n profile is added to $C^n_{(1)}$, the resulting profile has 2 voters for each rank order in the cyclic set $c^n_{(1)}$, 0 voter for each rank order in the reverse cyclic set $\rho(c^n_{(1)})$, and 1 for all other rank orders. A Condorcet profile has the feature that each candidate

[10]The induced rank order is also consistent over subsets of candidates, in other words, robust with respect to candidates dropping out of the race. This strong property has been described by Saari (1999, 2000a, 2000b) as the *additive transitive* property of basic profiles.

is supported in each position by an equal number of voters. Thus, it produces a complete tie across the candidates under all sum-scoring procedures.

Condorcet profiles when significantly present in a given profile, can generate an intransitive social rank order of the candidates under methods that use the pairwise score differences a_{ij} across all candidate pairs. A related and useful property of these scores is that although each a_{ij} is affected by the presence and size of the Condorcet cycles, the sum of the pairwise score differences for each candidate, $\sum_{j \neq i} a_{ij}$, is *not*. In other words, Condorcet effects cancel out when the pairwise score differences are aggregated for each candidate i over all his/her rivals (See Zwicker (1991), Saari (2000a)). This explains why the Borda count is unaffected by the presence of Condorcet cycles.

2.2.1 Basic profiles, Condorcet profiles and pairwise scores

The practical usefulness of the results of this paper, although not the results themselves, depends on there being a computationally simple way to extract the weights of the basic profile components when $n \geq 4$, without resorting to complete profile decomposition. Two existing results involving the relationship between the pairwise scores and the basic and the Condorcet profiles show that this is possible. The first of these is due to Saari.

Saari (Proposition 5, 2000a): On any given profile, $a_{ij} = a_{ij}^T + a_{ij}^C$ where the component a_{ij}^T is determined by the weights of basic profiles and the component a_{ij}^C is determined by the weights of all the distinct Condorcet profiles.

The proposition thus says that only the weights of the basic and the Condorcet profiles contribute to pairwise score differences. Other possible types of component profiles contribute nothing towards these values. Moreover, as noted earlier, the component, a_{ij}^T is equal to $a_i - a_j$, the difference in the weights of B_i^n and B_j^n only, in any given profile.

A subsequent result by Chandra and Roy, (2013) shows that the set of these differences, $\{a_i - a_j\}_{i \neq j}$, can be obtained using a recursive procedure on the pairwise scores $\{a_{ij}\}_{i \neq j}$ themselves. Thus, the process involves little computational complexity for arbitrary n when compared to what is involved in a complete profile decomposition. The interested reader is referred to both papers for details of this procedure.

As noted earlier, the Borda score of candidate i is an affine function of $\sum_{j \neq i} a_{ij}$. Combined with Saari's proposition above and the fact that the sum $\sum_{j \neq i} a_{ij}$ is free of Condorcet components, we have a useful property of the Borda count. Namely, the induced social rank order on *any* given profile under the Borda count, follows the weights of its basic profile components only. In looking for component profiles to explain the conflict between plurality and the Borda count, we must therefore look for component profiles that are orthogonal to basic profiles and influence plurality tallies. The next subsection introduces and characterizes such a class of profiles.

2.3 Reverse profiles

We begin by choosing an integer k such that $2 \leq k \leq \frac{n+1}{2}$. The choice k specifies a set of profiles which we describe as *class-k* reverse profiles. The family of all reverse profiles is the set of all class-k reverse profiles obtained by selecting all possible values of k satisfying $2 \leq k \leq \frac{n+1}{2}$. A class-$k$ reverse profile favoring candidate i, denoted $R_i^n(k)$, is defined as follows.

Definition 1 *(1) When n is even, such that,* $2 \leq k < \frac{n+1}{2}$, $R_i^n(k)$, *has 1 voter for each rank order in which candidate i is first ranked and the reversal of this rank order which has i ranked last; (-1) voter for each rank order in which candidate i is k-th ranked and the reversal of this rank order;* 0 *voters for all other rank orders. (2) If* $k = \frac{n+1}{2}$ *(n is odd),* $R_i^n(k)$ *has 1 voter for each rank order in which candidate i is first ranked and the reversal of this rank order which has i last ranked; (-2) voters for each rank order in which the candidate is k-th ranked;* 0 *voters for all other rank orders.*

To understand the structure of $R_i^n(k)$, assume $n > 3$ and $k = 2$. Then $R_i^n(2)$ is obtained from a $0 \cdot K^n$ profile by subtracting a voter from each rank order in which i is 2-nd or $(n-1)$-th ranked and adding a voter to a rank order in which i is first or last ranked. In other words, $R_i^n(2)$ is obtained by boosting support for rank orders in which i is placed at the two extremes and depleting support for rank orders in which i is placed in the intermediate 2-nd and $(n-1)$-th positions. The value of k in $R_i^n(k)$ thus specifies which rank orders with i in an intermediate position are depleted, while the rank orders in which i is placed first or last get a boost. If $k = 3$, support is taken away from rank orders in which i is in the 3-rd or $(n-2)$th place and so on.

In a $R_i^n(k)$ profile, each rank order that has i first and last ranked are supported by the same number of voters. Thus, such a profile represents an electorate that is extremely polarized around the i-th candidate.[11]

There are n number of reverse profiles in each class. By construction, $\sum_{i=1}^n R_i^n(k) = 0$ for each k, implying there are $(n-1)$ linearly independent reverse profiles in each class. The number of classes within the family

[11] The $R_i^n(k)$ profiles defined here are similar but not identical in structure to *Symmetric profiles* defined in Saari (2000b), although both types share some common properties. Saari's construction appears to be driven by a specific need to express positional tallies as deviations from the Borda Count (2000b, Proposition 1). Such a step in turn is necessary if the weights of the basic profiles are unknown and the Borda Count is used as a *surrogate*.

depends on the value of n. When $n = 4$, the only possible value of k is $k = 2$. The family of reverse profiles contains a single class of three distinct reverse profiles. These are listed in Table (8) in the Appendix. When $n = 5$, the family contains two classes of reverse profiles, one corresponding to $k = 2$ and the other corresponding to $k = 3$. Each class has four distinct profiles.

When $n = 3$, the set of two distinct basic profiles, one Condorcet profile, two distinct reverse profiles and a K^3 profile span the 6-dimensional space of preference profiles. A complete decomposition of any given profile is easy and straightforward. The next section illustrates the decomposition technology on a few interesting 3-candidate profiles. These also serve to clarify the underlying intuition of some of our main results.

The results of the paper assume a scenario which for lack of a better term, we describe as a *full-field* scenario. It assumes that candidates do not drop out in the middle of a race. That is, the set of candidates on which voters' preferences are defined is identical to the set from which the eventual outcome(s) of the race is determined. The extent to which these results generalize when candidates drop out in the middle of a race is left for future research.

3 Profile decomposition with three candidates

Table 2 lists the six possible rank orders with three candidates (A, B and C), the basic profiles B^3_A and B^3_B, the reverse profiles R^3_A and R^3_B, the Condorcet profile C^3, and the K^3 profile.

Table 2:

Rank order index	Rank order	B^3_A	B^3_B	R^3_A	R^3_B	C^3	K^3	Profile p
(1)	$A \succ B \succ C$	1	0	1	−2	1	1	1
(2)	$A \succ C \succ B$	1	−1	1	1	−1	1	0
(3)	$B \succ A \succ C$	0	1	−2	1	−1	1	0
(4)	$B \succ C \succ A$	−1	1	1	1	1	1	0
(5)	$C \succ A \succ B$	0	−1	−2	1	1	1	0
(6)	$C \succ B \succ A$	−1	0	1	−2	−1	1	0

Consider a profile p consisting of a single voter (or a unit mass of voters) with preference order $A \succ B \succ C$. That is, p is a unanimity profile. The profile can be expressed as $p = \frac{1}{6}K^3 + \frac{1}{3}B^3_A + (\frac{1}{6}B^3_B - \frac{1}{6}R^3_B) + \frac{1}{6}C^3$, that is as a deviation from the uniform distribution of voters, $\frac{1}{6}K^3$. The first notable feature of this decomposition is that all three types of component profiles are present with non-zero weights. In particular, a unanimity profile *cannot be* obtained as a linear combination of basic profiles only and moreover, has a reverse profile component.

The relative sizes and signs of the component weights provide insight into how p is obtained by decreasing support for some specific rank orders and increasing support for others. The B^3_A component profile has the greatest weight of all the component profiles and accounts for the shift away from rank orders (4) and (6) which has A last ranked to rank orders (1) and (2) which has A top ranked. The negative R^3_B component accounts for shift away from rank orders which has B first or last ranked to those which have B middle ranked. The unanimity profile also has a Condorcet component which does not favor any specific candidate for a specific position in the net (for example, a positive shift towards $A \succ B \succ C$ which has A top ranked is balanced by an equal negative shift towards $A \succ C \succ B$).

Furthermore, note that the expression $p = \frac{1}{6}K^3 + \frac{1}{3}B^3_A + (\frac{1}{6}B^3_B - \frac{1}{6}R^3_B) + \frac{1}{6}C^3$ can be alternatively written as $p = \frac{1}{6}K^3 + (\frac{1}{3}B^3_A + \frac{1}{6}R^3_A) + \frac{1}{6}B^3_B + \frac{1}{6}R^3_C + \frac{1}{6}C^3$ by substituting $R^3_B = -(R^3_A + R^3_C)$. This provides a second way to understand the profile. In this new expression, the combined component $(\frac{1}{3}B^3_A + \frac{1}{6}R^3_A) = (1/2, 1/2, -1/3, -1/6, -1/3, -1/6)$ indicates a net shift of voters away from rank orders in which A is last ranked or ranked in the middle and towards rank orders in which he/she is top ranked. The reverse profile R^3_C favors C either in the first or the last position. But the weight of the reverse profile is not strong enough to overcome the shift towards A for the first position.

3.1 Plurality and Borda count

A formal decomposition of the profiles of Examples 1 and 2 illustrate the role of reverse profiles in causing disagreement between the plurality and the Borda count rank orders. Consider the unanimity profile p in Table (3) with $B \succ A \succ C$ as the voters' rank order of the candidates and the profile, q, which has 3 voters supporting the rank order $A \succ B \succ C$ and 3 others supporting the reverse rank order, $C \succ B \succ A$. The sum of the two profiles, $p + q$, is the electorate described in Example 1.

Note that the two procedures agree on the social choice *outcome* of the unanimity profile p which has B as the unique winner (not surprising, since all voters want it). But they do *not* agree on the induced social rank order of the candidates. The social rank order is $B \succ A \sim C$ under plurality (as both A and C get 0 points) and $B \succ A \succ C$ under the Borda rule.

The disagreement in the induced rank orders can be traced to the reverse profile component of p which can be expressed as, $p = \frac{1}{6}K^3 + \frac{1}{3}B^3_B + (\frac{1}{6}B^3_A - \frac{1}{6}R^3_A) - \frac{1}{6}C^3$. Consider the rest of the expression without the $-\frac{1}{6}R^3_A$ term, namely, $\hat{p} = \frac{1}{6}K^3 + \frac{1}{3}B^3_B + \frac{1}{6}B^3_A - \frac{1}{6}C^3 = (1/6, 1/6, 2/3, 1/6, -1/3, 1/6)$. Both plurality and the Borda rule induce the same social rank order, $B \succ A \succ C$, on $\hat{p}$. Thus, the profile component $-\frac{1}{6}R^3_A$ (or alternatively, $(\frac{1}{6}R_B^3 + \frac{1}{6}R_C^3)$) is responsible for the difference in the induced rank orders on p.

We next show that on profiles in which the weights of the reverse profiles are large enough relative to those of basic profiles, the two methods differ on the social choice outcomes as well.

Table 3:

Rank order index	Rank order	Profile p	Profile q	Profile $p+q$	Profile s
(1)	$A \succ B \succ C$	0	3	3	5
(2)	$A \succ C \succ B$	0	0	0	0
(3)	$B \succ A \succ C$	1	0	1	0
(4)	$B \succ C \succ A$	0	0	0	3
(5)	$C \succ A \succ B$	0	0	0	0
(6)	$C \succ B \succ A$	0	3	3	0

Note that the component profile q can be expressed as $q = K^3 + R_A^3 + R_C^3$ (or alternately as $q = K^3 - R_B^3$). In other words, q is a sum of two equally strong reverse profiles which support A and C in the first and last places (or alternatively, a strong negative reverse profile which supports B in the second place). The combined profile $p+q$ has the decomposition $p+q = \frac{7}{6}K^3 + \frac{1}{6}B_A^3 + \frac{5}{6}R_A^3 + \frac{1}{3}B_B^3 + R_C^3 - \frac{1}{6}C^3$. When compared with p, the profile $p+q$ has larger, positive R_A^3 and R_C^3 components which account for $\{A,C\}$ being the plurality winner and $\{B\}$ being the Borda winner.

More generally, denote by $Pl(.)$ the plurality outcome and by $B(.)$ the Borda outcome on a profile. We have, $Pl(p) = B(p) = \{B\}$. Note that $B(mq) = \{A,B,C\}$ for any $m > 0$ by the Archimedean property and $B(p+mq) = \{A,B,C\} \cap \{B\} = \{B\}$ by reinforcement. On the other hand, $Pl(mq) = \{A,C\}$ for all $m > 0$, whereas for $m < \frac{1}{3}$, $Pl(p+mq) = \{B\}$. For $m = \frac{1}{3}$, $Pl(p+mq) = \{A,B,C\}$ and for $m > \frac{1}{3}$, $Pl(p+mq) = \{A,C\}$. Thus, reverse profile components need to be of significant weights to generate a conflict between the plurality and Borda outcomes. The results in section 5 establish the precise connection between rank reversals of candidate pairs under plurality and Borda and the relative weights of basic and reverse profiles, in a n-candidate field.

The profile in Example 2 is represented by s in Table 3 and has outcomes, $Pl(s) = \{A\}$ and $B(s) = \{B\}$. The profile s can be decomposed either as $s = \frac{8}{6}K^3 + \frac{7}{6}B_A^3 + \frac{5}{6}R_A^3 + \frac{8}{6}B_B^3 + \frac{2}{6}R_C^3 + \frac{8}{6}C^3$ or as $s = \frac{8}{6}K^3 + \frac{7}{6}B_A^3 + \frac{3}{6}R_A^3 + \frac{8}{6}B_B^3 - \frac{2}{6}R_B^3 + \frac{8}{6}C^3$ by using the relationship $-R_B^3 = R_A^3 + R_C^3$. The profile has strong basic profile components supporting A and B although one is less than the other. As the Borda count follows the weights of the basic profiles only, B with a larger basic profile weight is Borda ranked higher than A. The plurality winner is A because the weight of R_A^3 is significantly higher relative to the weight of R_B^3 ($\frac{5}{6}$ vs 0 by the first formulation, $\frac{3}{6}$ vs $-\frac{2}{6}$ by the second formulation). Specifically, the difference in the reverse profile weights are big enough to outweigh the difference in the basic profile weights.

Sections 4 and 5 present our main results.

4 Plurality tallies, basic and reverse profiles

Our first proposition describes some important properties of the $R_i^n(k)$ profiles.

Proposition 1 *For any given k,*

1. *The set of $\{R_i^n(k)\}_{i=1\ldots n}$ profiles span a $(n-1)$ dimensional subspace of the profile space and are not pairwise orthogonal to each other.*

2. *The set of $\{R_i^n(k)\}_{i=1\ldots n}$ profiles are pairwise orthogonal to the set of $\{B_i^n\}_{i=1\ldots n}$ profiles.*

3. *The plurality tallies of candidates under B_i^n and $R_i^n(k)$ profiles are identical, with candidate i receiving $(n-1)!$ points and every other candidate receiving $-(n-2)!$ points each. In particular, these tallies do not depend on the specific choice of k.*

4. *The pairwise score differences for each candidate pair, (i,j), on a reverse profile is zero which implies that procedures based on pairwise tallies have all candidates tied. Thus the Borda count on a $R_i^n(k)$ profile have all candidates tied.*

Proof: See Appendix.

Part 3 of the proposition is most important for our paper. While the nature of the hypothetical electorates represented by B_i^n and $R_i^n(k)$ are very dissimilar, both induce the same plurality rank order on the candidates. Specifically, the plurality tallies of candidate *i* may mask the fact that the candidate in question is polarizing.

The proposition also says that the pairwise score differences, the a_{ij} scores, are zero on reverse profiles, implying, in particular, that the Borda count has all candidates tied. This feature plays a key role in subsequent results.

The first main result of the paper shows that Definition 1 characterizes all possible component profiles that explain differences in the plurality and Borda count rank orders in a parsimonious way. No other type of component profiles need be considered.

Theorem 1 *The weights of basic and reverse profile components in any given profile are sufficient to explain the plurality tallies and the induced rank order of the candidates. In particular, weights of reverse profile components explain all conflict between the plurality and the Borda count rank orders.*

Proof: We try to characterize all possible component profiles that influence plurality tallies, are orthogonal to basic profiles and produce a zero pairwise score difference for each candidate pair so that the Borda count obtains a tie across all candidates on them.

Let p be any given profile. Note that by construction of B_i^n, the dot product,

$$p \cdot B_i^n = (\text{no. of voters in } p \text{ who rank } i \text{ first} - \text{no. of voters in } p \text{ who rank } i \text{ last})$$

Thus orthogonality implies that the total number of voters in the profile p who rank i first must equal the total number of voters who rank i last.

For pairwise tallies to be zero for each candidate pair on p, a rank order and its reversal must be supported by the same number of voters. The two statements together imply in particular, that rank orders that place a specific candidate in the first position in p and their reversals which place the same candidate in the last position must be supported by an equal number of voters.

The total number of voters in a component profile must add up to zero which implies that rank orders and their reversals which are supported by positive number of voters must be balanced by rank orders and their reversals supported by an equal but negative number of voters. In particular, this implies that positive support for rank orders which have candidate i in the first and last places must be balanced by an equal and negative support for rank orders and their reversals with i in an intermediate position.

Note that reverse profiles as described in Definition 1 satisfy all these required characteristics in a parsimonious way. Hence their weights and the weights of the basic profiles are sufficient to explain plurality tallies and all conflicts between plurality and the Borda count. Δ.

The theorem implies that the plurality tally of a candidate is the sum of two parts - a part contributed by the weights of the basic profiles and the other by the weights of the reverse profiles, only. Thus, consider a set of orthogonal basis vectors for the profile space and assume that the set includes all basic and reverse profiles as defined earlier. The theorem says that the plurality tallies of the candidates on any given profile are obtained on the part given by the combination of the basic and the reverse profiles only. Other basis profiles need not be considered to explain these tallies. The result lays out a pathway for extracting the weights of the reverse profiles without resorting to the computationally complex task of a complete profile decomposition.

5 Plurality tallies and weights of basic and reverse profiles

We establish the relationship between plurality tallies and weights of basic and reverse profiles (Theorem 2). The relative weights of these two types of profiles are used to explain rank reversals of candidate pairs under the plurality and Borda rank orders (Theorem 3). We also discuss potential measures of polarization based on the weights of the reverse profiles.

Denote by $\tau = \{\tau_i\}_{i=1}^{n}$, the vector of plurality tallies of the n candidates, on a given profile. Define by $\tau_{ij} = \tau_i - \tau_j$, the difference in the plurality tallies of candidates i and j.

By Theorem 1, τ is obtained from a linear combination of K^n and the basic and reverse profile components of the given profile. Assume $V = 1$. Then, τ is obtained on the part of the given profile which can be expressed as $\sum_{i=1}^{n} a_i B_i^n + \sum_{2\leq k\leq \frac{n+1}{2}} \sum_{i=1}^{n} r_i(k) R_i^n(k) + \frac{1}{n!} K^n$ where the sets $\{a_i\}$ and $\{r_i(k)\}$ represent the unknown coefficients of the basic and the reverse profiles.

As noted earlier, $(n-1)$ of the basic profiles and $(n-1)$ of the reverse profiles for each k are independent. We assume without loss of generality that $B_n^n = -\sum_{i=1}^{n-1} B_i^n$ and $R_n^n(k) = -\sum_{i=1}^{n-1} R_i^n(k)$ for each k. From Proposition 1, the plurality tallies of B_i^n and $R_i^n(k)$ are in identical direction, for each k. Together, these imply that the plurality tally of candidate i can be expressed as,

$$\begin{aligned}\tau_i &= \Big[(a_i - a_n) + (\sum_k r_i(k) - \sum_k r_n(k))\Big](n-1)! \\ &\quad - \sum_{j\neq i, j=1}^{n-1} \Big[(a_j - a_n) + (\sum_k r_j(k) - \sum_k r_n(k))\Big](n-2)! + \frac{1}{n},\ \forall i = 1\ldots n-1 \end{aligned} \tag{1}$$

and

$$\tau_n = -\sum_{j=1}^{n-1} \Big[(a_j - a_n) + (\sum_k r_j(k) - \sum_k r_n(k))\Big](n-2)! + \frac{1}{n}$$

.

Equation (1) says the plurality tally of a candidate can be expressed as an affine function of the weights of the basic and reverse profile components of a given profile. This leads us to our second main result and an analogue to Saari 2000a, Proposition 5.

Theorem 2 *The plurality tally difference between candidates i and m can be decomposed into two parts - a part given by the difference in the weights of the basic profiles and a part given by the difference in the* *weights of the reverse profiles. Specifically,*

$$\frac{\tau_{im}}{n(n-2)!} = \frac{\tau_i - \tau_m}{n(n-2)!} = (a_i - a_m) + (\sum_k r_i(k) - \sum_k r_m(k)) \tag{2}$$

Proof: Follows directly from (1) on simplification.

5.1 Plurality vs Borda ranking

Our last main result shows that the relative sizes of the differences, $\{(a_i - a_j)\}$ and $\{(\sum_k r_i(k) - \sum_k r_j(k))\}$, are sufficient to explain all conflicts in the relative rank order of two candidates under plurality and Borda count.

Theorem 3 *Let p be any given profile.*

a) All differences in the social rank order of candidates obtained under plurality and the Borda count on p are attributed to the presence of reverse profile components.

b) Assume $(a_i - a_j) \neq 0$. The relative rank order of the (i, j) pair under plurality and Borda count are in identical direction if the terms $(a_i - a_j)$ and $(\sum_k r_i(k) - \sum_k r_j(k))$ have the same sign or if $|\frac{(\sum_k r_i(k) - \sum_k r_j(k))}{a_i - a_j}| < 1$. The relative rank order is reversed if the terms are not of the same sign and $|\frac{(\sum_k r_i(k) - \sum_k r_j(k))}{a_i - a_j}| > 1$. If $(a_i - a_j) = 0$ but $(\sum_k r_i(k) - \sum_k r_j(k)) \neq 0$, the candidates are tied under the Borda method but not under plurality. Their relative rank order under plurality is determined by the sign of $(\sum_k r_i(k) - \sum_k r_j(k))$.

Proof: a) The Borda ranking of the candidates follow the differences $a_i - a_j$. Candidate i is Borda ranked above (below) candidate j if $a_i - a_j > 0$ ($a_i - a_j < 0$). The candidates are tied if $a_i - a_j = 0$. The plurality ranking follows the sum of the differences in the weights of the basic and reverse profiles as equation (2) shows. Hence any disagreement between the two rank orders is solely due to the presence of reverse profiles.

b) The relative rank order of the (i, j) pair in the social rank order are in the same direction under plurality and the Borda method if $(a_i - a_j)$ and $(\tau_i - \tau_j)$ have the same sign. The rest of the statement follows directly from equation (2). Δ.

Suppose without loss of generality that the n-th candidate is Borda last ranked, that is, $(a_i - a_n) \geq 0$ for all i. Then, if $(\sum_k r_j(k) - \sum_k r_n(k)) < -(a_j - a_n)$ for some j, by Theorem 2 we have $\tau_j - \tau_n < 0$ which implies that the n-th candidate is not plurality last ranked. In other words, a significantly larger and positive reverse profile component can boost the plurality tallies of the Borda last ranked candidate. Similarly, the Borda first ranked (m-th) candidate may be plurality lower ranked than another candidate because of a significantly smaller reverse profile component.

The practical usefulness of Theorems 2 and 3 stems from the fact that the terms $(a_i - a_j)$ and $(\sum_k r_i(k) - \sum_k r_j(k))$ can be obtained without recourse to complete profile decomposition. Chandra and Roy (2013) shows that the differences $(a_i - a_j)$, can be obtained from the pairwise scores a_{ij}, through a computationally simple recursive procedure the details of which are skipped here and the interested reader is referred to the paper. The differences $(\sum_k r_i(k) - \sum_k r_j(k))$ can then be obtained from the observed plurality tally differences τ_{ij} and the obtained $(a_i - a_j)$.

While it is possible to obtain the differences in the weights by using the Chandra-Roy algorithm, to obtain an individual a_i or the $\sum_k r_i(k)$ for a specific i would require a normalization. We discuss one in the next subsection where it is necessary. Moreover, equation (2) do not permit extraction of the weights of all the individual reverse profiles. It allows us to solve for the sum $\sum_k r_i(k)$ only for each i (subject to the normalization). Note however, that so long as the objective is to simply assess the extent to which preferences are extreme (like or dislike), it is not important to know the individual $r_i(k)$-s.

5.2 Polarization measures

The weights $\{a_i\}$ and the sums of the weights, $\{\sum_k r_i(k)\}$, are potentially useful for constructing a variety of measures. First, we need to normalize one of the a_i and one of the sums, $\sum_k r_i(k)$, to zero. This can be done in a way such that the remaining weights and the sums of weights are non-negative. Suppose without loss of generality, that $\max_{(i,j)}(a_i - a_j) = (a_m - a_n)$. As $(a_m - a_n) \geq 0$, the difference, $(a_n - a_m) = \min_{(i,j)}(a_i - a_j) \leq 0$. Set $a_n = 0$ to obtain the values of all the remaining a_i-s from the differences. Note that $a_n = 0$ implies $a_i \geq a_n = 0$ for all $i \neq n$. Since the Borda rank order of the candidates follows the differences $a_i - a_j$, this implies that the n-th candidate is Borda last ranked and the m-th candidate is Borda first ranked. Similar steps may be used to normalize $\sum_k r_i(k)$ to zero for some i, say $i = l$. Note that in general, $l \neq n$.

Denote $\sum_{i=1}^n \sum_k r_i(k) = \bar{r}$ and $\sum_{i=1}^n a_i = \bar{a}$. The ratios $\frac{\sum_k r_i(k)}{\bar{r}}$ and $\frac{\sum_k r_i(k)}{a_i}$, when defined, are useful as measures of how polarizing the specific candidate i is or how polarized the electorate is around the candidacy of i. The ratio $\frac{\sum_k r_i(k)}{\bar{r}}$ provides a measure relative to other candidates in the field. On the other hand, a higher value of $\frac{\sum_k r_i(k)}{a_i}$ implies a greater influence of the reverse profiles relative to the basic profile in determining the plurality tally of candidate i. A lower value of the ratio similarly implies a greater influence of the basic profiles in determining these tallies. If $a_i = 0$ and $\sum_k r_i(k) > 0$, the plurality tallies are entirely contributed by reverse profiles. Thus, the ratio (when defined) measures candidate i 's *polarity* relative to his/her *acceptability* within the electorate with a lower value indicating a higher acceptability.

The ratio $\frac{\bar{r}}{\bar{a}}$ measures how strong the reverse profiles are relative to the basic profiles on average in determining a social rank order of the candidates under plurality. A higher value of this ratio indicates that the induced rank order is obtained on an electorate that is more sharply divided into groups that simultaneously love or hate specific candidates. A lower value indicates that the induced rank order is more of a consensus and closer to the rank order induced by the Borda count. The ratio can therefore also be used as a broad measure of social divisions as reflected in the preferences for specific candidates or alternatives.

The last section of the paper applies some of these measures on ballot data from the Cambridge City Council elections.

6 Results from the Cambridge City Council Elections

The nine members of the Cambridge City Council are elected using a single transferable vote system over several counts of the ballots the first of which involves determining the plurality tallies of all the candidates. A candidate is elected if he/she wins a certain proportion of the votes, called a quota.[12] We test our decomposition method and measures on the ballot data over the period 1997-2013. These elections are held every two years providing us with nine years of data.

There are on average, 18 or19 *official* candidates, on the ballot. Voters also have the right to vote for unofficial candidates of their choice by writing their names in a designated space on the ballot. The write-in candidates appear mostly to be people well known within a very small group of voters (who rank them) but unknown outside of this circle with one exception for the year 2009. In the year 2009, a popular candidate who was successfully elected multiple times previously, failed to file the nomination papers on time and hence was not included in the official list of candidates. The candidate participated as a write-in candidate and actually ended up being elected. For our analysis for the year 2009, we treated this candidate as an official rather than as a write-in candidate. With the exception of 2009, the few ballots where a write-in candidate is ranked first in the other years are excluded from our analysis. The exclusion does not materially affect the results reported here and moreover, considerably reduces the problem of dimensionality[13].

The main limitation of the data set is that voters are *not* required to rank all candidates. Voters must rank at least one of them for the first place and are free to rank as many of the others as they like. Most voters

[12] All candidates who reach the quota after the first count are declared elected. Surplus votes received by them are transferred to the second choice candidates on the surplus ballots (A formula determines which ballots are selected as surplus ballots). After surplus votes are transferred, candidates who have fewer than fifty votes are eliminated and their votes are transferred to the second choice on these ballots. A new ranking is established of the continuing candidates, after this. The candidate with the lowest number of tallies after the two transfers is declared defeated and his/her ballots are transferred to the next continuing candidate marked on each ballot. Once a candidate reaches the quota, no more ballots are transferred to him/her. The process continues till all nine members are elected.

[13] There are typically 7-9 write-in candidates in every election raising the total number of official and unofficial candidates to 25-28. If included, this causes $n(n-2)!$ to be a large number and the left hand side of equation (2) to vanish.

rank only about 4 or 5 candidates. Thus the major limitation of the data set is that the individual voters' rank orders are not complete as required by traditional social choice theory. The following assumptions are made about the candidates who are not ranked by a voter. First, if a voter has not ranked a specific candidate, say A, then it is assumed that the voter strictly prefers all the candidates that he or she has actually ranked to candidate A. This enhances the pairwise tallies on candidates who are ranked relative to those who are not. Secondly, if a voter has not ranked two candidates A and B say, we assume that he or she is indifferent between A and B. The pairwise score difference for the (A,B) pair are thus effectively determined by the number of voters who have ranked both these candidates. These assumptions affect our results which are useful nevertheless, as a first attempt to apply the methods and measures discussed in the paper.

A second minor limitation of the data set is that, for some of the elections prior to 2005, we found several ballots with multiple candidates ranked in the same position ("over-votes"). The problem of over-votes is significantly less beginning with 2005, because of a new practice put in place by the Election Commission that automatically ejects all such ballots and gives the voters another chance to redo their ballots. We excluded all ballots with multiple candidates placed in the same position. Thus, for the years 1997-2003, on an average about 8-9% of the total ballots were discarded. For the years 2005-2011, this percentage is about 1-2%. The discards account for some slight discrepancies between our plurality tallies for the candidates and the official plurality tallies of the candidates after the first count for these years.

Table 4 presents the values obtained for the average measures, $\bar{r}$ and $\frac{\bar{r}}{\bar{a}}$, discussed in Section 5. We also include two other variants of these measures, $\bar{r_w}$ and $\frac{\bar{r_w}}{\bar{a_w}}$, which indicate these averages among the winning candidates (that is, the 9 candidates who were finally elected to the Council). We identified pairs of candidates whose relative ranks are reversed under plurality based on our estimates of the basic and reverse profile coefficients. The number of such pairs as a proportion of the total number of distinct pairs is denoted by Ψ in the table. Figure 1 plots the Table 4 values. Finally, Borda rank orders of the candidates were constructed on the basis of the basic profile coefficients for all years. Table 5 reports selected final outcomes in which a candidate who was Borda lower ranked by our estimates got elected but a candidate who was Borda higher ranked, didn't. Strong reverse profile components which boost plurality tallies in the first count may partially explain these oddities.

The main findings from this exercise are summarized below. The reader is also referred to the working paper version of this paper, (Roy, Wu and Chandra (2015)), for additional details that may be of limited interest and hence not reported here.

6.1 Summary results

The most significant finding is that with the exception of 1997, the values of $\bar{r}$, $\frac{\bar{r}}{\bar{a}}$, $\bar{r_w}$ and $\frac{\bar{r_w}}{\bar{a_w}}$ are generally higher for the period 2005-2013 than for the period 1999-2003. The ratio, $\frac{\bar{r}}{\bar{a}}$, $\bar{r_w}$, in particular shows steady increase from 2007. Moreover, after 2005, extreme preferences seem to have played a generally bigger role in determining the set of winners (compared to before 2005) as evidenced by the values of $\bar{r}_w$ and the ratio $\frac{\bar{r_w}}{\bar{a_w}}$. These results seem to confirm for local elections, evidence of increased political polarization obtained by other studies which are cited in the Introduction.

The measure Ψ does not show a trend over the years but is generally significant at an average of 36% across all years. Among the Table 5 entries, the year 2009 is specially interesting. A candidate who was Borda last ranked (21st) edged out Borda 4th and 9th ranked to get elected. This candidate was 1st ranked according to the $\sum_k r_i(k)$ coefficients and 5th ranked in the official first (plurality) count. Thus the candidate seems to have been elected because of a strong reverse profile component. There were no such rank reversals during the 2001 and 2003 elections. All the first nine Borda ranked candidates got elected to the Council. Interestingly enough, the measures $\frac{\bar{r}}{\bar{a}}$, $\bar{r_w}$ and $\frac{\bar{r_w}}{\bar{a_w}}$ were also noticeably lower during these two years than during the others. Furthermore, in 2013, four major rank reversals are estimated - higher than the number for any other year. Also noticeably, the values of $\bar{r}$, $\frac{\bar{r}}{\bar{a}}$ and $\bar{r_w}$ are significantly higher for this year than for the other years.

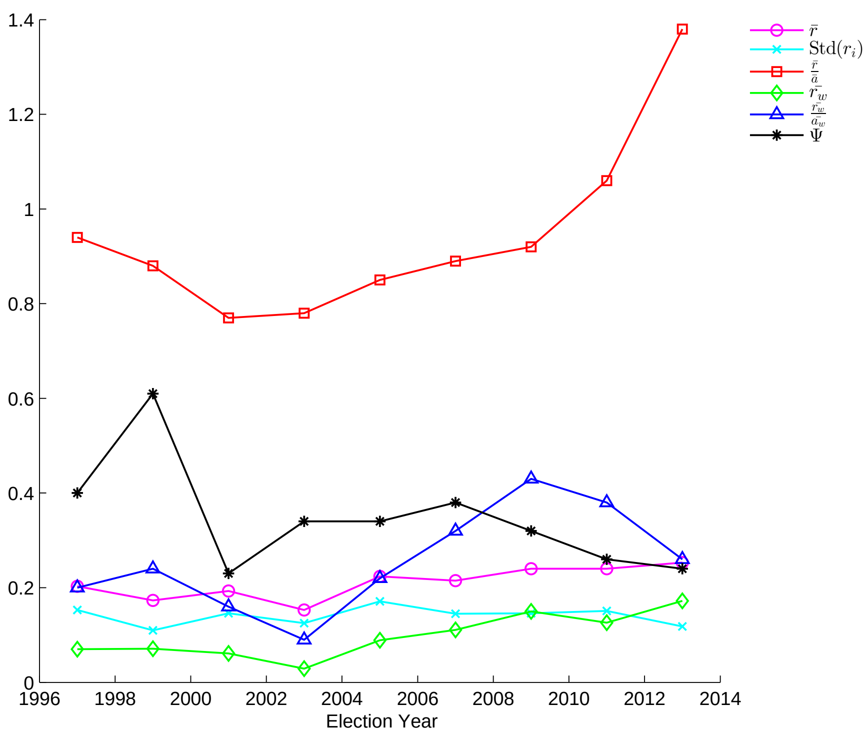
Table 4: SUMMARY RESULTS FOR THE PERIOD 1997-2013

Years	$\bar{r}$	Std(r_i)	$\frac{\bar{r}}{\bar{a}}$	$\bar{r_w}$	$\frac{\bar{r_w}}{\bar{a_w}}$	Ψ
1997	0.203	0.153	0.94	0.070	0.20	0.40
1999	0.173	0.110	0.88	0.071	0.24	0.61
2001	0.193	0.146	0.77	0.061	0.16	0.23
2003	0.153	0.125	0.78	0.029	0.09	0.34
2005	0.224	0.171	0.85	0.089	0.22	0.34
2007	0.215	0.145	0.89	0.111	0.32	0.38
2009	0.240	0.146	0.92	0.150	0.43	0.32
2011	0.240	0.151	1.06	0.126	0.38	0.26
2013	0.253	0.118	1.38	0.172	0.26	0.24

Table 5: SPECIFIC ELECTION ODDITIES

Years	Description
1997	Borda and a_i ranked 9th was edged out by Borda and a_i ranked 11th
1999	Borda and a_i ranked 2nd and 9th were edged out by Borda and a_i ranked 10th and 11th
2001	None
2003	None
2005	Borda and a_i ranked 7th was edged out by Borda and a_i ranked 11th
2007	Borda and a_i ranked 9th was edged out by Borda and a_i ranked 10th
2009	Borda and a_i ranked 4th and 9th were edged out by Borda and a_i ranked 10th and 21st
2011	Borda and a_i ranked 4th and 8th were edged out by Borda and a_i ranked 10th and 11th
2013	Borda and a_i ranked 2nd, 4th 8th and 9th were edged out by Borda and a_i ranked 11th

Figure 1: Summary results for the period 1997-2013



7 Appendix

7.1 Basic, reverse and Condorcet profiles with 4 candidates

The Tables (6), (7) and (8) list the three basic profiles, the three distinct Condorcet profiles and the three reverse profiles for a 4-candidate field.

Table 6:

Rank order	B_A^4	B_B^4	B_C^4	Rank order	B_A^4	B_B^4	B_C^4
1. $A > B > C > D$	1	0	0	7. $D > C > B > A$	-1	0	0
2. $A > B > D > C$	1	0	-1	8. $C > D > B > A$	-1	0	1
3. $A > C > B > D$	1	0	0	9. $D > B > C > A$	-1	0	0
4. $A > C > D > B$	1	-1	0	10. $B > D > C > A$	-1	1	0
5. $A > D > C > B$	1	-1	0	11. $B > C > D > A$	-1	1	0
6. $A > D > B > C$	1	0	-1	12. $C > B > D > A$	-1	0	1
13. $B > A > C > D$	0	1	0	19. $D > C > A > B$	0	-1	0
14. $B > A > D > C$	0	1	-1	20. $C > D > A > B$	0	-1	1
15. $C > A > B > D$	0	0	1	21. $D > B > A > C$	0	0	-1
16. $C > A > D > B$	0	-1	1	22. $B > D > A > C$	0	1	-1
17. $D > A > B > C$	0	0	-1	23. $C > B > A > D$	0	0	1
18. $D > A > C > B$	0	-1	0	24. $B > C > A > D$	0	1	0

Table 7:

Rank order	C_1^4	C_2^4	C_3^4	Rank order	C_1^4	C_2^4	C_3^4
1. $A > B > C > D$	1	0	0	7. $D > C > B > A$	-1	0	0
2. $A > B > D > C$	0	1	0	8. $C > D > B > A$	0	-1	0
3. $A > C > B > D$	0	0	1	9. $D > B > C > A$	0	0	-1
4. $A > C > D > B$	0	-1	0	10. $B > D > C > A$	0	1	0
5. $A > D > C > B$	-1	0	0	11. $B > C > D > A$	1	0	0
6. $A > D > B > C$	0	0	-1	12. $C > B > D > A$	0	0	1
13. $B > A > C > D$	0	-1	0	19. $D > C > A > B$	0	1	0
14. $B > A > D > C$	-1	0	0	20. $C > D > A > B$	1	0	0
15. $C > A > B > D$	0	1	0	21. $D > B > A > C$	0	-1	0
16. $C > A > D > B$	0	0	-1	22. $B > D > A > C$	0	0	1
17. $D > A > B > C$	1	0	0	23. $C > B > A > D$	-1	0	0
18. $D > A > C > B$	0	0	1	24. $B > C > A > D$	0	0	-1

Table 8:

Rank order	R_A^4	R_B^4	R_C^4	Rank order	R_A^4	R_B^4	R_C^4
1. $A > B > C > D$	1	-1	-1	7. $D > C > B > A$	1	-1	-1
2. $A > B > D > C$	1	-1	1	8. $C > D > B > A$	1	-1	1
3. $A > C > B > D$	1	-1	-1	9. $D > B > C > A$	1	-1	-1
4. $A > C > D > B$	1	1	-1	10. $B > D > C > A$	1	1	-1
5. $A > D > C > B$	1	1	-1	11. $B > C > D > A$	1	1	-1
6. $A > D > B > C$	1	-1	1	12. $C > B > D > A$	1	-1	1
13. $B > A > C > D$	-1	1	-1	19. $D > C > A > B$	-1	1	-1
14. $B > A > D > C$	-1	1	1	20. $C > D > A > B$	-1	1	1
15. $C > A > B > D$	-1	-1	1	21. $D > B > A > C$	-1	-1	1
16. $C > A > D > B$	-1	1	1	22. $B > D > A > C$	-1	1	1
17. $D > A > B > C$	-1	-1	1	23. $C > B > A > D$	-1	-1	1
18. $D > A > C > B$	-1	1	-1	24. $B > C > A > D$	-1	1	-1

7.2 Proof of Proposition 1

Part 1: Assume $k = 2$ to start with. Also, without loss of generality, consider the pair (R_1^n, R_2^n). R_1^n has non-zero voters for A in the 1-st, 2-nd, $(n-1)$-th and n-th places. R_2^n has non-zero voters for B in the 1-st, 2-nd, $(n-1)$-th and n-th places. The inner product of $(R_1^n)^T$ and R_2^n have non-zero components for all rankings in which (1) A is in the 1-st place and B is in the 2-nd, $(n-1)$-th or n-th place (2) A is in the 2-nd place and B is in the 1-st, $(n-1)$-th or n-th place (3) A is in the $(n-1)$-th place and B is in the 1-st, 2-nd or n-th place and (4) A is in the n-th place and B is in the 1-st, 2-nd or $(n-1)$-th place. In each of these cases (a total of twelve cases), A and B can be placed in their positions in $(n-2)!$ ways. These relevant components of R_1^n and R_2^n take values from the set $\{1, -1\}$. The non-zero components of the inner product sum to

$$-(n-2)! - (n-2)! + (n-2)! - (n-2)! + (n-2)! - (n-2)! - (n-2)! + (n-2)!$$
$$-(n-2)! + (n-2)! - (n-2)! - (n-2)! = -4(n-2)!$$

Hence R_1^n and R_2^n are not orthogonal. By way of illustration, for $n = 3$ and $n = 4$, $(R_1^3)^T R_2^3 = -6$ and $(R_1^4)^T R_2^4 = -8$. The argument extends to all pairs of R_i^n profiles for $k = 2$.

Next note that all the previous steps of the proof for the pair (R_1^n, R_2^n) apply to any $2 < k < \frac{n+1}{2}$ with the following changes: $R_1^n(k)$ has non-zero voters for A in the 1-st, k-th, $(n-k+1)$-th and n-th places. R_2^n has non-zero voters for B in the 1-st, k-th, $(n-k+1)$-th and n-th places. The inner product of $(R_1^n(k))^T$

and $R_2^n(k)$ have non-zero components for all rankings in which (1) A is in the 1-st place and B is in the k-th, $(n-k+1)$-th or n-th place (2) A is in the k-th place and B is in the 1-st, $(n-k+1)$-th or n-th place (3) A is in the $(n-k+1)$-th place and B is in the 1-st, k-th or n-th place and (4) A is in the n-th place and B is in the 1-st, k-th or $(n-k+1)$-th place. The sum of the non-zero components in the inner product is the same as before as the rank order tallies have not changed. Once again the argument extends to all pairs of $R_i^n(k)$ profiles for $2 < k < \frac{n+1}{2}$.

Now suppose we choose $k = \frac{n+1}{2}$ which can only happen if n is odd. In a $R_i^n(k)$ profile, candidate, i, is in the k-th place in $(n-1)!$ rankings and that half of these rankings are reversals of the other half. Each such ranking has (-2) voters by construction. The inner product of $(R_1^n)^T$ and R_2^n have non-zero components for all rankings in which (1) A is in the 1-st place and B is in the $\frac{n+1}{2}$-th or n-th place (2) A is in the $\frac{n+1}{2}$-th place and B is in the 1-st, or n-th place (3) A is in the n-th place and B is in the 1-st, $\frac{n+1}{2}$-th place. The non-zero components of the inner product equal

$$-2(n-2)! + (n-2)! - 2(n-2)! - 2(n-2)! + (n-2)! - 2(n-2)! \quad = \quad -6(n-2)!$$

which is not 0. Hence the non-orthogonality claim is true for any k and for all pairs of generic Reverse profiles.

Consider the sum $\sum_{i=1}^{n} R_i^n$ for $k = 2$. For each rank order, only four out of these n profiles at a time contribute non-zero voters. Two of the profiles contribute $(+1)$ voter each for the candidates in the first and last places. The other two profiles contribute (-1) each for candidates in the the 2-nd and $(n-1)$-th places. Hence the sum is 0. For example, with five candidates and for the rank order $A > B > C > D > E$, the profiles R_A^5 and R_E^5 contributes $(+1)$ voter each to the sum of the five reverse profiles. The profiles R_B^5 and R_D^5 contribute (-1) voter each to the same rank order.

Consider the sum of any $(n-1)$ profiles out of the n profiles. There are many rank orders to which only three out of $(n-1)$ reverse profiles contribute non-zero voters. As each profile contributes either $(+1)$ or (-1), the sum is not 0. Next consider any linear combination of the $(n-1)$ profiles and consider the first rank order. Assume without loss of generality that the profile R_i^n with coefficient a contributes $(+1)$ voter to this combination and profiles R_j^n and R_s^n with weights b and c respectively contribute (-1) voter each. Then the first component of the linear combination is $a-b-c$ which is zero if $a = b+c$. Consider a rank order in which candidate j occupies the first place and the other two candidates occupy the 2-nd and $(n-1)$-th places. The component corresponding to this rank order is zero if $b = a+c$. Similarly, the component corresponding to another such rank order in which candidate s is first ranked and the other two are in 2-nd and $(n-1)$-th places, is zero if $c = a+b$. It is straightforward to check that all three equalities are satisfied only if $a = b = c = 0$. Hence any $(n-1)$ reverse profiles are linearly independent and the set spans a $(n-1)$ dimensional subspace.

The arguments extend to any $2 < k < \frac{n+1}{2}$. When $k = \frac{n+1}{2}$, three out of these profiles contribute non-zero voters for each rank order at a time. Two of the profiles contribute $(+1)$ voter each for candidates in the first and last places. One profile contributes (-2) for candidate in the k-th place. Hence the sum is 0. It is straightforward to extend the remaining arguments to show linear independence of any subset of $(n-1)$ profiles.

Part 2: Consider the inner product of $(R_i^n)^T$ and B_i^n, for any given k. It has non-zero terms for all rank orders in which candidate i is in the first or last place. In the first case, both profiles contribute $(+1)$ voter to the product. In the second case, the reverse profile contributes $+1$ voter whereas the basic profile contributes (-1) voter to the product. The other terms in the product are all zero, given the structure of the two profiles. As there are $(n-1)!$ rankings in which candidate i is first ranked and another $(n-1)!$ rankings in which he/she is last ranked, the non-zero terms of the inner product equal $(n-1)!.(1).(1) - (n-1)!.(1).(-1) = 0$. Hence this pair is orthogonal to each other.

Next assume that $k = 2$ and consider the inner product of $(R_i^n)^T$ and B_j^n, where $i \neq j$. This has non-zero terms for all rankings in which (1) candidate j is in the 1-st place and i is in the 2-nd place (2) candidate j is in the 1-st place and i is in the $(n-1)$-th place (3) candidate j is in the n-th place and i is in the 2-nd place and (4) candidate j is in the n-th place and i is in the $(n-1)$-th place. The non-zero terms add up to $-(n-2)! - (n-2)! + (n-2)! + (n-2)! = 0$. Hence these two vectors are orthogonal and the claim is true.

Again, the arguments extend directly without any changes for any $k < \frac{n+1}{2}$. When $k = \frac{n+1}{2}$, the inner product has non-zero terms for all rankings in which (1) candidate j is in the 1-st place and i is in the $\frac{n+1}{2}$-th place (2) candidate j is in the n-th place and i is in the $\frac{n+1}{2}$-th place. The non-zero terms add up to $-2(n-2)! + 2(n-2)! = 0$. Hence claim is true for any given k.

Part 3: Under a B_i^n profile, candidate i is ranked first $(n-1)!$ times and hence receives as many points. Candidate j receives non-zero votes only for rankings in which he/she is ranked first and candidate i is ranked last. There are $(n-2)!$ such rankings each with (-1) voter. Thus every other candidate receives $-(n-2)!$ points. Under a R_i^n profile, with $k = 2$, candidate i is ranked first $(n-1)!$ times and receives as many points. Candidate j receives non-zero votes for every ranking in which (1) j is first ranked and i is second ranked (2) j is first ranked and i is $(n-1)$-th ranked (3) j is first ranked and i is n-th ranked. There are $(n-2)!$ rank orders in each case. Candidate j receives (-1) for each ranking in the first two cases and

$(+1)$ for each ranking in the last case. Hence j receives $-(n-2)!$ points.

These tallies remain unchanged for any choice of $2 < k < \frac{n+1}{2}$. For $k = \frac{n+1}{2}$, candidate j receives non-zero votes for every ranking in which (1) j is first ranked and i is $\frac{n+1}{2}$-th ranked (2) j is first ranked and i is n-th ranked. There are $(n-2)!$ rankings in each case. Candidate j receives (-2) for each ranking in the first case and $(+1)$ for each ranking in the last case. Hence j receives $-(n-2)!$ points.

Part 4: Under a R_i^n profile, each rank order and its reversal has the same number of voters. Thus the number of voters who rank a candidate i over candidate j is equal to the number of voters who rank j over i. Consider the profile $R_i^n + K^n$. The social rank order on R_i^n and $R_i^n + K^n$ are identical under Condorcet extensions and the Borda count. The total number of voters in a $R_i^n + K^n$ profile is $2(n-1)! + 2(n-1)! + (n-4)(n-1)! = n!$ for $n > 3$. Hence, $a_{ij} = 0$ for each (i, j) pair. Thus all candidates are tied under the Borda count and Condorcet extensions.

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