Partitioning forecast errors in numerical weather prediction models

David Giles McDonald
Iowa State University

Follow this and additional works at: https://lib.dr.iastate.edu/rtd

Part of the Atmospheric Sciences Commons, and the Statistics and Probability Commons

Recommended Citation
McDonald, David Giles, "Partitioning forecast errors in numerical weather prediction models " (1993). Retrospective Theses and Dissertations. 10250.
https://lib.dr.iastate.edu/rtd/10250

This Dissertation is brought to you for free and open access by the Iowa State University Capstones, Theses and Dissertations at Iowa State University Digital Repository. It has been accepted for inclusion in Retrospective Theses and Dissertations by an authorized administrator of Iowa State University Digital Repository. For more information, please contact digirep@iastate.edu.
INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.
Partitioning forecast errors in numerical weather prediction models

McDonald, David Giles, Ph.D.
Iowa State University, 1993
Partitioning forecast errors in
numerical weather prediction models

by

David Giles McDonald

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of the
Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Departments: Geological and Atmospheric Sciences
Statistics

Co-majors: Meteorology
Statistics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

For the Major Departments

Signature was redacted for privacy.

For the Graduate College

Iowa State University
Ames, Iowa

1993
This dissertation is dedicated to:

My family - My wife Sherrie and children: Mary and one unborn -
with the hope that twenty years (better make that thirty!)
down the road, we might truly be able to say
that it was worth the effort.

My Parents - who helped, both financially and by listening
to vented frustrations.

My brother, Jim - a little sibling rivalry helps.

My sister, Mary Ann - to hell with sibling rivalry!

Other non-traditional students: Keep trying -
it took me fourteen years from the start,
but I finally got my "Ph(u)D"!
TABLE OF CONTENTS

LIST OF FIGURES v
LIST OF TABLES xi
ACKNOWLEDGEMENTS xii

CHAPTER 1. OVERVIEW AND BACKGROUND 1
   Overview 1
   Background 2

CHAPTER 2. GENERAL DISCUSSION OF ERRORS IN NUMERICAL MODELS 8
   Sources of Errors 8
   Effect of Errors on Model Forecasts 10
   Summary 15

CHAPTER 3. ERROR PARTITIONING FOR DIFFERENTIAL EQUATIONS 16
   Partitioning: General Approach 16
   Partitioning Statistics 22
   Summary 24

CHAPTER 4. PRELIMINARY APPLICATION: INERTIAL OSCILLATION 25
   Inertial Oscillation Equations 25
   Comparison of Stochastic-Dynamic and Monte-Carlo Forecasts 27
   Partitioning Model Errors: Inertial Oscillation Equations 43
   Summary 54

CHAPTER 5. HIERARCHY OF NUMERICAL WEATHER MODELS 55
   Model Hierarchy Formulation 55
   Numeric Model Structure 64
   Monte-Carlo Forecast Description 72
   Monte-Carlo Forecast and Analysis System 74
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Four Possible Unique Forecasts</td>
<td>12</td>
</tr>
<tr>
<td>4.1</td>
<td>Complex MSE Magnitude (Model 0)</td>
<td>36</td>
</tr>
<tr>
<td>4.2</td>
<td>Complex MSE Magnitude (Model 1)</td>
<td>38</td>
</tr>
<tr>
<td>4.3</td>
<td>Complex MSE Magnitude (Model 2)</td>
<td>39</td>
</tr>
<tr>
<td>4.4</td>
<td>Complex MSE Magnitude (Model 3)</td>
<td>41</td>
</tr>
<tr>
<td>4.5</td>
<td>Magnitude of CMSE(SD)/CMSE(MC)</td>
<td>42</td>
</tr>
<tr>
<td>4.6</td>
<td>Complex MSE Magnitude</td>
<td>50</td>
</tr>
<tr>
<td>4.7</td>
<td>Magnitude of Sequential Complex MSE Ratios</td>
<td>51</td>
</tr>
<tr>
<td>4.8</td>
<td>Magnitude of Complex MSE Ratios (w.r.t. true model)</td>
<td>53</td>
</tr>
<tr>
<td>5.1</td>
<td>Vertical Structure of Models</td>
<td>60</td>
</tr>
<tr>
<td>6.1</td>
<td>240 Hour &quot;True&quot; Forecast: $\psi$</td>
<td>87</td>
</tr>
<tr>
<td>6.2</td>
<td>240 Hour Mean Forecast: $\text{MCBVX } \psi$</td>
<td>87</td>
</tr>
<tr>
<td>6.3</td>
<td>240 Hour Mean Forecast: $\text{MCBCO } \psi$</td>
<td>88</td>
</tr>
<tr>
<td>6.4</td>
<td>240 Hour Mean Forecast: $\text{MCLBX } \psi$</td>
<td>88</td>
</tr>
<tr>
<td>6.5</td>
<td>240 Hour Model Bias: $\text{MCBVX } \psi$</td>
<td>90</td>
</tr>
<tr>
<td>6.6</td>
<td>240 Hour Model Bias: $\text{MCBCO } \psi$</td>
<td>90</td>
</tr>
<tr>
<td>6.7</td>
<td>240 Hour Model Bias: $\text{MCLBX } \psi$</td>
<td>91</td>
</tr>
<tr>
<td>6.8</td>
<td>Global Mean of Model Variance: $\psi$</td>
<td>91</td>
</tr>
<tr>
<td>6.9</td>
<td>$\log_{10}$ (Zonal Mean Variance): $\text{MCBVX } \psi$</td>
<td>93</td>
</tr>
<tr>
<td>6.10</td>
<td>$\log_{10}$ (Zonal Mean Variance): $\text{MCBCO } \psi$</td>
<td>93</td>
</tr>
<tr>
<td>6.11</td>
<td>$\log_{10}$ (Zonal Mean Variance): $\text{MCLBX } \psi$</td>
<td>94</td>
</tr>
<tr>
<td>6.12</td>
<td>$\log_{10}$ (240 Hour Forecast Variance): $\text{MCBVX } \psi$</td>
<td>94</td>
</tr>
</tbody>
</table>
Figure 6.13  $\log_{10}(240 \text{ Hour Forecast Variance}): \text{MCBCO} \psi$ 95

Figure 6.14  $\log_{10}(240 \text{ Hour Forecast Variance}): \text{MCLBX} \psi$ 95

Figure 6.15  Global Mean of Model TMSO: $\psi$ 97

Figure 6.16  $\log_{10}(\text{Zonal Mean TMSO}): \text{MCBVX} \psi$ 97

Figure 6.17  $\log_{10}(\text{Zonal Mean TMSO}): \text{MCBCO} \psi$ 98

Figure 6.18  $\log_{10}(\text{Zonal Mean TMSO}): \text{MCLBX} \psi$ 98

Figure 6.19  $\log_{10}(240 \text{ Hour Forecast TMSO}): \text{MCBVX} \psi$ 100

Figure 6.20  $\log_{10}(240 \text{ Hour Forecast TMSO}): \text{MCBCO} \psi$ 100

Figure 6.21  $\log_{10}(240 \text{ Hour Forecast TMSO}): \text{MCLBX} \psi$ 101

Figure 6.22  Global Relative Frequency ($F\leq F_2^c$): TMSO($\psi$) 101

Figure 6.23  Zonal Relative Frequency ($F\leq F_2^c$): MCBVX TMSO($\psi$) 102

Figure 6.24  Zonal Relative Frequency ($F\leq F_2^c$): MCBCO TMSO($\psi$) 102

Figure 6.25  Zonal Relative Frequency ($F\leq F_2^c$): MCLBX TMSO($\psi$) 104

Figure 6.26  240 Hour Forecast ($F\leq F_2^c$): MCBVX TMSO($\psi$) 104

Figure 6.27  240 Hour Forecast ($F>F_2^c$): MCBCO TMSO($\psi$) 105

Figure 6.28  240 Hour Forecast ($F>F_2^c$): MCLBX TMSO($\psi$) 105

Figure 6.29  240 Hour "True" Forecast: $\tau$ 109

Figure 6.30  240 Hour Mean Forecast: MCBCO $\tau$ 109

Figure 6.31  240 Hour Mean Forecast: MCLBX $\tau$ 110

Figure 6.32  240 Hour Model Bias: MCBCO $\tau$ 110

Figure 6.33  240 Hour Model Bias: MCLBX $\tau$ 112

Figure 6.34  Global Mean of Model Variance: $\tau$ 112
Figure 6.35 $\log_{10}(\text{Zonal Mean Variance})$: MCBCO $\tau$ 113
Figure 6.36 $\log_{10}(\text{Zonal Mean Variance})$: MCLBX $\tau$ 113
Figure 6.37 $\log_{10}(\text{240 Hour Forecast Variance})$: MCBCO $\tau$ 114
Figure 6.38 $\log_{10}(\text{240 Hour Forecast Variance})$: MCLBX $\tau$ 114
Figure 6.39 Global Mean of Model TMSO: $\tau$ 115
Figure 6.40 $\log_{10}(\text{Zonal Mean TMSO})$: MCBCO $\tau$ 115
Figure 6.41 $\log_{10}(\text{Zonal Mean TMSO})$: MCLBX $\tau$ 117
Figure 6.42 $\log_{10}(\text{240 Hour Forecast TMSO})$: MCBCO $\tau$ 117
Figure 6.43 $\log_{10}(\text{240 Hour Forecast TMSO})$: MCLBX $\tau$ 118
Figure 6.44 Global Relative Frequency ($F_{FS}^T$): TMSO($\tau$) 118
Figure 6.45 Zonal Relative Frequency ($F_{FS}^T$): MCBCO TMSO($\tau$) 119
Figure 6.46 Zonal Relative Frequency ($F_{FS}^T$): MCLBX TMSO($\tau$) 119
Figure 6.47 240 Hour Forecast $F(\text{TMSO})$: MCBCO $\tau$ 120
Figure 6.48 240 Hour Forecast $F(\text{TMSO})$: MCLBX $\tau$ 120
Figure 6.49 240 Hour "True" Forecast: $\sigma$ 122
Figure 6.50 240 Hour Mean Forecast: MCLBX $\sigma$ 122
Figure 6.51 240 Hour Model Bias: MCLBX $\sigma$ 124
Figure 6.52 Global Mean of Model Variance: $\sigma$ 124
Figure 6.53 Zonal Mean Variance: MCLBX $\sigma$ 125
Figure 6.54 240 Hour Forecast Variance: MCLBX $\sigma$ 125
Figure 6.55 Global Mean of Model TMSO: $\sigma$ 126
Figure 6.56 Zonal Mean TMSO: MCLBX $\sigma$ 126
Figure 6.57 240 Hour Forecast TMSO: MCLBX σ 127
Figure 6.58 Global Relative Frequency (F≤F_c): TMSO(σ) 127
Figure 6.59 Zonal Relative Frequency (F≤F_c): MCLBX TMSO(σ) 129
Figure 6.60 240 Hour Forecast F(TMSO): MCLBX σ 129
Figure 7.1 Global Mean TMSO(ψ) Components 146
Figure 7.2 Global Mean TMSO(ψ) Sequential Composition 146
Figure 7.3 Global Relative Frequency [F_L≤F≤F_U]: TMSO(ψ) 146
Figure 7.4 Zonal Mean TMSO(ψ) Sequential Composition: τ 149
Figure 7.5 Zonal Mean TMSO(ψ) Sequential Composition: σ(μ) 150
Figure 7.6 Zonal Mean TMSO(ψ) Sequential Composition: σ(λ,t) 151
Figure 7.7 Zonal Relative Frequency [F_L≤F≤F_U]: TMSO(ψ) τ Component 152
Figure 7.8 Zonal Relative Frequency [F_L≤F≤F_U]: TMSO(ψ) σ(λ,t) Component 152
Figure 7.9 Zonal Mean TMSO(ψ) Sequential Composition: Nonlinear Bias 154
Figure 7.10 Zonal Relative Frequency [F_L≤F≤F_U]: TMSO(ψ) Nonlinear Bias Component 154
Figure 7.11 240 Hour Forecast Total TMSO(ψ) 156
Figure 7.12 240 Hour Forecast TMSO(ψ) Component: τ 156
Figure 7.13 240 Hour Forecast TMSO(ψ) Component: σ(μ) 157
Figure 7.14 240 Hour Forecast TMSO(ψ) Component: σ(λ,t) 157
Figure 7.15 240 Hour Forecast TMSO(ψ) Component: Nonlinear Bias 159
Figure 7.16 240 Hour Forecast TMSO(ψ) Component: Residual Error 159
Figure 7.17 240 Hour Sequential F[TMSO(ψ) Component]: τ 160
Figure 7.18 240 Hour Sequential F[TMSO(ψ) Component]: σ(λ,t) 161
Figure 7.19 Global Mean TMSO(τ) Components 164
Figure 7.20 Global Mean TMSO(τ) Sequential Composition 164
Figure 7.21 Global Relative Frequency $[F_L \leq F \leq F_U]$: TMSO(τ) 164
Figure 7.22 Zonal Mean TMSO(τ) Sequential Composition: $σ(μ)$ 166
Figure 7.23 Zonal Mean TMSO(τ) Sequential Composition: $σ(λ,t)$ 167
Figure 7.24 Zonal Relative Frequency $[F_L \leq F \leq F_U]$: TMSO(τ)
$σ(λ,t)$ Component 167
Figure 7.25 Zonal Mean TMSO(τ) Sequential Composition:
Nonlinear Bias 169
Figure 7.26 Zonal Relative Frequency $[F_L \leq F \leq F_U]$: TMSO(τ)
Nonlinear Bias Component 169
Figure 7.27 240 Hour Forecast Total TMSO(τ) 171
Figure 7.28 240 Hour Forecast TMSO(τ) Component: $σ(μ)$ 171
Figure 7.29 240 Hour Forecast TMSO(τ) Component: $σ(λ,t)$ 172
Figure 7.30 240 Hour Forecast TMSO(τ) Component: Nonlinear Bias 172
Figure 7.31 240 Hour Forecast TMSO(τ) Component: Residual Error 173
Figure 7.32 240 Hour Sequential $F[TMSO(τ) \text{ Component}]: σ(λ,t)$ 174
Figure 7.33 240 Hour Sequential $F[TMSO(τ) \text{ Component}]:$
Nonlinear Bias 175
Figure 7.34 Global Mean TMSO(σ) Components 178
Figure 7.35 Global Mean TMSO(σ) Sequential Composition 178
Figure 7.36 Global Relative Frequency $[F_L \leq F \leq F_U]$: TMSO(σ) 178
Figure 7.37 Zonal Mean TMSO(σ) Sequential Composition:
Nonlinear Bias 179
Figure 7.38 Zonal Relative Frequency $[F_L \leq F \leq F_U]$: TMSO(σ)
Nonlinear Bias Component 179
Figure 7.39 240 Hour Forecast Total TMSO(σ) 181
Figure 7.40  240 Hour Forecast TMSO(σ) Component: Nonlinear Bias  181
Figure 7.41  240 Hour Forecast TMSO(σ) Component: Residual Error  182
Figure 7.42  240 Hour Sequential F[TMSO(σ) Component]: Nonlinear Bias  182
Figure F.1  Raw 5° x 5° Gridded Surface Terrain  260
Figure F.2  Filtered T15 Surface Terrain  261
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table 1.1</td>
<td>The Number of Predictive Equations Required for Deterministic and Stochastic-Dynamic Models</td>
<td>6</td>
</tr>
<tr>
<td>Table 4.1</td>
<td>SD vs. MC Forecast Run-times</td>
<td>34</td>
</tr>
<tr>
<td>Table 4.2</td>
<td>INERTMSE Run-times and Final Kinetic Energy</td>
<td>48</td>
</tr>
<tr>
<td>Table 5.1</td>
<td>Model Hierarchy</td>
<td>60</td>
</tr>
<tr>
<td>Table 5.2</td>
<td>Maximum $C_x$ (m/s) for $\Delta t = 45$ minutes</td>
<td>69</td>
</tr>
<tr>
<td>Table 6.1</td>
<td>Time to &quot;Effectively&quot; Biased Forecasts (hours)</td>
<td>130</td>
</tr>
<tr>
<td>Table C.1</td>
<td>$P(\theta)$ and $P(\theta^*)$ versus Lapse Rate</td>
<td>227</td>
</tr>
<tr>
<td>Table D.1</td>
<td>Regression variables and coefficients</td>
<td>238</td>
</tr>
<tr>
<td>Table D.2</td>
<td>Size of Variance-Covariance arrays</td>
<td>238</td>
</tr>
</tbody>
</table>
ACKNOWLEDGEMENTS

I wish to express my sincere thanks to Professors T.C. "Mike" Chen and Herbert T. David for their advice and constant encouragement during some of the more discouraging portions of this research. Additional thanks go to Dr. Chen for suggesting the NCAR fellowship and the use of an office during my semiannual Iowa trips from Colorado.

I would also like to thank Dr. Joseph J. Tribbia, my advisor and sponsor at NCAR, whose insights and comments helped clear up some of the mysteries of spectral modeling.

I also thank the other members of my committee: Professors Noel A.C. Cressie, Douglas N. Yarger, and Theodore H. Okishi. Additional thanks go to Dr. Cressie for his suggestions to improve the statistical analysis portion of this research.

I am grateful to the Statistics and Meteorology secretarial staff for typing the numerous forms needed to complete the degree.

Above all I acknowledge the love, patience and support, during the last six years, of my dear wife, Sherrie, and the understanding of my daughter, Mary, who had to tolerate a dad who couldn't play when he was in the (com) "puter room".

This research was done while the author was a Graduate Research Assistant in the Advanced Study Program at the National Center for Atmospheric Research (NCAR) in Boulder, Colorado.

Additional financial support was received from the National Science Foundation under NSF grant ATM-9001747.
CHAPTER 1
OVERVIEW AND BACKGROUND

Overview

Description of study

The aim of this study was to develop techniques to partition forecast errors of numerical weather prediction models. When the errors, measured by the mean squared error statistic, are partitioned, it is possible to determine the amount of error due to a particular source. The sources of error used for partitioning in this study are errors due to deleted model terms and errors in the initial conditions. Once the error has been partitioned, it is possible to track the time evolution of each error component.

In addition to the basic partitioning, techniques are developed to test if the individual partitions are negligible. The error partitioning, analysis, and testing techniques, developed in this study, may be used as a guide for systematically evaluating model improvement efforts.

Chapter synopsis

Chapters 1 through 4 form an introduction to the approach used by this study to handle the problem of partitioning forecast errors. The remainder of this chapter presents a brief background and overview of the stochastic-dynamic and Monte-Carlo forecasting techniques. Chapter 2 discusses the sources of errors in numerical weather forecasting and the effect of these errors upon the forecasts. Chapter 3 applies a general
Taylor series expansion approach to partition the solution of differential equations. The results are used in partitioning some common statistics. Chapter 4 presents two example applications to the general inertial oscillation equations; one application compared stochastic-dynamic and Monte-Carlo forecasts, while the other application partitioned the mean squared error statistic.

Chapters 5 to 7 form the core of this study. Chapter 5 describes the hierarchy of numerical weather models, used in the remainder of the study; various model parameterizations and numerical model details; and the Monte-Carlo forecast system of programs developed for initialization, forecasting, and analysis. Chapters 6 presents the basic statistics from the Monte-Carlo forecasts using the model hierarchy described in Chapter 5. Chapter 7 presents the development of partitioning and partition-testing techniques and application to the Monte-Carlo forecasts.

Chapter 8 summarizes the study results and makes suggestions for improvements and areas of future research.

Background

Introduction

Numerical weather prediction (NWP) has been an ongoing endeavor since the first successful numerical weather forecast, based on a limited-area one-layer barotropic model, was run by Drs. Jule Charney, R. Fjörtoft, and John von Neumann (1950). Since that time, the numerical weather models have grown in complexity, from simple barotropic and baroclinic models, covering a limited area and having crude physical parameterizations, to
the global primitive equation models having sophisticated physical parameterizations accounting for such effects as radiational heating, moisture, surface conditions, etc.

Much of the current NWP research is directed toward improving the modeling of the various boundary layer, small scale (subscale), heating, moisture (i.e. cloud), and upper atmospheric processes. With all the improvements in NWP models over the years, the forecasts still lose much of their predictive skill after two to three days (model time). This loss of skill is due to imperfect process parameterizations, limited spatial representation (both vertical and horizontal), model initialization (based on spatially limited observations of limited accuracy), and ultimately on the basic chaotic nature of the atmosphere and the descriptive equations (Lorenz, 1960).

This study examines an extension of the problem presented in the paper, *The Reliability of Improvements in Deterministic Short-Range Forecasts in the Presence of Initial State and Modeling Deficiencies*, by J.J Tribbia and D.P. Baumhefner (1988). Instead of just partitioning forecast error into a single model deficiency component and a random error component, this study uses forecasts from a hierarchy of successively more complex models in the partitioning of the model deficiency portion into three (maximum) components and random error portion into two components.

**Incorporating data errors**

Currently, operational weather forecasts are *deterministic forecasts*. These are implicitly based upon the assumption that the initialized model
data are error-free. This type of forecast is made out of necessity since many of the current models strain the computational capabilities. As will be seen later, models incorporating the effect of initial data errors are computationally more demanding than their deterministic counterparts.

In the 1960's, scientists began to examine how to incorporate the observed initializing data variability into forecast models with hopes of improving long-term forecast skill. From that period until now, the main forecasting techniques, incorporating the initial data errors, may be classified as being either a stochastic-dynamic (SD) or a Monte-Carlo (MC) method.

The strengths of the stochastic-dynamic and Monte-Carlo forecasts are (1) they take into account the inaccuracies of the initializing data, (2) on the average, they give accurate forecasts further out into the future, and (3) they also give a forecast of the prediction error. Partitioning a particular statistic of this prediction error is the item of interest in this study.

Stochastic-dynamic method The stochastic-dynamic method has grown out of efforts by Thompson, Gleeson, Epstein, Fleming, Leith, and others to obtain equations to forecast, for each variable of interest, not only the mean value, but second order (variance and covariances) and, possibly, higher order moments. Thompson (1957) first examined the effects of initial condition errors on forecasts. Gleeson (1966, 1970) and Epstein (1969) generalized the effects of errors and posed the problem in terms of the prediction of statistical moments (i.e. stochastic-dynamic problem), while Fleming (1971a, 1971b) discussed the question of the closure of the
predictive equations, with respect to higher order statistical moments. It is this truncation of higher-order statistical moments that limits the length of forecasts made with the resulting predictive equations. Leith (1971) began to examine the growth of forecast errors, in terms of turbulence, and develop an empirical growth relation.

Within the last ten years, Thompson (1985, 1986, and 1988) has continued to examine the SD approach. He suggests, in his 1985 paper, eliminating certain error covariance terms and expressing remaining covariances in terms of variances; this would significantly reduce the number of predictive equations. In the later papers he builds upon this idea and, with assumptions about initial conditions, examines which terms are the predominant ones in error propagation for simple two and three dimensional models.

The cost of a SD forecast over a deterministic forecast is the increased complexity of the model, both in the number and complexity of the nonlinear differential equations. This translates into increases in manpower costs (program development, modification, and maintenance) and computer run time costs.

Table 1.1 presents the number of equations for deterministic (means only) forecasts (Deter), second order SD (means and second order statistical moments) forecasts (SD2), and third-order SD (means, second and third order statistical moments) forecasts (SD3). It is obvious that the number of SD forecast equations, increasing approximately as the square of the number of variables, soon becomes unmanageable. So, without some assumptions, such as those presented by Thompson (1985, 1986, 1988),
concerning relations among higher-order moments of the variables, application of the SD approach seems to be relegated to small models of academic interest.

Monte-Carlo method The power of the Monte-Carlo (MC) approach is its relative simplicity of implementation. It calculates the statistics from multiple forecasts so that modeling the time evolution of specific population statistics is not needed. Each forecast is based upon slightly different initial values, randomly selected (thus the term Monte-Carlo) from a population having specified parameters (mean, variance, etc.), and the original deterministic model.

Table 1.1 The Number of Predictive Equations Required for Deterministic and Stochastic-Dynamic Models

<table>
<thead>
<tr>
<th>Number of Predicted Variables</th>
<th>Number of Predictive Equations</th>
<th>Total Number of Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Order of Moments</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Means</td>
<td>Second</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>15</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>21</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>28</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>36</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>45</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>55</td>
</tr>
</tbody>
</table>
For a given model, the trade-offs between the MC and SD methods are for the MC method, a larger sample size is needed to increase accuracy; whereas, for the SD method, the prediction of higher order moments (increased number of variables and equation complexity) is needed to increase accuracy.

With Monte-Carlo forecasts, as the number of variables increases, the number of model equations will increase and the number of replications should also be increased to match the increase in model degrees of freedom. The additional costs of implementing Monte-Carlo model, over those of a deterministic model, would be the costs incurred in implementing modules to generate the initial sample data and to obtain the output statistics. The maintenance and modification costs of a Monte-Carlo model would be about the same as those for a deterministic model since both use the same predictive equations.
CHAPTER 2

GENERAL DISCUSSION OF ERRORS IN NUMERICAL MODELS

This chapter begins with a general discussion of sources of errors and their effects upon model forecasts. Next some basic ideas on model verification, relative to an error-free and "operational" analyses, are presented and finally the presentation of a mean squared error expansion introduces the idea of partitioning forecast errors.

Sources of Errors

One cannot exactly forecast atmospheric conditions for a number of reasons: the exact governing equations and physics are unknown, the existing models have finite spatial and temporal resolution and the initializing data has limited accuracy and coverage. Realizing these limitations, it is helpful to briefly examine the sources of error and the effects of these errors on the numerical forecasts. Finally, we need to begin to examine if the resulting forecast errors can be partitioned by source, and, if so, examine the results of this partitioning.

Three general sources of error effect the numerical forecasts: (1) errors in the initial conditions, (2) deficiencies in the model, and (3) deficiencies in the computations.

Errors in the initial conditions

The errors in the initial conditions are due to errors in observed data, errors introduced in the analysis and interpolation of this data,
and errors introduced in the model initialization phase of the numerical model forecast. Lorenz (1962) has shown that, due to the underlying chaotic nature of the predictive equations, slightly different initial conditions will evolve into totally different atmospheric states.

The exact nature of these errors are not examined in this study. The initialized analysis values had pre assigned random properties.

**Observation errors**  The original data observations may have errors due to (1) measurement error (instrument precision, response time, and bias), (2) data coverage (limited spatial resolution and coverage of measurement points and observation time), (3) data reduction/derivation (variables not observed directly). Effectively, the original data represents observations taken at irregularly spaced locations at different time with limited accuracy.

**Analysis errors**  The original data are inputs to an analysis program which spatially interpolates the variables to values on a fixed grid to be used by the numerical model. The analysis introduces errors due to (1) gridding (interpolating) the data, (2) smoothing small-scale features, (3) gridding technique, (4) the finite resolution of the grid, and (5) differences in observation times.

**Initialization errors**  Finally, the analyzed gridded data are inputted into the numerical model where they undergo further modification before they are used in making forecasts. This model initialization checks and adjusts the values of the variables so they are consistent with the model physics.
Deficiencies in the model

Model deficiencies may be divided into three groupings: (1) missing physical processes, (2) mathematical and physical process approximations, and (3) resolution deficiencies.

Examples of possible missing physical processes are moisture, cloud, and radiation processes.

The process approximations include mathematical approximations (of numerical functions, derivatives, and integrations) and parameterizations of physical processes (i.e. subscale parameterizations, cloud, moisture, and radiation). The effects of the parameterizations are difficult to evaluate since the parameterizations are often for poorly understood processes.

The resolution deficiencies include both spatial and temporal ones. The errors due to model deficiencies in directly handling small-scale features are resolution-related deficiencies.

Computational deficiencies

The main computational deficiency of note is the order of numerical approximation used in the solution of the differential equations. Another deficiency is the finite precision used in numerical representation. The computational deficiencies are assumed to be negligible in this study.

Effect of Errors on Model Forecasts

Basic definitions

Since the atmosphere cannot be modeled exactly, the best that can be hoped for is a reasonable approximation to actual atmospheric processes.
To define the problem, let $\mathbf{x}^M(t,e)$ represent a vector of forecast variables from model "M" at time "t" based upon initial data with error "e". For instance, define

$\mathbf{x}^T(0,0)$ - Initial: True atmospheric state
$\mathbf{x}^A(0,e)$ - Initial: Analyzed atmospheric state
$\mathbf{x}^M(0,0)$ - Initial: Model initialized from $\mathbf{x}^T(0,0)$
$\mathbf{x}^M(0,e)$ - Initial: Model initialized from $\mathbf{x}^A(0,e)$

$\mathbf{x}^T(t,0)$ - Time t: True atmospheric state
$\mathbf{x}^A(t,0)$ - Time t: Analyzed atmospheric state
$\mathbf{x}^A(t,e)$ - Time t: Analyzed atmospheric data
$\mathbf{x}^M(t,0)$ - Time t: Model forecast atmospheric state, based upon $\mathbf{x}^M(0,0)$
$\mathbf{x}^M(t,e)$ - Time t: Model forecast atmospheric state, based upon $\mathbf{x}^M(0,e)$

The respective model and true physical processes are

$\mathcal{F}_M(\mathbf{x})$ - physical processes in model atmosphere
$\mathcal{F}_T(\mathbf{x})$ - physical processes in "true" atmosphere

Four unique forecasts, diagramed in Figure 2.1, may be made using these definitions. These are defined as
Forecast I: Perfect forecast based upon perfect initial data  
(NOTE: this represents the actual atmosphere)

\[ \mathcal{X}_T(t,0) = \mathcal{X}_T(0,0) + \int_0^t H_T(\mathcal{X}_T(t',0)) \, dt' \]

Forecast II: Perfect forecast based upon analyzed initial data.  
(NOTE: this represents the actual atmosphere with a slightly different initial condition)

\[ \mathcal{X}_A(t,\varepsilon) = \mathcal{X}_A(0,\varepsilon) + \int_0^t H_T(\mathcal{X}_A(t',\varepsilon)) \, dt' \]

Forecast III: Model forecast based upon perfect initial data  
(NOTE: this represents a model forecast with a slightly different initial condition)

\[ \mathcal{X}_M(t,0) = \mathcal{X}_M(0,0) + \int_0^t H_M(\mathcal{X}_M(t',0)) \, dt' \]

Forecast IV: Model forecast based upon analyzed initial data  
(NOTE: this is the operational forecast)

\[ \mathcal{X}_G(t,\varepsilon) = \mathcal{X}_G(0,\varepsilon) + \int_0^t H_M(\mathcal{X}_G(t',\varepsilon)) \, dt' \]
Model verification

In actual operation, the model forecast is verified against the analysis. The resulting apparent (operational) model prediction error is

\[ e_0(t) = \hat{x}_H(t,e) - \hat{x}_A(t,e) . \]

Thus, the normal verification method measures the performance of the forecast model against the analysis; both have inherent errors. What is really desired is to verify the model forecast against the true atmosphere at time \( t \):

\[ e_T(t) = \hat{x}_H(t,e) - \hat{x}_T(t,0) . \]

While \( e_T(t) \) is the desired verification quantity, it is not practical since it implies the availability of perfect verifying observations at each grid point.
Partitioning forecast errors

A simple approach to the partitioning of forecast errors is to add and subtract $\hat{x}_n(t,0)$ and $\hat{x}_A(t,0)$ from the equation for the true error to get

$$e(t) = \hat{x}(t) - \hat{x}_A(t) + \hat{x}_A(t,0) - x(t,0).$$

The first term represents the error, for a given model, due to initial data errors. The second term represents the operational forecast bias, with respect to the verifying analysis, assuming no observational errors. The final term represents the bias of the verifying analysis against the true atmosphere, again assuming no observational errors.

The operational error may be partitioned in a similar manner:

$$e_o(t) = \left[ \hat{x}(t,e) - \hat{x}_A(t,0) \right] + \left[ \hat{x}_n(t,0) - \hat{x}_A(t,0) \right] + \left[ \hat{x}_A(t,0) - \hat{x}_T(t) \right] - \hat{x}_A(t,0) - \hat{x}_T(t)$$

$$= e_{eIM}(t) + \hat{e}_{MIA}(t) + \hat{e}_A(t) - \hat{e}_{AC}(t)$$

$$= e_T(t) - \hat{e}_{AC}(t)$$

the additional term represents error due to the analysis of observational data (combined effect of analysis and observational errors). Thus the
operational error is the total error minus the error in the verifying field due to the analysis of data observed at the verifying time.

Taking the expected values of the squared true error gives the true mean squared error:

$$\left\{ \text{MSE}_T(t) \right\} = E \left[ \left( \hat{\varepsilon}_T(t) \right) \left( \hat{\varepsilon}_T(t) \right)' \right]$$

$$\left\{ \text{MSE}_T(t) \right\} = \sum_T \left( \varepsilon_{1H}(t) + \left( \vec{B}_{HIA}(t) + \vec{B}_{A}(t) \right) \left( \vec{B}_{HIA}(t) + \vec{B}_{A}(t) \right)' \right)$$

The expected value of the squared operational error gives the operational mean squared error

$$\left\{ \text{MSE}_O(t) \right\} = \left\{ \text{MSE}_T(t) \right\} - 2 E \left[ \varepsilon_T \varepsilon_{AC}' \right] + \sum_{AC}$$

**Summary**

This chapter discussed some of the sources of forecast errors and began to look at the possible forecasts and how to partition the errors of these forecasts.

The next chapter examines the problems of partitioning the errors in models described by differential equations.
CHAPTER 3

ERROR PARTITIONING FOR DIFFERENTIAL EQUATIONS

Partitioning: General Approach

Forecast errors of a model may be partitioned by first expressing the
governing model equations as sums of component terms and then expanding
each equation as a Taylor series about the exact initial conditions. Each
of the equation component terms may be nonlinear; but, for partitioning
purposes, they are treated as "black boxes".

Suppose the set of governing differential equations (the model),
describing the time evolution of the vector, \( \dot{\mathbf{x}} = \dot{\mathbf{x}}(t, \varepsilon) \), is expressed as
the sum of components. If the full, or true, model is

\[
\frac{d\mathbf{x}}{dt} = \dot{A}_0(X) + \dot{A}_1(X) + \dot{A}_2(X) + \ldots + \dot{A}_N(X).
\]  

(3.1)

A hierarchy, or family, of progressively more complex models is

Model 0: \[ \frac{d\mathbf{x}}{dt} = \dot{A}_0(X) \]

Model 1: \[ \frac{d\mathbf{x}}{dt} = \dot{A}_0(X) + \dot{A}_1(X) \]

Model 2: \[ \frac{d\mathbf{x}}{dt} = \dot{A}_0(X) + \dot{A}_1(X) + \dot{A}_2(X) \]

etc.
Note the sequential nature of this hierarchy - each lower order model is contained in all higher order models. Also note that each component, \( \dot{A}_j(\dot{X}) \), may be a nonlinear function of \( \dot{X}(t,e) \).

Partitioning model component effects

The Taylor series expansion of the solution, \( \dot{X}(t,e) \), about the initial starting point, \( \dot{X}(0,e) \),

\[
\dot{X}(t,e) = \dot{X}(0,e) + \left. \frac{d\dot{X}}{dt} \right|_{t=0} t + \left. \frac{d^2\dot{X}}{dt^2} \right|_{t=0} \frac{t^2}{2!} + \left. \frac{d^3\dot{X}}{dt^3} \right|_{t=0} \frac{t^3}{3!} + \ldots
\]

is used to begin to examine the effect of successive model terms on the final true solution. Substituting the model differential equations for the time derivatives in the expansion gives

\[
\dot{X}_T(t,e) = \dot{X}(0,e) + \sum_{j=0}^{T} \left[ \dot{A}_j(J) \right] \left. \frac{t^j}{j!} \right|_{t=0} + \ldots
\]

This expansion may be partitioned by model and numerical truncation error, due to integration accurate to the \( k \)-th order time derivative,

\[
\dot{X}(t,e) = \dot{X}(0,e) + \sum_{j=0}^{T} \left[ \sum_{i=0}^{k} \left. \frac{d^i\dot{A}_j(J)}{dt^i} \right|_{t=0} \frac{t^i}{i!} \right] + \sum_{j=0}^{T} \left[ \sum_{i=k+1}^{\infty} \left. \frac{d^i\dot{A}_j(J)}{dt^i} \right|_{t=0} \frac{t^i}{i!} \right].
\]
The first two components of equation (3.4) are the model solution contributions; the last summation represents the numerical integration truncation error.

Subscript \( j \) is the model index. To this point, the solution, \( \hat{X}(t,\varepsilon) \), has been partitioned into the following components:

\[
\hat{X}(t,\varepsilon) = \hat{X}(0,\varepsilon) + \delta\hat{X}_0(t,\varepsilon) + \delta\hat{X}_{1:0}(t,\varepsilon) + \ldots + \delta\hat{X}_{T:T-1}(t,\varepsilon) + \]

\[
\text{+ Numerical truncation components} \quad (3.5)
\]

where

\[
\delta\dot{X}_{j:j-1}(t) = \sum_{i=0}^{k} \frac{d^i}{dt^i} \delta\dot{X}_{j}(X) \bigg|_{t=0} t^i/1! \quad (3.6)
\]

represents the unique contribution of model \( j \), over that of model \( j-1 \), in the calculation of \( \hat{X}(t,\varepsilon) \). This study assumed the numerical integration scheme is of sufficient accuracy to make the truncation contribution negligible.

The estimates of \( \hat{X}(t,\varepsilon) \) from the family of models are

\[
\hat{X}_0(t,\varepsilon) = \hat{X}(0,\varepsilon) + \delta\hat{X}_0(t,\varepsilon)
\]

\[
\hat{X}_1(t,\varepsilon) = \hat{X}(0,\varepsilon) + \delta\hat{X}_0(t,\varepsilon) + \delta\hat{X}_{1:0}(t,\varepsilon) = \hat{X}_0(t,\varepsilon) + \delta\hat{X}_{1:0}(t,\varepsilon)
\]

\[
\hat{X}_2(t,\varepsilon) = \hat{X}(0,\varepsilon) + \delta\hat{X}_0(t,\varepsilon) + \delta\hat{X}_{1:0}(t,\varepsilon) + \delta\hat{X}_{2:1}(t,\varepsilon) = \hat{X}_1(t,\varepsilon) + \delta\hat{X}_{2:1}(t,\varepsilon)
\]

etc.
The solution, \( \hat{X}_T(t, \epsilon) \), is equal to the initial value, \( \hat{X}(0, \epsilon) \), plus the sequential sum of the unique contributions of the sequence of subset models.

**Partitioning: Effect of initial data errors**

The previous section dealt with the problem of partitioning effects of the various model components on the final solution. This section takes a somewhat similar approach to partition out the effect of random errors in the initial conditions.

The vectors
\[
\begin{align*}
\vec{X} &= \hat{X}(t, \epsilon) = \left( x_i(t, \epsilon) \right)_{i=1, \ldots, N} \\
\vec{X}_0 &= \hat{X}(0, 0) \\
\vec{e}(0, \epsilon) &= \hat{X}(0, \epsilon) - \hat{X}(0, 0) = \left( \epsilon_i \right)_{i=1, \ldots, N}
\end{align*}
\]

represent the forecast vector, the initial vector, and the initial random error vector, respectively.

Expanding the model component, \( \hat{A}_j(X) \big|_{t=0} \) in a Taylor series about the component, \( \hat{A}_j(X_0) \), gives

\[
\hat{A}_j(X) = \hat{A}_j(X_0) + \sum_{i=1}^{N} \frac{\partial \hat{A}_j}{\partial x_i} \bigg|_{(t, \epsilon) = 0} \epsilon_i + \sum_{i=1}^{N} \sum_{m=1}^{N} \frac{\partial^2 \hat{A}_j}{\partial x_i \partial x_m} \bigg|_{(t, \epsilon) = 0} \epsilon_i \epsilon_m / 2! + \ldots
\]
Substituting the Taylor series expansion of $\hat{A}_j(\hat{X})$ into equation (3.4) gives the following series expansion which is infinite in two dimensions:

$$\hat{X}(t,\varepsilon) = \hat{X}(0,0) + \hat{\varepsilon}(0) +$$

$$+ \sum_{j=0}^{T} \left[ \frac{d^1}{dt^1} \hat{A}_j(\hat{X}_0) + \sum_{l=1}^{N} \frac{\partial \hat{A}_j}{\partial x_l} |_{(t,\varepsilon)=0} \varepsilon_l + \sum_{l=1}^{N} \sum_{m=1}^{N} \frac{\partial^2 \hat{A}_j}{\partial x_l \partial x_m} |_{(t,\varepsilon)=0} \varepsilon_l \varepsilon_m / 2! + \ldots \right] \frac{t^j}{j!} \right]. \quad (3.11)$$

The summations in equation (3.11) may be interchanged to partition out the error component

$$\hat{X}(t,\varepsilon) = \hat{X}(0,0) + \sum_{j=0}^{T} \left[ \frac{d^1}{dt^1} \hat{A}_j(\hat{X}_0) \right] +$$

$$+ \hat{\varepsilon}(0) + \sum_{j=0}^{T} \left[ \frac{d^1}{dt^1} \left( \sum_{l=1}^{N} \frac{\partial \hat{A}_j}{\partial x_l} |_{(t,\varepsilon)=0} \varepsilon_l + \sum_{l=1}^{N} \sum_{m=1}^{N} \frac{\partial^2 \hat{A}_j}{\partial x_l \partial x_m} |_{(t,\varepsilon)=0} \varepsilon_l \varepsilon_m / 2! + \ldots \right] \frac{t^j}{j!} \right]. \quad (3.12)$$

Define the following quantities: (1) the exact integrated change of $\hat{X}_0$,

$$M(\hat{X}_0) = \sum_{j=0}^{T} \left[ \frac{d^1}{dt^1} \hat{A}_j(\hat{X}_0) \right] \frac{t^j}{j!} , \quad (3.13)$$
(2) the effect of the initial condition errors,

\[ \vec{E}(0) = \vec{z}(0) \quad , \tag{3.14} \]

and (3) the additional integrated change of \( \vec{X} \) due to errors in the initial conditions (interaction of the model and initial errors),

\[ \overrightarrow{ME}(\vec{X}_0, \vec{z}) = \]

\[ \sum_{j=0}^{T} \left( \sum_{t=0}^{\infty} \frac{d^t}{dt^t} \left( \sum_{l=1}^{N} \frac{\partial^{2^j} \vec{M}}{\partial x_{l}^{2^j}} \right) \frac{\varepsilon_{l}}{l!} \right) \frac{t^j}{j!} \] \tag{3.15}

Using this shortened notation, the expansion of \( \vec{X}(t, \varepsilon) \), equation (3.12) may be expressed in a variety of ways:

\[ \vec{X}(t, \varepsilon) = \vec{X}(0, 0) + \vec{E}(0) + \vec{M}(\vec{X}_0) + \overrightarrow{EM}(\vec{X}_0, \vec{z}) \]

\[ = \vec{X}(0, \varepsilon) + \vec{M}(\vec{X}_0) + \overrightarrow{EM}(\vec{X}_0, \vec{z}) \]

\[ = \vec{X}(t, 0) + \vec{E}(0) + \overrightarrow{EM}(\vec{X}_0, \vec{z}) \]

**Partitioning:** Combining both model and initial data errors

The partitioning of both model and initial error effects in equation (3.12) is obvious since the the component parts, equations (3.13) and (3.15) are written as summations over the model components. Using this information equation (3.12) becomes
\[ \mathbf{x}(t, \varepsilon) = \mathbf{x}(0, 0) + \sum_{j=0}^{T} \mathbf{M}_j(\mathbf{x}_0) + \sum_{j=0}^{T} \mathbf{E}_j(\mathbf{x}_0) \]  

(3.16)

The resulting expansion is a linear sum of terms due to (1) model terms, (2) pure initial error effects, and (3) interactions between the model terms and initial error effects. It is this type of linearization that is used in this study to partition error by source.

**Partitioning Statistics**

**Mean**

Taking the expected value of equation (3.12), assuming the expected initial \( \mathbf{x} \) is \( \mathbf{x}_0 \), gives the mean vector

\[ \mathbf{\mu}_x(t, \varepsilon) = \mathbf{x}(0, 0) + \sum_{j=0}^{T} \sum_{l=0}^{d^t_1} \left[ \mathbf{A}_j \mathbf{x}_0 \right] t^l_1! + \sum_{j=0}^{T} \sum_{l=0}^{d^t_1} \left[ \sum_{m=1}^{N} \sum_{n=1}^{N} \frac{\partial^2 \mathbf{x}}{\partial x^l_1 \partial x^m_1} \right] \sigma_{lm} / 2! + \ldots \]  

(3.17)

**Bias**

The bias vector is just the difference between the mean and the true vectors

\[ \mathbf{b}_x(t, \varepsilon) = \mathbf{x}(t, 0) - \mathbf{\mu}_x(t, \varepsilon) \]
or

\[ \vec{B}_X(t, \varepsilon) = \sum_{j=0}^{T} \left( \sum_{l=0}^{\infty} \frac{d^l}{dt^l} \left( \sum_{l=1m=1}^{N} \frac{\partial^n A_j}{\partial x_1 \partial x_m} \left| (t, \varepsilon) = 0 \right. \right) \left. \right) \left( \sigma_{1m}/2! + \ldots \right) \right) \]

(3.18)

Note, if \( A_j \) is a linear function of \( \vec{x} \) then the model bias is zero; so the bias may be termed a nonlinear bias or bias due to the nonlinear model response to errors in the initial condition.

Variance-covariance

The variance-covariance matrix is determined by first defining the difference between the observed and expected vectors:

\[ \delta \vec{X}(t, \varepsilon) = \hat{\varepsilon}(0) + \]

\[ + \sum_{j=0}^{T} \left( \sum_{l=0}^{\infty} \frac{d^l}{dt^l} \left( \sum_{l=1m=1}^{N} \frac{\partial^n A_j}{\partial x_1 \partial x_m} \left| (t, \varepsilon) = 0 \right. \right) \left. \right) \left( \sigma_{1m}/2! + \ldots \right) \right) \]

(3.19)

The variance-covariance matrix is just the expected value of the difference vector and its transpose

\[ \Sigma = \mathbb{E} \left[ \left( \delta \vec{X}(t, \varepsilon) \right) \left( \delta \vec{X}(t, \varepsilon) \right)' \right] \]

(3.20)
Since the difference vector may be partitioned, the variance-covariance matrix components also may be partitioned.

Mean squared error

The mean squared error (MSE) matrix is simply the sum of the variance-covariance matrix and the product of the bias vector and its transpose,

\[ \text{MSE} = \sum + \left( \hat{\beta}_x(t, \epsilon) \right) \left( \hat{\beta}_x(t, \epsilon) \right)' \]  \hspace{1cm} (3.21)

Since the elements of the variance-covariance matrix and bias vector may be partitioned, the MSE matrix also may be partitioned.

Summary

In a model hierarchy described by differential equations consisting of sums of (possibly nonlinear) component terms, the effects of model terms, initial errors and model term-initial error interactions may be partitioned in the elements of the mean and bias vectors and the variance-covariance and mean squared error matrices.
CHAPTER 4

PRELIMINARY APPLICATION: INERTIAL OSCILLATION

This chapter presents an example application comparing stochastic-dynamic (SD) forecasts with Monte-Carlo (MC) forecasts and an example of forecast mean squared error partitioning. The inertial oscillation system of equations from the underlying model used in both examples. The first example shows the difficulties in implementing a SD forecast; the second example shows some of the problems in interpreting the error partitioning results. Both examples show how a hierarchy of models may be defined based upon a system of coupled non-linear differential equations.

Inertial Oscillation Equations

The inertial oscillation equations describe motion subject only to the apparent Coriolis and centrifugal forces. Compared with the full set of equations of motion, the inertial oscillation equations relatively simple, but they still retain the basic problems of partitioning forecast errors in a physical system described by simultaneous differential equations consisting of sums of linear and nonlinear terms.

The basic meteorological momentum equations are

\[
\frac{du}{dt} - 2\Omega \sin \phi \, v + 2\Omega \cos \phi \, w + \frac{uv}{a} - \frac{uv \tan \phi}{a} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + F_x \tag{4.1}
\]

\[
\frac{dv}{dt} + 2\Omega \sin \phi \, u + \frac{vw}{a} - \frac{u^2 \tan \phi}{a} = -\frac{1}{\rho} \frac{\partial P}{\partial y} + F_y \tag{4.2}
\]
\[
\frac{dw}{dt} + 2\Omega \cos \phi \ u + \frac{u^2 + v^2}{a} = \frac{1}{\rho} \frac{\partial \rho}{\partial z} - g + g_f \tag{4.3}
\]

where

\begin{align*}
  u & \quad \text{velocity component (east-west)} \\
  v & \quad \text{velocity component (north-south)} \\
  w & \quad \text{velocity component (vertical)} \\
  a & \quad \text{mean radius of the earth} \\
  g & \quad \text{acceleration of gravity} \\
  P & \quad \text{atmospheric pressure} \\
  \Omega & \quad \text{angular velocity of earth's rotation} \\
  \phi & \quad \text{latitude} \\
  g_f & \quad \text{friction}
\end{align*}

These equations, along with the perfect gas law, a conservation of mass equation, and a thermodynamic equation, form the basis for all numerical weather models. The inertial oscillation equations are just simplified versions of the basic momentum equations.

**Assumptions and resulting equations**

Pure inertial oscillation exists, under frictionless horizontal flow conditions, in a horizontally uniform pressure field, on a rotating surface. When these conditions are applied to the momentum equations, (4.1) to (4.3), the results are the inertial oscillation equations:

\[
\frac{du}{dt} = 2\Omega \sin \phi \ v + \frac{uv \tan \phi}{a} \tag{4.4}
\]

\[
\frac{dv}{dt} = -2\Omega \sin \phi \ u + \frac{u^2 \tan \phi}{a} \tag{4.5}
\]
The two terms on the right hand side of equations (4.4) and (4.5) are the Coriolis and centrifugal force, respectively. Combining these two terms, give

\[
\frac{du}{dt} = \left(2\Omega + \frac{u}{a \cos \phi}\right) \sin \phi \ v = f^* v \tag{4.6}
\]

\[
\frac{dv}{dt} = -\left(2\Omega + \frac{u}{a \cos \phi}\right) \sin \phi \ u = -f^* u \tag{4.7}
\]

where \(f^*\) may be termed the effective Coriolis parameter.

Using equations (4.6) and (4.7), it is easy to show that inertial oscillation conserves the kinetic energy of a particle. Using this kinetic energy conservation, the following conservation relation of the population means and variances of particle velocities is obtained

\[
\sigma_u^2(t) + \sigma_v^2(t) + \mu_u^2(t) + \mu_v^2(t) = \text{constant} \tag{4.8}
\]

The derivation details are presented in Appendix B.

The kinetic energy conservation relation and equation (4.8) form a convenient check of the derived SD predictive equations and the numerical results from MC forecasts.

**Comparison of Stochastic-Dynamic and Monte-Carlo Forecasts**

The SD equations are obtained by applying the Delta-Method technique to equations (4.6) and (4.7). The details of the technique, as applied to the SD equation derivation, are in Appendix B. The resulting
differential equations comprise a second order SD predictive set of equations for the means, variances, and covariances.

Model Equations

Basic model equations To simplify the derivation and resulting SD equations, this analysis used, as a full model, equations of motion describing inertial oscillation motions on a beta plane with the inclusion of centrifugal force due to the relative motion. The basic equations of the full model are

\[
\begin{align*}
\frac{du}{dt} &= a_0 u + a_1 vy + a_2 uv + a_3 uvy \\
\frac{dv}{dt} &= b_0 u + b_1 uy + b_2 u^2 + b_3 u^2 y \\
\frac{dy}{dt} &= v
\end{align*}
\] (4.9) (4.10) (4.11)

The equation constants are

\[
\begin{align*}
a_0 &= -b_0 = 2\Omega \sin \phi_0 \\
a_1 &= -b_1 = \frac{2\Omega \cos \phi_0}{a} \\
a_2 &= -b_2 = \frac{u}{a} \tan \phi_0 \\
a_3 &= -b_3 = \frac{1}{(a \cos \phi_0)^2}
\end{align*}
\] (4.12) (4.13) (4.14) (4.15)

where \( \phi_0 \) is a reference latitude.
The first right hand term of equations (4.9) and (4.10) represents oscillation with a constant magnitude Coriolis force, the *f-plane* approximation, and the second term on the right represents the first order effects of the change of Coriolis force with latitude; together, these terms represent the *β-plane* approximation to the actual inertial oscillation. The final two right hand terms represent the centrifugal force due to motion in the east-west direction, on the *β-plane*.

A hierarchy of forecast models may be constructed by setting pairs of constants, \((a_i, b_i)\), to zero:

**Model 0:**
\[
(a_1, b_1) = (0, 0) \\
(a_2, b_2) = (0, 0) \\
(a_3, b_3) = (0, 0)
\]

**Model 1:**
\[
(a_2, b_2) = (0, 0) \\
(a_3, b_3) = (0, 0)
\]

**Model 2:**
\[
(a_3, b_3) = (0, 0)
\]

**Model 3:** all pairs are not equal to \((0, 0)\).

Model 0 represents the classic *f-plane* inertial oscillation, model 1 represents inertial oscillation on the *β-plane*. Models 2 and 3 adds the effect of the relative motion.
Application of the Delta-Method to equations (4.9) - (4.11) give the stochastic-dynamic (SD) predictive equations for the means, variances, and covariances.

**SD predictive equations: means**  
The SD equations for the means are

$$\frac{d}{dt} \mu_u = a_0 \mu_v +$$

$$+ a_1 \left( \mu_u \mu_y + \sigma_{vy} \right) +$$

$$+ a_2 \left( \mu_u \mu_v + \sigma_{uv} \right) +$$

$$+ a_3 \left( \mu_u \mu_y + \mu_v \sigma_{vy} + \mu_v \sigma_{uy} + \mu_y \sigma_{uv} + \sigma_{uvy} \right)$$  \(\text{(B.10)}\)

$$\frac{d}{dt} \mu_v = b_0 \mu_u +$$

$$+ b_1 \left( \mu_u \mu_y + \sigma_{uy} \right) +$$

$$+ b_2 \left( \mu_u \mu_v + \sigma_{uv} \right) +$$

$$+ b_3 \left( \mu_u \mu_y + \mu_u \sigma_{uy} + \mu_v \sigma_{uy} + \mu_y \sigma_{uv} + \sigma_{uvy} \right)$$  \(\text{(B.11)}\)

$$\frac{d}{dt} \mu_y = \mu_v$$  \(\text{(B.12)}\)
The SD equations for the variances are

$$\frac{d\sigma^2}{dt} = 2a_0 \sigma_{uv} + 2a_1 \left( \mu_v \sigma_{uv} + \mu_y \sigma_{uv} + \sigma_{uvy} \right) \quad \text{(B.16)}$$

$$\frac{d\sigma^2}{dt} = 2b_0 \sigma_{uv} + 2b_1 \left( \mu_v \sigma_{uv} + \mu_y \sigma_{uv} + \sigma_{uvy} \right) \quad \text{(B.17)}$$

The SD equations for the covariances are

$$\frac{d\sigma}{dt} = a_0 \sigma_{vv} + b_0 \sigma_{uu} + \frac{d\sigma}{dt} \quad \text{(B.18)}$$
\[ + a_1 \left( \mu \sigma_{\nu\nu} + \mu \sigma_{\nu\nu} + \sigma_{\nu\nu} \right) + b_1 \left( \mu \sigma_{\nu\nu} + \mu \sigma_{\nu\nu} + \sigma_{\nu\nu} \right) + \\
+ a_2 \left( \mu \sigma_{\nu\nu} + \mu \sigma_{\nu\nu} + \sigma_{\nu\nu} \right) + b_2 \left( \mu \sigma_{\nu\nu} + \mu \sigma_{\nu\nu} + \sigma_{\nu\nu} \right) + \\
+ a_3 \left( \mu \mu_{\nu\nu} + \mu \mu_{\nu\nu} + \mu \mu_{\nu\nu} + \mu \mu_{\nu\nu} \right) + \\
+ \mu_{\nu\nu} \sigma_{\nu\nu} + \mu_{\nu\nu} \sigma_{\nu\nu} + \mu_{\nu\nu} \sigma_{\nu\nu} + \sigma_{\nu\nu} \right) + \\
+ b_3 \left( \mu \mu_{\nu\nu} + \mu \mu_{\nu\nu} + \mu \mu_{\nu\nu} + \mu \mu_{\nu\nu} \right) + \\
+ \mu_{\nu\nu} \sigma_{\nu\nu} + \mu_{\nu\nu} \sigma_{\nu\nu} + \mu_{\nu\nu} \sigma_{\nu\nu} + \sigma_{\nu\nu} \right) \]  
(B.19)

\[ \frac{d}{dt} \sigma_{\nu\nu} = \sigma_{\nu\nu} + a_0 \sigma_{\nu\nu} + a_1 \left( \mu \sigma_{\nu\nu} + \mu \sigma_{\nu\nu} + \sigma_{\nu\nu} \right) + \\
+ a_2 \left( \mu \sigma_{\nu\nu} + \mu \sigma_{\nu\nu} + \sigma_{\nu\nu} \right) + \\
+ a_3 \left( \mu \mu_{\nu\nu} + \mu \mu_{\nu\nu} + \mu \mu_{\nu\nu} + \mu \mu_{\nu\nu} \right) + \\
+ \mu_{\nu\nu} \sigma_{\nu\nu} + \mu_{\nu\nu} \sigma_{\nu\nu} + \mu_{\nu\nu} \sigma_{\nu\nu} + \sigma_{\nu\nu} \right) \]  
(B.20)

\[ \frac{d}{dt} \sigma_{\nu y} = \sigma_{\nu y} + b_0 \sigma_{\nu y} + b_1 \left( \mu \sigma_{\nu y} + \mu \sigma_{\nu y} + \sigma_{\nu y} \right) + \\
+ b_2 \left( \mu \sigma_{\nu y} + \mu \sigma_{\nu y} + \sigma_{\nu y} \right) + \\
+ b_3 \left( \mu \mu_{\nu y} + \mu \mu_{\nu y} + \mu \mu_{\nu y} + \mu \mu_{\nu y} \right) + \\
+ \mu_{\nu y} \sigma_{\nu y} + \mu_{\nu y} \sigma_{\nu y} + \mu_{\nu y} \sigma_{\nu y} + \sigma_{\nu y} \right) \]
Notice the equations for the variances and covariances are in the linear component form.

The above SD forecast equations conserve the quantity

$$\sigma_u^2 + \sigma_v^2 + \mu_u^2 + \mu_v^2$$

Numerical model: Stochastic-dynamic and Monte-Carlo comparison

A computer program, STOCHAST, was written to compare forecasts based upon the SD predictive equations with MC forecasts and deterministic (DET) forecasts.

STOCHAST initially generated a sample of 100 random independent normal velocity component values \((u,v)\); these values were systematically adjusted to give a sample having user specified means, \((\mu_u, \mu_v)\) and variances \((\sigma_u^2, \sigma_v^2)\), while maintaining a random sample correlation. The initial latitude was assumed to be exact.

The differential equations were solved using a fourth-order Runge-Kutta integration scheme and 400 time steps per reference period. The reference period, defined as the \(f\)-plane inertial oscillation period, was set to 60000 seconds by assigning the appropriate reference latitude.
The compiled code was written using Microsoft QuickBASIC® Version 4.0 and run on a 33MHz 80486-based personal computer under MS-DOS® 5.0. The model run-times are tabulated in Table 4.1.

Table 4.1. Forecast Run-Times

<table>
<thead>
<tr>
<th>Type of Forecast</th>
<th>Run-Time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deterministic</td>
<td>4</td>
</tr>
<tr>
<td>Monte-Carlo</td>
<td>240</td>
</tr>
<tr>
<td>Stochastic Dynamic</td>
<td>15</td>
</tr>
</tbody>
</table>

SD forecasts  The SD forecast portion of STOCHAST used the sample means, variances, and covariances of (u,v,y) as initial values in the numerical solution of the SD predictive equations. The forecast values of variance and covariance were checked so the correlation coefficient always had a magnitude equal to, or less than one; if necessary, proportional adjustments were made to the variances to maintain this condition. The problem of statistical moment closure was not addressed.

MC forecasts  The MC forecast portion of STOCHAST used the 100 individual sampled values of (u,v) as initial values in the numerical solution of the original simplified equations. Based upon the resulting 100 forecasts, the forecast means, variances, and covariances were calculated.
Data analysis  The values which were stored for later use were (1) forecast time, (2) mean u and v, (3) mean y, (4) u and v standard deviations (MC and SD only), and (5) Correlations between u, v, y (MC and SD only).

The verifying forecast was based upon the full inertial oscillation model given by equations (4.6) and (4.7), along with

\[ \frac{d\phi}{dt} = \frac{v}{a} \quad (4.16) \]

The verifying forecast was calculated in the program, INERTMSE, described in the error partitioning section of this chapter.

Since the solutions of the two velocity components, (u,v), are coupled, the velocity components were treated as a single complex quantity, \( V_c = u + iv \). From these values, the complex MSE (CMSE) for the SD and MC forecasts, CMSE(SD) and CMSE(MC), and the magnitude of the complex ratio, CMSE(SD)/CMSE(MC) were calculated using the statistical analysis program, MINITAB. The basic statistics of complex quantities are discussed in Appendix A.

Final comparative plots, one set for each of the four models and one set for the magnitude of the complex ratio CMSE(SD)/CMSE(MC), were plotted with HARVARD GRAPHICS® Version 2.1.

Numerical results: Stochastic-dynamic and Monte-Carlo comparison

Model 0  Figures 4.1a-d present the magnitudes of CMSE(SD) and CMSE(MC) for the f-plane model. The model gave identical values for CMSE(SD) and CMSE(MC). Due to the linearity of the basic equations for
Figure 4.1 Complex MSE Magnitude (Model 0)
the $f$-plane, the SD predictive equations for means are identical to the
MC equations; this model's linearity also gives closed SD predictive
equations for the variances and covariances (i.e., not functions of
higher order statistical moments), resulting in the exact match with the
MC forecast variances and covariances.

The decrease in the magnitude of the CMSE was due to the "real"
inertial oscillation migrating westward around the globe and beginning to
again approach the stationary $f$-plane oscillation.

**Model 1**  Figures 4.2a-d present the magnitude of CMSE(SD) and
CMSE(MC) for the $\beta$-plane model. Initially, both the magnitudes of the
CMSE(SD) and CMSE(MC) had similar oscillation pattern. The magnitude of
CMSE(SD) was about two to three times that of CMSE(MC). After about 15
reference periods, the CMSE(SD) decreased in a similar manner to that in
model 1 (Figures 4.1a-d); this decrease was absent in the CMSE(SD) of
Model 2 (Figures 4.3a-d). The CMSE(MC) showed only a slight decrease
during this same period. The CMSE(SD) decrease is possibly due to the
cumulative effect of the truncation of higher order statistical moments
and centrifugal terms from the predictive equations. It is of interest
to note the similarity of the CMSE(SD) magnitude plots for the $f$-plane
and $\beta$-plane models, Figures 4.1a-d and 4.2a-d, respectively. Except for
the gradual $\beta$-plane phase shift, the plots are virtually identical.

**Model 2**  Figures 4.3a-d present the magnitude of the CMSE(SD) and
CMSE(MC) for the the partial $\beta$-plane model. The marked decrease in
CMSE(SD), present in models 0 and 1, was absent in the models which
include the centrifugal terms.
Figure 4.2 Complex MSE Magnitude (Model 1)
Figure 4.3  Complex MSE Magnitude (Model 2)
By seven reference periods into the forecast, the CMSE(SD) was about two to three times as large as the CMSE(MC). This continued throughout the rest of the forecast period.

**Model 3** Figures 4.4a–d present the magnitude of the CMSE(SD) and CMSE(MC) for the full model. The growth of CMSE(MC) was slower than that of CMSE(SD), but they approached the same levels after about 20 periods.

Throughout most of the forecast period, the CMSE(SD) was up to five times as large as the CMSE(MC). This gradually reversed after 24 periods and by 30 periods, the CMSE(SD) was about half that of the CMSE(MC). The CMSE(SD) oscillations led the CMSE(MC) oscillations out to 17 reference periods. There was a gradual phase reversal thereafter.

**Complex ratio of CMSE(SD)/CMSE(MC)** Figure 4.5a–d presents the magnitude of the complex ratio, CMSE(SD)/CMSE(MC). Since CMSE(SD) equals CMSE(MC) in the f-plane approximation, their ratio is a constant value of 1. This served as a convenient reference line for the other models. As the forecast progressed, models 2 and 3, containing the centrifugal force terms, showed similar CMSE behavior.

**Conclusions: Stochastic-dynamic and Monte-Carlo comparison**

If the SD and MC forecasts were exactly the same, as in model 1, the CMSE(SD) would be equal to the CMSE(MC). But since the SD predictive equations are a truncated set, any nonlinearities in the underlying physical model will introduce the effects of higher order statistical moments. These unaccounted effects lead to the differences in the forecast CMSEs.
Figure 4.4 Complex MSE Magnitude (Model 3)
Figure 4.5 Magnitude of CMSE(SD)/CMSE(MC)
With this simple model, the computational load favors the SD approach over the MC (9 forecasts .vs. 100 forecasts). This would change rapidly if the number of basic equations were increased or the order of the SD equations were increased, as shown in Table 1.1 of chapter 1. For short forecasts, both the SD and MC were similar; but, as the forecast time lengthens, the MC forecasts are generally superior to the SD forecasts since they, by their nature, incorporate the effects of the higher order moments.

One final word of caution concerning the Monte-Carlo forecasts is, for a given sample size, the accuracy falls off as the order of the estimated moment increases. This sets a limit of accuracy in the Monte-Carlo method, just as the truncation of the SD equations sets the accuracy limit; however, the Monte-Carlo limit, conceptually, is easier to overcome - just increase the sample size.

Partitioning Model Errors: Inertial Oscillation Equations

Hierarchy of numerical models

For this section, equations (4.6) and (4.7) were assumed to be the exact equations describing inertial oscillation. Three different sets of model equations were obtained using successively higher order truncations of the Taylor series expansion of \( \sin \phi \)

\[
\sin \phi = \sin \phi_0 + \cos \phi_0 (\phi - \phi_0) - \sin \phi_0 \frac{(\phi - \phi_0)^2}{2!} + \ldots \quad (4.17)
\]
The additional predictive equation needed for latitude, \( \phi \), is equation (4.16)

\[
\frac{d\phi}{dt} = \frac{v}{a} \quad \text{(4.16)}
\]

The three predictive equations for the velocity components \((u, v)\) and latitude \((\phi)\) form a closed system. If the longitude \((\lambda)\) is desired, its predictive equation is

\[
\frac{d\lambda}{dt} = \frac{u}{a \cos \phi} \quad \text{(4.18)}
\]

**Model A: f-plane approximation**  
The first and simplest inertial oscillation model to be considered is the \(f\)-plane approximation model. The simplified Coriolis term uses the first term of the Taylor series expansion, equation (4.17)

\[
f_A = 2\Omega \sin \phi 
\]

(4.19)

Based upon this model, the exact solutions for velocity components and latitude are:

\[
u = u_0 \cos(f_A t) + v_0 \sin(f_A t) \quad \text{(4.20)}
\]

\[
v = v_0 \cos(f_A t) - u_0 \sin(f_A t) \quad \text{(4.21)}
\]
\[ \phi = \frac{u}{a f_A} + \phi_0 \quad (4.22) \]

These exact solutions may be compared with the numerical solutions to judge the accuracy of the numerical solution technique.

**Model B: \( \beta \)-plane approximation** The second model is the \( \beta \)-plane approximation model. The simplified Coriolis term, \( f_B \), uses the first two terms of the Taylor series expansion, equation (4.17)

\[ f_B = 2\Omega \sin \phi_0 + 2\Omega \cos \phi_0 (\phi - \phi_0) = f_A + \delta f_B. \quad (4.23) \]

**Model C: Exact simplified \( f \)** The third model is based upon the full Taylor series expansion of \( \sin \phi \). The effective Coriolis term of model C is

\[
\begin{align*}
 f_C &= 2\Omega \sin \phi \\
 &= f_B + \delta f_C \\
 &= f_A + \delta f_B + \delta f_C.
\end{align*}
\]

**Model D: Full model** The fourth model is based upon the full expression for the effective Coriolis term, \( f^* \),

\[
\begin{align*}
 f_D &= \left( 2\Omega + \frac{u}{a \cos \phi} \right) \sin \phi = 2\Omega \sin \phi + \frac{\tan \phi}{a} u \\
 &= f_C + \delta f_D \\
 &= f_A + \delta f_B + \delta f_C + \delta f_D. \quad (4.25)
\end{align*}
\]
General form of models

Examination of the four models shows they may be written as:

\[
\frac{d}{dt}u_{(1)} = +f_{(1)}v_{(1)}
\]

(4.26)

\[
\frac{d}{dt}v_{(1)} = +f_{(1)}u_{(1)}
\]

(4.27)

\[
\frac{d}{dt}\phi_{(1)} = \frac{1}{a}v_{(1)}
\]

(4.28)

\[
\frac{d}{dt}\lambda_{(1)} = \frac{1}{a\cos\phi_{(1)}}u_{(1)}
\]

(4.29)

where the \((i)\), \(i = A,B,C,D\), subscript indicates the model used to calculate the variable. Notice that each successive model contains the lower order models plus corrective terms. It is this fact that allows error partitioning by model terms.

Numerical model: Model error partitioning

General program description

The computer program, INERTMSE, was written to compare forecast CMSE based upon each of the four models; A, B, C, and D. INERTMSE initially generates a sample of 100 random values of the velocity components; the initial latitude is assumed to be exact. Each velocity component is sampled from a normal distribution having user specified means and variances. The final sample values are systematically adjusted to give the specified means and variances, while maintaining the random relation with each other. In order to have all models
working with the same data, each model was started with the same random number seed: 131. The initial starting means and standard deviations were

\[(\mu_u, \mu_v, \mu_\phi) = (0, 100, 0.800973267937)\]

\[(\sigma_u, \sigma_v, \sigma_\phi) = (0, 10, 0)\]

where the \((u,v)\) units are meters per second and \(\phi\) is in radians.

All differential equations were solved using a fourth-order Runge-Kutta integration scheme and 400 time steps per reference period. The reference period, defined as the period for an f-plane inertial oscillation, is set to 60000 seconds (16.66 hours) by assigning the appropriate reference latitude.

The model was run on a personal computer and, as a diagnostic check of model stability, the initial mean kinetic energy was compared with the final mean kinetic energy, which should be exactly the same. The initial mean kinetic energy for all of the models was 5049.995. The model run-times, of the compiled Microsoft QuickBASIC® code on a 33 MHz 80486-based personal computer, and final kinetic energy are presented in Table 4.2. The overall numerical solution, as measured by the mean kinetic energy appears stable since the mean kinetic energy decreased by \(4 \times 10^{-7}\%\). Another check of overall stability showed the Model A (f-plane) was behaving precisely as predicted by the exact solution.
Table 4.2. INERTMSE Run-Times and Final Kinetic Energy

<table>
<thead>
<tr>
<th>Model</th>
<th>Run-Time (sec)</th>
<th>Final Kinetic Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>300</td>
<td>5049.994987</td>
</tr>
<tr>
<td>B</td>
<td>330</td>
<td>5049.994986</td>
</tr>
<tr>
<td>C</td>
<td>459</td>
<td>5049.994987</td>
</tr>
<tr>
<td>D</td>
<td>587</td>
<td>5049.994979</td>
</tr>
</tbody>
</table>

Monte-Carlo forecasts For each of the four models (A–D), random samples of 100 values of (u,v) were used as initial values in the numerical solution of the model equations. The 100 resulting forecasts for each model were then used to calculate the model forecast means, variances, and covariances. These forecast statistics, along with the verifying forecast, were used to calculate the complex mean-squared error (CMSE) components of each model. From these values, the CMSE of velocity were calculated using MINITAB and HARVARD GRAPHICS™ was used to obtain the final comparative plots.

Numerical results: Model error partitioning

The primary emphasis of the analysis was to examine the CMSE of the various models and their relations to each other. While a more in depth study may be done, this study examined only the magnitudes.

CMSE magnitudes Figures 4.6a–d present the actual magnitudes of the forecast CMSE from the various models. The CMSE for model D, CMSE(D), represents the effect of initial data error on a forecast made
with an otherwise perfect model. There were strong oscillations of one reference period throughout the forecast for all of the CMSEs.

The expected relationship of $CMSE(A) > CMSE(B) > CMSE(C) > CMSE(D)$ established itself 3 to 4 reference periods into the model and continued out to 16 to 18 reference periods when the $CMSE(A)$ began to drop. Models B, C, and D generally maintained the expected CMSE relation, with the exception that $CMSE(B)$ and $CMSE(C)$ merged as the forecast progressed, indicating the difference between the two models becomes negligible.

The explanation for the behavior of $CMSE(A)$ is that Model A, with its constant effective Coriolis parameter, produces stationary oscillations as opposed to the other three models whose variable Coriolis parameter produces a westward drift of their oscillations. The drop in the magnitude of $CMSE(A)$ was due to the true oscillation migrating westward around the globe until it approached the stationary Model A oscillation.

Since Model A does not support the migration of the inertial oscillations, it is neglected in further discussions.

**Sequential CMSE ratios**  
Figures 4.7a-d present the magnitudes of the complex sequential ratios, $CMSE(A)/CMSE(B)$, $CMSE(B)/CMSE(C)$ and $CMSE(C)/CMSE(D)$. These are a measure of how well each additional term in the models help to improve the forecast.

Most of the improvement in the forecast came from the inclusion of the centrifugal term in Model D. This was not surprising since the magnitude of the term was on the order of 15% of $2\Omega \sin \phi$; smaller velocities would decrease the effect of this term.
Figure 4.6 Complex MSE Magnitude

(a) 0 to 7.375 Reference Periods

(b) 7.5 to 14.875 Reference Periods

(c) 15 to 22.375 Reference Periods

(d) 22.5 to 29.875 Reference Periods
Figure 4.7 Magnitude of Sequential CMSE Ratios
There was not much improvement in using $2\Omega \sin\phi$ over the simpler $\beta$-plane approximation (Model C vs. Model B). This shows in all plots and the similarities in forecasts became greater the further into the forecast. The difference between the two models would become greater for larger values of $v$ (i.e. large variations in $\sin \phi$ over the oscillation path).

**CMSE ratios with full model**

Figures 4.8a-d present the magnitude of the complex ratios of model (A, B, C) CMSE with the full model (D) CMSE. These plots show the total effect of deleting model terms. The results are similar to the sequential CMSE ratio plots.

Since the $\beta$-plane approximation, Model B, did as well as Model C, one would wonder about the performance of a model with just the beta plane approximation and the centrifugal acceleration. This model is Model 3, discussed in the section on Stochastic-Dynamic forecasts. Figure 4.3a-d presents the magnitude of CMSE(3). Comparing the CMSE(3) with that of CMSE(D) shows that the terms not included in the beta plane approximation are still important. This is indicated by CMSE(3) having a magnitude of about 12000 during 7.5 and 22 reference periods whereas the magnitude of CMSE(D) was on the order of 1300 during the same period. This emphasizes the sequential nature of the model terms; some terms may not become important until other terms are in the model.

**Conclusions: model error partitioning**

The sequential inclusion of terms into a model and the use of CMSE to measure forecast performance is a useful tool in model term selection.
Figure 4.8 Magnitude of Complex MSE Ratios (w.r.t. true model)
and error partitioning. However, it must be kept in mind that the performance of the candidate terms may be sensitive to what is currently in the model. This sequential sensitivity of model terms was exemplified by the relative performance of Models B, C, and 1 (SD section) with respect to the full model, Model D.

Summary

This chapter presented two numerical examples based upon the inertial oscillation equations: SD and MC forecast comparison and model error partitioning. Both examples showed how a hierarchy of models may be defined based upon a system of coupled non-linear differential equations. The SD and MD forecast comparison example showed how difficult the SD equations are for even the simplest model and how easily a MC forecast is to implement. The model error partitioning example showed some of the problems encountered with the partitioning of forecast errors, both in presentation and interpretation.

Chapter 5 extends the ideas of this chapter to actual (although simplified) meteorological models; A hierarchy of models is defined and the MC forecast program and supporting programs are discussed. Results of analysis of the MC forecasts, using the model hierarchy, are presented in Chapters 6 and 7.
CHAPTER 5
HIERARCHY OF NUMERICAL WEATHER MODELS

Chapters 2 and 3 show, in a hierarchy of models composed of additive, possibly nonlinear, terms, how model forecast errors may be sequentially partitioned by source. Chapter 4 presents a simple application of error partitioning to a hierarchy of inertial oscillation models using both stochastic-dynamic and Monte-Carlo techniques.

The inertial oscillation models are examples of modeling with a lagrangian frame of reference; i.e. modeling properties of a control volume moving with the flow. Most meteorological models use a eulerian frame of reference. In an eulerian frame of reference, the control volume is fixed with respect to the particular coordinate axes.

This chapter describes the hierarchy of eulerian models used in the remainder of the study, the various model parameterizations and other numerical modeling details and the Monte-Carlo technique. Subsequent chapters discuss the results of error partitioning, based upon Monte-Carlo forecasts using the model hierarchy.

Model Hierarchy Formulation

Introduction

In his 1960 paper, "Energy and Numerical Weather Prediction", Edward Lorenz presented a systematic technique to simplify the full primitive equations, while maintaining the property of total energy conservation.
The resulting set of models was a guide in selecting the model hierarchy used in the remainder of this study.

Lorenz began with the full set of nonhydrostatic primitive equations for a dry atmosphere

\[
\frac{du}{dt} = \left(2\Omega + \frac{u}{a \cos \phi}\right) \sin \phi \ v - \frac{uw}{a} - \frac{1}{\rho} \frac{\partial P}{\partial x} \tag{5.1}
\]

\[
\frac{dv}{dt} = -\left(2\Omega + \frac{u}{a \cos \phi}\right) \sin \phi \ u - \frac{vw}{a} - \frac{1}{\rho} \frac{\partial P}{\partial y} \tag{5.2}
\]

\[
\frac{dw}{dt} = \frac{u^2 + v^2}{a} - g - \frac{1}{\rho} \frac{\partial P}{\partial z} \tag{5.3}
\]

\[
\frac{1}{\rho} \frac{d\rho}{dt} = -\nabla_j (u \partial_j + v \partial_j + w \partial_j) \tag{5.4}
\]

\[
C_p \frac{dT}{dt} = -\frac{1}{\rho} \frac{dP}{dt} + \dot{q} \tag{5.5}
\]

\[
P = \rho RT \tag{5.6}
\]

where

- \(C_p\) specific heat at constant pressure
- \(f\) Coriolis parameter, \(2\Omega \sin \phi\)
- \(g\) acceleration of gravity
- \(P\) pressure
- \(\rho\) density
- \(\dot{q}\) heating
- \(T\) temperature
- \((u,v,w)\) \((x, y, z)\) velocity components
Note, in accordance to Holton (1979), terms proportional to \( \cos \phi \) were dropped from equations (5.1) and (5.3) to preserve angular momentum conservation when \( r \) is replaced by \( a \) (earth's radius, a constant).

**Primitive equations**

To obtain the hydrostatic primitive equations, the vertical momentum equation (5.3) is replaced with the hydrostatic equation,

\[
\frac{\partial P}{\partial z} = -\rho g \tag{5.7}
\]

and the vertical motion components in the horizontal momentum equations, (5.1) and (5.2), are deleted.

Next, the horizontal momentum equations are combined and rewritten as vorticity and divergence equations. The resulting equations, expressed in the \((x,y,P)\) coordinate system, are the hydrostatic primitive equations in vorticity-divergence form.

\[
\frac{\partial \zeta}{\partial t} = -J(\psi, \xi) - J(\psi, f) - \nabla \cdot \bar{\nabla} - \bar{\nabla} \cdot \xi - \zeta D - \omega \frac{\partial \xi}{\partial P} - \\
- \nabla \omega \cdot \nabla \frac{\partial \psi}{\partial P} - \left[ \omega, \frac{\partial \xi}{\partial P} \right] \tag{5.8}
\]

\[
\frac{\partial D}{\partial t} = -\nabla^2 \phi + \nabla \cdot (f \nabla \psi) - J(f, \chi) - \nabla \cdot \left( \bar{\nabla} \right) - \nabla \cdot \left( \bar{\nabla} \right) - \\
\nabla \cdot \left( \bar{\nabla} \right) - \nabla \cdot \left( \bar{\nabla} \right) - \nabla \omega \cdot \frac{\partial \psi}{\partial P} - \nabla \omega \cdot \frac{\partial \psi}{\partial P} \tag{5.9}
\]
\[ \frac{\partial \theta}{\partial t} = - J(\psi, \theta) - \nabla \cdot \nabla \theta - \omega \frac{\partial \theta}{\partial P} \quad (5.10) \]

\[ \frac{\partial \phi}{\partial P} = - \alpha \quad (5.11) \]

where

\[ \theta = \frac{P}{P^*} \left( \frac{P^*}{P} \right)^{R/C_p} \quad (5.12) \]

\[ \alpha = \frac{1}{\rho} = \frac{RT}{P} \]

\[ \zeta = \nabla^2 \psi \]

\[ D = \nabla \cdot \vec{v} = \nabla^2 \chi = - \frac{\partial \omega}{\partial P} \]

\[ \vec{v} = \vec{v}_\psi + \vec{v}_\chi \]

\[ \vec{v}_\psi = \vec{r} \times \nabla \psi \]

\[ \vec{v}_\chi = \nabla \chi \]

Strictly speaking, the effective Coriolis parameter should be

\[ f = \left( 2\Omega + \frac{u}{a \cos \phi} \right) \sin \phi \quad ; \quad (5.13) \]

but the common approximation,

\[ f = 2\Omega \sin \phi \quad , \quad (5.14) \]

will be used instead in the models.
Balance equations  Simplifications may be made to the hydrostatic primitive equations by setting the local time derivative of divergence, $\frac{\partial D}{\partial t}$, in the divergence equation (5.9), to zero. To conserve total energy, all divergence equation terms, related to the velocity potential function, $\chi$, are set to zero and all vorticity equation, $(\chi \partial)^c$ terms are set to zero. The thermodynamic equation (5.10) retains all original terms. The resulting balance predictive equations are

\[
\begin{align*}
\frac{\partial \xi}{\partial t} &= - J(\psi, \zeta) - J(\psi, \rho) - \nabla \cdot f \nabla \phi - \nabla \cdot \nabla \zeta - \zeta D - \omega \frac{\partial \xi}{\partial P} \quad (5.15) \\
\nabla^2 \phi &= \nabla \cdot (f \nabla \psi) - \nabla \left( \nabla \cdot \nabla \psi \right) \\
\frac{\partial \theta}{\partial t} &= - J(\psi, \theta) - \nabla \cdot \nabla \theta - \omega \frac{\partial \theta}{\partial P} \quad (5.17)
\end{align*}
\]

Linear balance equations  Further simplifications may be made to the balance predictive equations. The elimination of the $(\psi \partial)^c$ terms from the balance equation (5.16) and the elimination of the $(\chi \partial)^c$ terms from the vorticity equation (5.15) give the linear balance predictive equations

\[
\begin{align*}
\frac{\partial \xi}{\partial t} &= - J(\psi, \zeta) - J(\psi, \rho) - \nabla \cdot f \nabla \phi \\
\nabla^2 \phi &= \nabla \cdot (f \nabla \psi) \\
\frac{\partial \theta}{\partial t} &= - J(\psi, \theta) - \nabla \cdot \nabla \theta - \omega \frac{\partial \theta}{\partial P} \quad (5.20)
\end{align*}
\]

These equations are the basis of the model hierarchy used in this study.
Model hierarchy: overview

Based upon the set of linear balance equations (5.18) to (5.20), this study used the hierarchy of models outlined in Table 5.1. The variable, \( \sigma \), represents the static stability, proportional to \( \partial \theta / \partial p \).

Table 5.1 Model Hierarchy

<table>
<thead>
<tr>
<th>Model</th>
<th>Type</th>
<th>Forecast Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Barotropic</td>
<td></td>
<td>( \psi )</td>
</tr>
<tr>
<td>Baroclinic</td>
<td>( \sigma = \sigma_0 )</td>
<td>( \psi, \tau )</td>
</tr>
<tr>
<td></td>
<td>( \sigma = \sigma_0(\mu) )</td>
<td>( \psi, \tau )</td>
</tr>
<tr>
<td>Linear Balance</td>
<td>( \partial \theta / \partial p \rightarrow \sigma )</td>
<td>( \psi, \tau, \sigma )</td>
</tr>
</tbody>
</table>

The barotropic model is a one layer model, while the baroclinic and linear balance models are two-layer models. The vertical structure of the baroclinic and linear balance models is diagramed in Fig. 5.1.

![Figure 5.1 Vertical Structure of Models](image-url)
The model equations allow for the effects of terrain and heating, even though only terrain forcing was included in this study.

Following the lead of Lorenz, instead of using the level variables, as they appear in Fig. 5.1, the following model variables were used

Streamfunction

\[ \psi = \frac{\psi_1 + \psi_3}{2} \]  
(5.21)

\[ \tau = \frac{\psi_1 - \psi_3}{2} \]  
(5.22)

Velocity Potential

\[ \chi = \frac{\chi_1 + \chi_3}{2} \]  
(5.23)

\[ \delta = \frac{\chi_1 - \chi_3}{2} \]  
(5.24)

Potential Temperature

\[ \theta = \frac{\theta_1 + \theta_3}{2} \]  
(5.25)

\[ \sigma = \frac{\theta_1 - \theta_3}{2} \]  
(5.26)

Diabatic Heating

\[ H_1 = \frac{R}{P} \left[ \frac{P}{P_0} \right] \frac{\dot{q}_1}{C_p} \]  
(5.27)

\[ H_\theta = \frac{H_1 + H_3}{2} \]  
(5.28)

\[ H_\sigma = \frac{H_1 - H_3}{2} \]  
(5.29)

Vertical Velocity

\[ \frac{\omega_4}{2\Delta P} = -\nabla^2 \chi \]  
(5.30)
\[ \frac{\omega^2}{\Delta P} = - \left( \nabla^2 \delta + \nabla^2 \chi \right) \] (5.31)

The subscripts, \( , \), denote the model level.

**Linear balance model**

The linear balance model is based upon the linear balance equations (5.18) through (5.20). The vorticity equation (5.18) was modified by the substitution of \( \nabla \chi \) for \( \nabla \phi \) and the expansion of the \( \nabla \cdot (f \nabla \chi) \) term; the linear balance equation (5.19) was rewritten in terms of \( \theta \) and \( \partial \phi / \partial P \); and, the thermodynamic equation (5.20) was modified by adding a diabatic heating term (5.27). The resulting linear balance equations are

\[ \frac{\partial \psi^2}{\partial t} = - J(\psi, \psi^2 + f) - \nabla f \cdot \nabla \chi - f \nabla^2 \chi \] (5.32)

\[ - \frac{R}{P} \left( \frac{C_p}{P} \right) \nabla \theta = \nabla f \cdot \frac{\partial \psi}{\partial P} + f \nabla^2 \frac{\partial \psi}{\partial P} \] (5.33)

\[ \frac{\partial \theta}{\partial t} = - J(\psi, \theta) - \nabla \chi \cdot \nabla \theta - \omega \frac{\partial \theta}{\partial P} + \left( \frac{P_*}{P} \right) \frac{R}{P} \frac{q}{C_p} \] (5.34)

The model equations (5.35) to (5.39) were obtained the manner outlined in Lorenz (1960) and Appendix B. The thermodynamic equations for \( \theta \) and \( \sigma \) were derived under the assumption

\[ \frac{\partial \theta}{\partial P} = f(\mu, \lambda, t) = \frac{2\sigma}{\Delta P} . \]
Appendix B discusses the application of other possible assumptions concerning the variation of $\theta$ with pressure.

\[
\frac{\mathcal{R} \Delta P}{2P_2}\left(\frac{P}{P_*}\right)^{\mathcal{R}/C_p} \frac{V^2\theta}{fV^2\tau} = \nabla f \cdot \nabla \tau + fV^2\tau \tag{5.35}
\]

\[
\frac{\partial}{\partial t} V^2\psi = - J(\psi, \nabla^2\psi + f) - J(\tau, V^2\tau) - \nabla \chi^* \nabla f - fV^2\chi \tag{5.36}
\]

\[
\frac{\partial}{\partial t} V^2\tau = - J(\psi, \nabla^2\tau) - J(\tau, \nabla^2\psi + f) - \nabla \delta \nabla f - fV^2\delta \tag{5.37}
\]

\[
\frac{\partial \theta}{\partial t} = - J(\psi, \theta) - J(\tau, \sigma) - \nabla \chi^* \nabla \theta - \nabla \delta \nabla \sigma - 2\sigma \nabla^2 \chi - \sigma V^2\delta + H* \theta \tag{5.38}
\]

\[
\frac{\partial \sigma}{\partial t} = - J(\psi, \sigma) - J(\tau, \theta) - \nabla \chi^* \nabla \sigma - \nabla \delta \nabla \theta - \sigma V^2\chi + H* \sigma \tag{5.39}
\]

The $\theta$ equation (5.38) is not needed since, with the exception of the global mean, $\theta$ may be recovered from $\tau$ through the linear balance equation (5.35). Components of the predictive $\theta$ equation are needed in the diagnostic estimation of $\delta$ using the "omega" equation.

The "omega" equation (5.40) is a diagnostic equation for determining $\delta$ from $(\psi, \tau, \theta, \sigma)$. The procedure used in deriving equation (5.40), from (5.35), (5.36), and (5.38), is outlined in Appendix B.

\[
\nabla^2 (\sigma V^2\delta) - \frac{f^2}{C_{Bat}} V^2\delta = \nabla^2 \left(\mathcal{A} \theta + \mathcal{D} \theta + H* \theta\right) - \frac{f}{C_{Bat}} \left(\mathcal{A} \tau + \mathcal{D} \tau\right) - \frac{1}{C_{Bat}} \nabla f \cdot \nabla \frac{\partial \theta}{\partial t} \tag{5.40}
\]
The individual components are

\[ C_{\text{Bal}} = \frac{R\Delta P}{2P_2} \left( \frac{P_2}{P_s} \right)^{R/C_p} \]  

\[ \Delta F_\tau = - J(\psi, \nabla^2 \tau) - J(\tau, \nabla^2 \psi + f) \]  

\[ D_\tau = - \nabla \delta \cdot \nabla f \]  

\[ \Delta F_\theta = - J(\psi, \theta) - J(\tau, \theta) - \nabla \chi \cdot \nabla \theta - 2\sigma \nabla \chi \]  

\[ D_\theta = - \nabla \delta \cdot \nabla \sigma \]

**Numerical Model Structure**

The computer utility subroutines, used in this study, performed spectral operations on a unit radius sphere. To ease the task of model implementation, the equations were converted to equations having unit sphere operations, denoted by \((\_)_s\). The constant \(a\) denotes the radius of the Earth.

The global means of \(\theta\) and \(\sigma\) were held constant and no heating was used.

**Model equations**

**Linear balance model (MCLBX)** The linear balance model (MCLBX) equations, expressed in terms of unit sphere operations, are
\[ v_s^2 \frac{\partial \psi}{\partial t} = - \frac{1}{a^2} \left\{ J_s(\psi, v^2_s\psi) + J_s(\tau, v^2_s\psi) \right\} - 2\Omega \left\{ \frac{\partial \psi}{\partial \lambda} + (1-\mu^2) \frac{\partial \chi}{\partial \mu} + \mu v^2_s \chi \right\} \] (5.46)

\[ v_s^2 \frac{\partial \tau}{\partial t} = - \frac{1}{a^2} \left\{ J_s(\psi, v^2_s\tau) + J_s(\tau, v^2_s\psi) \right\} - 2\Omega \left\{ \frac{\partial \tau}{\partial \lambda} + (1-\mu^2) \frac{\partial \delta}{\partial \mu} + \mu v^2_s \delta \right\} \] (5.47)

\[ \frac{\partial \sigma}{\partial t} = - \frac{1}{a^2} \left\{ J_s(\psi, \sigma) + J_s(\tau, \sigma) + \nabla_s \chi \cdot \nabla_s \sigma + \nabla_s \delta \cdot \nabla_s \theta - \sigma v^2_s \chi \right\} \] (5.48)

\[ \frac{R\Delta P}{2P_2} \left( \frac{P_2}{P_s} \right)^{R/C_p} v_s^2 \theta = 2\Omega \left\{ (1-\mu^2) \frac{\partial \tau}{\partial \mu} + \mu v^2_s \tau \right\} \] (5.49)

\[ v_s^2 (\sigma v^2_s \delta) - \frac{4na^2}{C_{Bal}} \mu v^2_s \delta = a^2 v_s^2 \left\{ A\theta + D\theta \right\} - \frac{2\Omega a^2}{C_{Bal}} \left\{ \mu \left( A\tau + D\tau \right) - (1-\mu^2) \frac{\partial}{\partial \mu} A \frac{\partial \tau}{\partial t} \right\} \] (5.50)

where the components of equation (5.48) are

\[ A\tau = - \frac{1}{a^2} \left\{ J_s(\psi, v^2_s\tau) + J_s(\tau, v^2_s\psi) \right\} - 2\Omega \left\{ \frac{\partial \tau}{\partial \lambda} \right\} \] (5.51)

\[ D\tau = - 2\Omega \left\{ (1-\mu^2) \frac{\partial \delta}{\partial \mu} \right\} \] (5.52)

\[ A\theta = - \frac{1}{a^2} \left\{ J_s(\psi, \theta) + J_s(\tau, \sigma) + \nabla_s \chi \cdot \nabla_s \theta + 2\sigma v^2_s \chi \right\} \] (5.53)

\[ D\theta = - \frac{1}{a^2} \left\{ \nabla_s \delta \cdot \nabla_s \sigma \right\} \] (5.54)
The MCLBX model is assumed to be the "true" forecast model in this study. These equations form the basis for the remaining models in the hierarchy.

**Baroclinic model**  The two baroclinic models (MCBCx) used in this study differ from traditional baroclinic models in that these are fully global models and, as a consequence, a fully variable Coriolis parameter, $f$, is used instead of the usual beta-plane approximation. Allowing a fully variable $f$ necessitates the use of the linear balance equation to relate $\theta$ to $\tau$ and the retention of the advection of planetary vorticity by the divergent wind in the vorticity equation (Holton, 1979).

The baroclinic models, with the exception of $\partial \sigma/\partial t$ terms, use the linear balance equations (5.46) through (5.54). Since $\sigma$ is not predicted, equation (5.48) is not used. The baroclinic model equations are

\[
\nabla_s^2 \psi = - \frac{1}{a^2} \left\{ J_s(\psi, \nabla_s^2 \psi) + J_s(\tau, \nabla_s^2 \tau) \right\} - 2\Omega \left\{ \frac{\partial \psi}{\partial \lambda} + (1-\mu^2) \frac{\partial \chi}{\partial \mu} + \mu \nabla_s^2 \chi \right\} \tag{5.55}
\]

\[
\nabla_s^2 \tau = - \frac{1}{a^2} \left\{ J_s(\psi, \nabla_s^2 \psi) + J_s(\tau, \nabla_s^2 \psi) \right\} - 2\Omega \left\{ \frac{\partial \tau}{\partial \lambda} + (1-\mu^2) \frac{\partial \delta}{\partial \mu} + \mu \nabla_s^2 \delta \right\} \tag{5.56}
\]

\[
\frac{R\Delta p}{2P_2} \left[ \frac{P_2}{P_s} \right]^{R/C_p} v_s^2 \theta = 2\Omega \left\{ (1-\mu^2) \frac{\partial \tau}{\partial \mu} + \mu v_s^2 \right\} \tag{5.57}
\]

\[
\nabla_s^2 (\sigma v_s^2 \delta) - \frac{4\Omega a^2}{C_{B1}} \mu^2 v_s^2 \delta = a^2 v_s^2 \left( A_{\theta} \right) - \frac{2\Omega a^2}{C_{B1}} \left\{ \mu \left( \frac{A_{\tau} + D_{\tau}}{\tau} \right) - (1-\mu^2) \frac{\partial}{\partial \mu} \frac{\partial \delta}{\partial t} \right\} \tag{5.58}
\]

where the components are
\[ AF_\tau = - \frac{1}{a^2} \left\{ J_\theta(\psi, \nabla_\theta^2 \psi) + J_\tau(\tau, \nabla_\tau^2 \psi) \right\} - 2\Omega \left\{ \frac{\partial \tau}{\partial \lambda} \right\} \] (5.59)

\[ D_\tau = - 2\Omega \left\{ (1-\mu^2) \frac{\partial \delta}{\partial \mu} \right\} \] (5.60)

\[ AF_\theta = - \frac{1}{a^2} \left\{ J_\theta(\psi, \theta) + \nabla_\theta \chi \cdot \nabla_\theta \theta + 2\theta \nabla_\theta^2 \chi \right\} \] (5.61)

**MCBCO:** \( \sigma = \sigma_0 \) In this model, \( \sigma \) was set to the initial observed global mean of \( \sigma \). This was accomplished by zeroing all initial \( \sigma \) spectral coefficients, \( \sigma_n^m = 0 \) for \( n \geq 1 \).

**MCBC1:** \( \sigma = \sigma_0(\mu) \) In this model, \( \sigma \) was allowed to vary with latitude only. This was accomplished by zeroing all initial non-zonal \( \sigma \) spectral coefficients, \( \sigma_n^m = 0 \) for \( m \neq 0 \).

**Barotropic model (MCBVX)** The barotropic model (MCBVX) \( \psi \) equation was the same as the linear balance \( \psi \) equation, except for the deletion of the \( \tau \) advection term. The \( \theta \) equation (5.51), is a simple advective equation incorporating the mean \( \sigma \) in the surface forcing term. The surface forcing component, \( \nabla_\theta^2 \chi \), was determined in the same manner as with other models.

The barotropic model equations are

\[ \nabla_\theta^2 \psi = - \frac{1}{a^2} \left\{ J_\psi(\psi, \nabla_\psi^2 \psi) \right\} - 2\Omega \left\{ \frac{\partial \psi}{\partial \lambda} + (1-\mu^2) \frac{\partial \chi}{\partial \mu} + \mu \nabla_\psi^2 \chi \right\} \] (5.62)

\[ \frac{\partial \theta}{\partial t} = - \frac{1}{a^2} \left\{ J_\theta(\psi, \theta) + \nabla_\theta \chi \cdot \nabla_\theta \theta + 2\theta \nabla_\theta^2 \chi \right\} \] (5.63)
Numerical model detail

A global spectral two layer model was the general model form used in this study. The barotropic model, while only one layer, was still a global spectral model.

Vertical resolution The vertical structure, diagramed in Figure 5.1, consists of a layer from 900 mb to 500 mb and a layer from 500 mb to 100 mb. These bounds were chosen because the 900 mb lower bound roughly corresponds to the top of the sea level boundary layer, the 100 mb top bound roughly corresponds to the tropopause, and the midlayer levels are the standard 700mb and 300mb levels. Standard mid-layer levels made initialization much easier.

Horizontal resolution Spectral triangular truncation was selected over rhomboidal truncation because the coefficients in the triangular truncation may be assigned so that changes in model resolution involves simply changing the parameters describing vector and array sizes. The rhomboidal truncation would involve more cumbersome code to achieve this capability.

The horizontal resolution of the models was limited to triangular truncation of T15. This limitation was due to the execution time and amount of generated data. This spectral resolution is equivalent to a horizontal resolution of about 7.5 degrees longitude by 6.5 degrees latitude.

Temporal resolution The spectral resolution sets a limit to the size of the time step; this limit is approximately
\[ \Delta t = \frac{a \cos \phi}{\sqrt{2} m_{\text{max}} c_x} \]

where \( a \) is the radius of the earth, \( \phi \) is the latitude, \( m_{\text{max}} \) is the maximum longitudinal wave number, and \( c_x \) is the east-west wind component. Using this as a guide, Table 5.2 gives the maximum values of \( c_x \) allowed for the 45 minute time step used in the models.

<table>
<thead>
<tr>
<th>Resolution</th>
<th>0°</th>
<th>20°</th>
<th>40°</th>
<th>60°</th>
<th>80°</th>
<th>85°</th>
</tr>
</thead>
<tbody>
<tr>
<td>T 9</td>
<td>185</td>
<td>174</td>
<td>142</td>
<td>92</td>
<td>32</td>
<td>16</td>
</tr>
<tr>
<td>T12</td>
<td>139</td>
<td>130</td>
<td>106</td>
<td>69</td>
<td>24</td>
<td>12</td>
</tr>
<tr>
<td>T15</td>
<td>111</td>
<td>104</td>
<td>85</td>
<td>55</td>
<td>19</td>
<td>9</td>
</tr>
<tr>
<td>T18</td>
<td>92</td>
<td>87</td>
<td>71</td>
<td>46</td>
<td>16</td>
<td>8</td>
</tr>
<tr>
<td>T21</td>
<td>79</td>
<td>74</td>
<td>60</td>
<td>39</td>
<td>13</td>
<td>7</td>
</tr>
</tbody>
</table>

The equations are solved using, for the first time step, a second order Runge-Kutta scheme, and for subsequent time steps, a leap-frog scheme. Specific details of the actual time stepping procedure are found in the next section dealing with model parameterization of diffusion.

The Runge-Kutta scheme is repeated every 24th time step thereafter to control the growth of the computational mode associated with the leap-frog scheme. The Runge-Kutta scheme consists of two steps: a forward time step followed by a leap-frog time step; each of these steps use a time increment equal to \( \Delta t/2 \).
Model parameterizations

Terrain forcing The surface terrain spectral coefficients were truncated to the model resolution of T15. The remaining coefficients were filtered using a Lanczos filter (see Section 4.2 of The NOGAPS Forecast Model: A Technical Description). The effect of terrain forcing was to generate vertical motion at the lowest level, \( \omega_4 \), through the approximation

\[
\omega_4 \approx - \left( \frac{g_{sfc}^P}{RT_{sfc}} \right) \cdot \nabla_{sfc} \cdot \nabla_{sfc} Z_{sfc}.
\] (5.64)

To make this approximation work for all models, the surface pressure-temperature term was replaced by a latitudinal varying approximation and the surface wind was approximated with 70% of the 500mb wind. This simplification allowed the surface forcing to be similar for all models (including the one-layer barotropic model) so that any forecast differences would be due to the model and not due to, in part, differences in surface forcing.

Terrain spectral coefficients, at a T21 resolution, are tabulated in Appendix F.

Horizontal and vertical diffusion The effects of diffusion in the models is introduced in the updating module. The general differential equation is assumed to be of the form

\[
\frac{d\psi_n^m}{dt} = F_n^m + D_n \psi_n^m
\] (5.65)

where \( \psi_n^m \) is the spectral coefficient to be updated, \( F_n^m \) is the net forcing term, and \( D_n \) is the spectral diffusion. Using the approach given in
Section 8.3 of *The NOGAPS Forecast Model: A Technical Description*, the spectral diffusion, $D_n$, is calculated as

$$D_n = \frac{n(n+1)}{N(N+1)} D_{sw}$$  \hspace{1cm} (5.66)$$

where $D_{sw}$ is the diffusion coefficient of the shortest wave. The models set $D_{sw}$ equal to $2.3 \times 10^{-6} \text{ s}^{-1}$; this value gives an "e-folding" time of five days.

The differential equation, equation (6.), is solved by applying a Crank-Nicholson scheme to the forward time step, to get

$$\psi_n^{m(t+\Delta t)} = \frac{2 + D_n \Delta t}{2 - D_n \Delta t} \psi_n^{m(t)} + \frac{2\Delta t}{2 - D_n \Delta t} F_n^{m(t)},$$ \hspace{1cm} (5.67)$$

and to the Leap-Frog time step, to get

$$\psi_n^{m(t+\Delta t)} = \frac{1 + D_n \Delta t}{1 - D_n \Delta t} \psi_n^{m(t-\Delta t)} + \frac{2\Delta t}{1 - D_n \Delta t} F_n^{m(t)}.$$ \hspace{1cm} (5.68)$$

Combining the above results with the previously outlined time-stepping schemes, the Runge-Kutta scheme becomes the forward $\Delta t/2$ time step,

$$\psi_n^{m(t+\Delta t/2)} = \left[\frac{4 + D_n \Delta t}{4 - D_n \Delta t}\right] \psi_n^{m(t)} + \left[\frac{2\Delta t}{4 - D_n \Delta t}\right] F_n^{m(t)},$$ \hspace{1cm} (5.69)$$

and the $\Delta t/2$ time step leap-frog,

$$\psi_n^{m(t+\Delta t)} = \left[\frac{2 + D_n \Delta t}{2 - D_n \Delta t}\right] \psi_n^{m(t)} + \left[\frac{2\Delta t}{2 - D_n \Delta t}\right] F_n^{m(t+\Delta t/2)}.$$ \hspace{1cm} (5.70)$$
The full time step leap-frog scheme is equation (5.72). Similar equations are used for the variables τ and σ.

Monte Carlo Forecast Description

To avoid the complexities introduced when using a stochastic-dynamic approach to error estimation, a Monte-Carlo approach was used to examine the partitioning of forecast errors. The difficulty encountered with both approaches is the problem of a non-arbitrary specification of spectral coefficient variances and covariances. To solve this problem, a multi-step process was devised. This process is described in Appendix D.

Monte-Carlo initialization

To make the Monte-Carlo forecasts, used in this study, a random sample of 50 sets of spectral coefficients were generated for each of the primary variables, (ψ,τ,σ). The random sample generation was a multi-step process. The basic steps were [1] the estimation the mean vector and variance-covariance matrix of zonal spectral coefficients of (ψ,τ,θ,σ) from zonal means of (u,T), [2] the generation of a multivariate normally distributed sample of zonal coefficients from the mean and variance-covariance matrix, and [3] the generation of matching non-zonal spectral coefficients through the use of a modified forecast model program. The model is the generator of the complex non-zonal spectral coefficients and their underlying variance-covariance matrix.

Step 1 The zonal spectral coefficients and their variance-covariance matrix were estimated using generalized regression (Searle,
1971), as described in Appendix D, on the 300 and 700 mb zonal means of the u wind component and the temperature. The 1963-73 December-February (NH winter) zonal means and standard deviations, taken from Global Atmospheric Circulation Statistics 1958-1973, were used. The zonal means and standard deviations are tabulated in Appendix F. This study assumed these zonal means were independent of each other.

**Step 2**

The zonal coefficient mean vector and variance-covariance matrix, along with a vector of independent standard normal random numbers were used to generate a sample of correlated zonal spectral coefficients. The technique, described in Appendix D, involved the sum of the estimated mean vector with the product of a standard normal random vector and the Cholesky decomposition of the estimated variance-covariance matrix.

**Step 3**

The sampled zonal spectral coefficients were inputted into a modified Monte-Carlo forecast model program to generate a 10 day forecast of the spectral coefficients, while holding the zonal coefficients constant. These forecast coefficients formed the final initial sample used in the remainder of the study.

**Monte-Carlo forecast**

The generated sample of initial spectral coefficients of \((\psi, \tau, \sigma)\) were used as initial values by each model, as needed. The forecast models generated, for each of the 50 sampled set of coefficients, forecasts out to 10 days, at one day intervals. The forecast models used in this study were the [1] barotropic model (MCBVX), [2] a baroclinic model (MCBC0), where \(\sigma\) was set to the initial global mean; [3] a baroclinic model (MCBC1), where \(\sigma\)
was set to the initial zonal mean; and [4] the linear balance model (MCLBX), where complete spatial and temporal variation of \( \sigma \) was allowed.

The "true" forecast was assumed to be the linear balance model forecast based upon the assumed "true" initial conditions. The "true" initial conditions were assumed to be represented by the estimated mean of the initial spectral coefficients.

Monte-Carlo Forecast and Analysis System

The Monte-Carlo approach, described above, required a number of support subsystems/programs to be written. The subsystems/programs were: (1) the initialization subsystem, (2) the forecast program, and (3) the analysis/display subsystem.

Initialization subsystem

The generation of reasonable initial sample of correlated spectral coefficients is a major hurdle in making a Monte-Carlo forecast. The initialization subsystem generates a correlated sample of complex spectral coefficients from the set of standard level zonal means and standard deviations of \((u, \theta)\). The subsystem consists of the four programs, LINBAL, MODLZONE, MODLINIT, and CALCINIT.

LINEAL The program LINEAL is actually a list of commands to be executed by MINITAB, a statistical analysis package. LINBAL takes the 300 and 700 mb zonal means of temperature, \(T\), and the wind component, \(u\), and estimates the vector of means and variance-covariance matrix of \(\Psi_n^0\), \(T_n^0\), \(\Theta_n^0\), and \(\Sigma_n^0\), the zonal spectral coefficients of \(\psi\), \(\tau\), \(\theta\), and \(\sigma\), respectively.
The estimates are the output of weighted least-squares; as outlined in Appendix D.

The program outputs, for each of the variables \((\psi, \tau, \theta, \sigma)\), two files; one file contains the mean vector and variance-covariance matrix of the zonal coefficients and the other file contains a vector of standard normal random numbers, adjusted to have the sample mean and variance equal to 0 and 1, respectively. These files are used as input by the program MODLZONE.

MODLZONE The program MODLZONE uses files containing the mean vectors, variance-covariance matrices and vectors of standard normal random numbers to generate a sample of multivariate normally distributed random vectors of zonal spectral coefficients of 700 and 300mb \(\psi\) and \(\theta\). The exact procedure is outlined in Appendix D. These coefficients, along with the zeroed non-zonal spectral coefficients are stored in a file which is used as input into the initializing forecast model program, MODLINIT.

The mean zonal spectral coefficients are stored as sample 0 and NSAMPLE + 1.

MODLINIT The program MODLINIT takes the sample of zonal spectral coefficients and generates a sample of 240 hour (10 day) forecasts. In generating the forecasts, zonal coefficients are held constant, but all others are allowed to vary. The end result is a sample of correlated complex spectral coefficients to be used, after calculating the mean and assigning it sample number zero, as input for the forecast models. MODLINIT is identical to MODLFCST with exception of the data being output only once, at the end of the initializing period, in MODLINIT, as opposed
to every 24 hours in MODLFCST. The user has the option of splitting the initialization into portions; for instance, a 240 hour initialization may be done with a baroclinic model the first 120 hours and the linear balance model the final 120 hours.

**CALCINIT**  The program CALCINIT simply replaces sample 0 and (NSAMPLE+1) spectral coefficients with the mean of the sampled spectral coefficients.

**Forecast program:** MODLFCST

The forecast program, MODLFCST, takes the sampled initialized spectral coefficients, from the initialization subsystem, and calculates forecasts, based on the user-specified model. The forecasts were made holding the global means of $\theta$ and $\sigma$ constant.

The primary variables forecasted by each model are (1) barotropic model: $\psi$, (2) baroclinic models: $\psi, \tau$, and (3) linear balance model: $\psi, \tau, \sigma$.

**Analysis subsystem**

The analysis subsystem consists of the analysis programs: STATANAL and TMSOANAL, and the gridding programs: GLOBMEAN, ZONEMEAN, GLOBPART, ZONEPART, and GRIDSTAT. The analysis programs calculate the basic forecast means, variance, total mean squared error (TMSO), and TMSO F-ratios. The gridding programs generate the gridded data to be used by the commercial display programs, HARVARD GRAPHICS® and SURFER®.
STATANAL  The program, STATANAL, calculates global and zonal means and spectral coefficients of statistics comparing two user-specified models. The statistics are described in Chapter 6. The global means allow an overall comparison of all models, over time, on a single chart. The zonal means present the mean zonal changes of a single model statistic over time. Finally, the spectral coefficients of the statistics, using output from GRIDSTAT, may be used to view the spatial value at a give point in the forecast.

To avoid a Gibbs phenomenon in the recovered gridded variance and TMSO fields, the spectral coefficients of the common logarithm of these statistics are stored.

TMSOANAL  The program, TMSOANAL, calculates the gridded fields of partitioned TMSO directly from the forecast sample; the partitioning technique is discussed in Chapter 6. The gridded fields are stored in a SURFER® readable file for later use in latitude-longitude (lat-lon) contour plots of the partitioned TMSO fields.

GRIDSTAT  The program, GRIDSTAT, generates gridded fields of the statistics of primary variables ($\psi, \tau, \sigma$). The user specifies the desired model and forecast time of these fields. The output is stored in SURFER readable data files.

GLOBMEAN and GLOBPART  The program GLOBMEAN combines the STATANAL-generated global means of the forecast statistics, from all models and forecast times, into a single file; one file for each statistic. The program GLOBPART combines the STATANAL-generated global mean TMSO
statistics, from all models and forecast times, and calculates the global mean partitioned TMSO components.

The outputs from GLOBMEAN and GLOBPART are stored in HARVARD GRAPHICS® readable data files.

ZONEMEAN and ZONEPART The program ZONEMEAN combines the STATANAL-generated zonal means of the forecast statistics, from all models and forecast times, into a single file for each statistic. The program ZONEPART combines the STATANAL-generated zonal mean TMSO statistics, from each model and forecast times, and calculates the zonal mean partitioned TMSO components.

The outputs from ZONEMEAN and ZONEPART are stored in SURFER® readable data files.

Display subsystem

The display subsystem consists of two commercial programs, HARVARD GRAPHICS® and SURFER®.

HARVARD GRAPHICS® This program, by Software Publishing Corp., was used to create the x-t plots of the comparative time evolutions of the global means of model statistics and TMSO partitions.

SURFER® This program, by Golden Software, Inc., was used to create contour plots of the time evolution of zonal mean statistics and lat-lon plots of the various model statistics and partitioned TMSO at a particular forecast time.
Summary

The model hierarchy used in this study is based upon the linear balance equations and the assumption of $\partial \theta / \partial p = f(\mu, \lambda, t)$. The models were global spectral models and, with the exception of the one-layer barotropic model, had two layers. In all models, the global mean potential temperatures ($\theta_1$ and $\theta_3$) were held constant, implying the global mean of the model variable, $\sigma$, was also held constant. The model equations are summarized in Appendix F.

The Monte-Carlo forecast model system, developed for this study, consists of three main subsystems, or sets of programs: (1) the initialization subsystem, (2) the Monte-Carlo forecast program, and (3) the analysis/display subsystem. These subsystems generate the initial sample of random spectral coefficients, make a Monte-Carlo forecast using the sampled spectral coefficients and a user-specified model, and generate sample statistics from the sample of forecasts.

Chapters 6 presents the analysis of the basic forecast statistic and Chapter 7 presents the partitioning analysis results.
CHAPTER 6
FORECAST ANALYSIS: BASIC MODEL STATISTICS

The next two chapters present the analysis of Monte-Carlo forecasts generated using the model hierarchy described in Chapter 5. The analysis is split into two parts: the basic model statistics (Chapter 6) and the partitioning of total mean squared error (Chapter 7).

The size of the analysis was limited to the examination of only basic statistics of the primary variables, \((\psi, \tau, \sigma)\). These basic statistics are (1) mean and bias of the model forecasts, (2) model forecast variance, (3) model TMS0, and an (4) "F ratio" statistic for the model TMS0. Analysis discussions do not cover all of the models, since some statistics are similar for the various models.

This study assumed the "true" model was the linear balance (MCLBX) model and the "true" initial data were the mean initial conditions.

Description of Basic Statistics

This section reviews the basic model forecast statistics presented in this chapter.

Mean and bias

The sample mean model forecast, \(\bar{X}_m(t, \varepsilon)\), is the average, over the sample of model \((M)\) forecasts, at time \(t\). The population mean model forecast, \(\mu_m(t, \varepsilon)\), is the expected value of the sampled forecasts.
The model bias is the difference between the mean model forecast and the "true" forecast, $\mu_t(t,0)$.

$$B_{\text{pop}} \left( X_H(t,\varepsilon) \right) = \mu_H(t,\varepsilon) - \mu_t(t,0) \quad \text{Population bias}$$

$$B \left( X_H(t,\varepsilon) \right) = \bar{X}_H(t,\varepsilon) - \mu_t(t,0) \quad \text{Sample bias}$$

Variance

The population model variance is denoted as $\sigma^2 \left( X_H(t,\varepsilon) \right)$ and the sample model variance is denoted as $s^2 \left( X_H(t,\varepsilon) \right)$. The model variance measures the forecast variability relative to the mean forecast. The model standard deviation is the square root of the model variance.

$$\sigma^2 \left( X_H(t,\varepsilon) \right) = E \left( X_H(t,\varepsilon) - \mu_H(t,\varepsilon) \right)^2 \quad \text{Population variance}$$

$$s^2 \left( X_H(t,\varepsilon) \right) = \frac{1}{n-1} \sum_{i=1}^{n} \left( X_H(t,\varepsilon) - \bar{X}_H(t,\varepsilon) \right)^2 \quad \text{Sample variance}$$
Total mean squared error (TMSO)

TMSO, total mean squared error with reference to the "true" forecast, is defined as the mean squared difference between the model forecast, based upon initial conditions with errors, and the "true" forecast. The TMSO measures the model forecast variability, relative to the "true" forecast and is equal to the sum of the model variance and the square of the model bias. This is the only analyzed statistic measuring the overall model forecast performance against the "true" forecast.

\[
\text{TMSO}_{\text{pop}} \left( X_H(t,e) \right) = \mathbb{E} \left( X_H(t,e) - \mu_T(t,0) \right)^2
\]

\[
= \sigma^2 \left( X_H(t,e) \right) + B_{\text{pop}} \left( X_H(t,e) \right)^2 \quad \text{Population TMSO}
\]

\[
\text{TMSO} \left( X_H(t,e) \right) = \frac{1}{n} \sum_{i=1}^{n} \left( X_H(t,e) - \mu_T(t,0) \right)^2
\]

\[
= \frac{n-1}{n} \sigma^2 \left( X_H(t,e) \right) + B \left( X_H(t,e) \right)^2 \quad \text{Sample TMSO}
\]
Chapter 7 presents results of partitioning TMSO into components representing the effects of the sequential deletion of model variables in the hierarchy.

**F ratio statistic**

**Definition** The two analyzed measures of model variability were the model variance and TMSO. The difference between the two statistics is the inclusion of the square of the model bias in the TMSO. The F ratio allows the evaluation of whether or not TMSO is "effectively" equal to the model variance, which is equivalent to evaluating whether or not the model bias is "effectively" zero. If the model bias is "effectively" zero, the differences between the model and "true" forecasts are negligible.

The "F ratio" statistic of the TMSO is simply the ratio of the model TMSO over the model variance,

\[
F_{\text{pop}} \left( X_{\text{H}}(t, \varepsilon) \right) = \frac{\text{TMSO}_{\text{pop}} \left( X_{\text{H}}(t, \varepsilon) \right)}{\sigma^2 \left( X_{\text{H}}(t, \varepsilon) \right)} = 1 + \frac{\left( B_{\text{pop}} \left( X_{\text{H}}(t, \varepsilon) \right) \right)^2}{\sigma^2 \left( X_{\text{H}}(t, \varepsilon) \right)}
\]

The F ratio is estimated using the sample estimates of TMSO and \( \sigma^2 \)

\[
F \left( X_{\text{H}}(t, \varepsilon) \right) = \frac{\text{TMSO} \left( X_{\text{H}}(t, \varepsilon) \right)}{\frac{n-1}{n} s^2 \left( X_{\text{H}}(t, \varepsilon) \right)} = 1 + \frac{\left( B \left( X_{\text{H}}(t, \varepsilon) \right) \right)^2}{\frac{n-1}{n} s^2 \left( X_{\text{H}}(t, \varepsilon) \right)}
\]
The F ratio measures the size of the model bias relative to the model standard deviation and is never less than 1 since it is equal to 1 plus the ratio of two positive quantities.

Selection of critical F values  The F ratio values used to evaluate model biases are equal to, or greater than, one. If an F ratio was near one, the model bias was still considered negligible; but, if the F ratio was much larger than one, the model bias was considered to be "effectively" not equal to zero.

For this study, the F ratios were allowed to be up to 20% larger than 1 before being considered "effectively" different from 1. This is equivalent to requiring the root-mean-squared bias to be larger than approximately 10 percent of the model standard deviation before the model mean forecast was considered to be "effectively" biased. Using this criterion, the critical F ratio value was

\[ F_c = 1.2 \]

Because of the sharpness of the transition between spatial regions of "effectively" biased and unbiased model forecasts, boundaries of these areas were insensitive to minor changes in the selected value of \( F_c \).

Application: Grid points  The gridded TMSO and \( s^2 \) values were used to calculate the gridded F ratio values. These F ratios were compared with \( F_c \) and regions of "effectively" biased model forecasts were mapped.
Application: Zonal comparison  
To allow overall zonal evaluation of the model bias, the relative frequency of F ratios, less than or equal to $F_c$, were calculated at each latitude. If the relative frequency was less than a critical limit, 0.8 in this study, the mean model forecast was considered to be "effectively" biased and therefore different from the "true" forecast at that latitude.

Latitude-time contour plots of the relative frequencies allow the evaluation of which latitudes were most sensitive to model differences and at what time the mean model forecast became "effectively" different from the "true" forecasts.

Application: Global comparison  
The overall global evaluation of the model bias was performed in the same manner as the zonal evaluations except the global relative frequency is the area-weighted mean of zonal relative frequencies. The weights are the same latitudinal Gaussian weights used in the spectral transformations.

The plot of global relative frequency with time, for each model, allow simultaneous comparison of all model mean forecasts with the "true" forecast over time.

Basic Model Statistics: $\psi$

240 hour "true" forecast: $\psi$

The 240 hour "true" MCLBX forecast, Figure 6.1, showed the expected lee side troughing in eastern China. A very strong jet core was located over in the area of Japan and Korea, but the flow diverged over the ridge located at 160°E longitude. The jet core was located between a cutoff low
over Siberia and a strong high in the South China Sea. There was a strong trough at 180°E longitude with the jet rebuilding as the flow moved eastward over the United States. Strong ridging was found off of the eastern U.S. and in Central Russia. Other cutoff lows were found over northern Canada and the Scandinavian Peninsula.

The Southern Hemisphere mid latitudes had moderate zonal flow with a superimposed wave 8 component.

240 hour mean forecasts: $\psi$

The long-wave structure of the 240 hour mean barotropic (MCBVX) forecast was similar to the long wave structure of the "true" forecast, but, as shown in Figure 6.2, the MCBVX mean forecast lacked the shorter waves. This lack of short wave structure at 240 hours was due to the cumulative effects of the diffusion scheme which smoothed the short waves more than the long waves and the lack of dynamic growth mechanisms in the barotropic model. In the Southern Hemisphere, the lack of short waves gave almost pure zonal flow.

The wave structure of the baroclinic (MCBCO) and the linear balance (MCLBX) mean forecasts, Figures 6.3 and 6.4, respectively, was very similar to the "true" forecast. The baroclinic model had stronger flows than the linear balance model, indicating the moderating influence of a fully variable $\sigma$ alluded to by Gates (1961) in his study of a linearized linear balance model. There was some smoothing of the shorter waves due to the averaging of forecasts with slightly different short wave phasing.
Figure 6.1 240 Hour "True" Forecast: $\psi$  

Figure 6.2 240 Hour Mean Forecast: MCBVX $\psi$
Figure 6.3 240 Hour Mean Forecast: MCBCO $\psi$

Figure 6.4 240 Hour Mean Forecast: MCLEX $\psi$
The lack of short wave structure was evidenced in the plot of MCBVX model bias, Figure 6.5, with regions of large biases located throughout the Northern Hemisphere and a weaker wave 7-8 bias structure in the Southern Hemisphere. The two-layer model biases, Figure 6.6 for MCBC0 and Figure 6.7 for MCLBX, both had a strong maximum positive value in the core of the jet over Japan and Korea due to a slightly different orientation of the jet core and looser contour packing in the mean forecast. The remaining bias maxima/minima appeared to be due to the phase-caused smoothing of the shorter waves in the mean forecasts.

Variance: $\psi$

Examination of the time plot of the global mean of the model variances, Figure 6.8, shows two unexpected relations: (1) the MCBVX variance was constant over time, with the exception of a slight diffusion-caused decrease, and (2) the two-layer model variances were nearly equal.

The barotropic model does not contain a mechanism for dynamic error growth that is present in the two-layer MCBCx and MCLBX models. It is essentially a purely advective model for vorticity, conserving mean vorticity and enstrophy. So in retrospect, it was not surprising that the mean variance is constant over time. The observed slight decrease in variance over time was due to diffusion built into the time stepping portion of the forecast program.

The near equality of the global mean variance of the two-layer models (MCBCx and MCLBX) indicates, at least on a globally averaged level, the
Figure 6.5  240 Hour Model Bias: MCBVX $\psi$

Figure 6.6  240 Hour Model Bias: MCBCO $\psi$
Figure 6.7  240 Hour Model Bias: MCLBX

Figure 6.8  Global Mean of Model Variance: ψ
physical processes, causing the growth of variances, are common to both the baroclinic and linear balance models. Equality of model variances is a useful assumption to make when partitioning the total mean squared error (TMSO).

**Zonal mean** It was of interest to see if the constancy of global mean barotropic variance and equality of two-layer model global mean variance carried over to the analogous zonal mean variances. Time plots of the common logarithm of the zonal mean of model variance for the MCBVX, MCBCO, and MCLBX models (Figures 6.9 to 6.11, respectively) show that, as with the global mean variances, the barotropic variance didn’t change much over time and the two-layer model variances were approximately equal. The MCBCO and MCLBX zonal plots also showed the middle latitudes experienced the largest growth of model variance. This growth, baroclinic in nature, was absent in the MCBVX model.

**240 hour forecast** Examination of the plots of 240 hour forecast model variances for the MCBVX, MCBCO, and MCLBX models (Figures 6.12 to 6.14, respectively) showed distortions in a zonally-uniform barotropic variance field, due to advection of the variance; but, there was not the growth evidenced in the two-layer models. The overall patterns of MCBCO and MCLBX variances were similar, although not equal by any means.

**TMSO:** \( \psi \)

**Global mean** Similarities between the global mean TMSO (Figure 6.15) and the global mean variance (Figure 6.8) indicated a large portion of the model TMSO was due to the errors introduced by the initial data.
Figure 6.9 $\log_{10}(\text{Zonal Mean Variance}): \text{MCBVX} \ \psi$

Figure 6.10 $\log_{10}(\text{Zonal Mean Variance}): \text{MCBCO} \ \psi$
Figure 6.11 $\log_{10}$ (Zonal Mean Variance): MCLBX $\psi$

Figure 6.12 $\log_{10}$ (240 Hour Forecast Variance): MCBVX $\psi$
Figure 6.13 $\log_{10}(240 \text{ Hour Forecast Variance})$: MCBC0 $\psi$

Figure 6.14 $\log_{10}(240 \text{ Hour Forecast Variance})$: MCLBX $\psi$
This observation was verified, in Chapter 7, by the analysis of TMSO partitions.

What is striking about Figure 6.15 is the slowing of MCBVX TMSO growth later in the forecast period. Since the MCBVX global variance was approximately constant, the slowing of MCBVX TMSO growth was due to a decrease in the model bias growth rate.

The two-layer global mean TMSOs had similar growth rates after about 3 days. Although the MCBCO model had a slightly smaller TMSO than the MCBCL model, an F ratio evaluation of the partitioned component was needed to determine if this difference was "effectively" not equal to zero.

Zonal mean: Figures 6.16 to 6.18 present time plots of the common logarithm of the zonal mean model TMSO for the MCBVX, MCBCO, and MCLBX models, respectively. Because the model variance dominated the two layer model TMSO, they had similar zonal TMSO and variance plots and are not discussed.

The barotropic zonal TMSO plot, Figure 6.16, is similar to the two-layer model plots, Figures 6.17 and 6.18, due to the structure of the MCBVX bias. The bias is due, in part, to the lack of a baroclinic development mechanism in the barotropic model. What is interesting about the MCBVX TMSO zonal plot is the low values in the Northern Hemisphere polar region, north of about 75°N. The two-layer model TMSO values were almost an order of magnitude larger in the late portions of the forecast period. A similar pattern was not seen in the Southern Hemisphere.

240 hour forecast: Figures 6.19 to 6.21 present the lat-long plots of the 240 hour forecast of the common logarithm of TMSO for the MCBVX,
Figure 6.15 Global Mean of Model TMSO: $\psi$

Figure 6.16 $\log_{10}$ (Zonal Mean TMSO): MCBVX $\psi$
Figure 6.17 $\log_{10}$ (Zonal Mean TMSO): MCBCO $\psi$

Figure 6.18 $\log_{10}$ (Zonal Mean TMSO): MCLBX $\psi$
MCBCO, and MCLBX models, respectively. As with the global and zonal mean TMSOs, the model variance dominated the two-layer model TMSO. As was expected, due to lack of shortwave structure, the MCBVX TMSO showed numerous maxima along the Northern Hemisphere mid latitude storm track.

F ratio: $TMSO(\psi)$

The critical minimum relative frequency of $(F \leq F_c)$ was set to 0.8; meaning, if more than 20% of the F ratios were greater than $F_c$, the model in the area (zonal or global) was considered to "effectively" biased.

Global mean Figure 6.22 presents a time plot of the global relative frequency of $(F \leq F_c)$. The barotropic (MCBVX) mean forecast was "effectively" biased after 1 1/2 days; the baroclinic (MCBCO and MCBCl) models after 4 days and the linear balance (MCLBX) model after 7 1/2 days.

Zonal mean Figures 6.23 to 6.25 present time plots of the zonal relative frequency of $(F \leq F_c)$ for the MCBVX, MCBCO, and MCLBX models, respectively. The fastest deterioration of the model forecasts, relative to the "true" forecast occurred in the low to mid latitudes of both hemispheres, with the fastest deterioration in the Northern Hemisphere.

The MCBVX mean forecast was "effectively biased after only 10 hours in the Northern Hemisphere and 72 hours for the Southern Hemisphere. The baroclinic (MCBCx) became "effectively" biased after 40 hours in the Northern Hemisphere and 3 1/2 days in the Southern Hemisphere. The baroclinic model bias was most "significant" in low latitudes, around 20°.

The growth of the linear balance bias was much slower than in the baroclinic model. Note that the slow growth rate may not be the case if
Figure 6.19  $\log_{10}(240 \text{ Hour Forecast TMSO})$: MCBVX

Figure 6.20  $\log_{10}(240 \text{ Hour Forecast TMSO})$: MCBCO
Figure 6.21  $\log_{10}(240$ Hour Forecast TMS0): MCLBX $\psi$

Figure 6.22  Global Relative Frequency ($F \leq F_c$): TMS0($\psi$)
Figure 6.23  Zonal Relative Frequency ($F \leq F_c$): MCBVX TMS0($\psi$)

Figure 6.24  Zonal Relative Frequency ($F \leq F_c$): MCBCO TMS0($\psi$)
the "true" model was changed from the linear balance to a primitive equation model. The MCLBX bias became "significant" after about 5 days in the Northern Hemisphere and 9 days in the Southern Hemisphere.

240 hour forecast Figures 6.26 to 6.28 present lat-long plots of the 240 hour forecast F(TMS0) for the MCBVX, MCBCO, and MCLBX models, respectively.

The black areas in Figure 6.26 show where the F ratio was less than the critical F, indicating the barotropic model forecast was "effectively" the same as the "true" forecast. Except for some scattered areas in the tropics, most of the 240 hour barotropic forecast had "significant" biases. There were two polar areas, one just north of the Antarctic highlands at 90°E latitude and one region north of Russia, Siberia and Alaska, where the biases were not "significant". The extreme polar regions were not resolved well with the T15 resolution of the model.

The black areas in Figures 6.27 and 6.28 show where the F ratio was greater than the critical F, indicating the particular mean model forecast was "effectively" different from the "true" forecast. Generally the linear bias model had smaller areas of "significant" biases than the baroclinic model, especially in the Southern Hemisphere.

Summary

Means and biases The mean forecasts tended to be smoother than the "true" forecast because of the smoothing effect which occurs when averaging individual forecasts having slightly different short wave phasing. After 240 hours all of the barotropic shortwave structure was
Figure 6.25  Zonal Relative Frequency ($F \leq F_c$): MCLBX TMS0($\psi$)

Figure 6.26  240 Hour Forecast ($F \leq F_c$): MCBVX TMS0($\psi$)
Figure 6.27  240 Hour Forecast ($F \geq F_c$): MCBCO  TMSO($\psi$)

Figure 6.28  240 Hour Forecast ($F \geq F_c$): MCLBX  TMSO($\psi$)
diffused out leaving the long wave structure which was similar to that in the "true" forecast. The baroclinic models tended to have stronger flow than the linear balance model, indicating the moderating influence of a fully variable $\sigma$ on system growth.

**Variance** Two approximations may be made for the model forecast variances. The first approximation is that the variance of the barotropic forecasts is constant. This is due to advective nature of the model and the lack of model physics allowing baroclinic growth. The second approximation is that the variances are equal in the two-layer models. This is due to both model types (baroclinic and linear balance) having the essential physics to describe the baroclinic growth. The approximations hold extremely well with the global mean variances, fairly well with the zonal mean variances and relatively poorly with the gridded variances. They are used in Chapter 7 to aid in understanding various aspects of the TMSO partitioning.

**TMSO** For the two-layer models, most of the TMSO is due to the model variance, so both had similar patterns.

The MCBVX global mean TMSO showed a decrease in growth toward the end of the forecast period. This was due to a decrease in the bias growth. Partitioning is needed to pinpoint the exact component causing the slowing of growth.

Large differences between the MCBVX and two-layer model TMSO showed in the Northern Hemisphere polar regions. Whether or not this is unique with the northern pole or only with the winter hemisphere, remains to be
determined. Simulations made during the Southern Hemisphere winter would help to determine the cause.

**F ratios** While the lat-long plots of the F ratios are of limited value, they do show where there are "significant" differences between the model and "true" forecasts.

The zonal and global relative frequency plots show, globally, the barotropic model produced reasonable forecasts out to 36 hours; zonally, this varied from less than 12 hours in the most active Northern Hemisphere latitudes to over 7 days in the NH polar regions. The baroclinic and linear balance models, as expected, produced forecasts which were reasonable further out in time.

**Basic Model Statistics: \(\tau\)**

The model variable \(\tau\) is related to the mean potential temperature, \(\Theta\), through the linear balance equation, so \(\tau\) and \(\Theta\) are used interchangeably in the discussion to achieve better clarity and understanding.

**240 hour "true" forecast: \(\tau\)**

The 240 hour "true" forecast, Figure 6.29, showed a strong \(\tau\) gradient in the Northern Hemisphere mid latitudes through subtropics. The gradient corresponds to the mean potential temperature gradient existing between the cold polar and warm tropical air masses.

The gradient was especially strong in the region of the Japan-Korean jet. Cold air pockets were located over Scandinavia, west of Korea and in northwestern Canada. An area of warm air was located in an area extending
from eastern Labrador to southwestern Greenland. As was expected for a summer hemisphere, the Southern Hemisphere $\tau$ gradients were much smaller than the Northern Hemisphere "winter" gradients.

**240 hour mean forecasts: $\tau$**

The MCBCO and MCLBX models had similar 240 hour mean forecasts (Figures 6.30 and 6.31, respectively) and both showed large biases (Figures 6.32 and 6.33) when compared to the "true" forecast (Figure 6.29). The existence of the Labrador-Greenland warm pocket, shown in the "true" forecast, seemed to be especially sensitive to initial conditions; the pocket was absent in the mean forecast plots. The cold air pockets, as far as size and placement, also showed sensitivities to the initial conditions.

The 240 hour MCBCO and MCLBX forecast model biases showed a large bias maximum over the location of the cold pocket west of Korea, with smaller minimum over southern Tibet and maximum over the east coast of North America. The southern Tibet minimum showed a marked structural difference between the baroclinic and linear balance models, indicating that it was possibly due to advection or time variations of stability, $\sigma$. The maximum over the eastern coast of North America was similar for the two models, indicating the bias probably was due to the effect of errors in the initial conditions.
Figure 6.29 240 Hour "True" Forecast: \( \tau \)

Figure 6.30 240 Hour Mean Forecast: MCBO \( \tau \)
Figure 6.31 240 Hour Mean Forecast: MCLBX $\tau$

Figure 6.32 240 Hour Model Bias: MCBCO $\tau$
Variance: $\tau$

As with the model variance of $\psi$, the global mean variance of $\tau$, from the baroclinic and linear balance models (Figure 6.34), were almost equal. The near equality of the global mean variance of the two-layer models indicates that, at least for globally averaged level, the physical processes causing the growth of variances are common to both the baroclinic and linear balance models.

Time plots of the common logarithm of zonal mean variance of $\tau$ for the MCBC0 and MCLBX models (Figures 6.35 and 6.36, respectively) show this near equality of model variances carries over to the zonal mean variance of $\tau$ for the baroclinic and linear balanced models. Both zonal plots show the mid latitudes experienced the largest growth of model variance.

The overall patterns of 240 hour MCBC0 and MCLBX model variances of $\tau$ (Figures 6.37 and 6.38, respectively) were similar, although not equal by any means.

TMS0: $\tau$

The relative differences between the global means of the baroclinic (MCBCx) and the linear balance TMS0 values (Figure 6.39) were much larger for $\tau$ than they were for $\psi$. After 2 to 3 days, both models had about the same rate of TMS0 growth. The MCBC1 model had a slightly smaller global mean TMS0 than the MCBC0 model.

As with $\psi$, variance dominated the TMS0, so the MCBC0 and MCLBX model zonal $\tau$ TMS0 (Figures 6.40 and 6.41, respectively) were similar to the variance (6.35 and 6.36, respectively)
Figure 6.33  240 Hour Model Bias: MCLBX $\tau$

Figure 6.34  Global Mean of Model Variance: $\tau$
Figure 6.35  $\log_{10}(\text{Zonal Mean Variance})$: MCBCO $\tau$

Figure 6.36  $\log_{10}(\text{Zonal Mean Variance})$: MCLBX $\tau$
Figure 6.37 $\log_{10}(240 \text{ Hour Forecast Variance})$: MCBCO $\tau$

Figure 6.38 $\log_{10}(240 \text{ Hour Forecast Variance})$: MCLBX $\tau$
Figure 6.39  Global Mean of Model TMSO: $\tau$

Figure 6.40  $\log_{10}$ (Zonal Mean TMSO): MCBCO $\tau$
The largest 240 hour forecast $\tau$ TMSO values, for the MCBCO (Figure 6.42) and MCLBX (Figure 6.43) models, occurred in the areas of the cold pockets over eastern Asia. Differences between the MCLBX and MCBCO TMSO maximum in southern Tibet indicate that the maximum was possibly due to not allowing advection or time variations of $\sigma$ in the baroclinic model.

F ratio: TMSO($\tau$)

The plot of global relative frequency of ($F \leq F_c$), Figure 6.44, shows the baroclinic $\tau$ forecast became "effectively" biased after about 1 1/2 days and the linear balance $\tau$ after 8 1/2 days.

The zonal relative frequency of ($F \leq F_c$), Figure 6.45 for the MCBCO model and Figure 6.46 for the MCLBX model, show the MCBCO forecast was "effectively" biased after only 16 hours in at 30°N and the MCLBX forecast was "effectively" biased after almost 4 days at 45°N. The SH MCBCO forecast became "effectively" biased after about 26 hour and the SH MCLBX forecast did not become "effectively" biased until the end of the period.

Lat-long plots of the 240 hour forecast $F[TMSO(\tau)]$ for the MCBCO and MCLBX models (Figures 6.47 and 6.48, respectively) show most "significant" MCBCO biases were associated with the area just south of Tibet and were not observed in the MCLBX model. Other MCBCO-unique bias regions were located in the Southern Hemisphere south of Madagascar, north and east of Australia, one to the west and one east of South America, and one in the South Atlantic. The Japan-Korean area continued to be the location of a region that was very sensitive to errors in the initial data.
Figure 6.41 $\log_{10}(\text{Zonal Mean TMSO})$: MCLEX $\tau$

Figure 6.42 $\log_{10}(\text{240 Hour Forecast TMSO})$: MCBCO $\tau$
Figure 6.43 $\log_{10}(240$ Hour Forecast TMS0): MCLBX $\tau$

Figure 6.44 Global Relative Frequency ($F \leq F_c$): TMS0($\tau$)
Figure 6.45 Zonal Relative Frequency ($F \leq F_c$): MCECO TMSO($\tau$)

Figure 6.46 Zonal Relative Frequency ($F \leq F_c$): MCLBX TMSO($\tau$)
Figure 6.47  240 Hour Forecast F(TMS0): MCBCO τ

Figure 6.48  240 Hour Forecast F(TMS0): MCLBX τ
Summary

The basic statistics of $\tau$ showed similarities with those of $\psi$. The size and placement of warm and cold "pockets" was sensitive to initial conditions. As for the $\psi$ variances, equality approximations may be made for the two-layer model $\tau$ variances.

There were marked differences in the 240 hour forecast TMSO plots of the MCBCO and MCLBX models. These differences indicate that not all of the maxima had the same cause. Chapter 7 examines this more closely.

Basic Model Statistics: $\sigma$

240 hour "true" forecast: $\sigma$

The 240 hour "True" $\sigma$ forecast, Figure 6.49, showed strong gradients around and downwind from Tibet. The area of greatest instability (smallest $\sigma$) was located just east of Japan and Korea. There was very little variation of $\sigma$ in the Southern Hemisphere which is consistent with a summer hemisphere.

240 hour mean forecasts: $\sigma$

The 240 hour mean MCLBX $\sigma$ forecast, Figure 6.50, showed little overall pattern differences from the "true" forecast. The 240 hour forecast bias, Figure 6.51, shows areas of relatively large biases (> 1 degree K) in eastern Asia, off of its coast, and in the eastern and Gulf Coast portions of the U.S.
Figure 6.49  240 Hour "True" Forecast: $\sigma$

Figure 6.50  240 Hour Mean Forecast: MCLBX $\sigma$
Variance: $\sigma$

Global mean The global mean variance of $\sigma$, as shown in Figure 6.52, decreased during the first four forecast days then increased for the remainder of the period. This was probably due to "spin up", the forecast model, with variable zonal coefficients, adjusting to data initialized with a model with constant zonal coefficients.

Zonal mean The early overall decrease in model variance showed up in the non-polar regions of the time plot of zonal $\sigma$ variance, Figure 6.53. Later variance growth occurred mainly around $30^\circ$N latitude.

240 hour forecast Figure 6.54 is a lat-long plot of the 240 hour forecast $\sigma$ variance. There were three areas of maximum variance: over the Middle East, upwind from Tibet; on the east coast of China, downwind from Tibet; and over the mid western US.

TMSO($\sigma$)

The global mean TMSO, Figure 6.55, was similar to the forecast variance, Figure 6.52. The time plot of the zonal mean TMSO($\sigma$), Figure 6.56, showed the early overall TMSO decrease occurred mainly in the Northern Hemisphere mid latitudes. The later TMSO growth occurred around $30^\circ$N latitude. The lat-long plot of the 240 hour forecast TMSO($\sigma$), Figure 6.57, showed maximums in the same three areas as the variance.

F Ratio: TMSO($\sigma$)

The plot of global relative frequency of $(F \leq F_c)$, Figure 6.58, shows the linear balance $\sigma$ forecast became "effectively" biased after about
Figure 6.51 240 Hour Model Bias: MCLBX $\sigma$

Figure 6.52 Global Mean of Model Variance: $\sigma$
Figure 6.53  Zonal Mean Variance: MCLBX $\sigma$

Figure 6.54  240 Hour Forecast Variance: MCLBX $\sigma$
Figure 6.55 Global Mean of Model TMSO: $\sigma$

Figure 6.56 Zonal Mean TMSO: MCLBX $\sigma$
Figure 6.57  240 Hour Forecast TMSO: MCLBX $\sigma$

Figure 6.58  Global Relative Frequency ($F \leq F_c$): TMSO($\sigma$)
10 1/2 days. The zonal relative frequency of \( F \leq F_c \), Figure 6.59 shows the MCBCO \( \sigma \) forecast was "effectively" biased after 8 days at about 40°N and the forecast in the remaining Northern Hemisphere mid latitudes became "effectively" biased after 9 to 9 1/2 days.

The lat-long plot of the 240 hour forecast TMSO(\( \sigma \)), Figure 6.60, shows most of the "effectively" biased regions were found immediately north and south of Tibet, in a region east of Japan and south of Korea, over Mexico, and in a region east of the US and south of Greenland.

**Summary**

Examination of the growth and geographical patterns of the basic model forecast statistics gave insight into where each model is most sensitive to errors in the initial conditions, areas of similar statistics and areas sensitive to model differences.

The 240 hour mean forecast plots of the primary variables were similar to the "true" forecast except for the barotropic model which had the shorter waves diffused out of the forecast by the 10-day point. The area over Japan and Korea, downwind from Tibet, seemed to be especially sensitive to variations in the initial conditions.

The barotropic model global mean of model variance and, to a somewhat lesser extent, zonal mean of model variance were constant over time. The two-layer models had almost identical global and zonal mean model variances for both \( \psi \) and \( \tau \). The near equality of two-layer model variances indicate that the physics affecting the variance growth was present in all two-layer models.
Figure 6.59  Zonal Relative Frequency \( (F \leq F_c) \): MCLBX TMSO(\( \sigma \))

Figure 6.60  240 Hour Forecast \( F(TMSO) \): MCLBX \( \sigma \)
The TMSO showed greatest growth in the mid latitudes of the Northern Hemisphere. Table 6.1 summarizes the approximate time when the forecasts of each model became "effectively" biased, both globally and zonally (NH mid latitudes).

Table 6.1  Time to "Effectively" Biased Forecasts (hours)

<table>
<thead>
<tr>
<th></th>
<th>$\psi$ Global</th>
<th>$\tau$ Global</th>
<th>$\sigma$ Global</th>
<th>$\psi$ Zonal</th>
<th>$\tau$ Zonal</th>
<th>$\sigma$ Zonal</th>
</tr>
</thead>
<tbody>
<tr>
<td>MCBVX</td>
<td>36</td>
<td>10</td>
<td>---</td>
<td>36</td>
<td>16</td>
<td>---</td>
</tr>
<tr>
<td>MCBCx</td>
<td>96</td>
<td>40</td>
<td>36</td>
<td>16</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>MCLBX</td>
<td>180</td>
<td>120</td>
<td>205</td>
<td>48</td>
<td>250</td>
<td>190</td>
</tr>
</tbody>
</table>

The next chapter takes the model TMSO statistics and partitions them into components due to sequentially deleted model terms. F ratio evaluations of the partitioned components helped in determining when they became "effectively" different from zero.
Basic Partitioning and Evaluation

The primary purpose of this study was to develop a practical method for partitioning model forecast errors by model source term. Building upon the background presented in earlier chapters, this section develops a simple technique to sequentially partition the forecast errors and a simple method to check if the resulting partitions "effectively" differ from zero. The application to an operational forecast model hierarchy, with an unknown "true" forecast, is presented at the end of this section.

Differences between the model and "true" forecasts are quantified using the TMSO statistic. The error partitioning technique used in this study involves the sequential partitioning of the mean squared error. The technique derivation is based on three model types; two subset models, A and B, and the "True" model, T; but may be easily extended to more. These model types correspond to the barotropic (A), baroclinic (B), and linear balance (T) models of this study.

For the three model types, define the following quantities: the individual model forecasts

\[
\begin{align*}
\text{Model A: } X_A(t,c) &= X_A \\
\text{Model B: } X_B(t,c) &= X_B \\
\text{Model T: } X_T(t,c) &= X_T
\end{align*}
\]

the mean of the model forecasts,
Model A: $\mu_A(t, \varepsilon) = \mu_A(X)$
Model B: $\mu_B(t, \varepsilon) = \mu_B(X)$
Model T: $\mu_T(t, \varepsilon) = \mu_T(X)$

and the true forecast,

Model T: $\mu_T(t, 0) = \mu_{T_0}(X)$

Total mean squared error partitioning

To partition $TMSO_A(X)$, the total MSE of variable $X$ in model $A$, with reference to the true forecast, $\mu_{T_0}(X)$, the individual forecast error is first expanded.

$$\varepsilon_{AT_0}(X) = \begin{pmatrix} X_A - \mu_{T_0}(X) \end{pmatrix} = \begin{pmatrix} (X_A - X_B) + (X_B - X_T) + (X_T - \mu_T(X)) + (\mu_T(X) - \mu_{T_0}(X)) \end{pmatrix}$$

$$= (X_A - X_B) + (X_B - X_T) + \delta_I(X) + B_{TT_0}(X)$$

$$= (X_A - X_B) + (X_B - X_T) + \varepsilon_{TT_0}(X)$$

$$= (X_A - X_B) + \varepsilon_{BT_0}(X)$$

is rewritten as

$$\varepsilon_{AT_0}(X) = \Delta_{AB}(X) + \Delta_{BT}(X) + \delta_I(X) + B_{TT_0}(X)$$
where
\[ \Delta_{1j}(X) = X_1 - X_j \]
\[ \delta_i(X) = X_i - \mu_i(X) \]
\[ B_{ij}(X) = \mu_i(X) - \mu_j(X) . \]

Define the TMSO of variable X, for model \( \eta \), to be

\[ \text{TMSO}_{\eta T_0}(X) = E \left[ \left( \frac{c_{\eta T_0}(X)}{T_0} \right)^2 \right] . \]

The TMSO of variable X, for model A, becomes

\[
\text{TMSO}_{AT_0}(X) = \left( E \left[ \delta_T^2(X) \right] + B_{TT_0}^2(X) \right) + \\
E \left[ \Delta_{BT}^2(X) + 2\Delta_{BT}(X) \delta_T(X) \right] + 2B_{BT}(X) B_{TT_0}(X) + \\
E \left[ \Delta_{AB}^2(X) + 2\Delta_{AB}(X) \left( \Delta_{BT}(X) + \delta_T(X) \right) \right] + 2B_{AB}(X) B_{TT_0}(X) .
\]

Using the expected values,

\[
E \left[ \Delta_{1j}^2(X) \right] = \sigma_i^2(X) + \sigma_j^2(X) - 2\sigma_{1j}(X) + B_{1j}^2(X)
\]

\[
E \left[ \Delta_{1j}(X) \Delta_{kl}(X) \right] = \left( \sigma_{1k}(X) - \sigma_{11}(X) \right) - \left( \sigma_{jk}(X) - \sigma_{j1}(X) \right) + B_{1j}(X) B_{kl}(X)
\]
the TMSO equation simplifies to

\[
TMSO_{AT}^{(X)} = \sigma_T^2(X) + B_{BT}^2(X) + \left( \sigma_B^2(X) - \sigma_T^2(X) \right) + B_{BT}(X) \left( B_{BT}(X) + 2B_{TT}(X) \right) + \\
+ \left( \sigma_A^2(X) - \sigma_T^2(X) \right) + B_{AB}(X) \left( B_{AB}(X) + 2B_{TT}(X) + 2B_{BT}(X) \right)
\] (7.1)

This may be written as the sequential sum of partition components

\[
TMSO_{AT}^{(X)} = TMSO_{TIT}^{(X)} + \delta TMSO_{BT}^{(X)} + \delta TMSO_{AB}^{(X)}
\]

\[
= TMSO_{BT}^{(X)} + \delta TMSO_{AB}^{(X)}
\]

TMSE\(_{AT}^{(X)}\), the total mean squared error of model A, relative to the mean forecast of the true model, for variable X may be partitioned in a similar manner. The resulting partition is

\[
TMSE_{AT}^{(X)} = \sigma_T^2(X) + \\
+ \left( \sigma_B^2(X) - \sigma_T^2(X) \right) + B_{BT}^2(X) + \\
+ \left( \sigma_A^2(X) - \sigma_T^2(X) \right) + B_{AB}(X) \left( B_{AB}(X) + 2B_{TT}(X) + 2B_{BT}(X) \right)
\]
The TMSE may be written as the sequential sum of partition components

\[ \text{TMSO}^\alpha(X) = \sigma^2(X) + \delta \text{TMSO}^\beta(X) + \delta \text{TMSO}^\omega(X) \]

The partitioning of TMSE is useful if the "true" model is known, but the "true" initial conditions are not. In this case, the mean of the "true" model may be determined, but the "true" forecast remains unknown.

**Interpretation** Application of the results from the analysis of the forecast variance, presented in Chapter 6, gives some insights into relations between the TMSO (and TMSE) of the various models.

Analysis of the global and zonal mean forecast variances showed that (1) the barotropic variances were roughly constant over time and (2) the two-layer models (MCBC0, MBC1, and MCLBX) had roughly equal variances. Based upon these observations, letting Model A denote the barotropic model, model B a baroclinic model, and model T a linear balance model, TMSO^\alpha(X) is approximately

\[ \text{TMSO}^\alpha(X) = \sigma^2(X) + B^2(X) + \]

\[ + B^\beta(X) \left( B^\beta(X) + 2B^\omega(X) \right) + \]
\[ \delta \text{TMSO}_{B}(T, T_0) = \left( \sigma_A^2(X) - \sigma_T^2(X) \right) + B_{\text{AB}}(X) \left( B_{\text{AB}}(X) + 2B_{\text{TB}}(X) + 2B_{\text{BT}}(X) \right) \]

The \( \delta \text{TMSO}_{B}(T, T_0) \) component is strictly positive if the bias due to nonlinear model response of the linear balance model (T) to initial errors is relatively small, \( (2|B_{\text{TB}}| < |B_{\text{BT}}|) \), when compared with the bias of the baroclinic model (B), with respect to model T.

The \( \delta \text{TMSO}_{A}(B, T, T_0) \) component becomes negative if the growth of the square of the biases is slower than the two-layer model variance growth. In this case, even though the model B forecast may be closer, on the average, to the "true" forecast than model A, because of the larger model B variance, model A would be the better forecast, as judged by the TMSO statistic.

**TMSO partition components**

As presented in equations (7.1) and (7.2), the TMSO statistics are partitioned by model. In a model hierarchy, some model TMSO values may be considered equivalent to the TMSO values of the primary variables. For the hierarchy used in this study, the TMSO values of the primary variables \((\psi, \tau, \sigma)\) are equivalent to

\[ \text{TMSO}(\psi) = \text{TMSO}_{\text{MCBVX}}(\psi) \]

\[ \text{TMSO}(\tau) = \text{TMSO}_{\text{MCBCO}}(\tau) \]
and may be partitioned by model terms.

Description of partition components Tribbia and Baumhefner (1988) examined the partitioning of forecast error into a model deficiency and random error component. In this study, the model deficiency is split into as many as three (maximum) and the random error into two components. Depending on the primary variable, TMSO is partitioned into as many as five components: three model deficiency components: (1) a $\tau$ component, (2) a $\sigma(\mu)$ component, (3) a $\sigma(\lambda,t)$ component, and two random error components: (4) a nonlinear bias component, and (5) a residual error component.

The $\tau$ component, effectively the difference between the MBCO and MCBVX models, is the portion due to the deletion of $\tau$ from the model; the effect of going from a two-layer (MBCO baroclinic) to a one layer (MCBVX barotropic) model. Only the TMSO($\psi$) has this component since at least a two-layer model (MBCO) is needed to forecast $\tau$ and the MCLBX model is needed to forecast $\sigma$.

The $\sigma(\mu)$ component, effectively the difference between the MBC1 and MBCO, models, is the portion due to assuming $\sigma$ is a global, instead of a zonal, constant in a two-layer baroclinic model. This partition may be thought of as being used to evaluate an incremental improvement to a classical (global constant $\sigma$) baroclinic model. Both TMSO($\psi$) and TMSO($\tau$) have this partition component. Since a MCLBX model is needed to forecast $\sigma$, TMSO($\sigma$) does not have this component.
The $\sigma(\lambda,t)$ component, effectively the difference between the MCLBX and MCBC1 models, is the partition due to allowing advection and full time variation of $\sigma$ in the model. Both TMSO($\psi$) and TMSO($\tau$) have this partition component. Since a MCLBX model is needed to forecast $\sigma$, TMSO($\sigma$) does not have this component.

The nonlinear bias component is the error-related partition due to the nonlinear linear balance model response to errors in the initial conditions. As the initial errors decrease, the nonlinear bias would decrease. The TMSO of all three primary variables; TMSO($\psi$), TMSO($\tau$), and TMSO($\sigma$); have this partition.

Finally, the residual error component is the error-related component remaining after subtracting all of the other components from the TMSO. As with the nonlinear bias, this component would decrease as initial errors decrease. The TMSO of all three primary variables; TMSO($\psi$), TMSO($\tau$), and TMSO($\sigma$); have this partition.

**Calculation of TMSO components**

The TMSO of the three primary variables are partitioned as:

\[
\text{TMSO}(\psi) = \left\{ \text{TMSO}_{\text{MCBX}}(\psi) - \text{TMSO}_{\text{MCBC1}}(\psi) \right\} \sigma(\mu) \\
+ \left\{ \text{TMSO}_{\text{MCBC1}}(\psi) - \text{TMSO}_{\text{MCLBX}}(\psi) \right\} \sigma(\lambda,t) \\
+ \left\{ \text{TMSO}_{\text{MCLBX}}(\psi) - \sigma_{\text{MCLBX}}^2(\psi) \right\} \text{nonlinear bias}
\]
\begin{align*}
TMSO(\tau) &= \left[ TMSO_{\text{MCBC0}}(\tau) - TMSO_{\text{MCBC1}}(\tau) \right] \sigma(\mu) \\
&\quad + \left[ TMSO_{\text{MCBC1}}(\tau) - TMSO_{\text{MCCLBX}}(\tau) \right] \sigma(\lambda, t) \\
&\quad + \left[ TMSO_{\text{MCCLBX}}(\tau) - \sigma_{\text{MCCLBX}}^2(\tau) \right] \text{nonlinear bias} \\
&\quad + \left[ \sigma_{\text{MCCLBX}}^2(\tau) \right] \text{residual error} \quad (7.4)
\end{align*}

\begin{align*}
TMSO(\sigma) &= \left[ TMSO_{\text{MCCLBX}}(\sigma) - \sigma_{\text{MCCLBX}}^2(\sigma) \right] \text{nonlinear bias} \\
&\quad + \left[ \sigma_{\text{MCCLBX}}^2(\sigma) \right] \text{residual error} \quad (7.5)
\end{align*}

Component interpretation: TMSO(\psi) TMSO(\psi) partition components are, beginning with the last term in Equation (7.3) and working up, [1] residual error due to initializing errors; [2] nonlinear mean model response (model bias) to variations in the initial conditions, adjusting for term 1; [3] errors due to the deletion of time and advective terms of \( \sigma \), adjusting for terms 2 and 1; [4] errors due to using a constant \( \sigma \) over \( \sigma(\mu) \), after adjusting for terms 3, 2, and 1; and finally [5] errors due to
the deletion of $\tau$ (going from a two layer to one layer model), after adjusting for terms 4, 3, 2, and 1.

Component interpretation: $\text{TMSO}(\tau)$ $\text{TMSO}(\tau)$ partition components are, beginning with the last term in Equation (7.4) and working up, [1] residual error due to initializing errors; [2] nonlinear mean model response (model bias) to variations in the initial conditions, adjusting for term 1; [3] errors due to the deletion of time and advective terms of $\sigma$, adjusting for terms 2 and 1; and finally [4] errors due to using a constant $\sigma$ over $\sigma(\mu)$, after adjusting for terms 3, 2, and 1.

Component interpretation: $\text{TMSO}(\sigma)$ $\text{TMSO}(\sigma)$ partition components are, beginning with the last term in Equation (7.5) and working up, [1] residual error due to initializing errors and [2] nonlinear mean model response (model bias) to variations in the initial conditions, adjusting for term 1.

Partition component evaluation

The partitioning of the forecast errors is only half of the problem; the other half is the assessment to determine which of the partitioned components are "effectively" different from zero and reflect actual model differences. The F ratios, derived below, aid in this assessment of partitioned components. The F ratios are calculated for each grid point.

Beginning with the sequential form of the partitioned TMSO and TMSE,

$$\text{TMSO}_{\Delta T_0}(X) = \text{TMSO}_{\triangle T_0}(X) + \delta \text{TMSO}_{B1(T,T_0)}(X) + \delta \text{TMSO}_{A1(B,T,T_0)}(X)$$
\[
TMSE_{AT}(X) = \text{VAR}_T(X) + \delta \text{TMSE}_{BT}(X) + \delta \text{TMSE}_{A1(B,T)}(X),
\]

the following F-ratios may be used to compare model A with model B

\[
F\left[\frac{\text{TMSO}_{AT}(X)}{\text{TMSO}_{BT}(X)}\right] = 1 + \frac{\delta \text{TMSE}_{A1(B,T)}(X)}{\text{TMSO}_{BT}(X)}
\]

(7.3)

or

\[
F\left[\frac{\text{TMSE}_{AT}(X)}{\text{TMSE}_{BT}(X)}\right] = 1 + \frac{\delta \text{TMSE}_{A1(B,T)}(X)}{\text{TMSE}_{BT}(X)}
\]

(7.4)

An F-ratio for the true model (T) bias, due to the nonlinear model response to errors in the initial conditions, is

\[
F\left[\frac{\text{TMSO}_{TT}(X)}{\text{VAR}_T(X)}\right] = 1 + \frac{\sigma^2_T(X)}{\sigma^2_T(X)}
\]

(7.5)

Selection of critical F values

Typical statistical tests, such as in analysis of variance (ANOVA), attach significance to F values much larger than 1. Unlike ANOVA, the evaluation of partition components attaches meaning to F ratio values both larger and smaller than 1.

Referring to equations (7.1) and (7.2), the incremental TMSO and TMSE changes,
\[ \delta_{\text{TMSO}_{A_1(B,T,T_o)}}(X) = \left( \sigma^2_A(X) - \sigma^2_B(X) \right) + B_{A_1}(X) \left( B_{A_1}(X) + 2B_{T_1}(X) + 2B_{T_o}(X) \right) \]

\[ \delta_{\text{TMSO}_{A_1(B,T)}}(X) = \left( \sigma^2_A(X) - \sigma^2_B(X) \right) + B_{A_1}(X) \left( B_{A_1}(X) + 2B_{T}(X) \right) \]

may be negative.

Significance is attached to partition components only if their F ratios are much larger or smaller than 1. Critical F values represent the limits to which F ratios may deviate from 1 without being considered "effectively" different from 1; values of the F ratios between upper and lower critical F values, denoted by \( F_L \) and \( F_U \), respectively, indicate the partition component being evaluated is not "effectively" different from zero.

For this study, F ratios were allowed to differ by up to 20% from 1 and still be considered "effectively" 1. This is equivalent to requiring the numerator model total root-mean-squared error (RMSE) to differ from the denominator total RMSE by more than 10 percent (approximately) before considering the forecasts from the two models to be "effectively" different (i.e. the TMSO component "effectively" different from zero). This study used lower and upper critical F values of

\[ (F_L, F_U) = (0.8, 1.2) \]

Because of the sharpness of the transition between model superiority, the F ratio global and zonal analyses results insensitive to minor changes in
the critical F values; plots of areas with "effectively" different forecasts, based upon critical F values of (0.8, 1.2) were almost identical to those based upon critical F values of (0.7, 1.4).

The following is an example of the interpretation of F ratios. When evaluating the TMSO partition associated with the differences between models A and B, the F ratio is

\[
F \left[ \text{TMSO}_{AB}(X) \right] = \frac{TMSO_{AT}(X)}{TMSO_{BT}(X)}.\]

If the F ratio is smaller than the lower critical F value, \(F < F_L\), model A is said to give an "effectively" better forecast than model B. If the F ratio is between the lower and upper critical F values \(F_L < F < F_U\), the models are considered to give "effectively" the same forecast. Finally, if the F ratio is larger than the upper critical F value, \(F > F_U\), model B is said to give an "effectively" better forecast than model A.

**Application: Grid points** The F ratio values are calculated using the gridded TMSO values. These F ratios are compared with the critical F values and areas of model superiority mapped out.

**Application: Zonal comparison** To evaluate zonal comparisons of the partition components, the relative frequency of F ratio values; for each of the three categories (1) \(F < F_L\), (2) \(F_L < F < F_U\), and (3) \(F > F_U\); are calculated at each latitude. If the relative frequency of the second category is less than a critical limit, 0.8 in this study, the
partition component is considered to be "effectively" different from zero and the two models are considered to be "effectively" different.

Latitude-time contour plots of the relative frequencies allow the evaluation of which latitudes are most sensitive to model differences and at what time the forecast models become "effectively" different.

**Application: Global comparison**

The global comparisons of the partition components are performed in the same manner as with the zonal comparisons except the three global relative frequencies were the area-weighted means of the respective zonal relative frequencies. The weights are the latitudinal Gaussian weights used in the spectral transformations.

The plot of global relative frequencies, evaluated in the same manner as the zonal relative frequencies, allow simultaneous comparison of all partition components and essentially presents the portion of the surface having F ratios in one of the three categories.

**Application to operational forecast models**

The partitioning and evaluation techniques may be applied to operational models where the true model and forecast are unknown. If an estimate of the variance of the analysis is available, then the analysis may be as a proxy for the "true" model.

Once this assumption is made, TMSE partitioning and evaluation of the components may be done with the analysis representing the true model. Interpretation of the F ratio values is identical to the TMSO analysis.
Partitioning: TMSO(ψ)

The global and zonal evaluations of the TMSO partitions were useful in gaining an understanding of which components were "effectively different from zero and their latitudinal variations in size and effect. Evaluations, such as these, may be used as a first pass in assessing model improvements; detailed examination of lat-long plots of the nonzero partitions may follow this "first pass".

Global evaluation

The global mean TMSO(ψ) components, sequential composition, and F ratio relative frequency of (F_s/F_0) are plotted in Figures 7.1 to 7.3, respectively. The global mean of TMSO(ψ) partition components, Figure 7.1, shows the largest portion of TMSO(ψ) was due to the residual error component, followed by the τ component, the σ(λ,t) component, the nonlinear bias component, and the σ(μ) component. All components, except for τ, appeared to increase exponentially over time.

τ component Figure 7.1 shows the size of the τ component reached a maximum, 8 days into the forecast, and decreased rapidly thereafter. The composition plot, Figure 7.2, indicates the τ composition varied from a low of -4%, after one day, to a high of 30% of TMSO(ψ) at the 7 day point. The 0.8 contour of the F ratio relative frequency plot, Figure 7.3, shows that after 36 hours, less than 80% of the surface had F ratios not "effectively" different from one, implying over 20% of the surface had a non-negligible τ component of TMSO(ψ).
Figure 7.1 Global Mean TMSO(ψ) Components

Figure 7.2 Global Mean TMSO(ψ) Sequential Composition

Figure 7.3 Global Relative Frequency [F_L ≤ F ≤ F_U]: TMSO(ψ)
\( \sigma(\mu) \) component  The \( \sigma(\mu) \) component accounted for only about 2\% of the total \( TMS0(\psi) \). After 3 days, it accounted for only 2 to 3\% of the \( TMS0(\psi) \) of the MCBC0 model. Less than 1\% of the surface had F ratios "effectively" different from zero during the forecast period. The incremental "improvement" to the classic baroclinic model gave negligible forecast improvements.

\( \sigma(\lambda,t) \) component  The \( \sigma(\lambda,t) \) component accounted for about 10 to 12\% of the total \( TMS0(\psi) \). The forecast \( TMS0(\psi) \) of the MCBC1 model would have decreased by 11 to 14\% by using the MCLBX model forecasts. After 4 days, more than 20\% of the surface had F ratios "effectively" different from 1 which indicated differences between the MCBC1 and MCLBX model forecasts.

Nonlinear bias component  The nonlinear bias component accounted for an increasing portion of the \( TMS0(\psi) \). This error-related component accounted for about 20\% of the total \( TMS0(\psi) \) and 23\% of the MCLBX \( TMS0(\psi) \) by the end of the forecast period. The relative frequency plot indicates the MCLBX model was "effectively" biased after 7 1/2 days. The only way to decrease the nonlinear bias component is to decrease the errors in the initial conditions.

Residual error component  The residual error component accounted for about 60\% of the total \( TMS0(\psi) \); as with the nonlinear bias component, only way to decrease the residual error component is to decrease the errors in the initial conditions.

Summary  The global evaluation of the four \( TMS0(\psi) \) components indicated only the \( \sigma(\mu) \) component was negligible throughout the forecast
period. This means the there were negligible differences between forecasts made by the MCBC1 (a zonally constant $\sigma$) and MCBC0 (globally constant $\sigma$) models.

Zonal evaluation

$\tau$ component Figure 7.4 shows the primary variable $\tau$ contributes a negative component to TMSO($\psi$) in the NH polar and SH tropical regions throughout the forecast period. The NH mid latitudes have a negative $\tau$ component, amounting to over $-10\%$ of the TMSO($\psi$), during the first two days of the forecast. By the third day, the $\tau$ component is positive in the NH mid latitudes and SH extratropical latitudes.

As indicated by the 0.8 line of Figure 7.7, out to 3 to 4 days into the forecast, the $\tau$ component was negligible everywhere except in NH mid latitudes where the two models were "effectively" different after only 10 hours.

$\sigma(\mu)$ component Figure 7.5 shows the $\sigma(\mu)$ component accounted for, at most, 5$\%$ of the total TMS0($\psi$); with the main improvement showing in the tropics. All zonal F ratio relative frequencies were above 95$\%$ showing there was negligible model improvement over the classic baroclinic model.

$\sigma(\lambda,t)$ component Figure 7.6 shows the $\sigma(\lambda,t)$ component accounted up to 35 percent of the MCBC1 TMSO($\psi$), with highest composition percentage in the subtropical regions. The figure shows all of the improvement to the forecast occurred in the non polar latitudes of both hemispheres.

The 0.8 line in Figure 7.8 shows a nonzero $\sigma(\lambda,t)$ component in the NH mid latitudes and tropics by day 3 of the forecast, in the SH tropics by
Figure 7.4 Zonal Mean TMSO(\(\psi\)) Sequential Component: \(\tau\)
Figure 7.5   Zonal Mean TMS0(ψ) Sequential Component:  σ(μ)
Figure 7.6  Zonal Mean TMS0(ψ) Sequential Component:  σ(λ,t)
Figure 7.7 Zonal Relative Frequency $[F_L \leq F \leq F_U]$: TMS0($\psi$) $\tau$ component

Figure 7.8 Zonal Relative Frequency $[F_L \leq F \leq F_U]$: TMS0($\psi$) $\sigma(\lambda, t)$ component
the middle of day 4, and SH mid latitudes by the middle of day 6. The polar regions, within 15° of the poles, had a negligible $\sigma(\lambda,t)$ component throughout the forecast period.

**Nonlinear bias component**

The nonlinear bias component, Figure 7.9, accounted for an increasing portion of the MCLBX TMSO($\psi$). By 240 hours, the nonlinear bias accounted for over 25% in the NH mid latitudes.

The 0.8 line in Figure 7.10 shows the NH mid latitude MCLBX model did not become "effectively" biased until the fifth to sixth day into the forecast and it took 8 days for the SH forecast to become biased.

**Summary**

The primary MCBCO model improvements, as indicated by the $\tau$ component F ratios occurred in the NH mid latitudes after the second day. The improvements were limited to the NH non polar regions and the SH extratropical regions.

Consistent with the global evaluation, the zonal evaluation showed the $\sigma(\mu)$ component was "effectively" zero, at all latitudes, throughout the forecast period, thus the MCBC1 model showed negligible improvements over the MCBCO model.

The primary MCLBX model improvements, as indicated by the $\sigma(\lambda,t)$ component F ratios occurred in the NH after 2 days and after 4 days in the SH. The improvement was greatest in the tropical regions and non existent in the polar regions.

The growth of the nonlinear bias component, tied to the error, was mainly in the active regions, where the NH MCLBX forecast became biased after 5 days and the SH forecast after 8 days.
Figure 7.9  Zonal Mean TMSO(ψ) Sequential Composition: Nonlinear Bias

Figure 7.10  Zonal Relative Frequency \([F_L \leq F \leq F_U]\): TMSO(ψ)
Nonlinear Bias component
Lat-long evaluation

Examination of the 240 hour lat-long plots of TMSO($\psi$) components was helpful in beginning to assess the cause of the various TMSO maxima.

240 hour forecast Figures 7.11 to 7.16 are the 240 hour forecast plot of the TMSO($\psi$) and its components, respectively. The 240 hour forecast TMSO($\psi$), Figure 7.11, shows maxima were located (1) over Scandinavia, (2) over Russia, (3) in a broad band arcing through Siberia to (4) a very high maximum over the Korea-Japan area, (5) south of Tibet, (6) in a band arcing south from eastern China and around the Korea-Japan maximum, (7) off of the west coast of the U.S., (8) in western Canada and over Hudson Bay, (9) in a band arcing from the Gulf Coast northward to southern Greenland and (10) west of Africa. The Southern Hemisphere did not have any large TMSO($\psi$) maxima.

Examination of the TMSO($\psi$) component plots, Figures 7.12 to 7.16, helped in assigning a cause to each of the maxima. Figure 7.12 shows by including $\tau$ in the model (accounting for baroclinic growth by going from a MCBVX to a MCBCO model), most of the TMSO($\psi$) maxima are accounted for, with the exception of problems in the Northern Hemisphere polar region north of the Russian-Siberian land mass and in an area between maxima (4) and (6).

Referring back to Figure 6.49, the area between maxima (4) and (6) was in an area of large $\sigma$ gradients and maximum instability. Figure 7.13 shows there was a small positive $\sigma$($\mu$) contribution over the northern portion (having largest $\partial \sigma / \partial \mu$). The large $\sigma$($\lambda$,t) contribution, shown in Figure 7.14, centered over the area between maxima (4) and (6), was a
Figure 7.11  240 Hour Forecast TMSO(ψ)

Figure 7.12  240 Hour Forecast TMSO(ψ) Component: τ
Figure 7.13 240 Hour Forecast TMS0(ψ) Component: \( \sigma(\mu) \)

Figure 7.14 240 Hour Forecast TMS0(ψ) Component: \( \sigma(\lambda,t) \)
combination of advective and $\delta \sigma/\delta t$ contributions. Even with the variations in $\sigma$ accounted for (i.e. using a MCLBX model), both the nonlinear bias component, Figure 7.15, and the residual error, Figure 7.16, showed moderate maxima in this area, indicating a particular sensitivity to initial condition errors.

240 hour sequential F ratios  The F ratio plots, Figures 7.17 and 7.18, plot individual areas of superior forecasts for the (MCBVX, MCBCO) and (MCBC1, MCLBX) models, respectively.

Figure 7.17 shows that by 240 hours, the barotropic model produced better forecasts than the baroclinic model in the meteorologically quiet northern polar regions. Not surprisingly the areas where the baroclinic model gives better forecasts, matched the areas with a strongly positive $\tau$ component.

Summary

The global evaluation of partition components is a convenient way to determine if overall model performance is changed by incremental model changes. Evaluation of the four $\nabla \mathbf{MSO}(\psi)$ components indicated only the $\sigma(\mu)$ component was negligible throughout the forecast period, meaning the baroclinic model forecast, based upon a zonally constant $\sigma$ (MCBC1), was essentially unchanged from a classic baroclinic model forecast, based upon a globally constant $\sigma$ (MCBCO).

The zonal evaluation of partition components is a convenient way to determine how the incremental model changes effect on forecasts varies with latitude over time. The primary MCBCO model improvements, indicated
Figure 7.15 240 Hour Forecast TMS0(\psi) Component: Nonlinear Bias

Figure 7.16 240 Hour Forecast TMS0(\psi) Component: Residual Error
Figure 7.17 240 Hour Sequential $F[TM_{SO}(\psi)\ Component]$
Figure 7.18 240 Hour Sequential $F[TMS0(\psi) \text{ Component}]$: $\sigma(\lambda,t)$
by the $\tau$ component $F$ ratios, occurred in the NH mid latitudes after the second day. Improvements were limited to the NH non polar regions and the SH extratropical regions.

Consistent with the global evaluation, the zonal evaluation showed the $\sigma(\mu)$ component was "effectively" zero throughout the forecast period at all latitudes, thus the MCBC1 model showed negligible improvements over the MCBC0 model.

The primary MCLBX model improvements, as indicated by the $\sigma(\lambda,t)$ component $F$ ratios, occurred after 2 days in the NH and after 4 days in the SH. The improvement was greatest in the tropical regions and non existent in the polar regions.

The growth of the nonlinear bias component, tied to the error, was mainly in the active regions, where the NH MCLBX forecast became biased after 5 days and the SH forecast after 8 days.

The lat-long plots are helpful in the initial cause assessment of the various TMSO maxima. The $\tau$ component accounted for the dominant portion of TMSO$^\psi$. The $\sigma(\lambda,t)$ and nonlinear bias components explained most of the remaining TMSO in the large Korea-Japan maxima.

**Partitioning: TMSO($\tau$)**

**Global evaluation**

The global mean TMSO($\tau$) components, sequential composition, and $F$ ratio relative frequency of $(F=\hat{F})$ are plotted in Figures 7.19 to 7.21, respectively. The global mean of TMSO($\tau$) partition components, Figure 7.19, shows the largest portion of TMSO($\tau$) was due to the residual error.
component, followed by the $\sigma(\Lambda,t)$ component, the nonlinear bias component, and the $\sigma(\mu)$ component. The nonlinear bias component appears to increase exponentially over time.

**$\sigma(\mu)$ component** Figure 7.20 shows the $\sigma(\mu)$ component accounted for only about 3 to 4 percent of the total TMS0($\tau$). Less than 1 percent of the surface had F ratios "effectively" different from zero during the forecast period.

This incremental change to the classic baroclinic model resulted in negligible improvements to the forecasts of $\tau$.

**$\sigma(\Lambda,t)$ component** The $\sigma(\Lambda,t)$ component accounted for approximately 15 percent of the total TMS0($\tau$). The forecast TMS0($\tau$) would have been decreased by 15 to 30 percent by using the MCLBX model instead of MCBC1 model forecasts. After 3 days, more than 20% of the surface had F ratios "effectively" different from 1 which indicated differences between the MCBC1 and MCLBX model forecasts.

**Nonlinear bias component** The nonlinear bias component accounted for an increasing portion of the TMS0($\tau$). This error-related component accounted for about 20% of the total TMS0($\tau$) and 29% of the MCLBX TMS0($\tau$) by the end of the forecast period. The relative frequency plot indicates the MCLBX model was "effectively" biased after 8 1/2 days.

The only way to decrease the nonlinear bias component is to decrease the errors in the initial conditions.

**Residual error component** The residual error component accounted for about 60 percent of the total TMS0($\tau$); as with the nonlinear bias
Figure 7.19 Global Mean TMSO(τ) Components

Figure 7.20 Global Mean TMSO(τ) Sequential Composition

Figure 7.21 Global Relative Frequency \( [F_L \leq F \leq F_u] \): TMSO(τ)
component, only way to decrease the residual error component is to decrease the errors in the initial conditions.

**Summary** The global evaluation of the four TMSO(τ) components indicated only the σ(μ) component was negligible throughout the forecast period. This means there were negligible differences between forecasts made by the MCBC1 (a zonally constant σ) and MCBC0 (globally constant σ) models.

**Zonal evaluation**

**σ(μ) component** Figure 7.22 shows the σ(μ) component accounted for, at most, 7% of the total TMSO(τ); with the main improvement showing in the lower NH mid latitudes. All zonal F ratio relative frequencies were above 95% showing there was negligible model improvement over the classic baroclinic model.

**σ(λ,t) component** Figure 7.23 shows the σ(λ,t) component accounted up to 50 percent of the MCBC1 TMSO(τ), with highest composition percentage in the subtropical regions. The figure shows most of the improvement to the forecast occurred in the non polar latitudes of both hemispheres.

The 0.8 line in Figure 7.24 shows a nonzero σ(λ,t) component 15 hours into the forecast period in the NH mid latitudes, by 2 1/2 days in the SH subtropics, and by day 6 for the SH tropics and NH polar regions. The SH polar regions, within 15° of the pole, had a negligible σ(λ,t) component throughout the forecast period.
Figure 7.22  Zonal Mean TMSO(\tau) Sequential Composition: $\sigma(\mu)$
Figure 7.23  Zonal Mean TMSO(\(\tau\)) Sequential Composition: \(\sigma(\lambda,t)\)

Figure 7.24  Zonal Relative Frequency \([F_L \leq F \leq F_U]\): TMSO(\(\tau\)) \(\sigma(\lambda,t)\) component
Nonlinear bias component The nonlinear bias component, Figure 7.25, accounted for an increasing portion of the MCLBX TMSO(τ). By 240 hours, the nonlinear bias accounted for over 30% in the NH mid latitudes.

The 0.8 line in Figure 7.26 shows the NH mid latitude MCLBX model became "effectively" biased after the fourth day into the forecast and it took 9 days for part of the SH tropics forecast to become biased.

Summary Consistent with the global evaluation, a zonal evaluation showed, at all latitudes, the σ(μ) component was "effectively" zero during the forecast period, thus the MCBC1 model showed negligible improvements over the MCBC0 model.

The primary MCLBX model improvements, as indicated by the σ(λ,t) component F ratios occurred in the NH after only 15 hours and after 2 1/2 days in the SH. The improvement was greatest in the subtropical regions and non existent in the SH polar regions.

The growth of the nonlinear bias component, tied to the error, was mainly in the active regions; the NH MCLBX forecast became biased after 4 days and the SH tropics began to show biased forecasts after 9 days.

Lat-long evaluation

240 hour forecast Figures 7.27 to 7.31 are the 240 hour forecast plot of the TMSO(τ) and its components, respectively. The 240 hour forecast TMSO(τ), Figure 7.27, shows maxima were located [1] over southern Scandinavia (weak), [2] northern Tibet (weak), [3] southern Tibet (strong) [4] in the area of Korea and Japan (strong), [5] northern Canada and Alaska (weak), [6] mid western U.S. (weak), [7] mid Atlantic U.S. coast
Figure 7.25 Zonal Mean TMS0(τ) Sequential Composition: Nonlinear Bias

Figure 7.26 Zonal Relative Frequency $[F_L ≤ F ≤ F_U]$: TMS0(τ)
Nonlinear Bias Component
(moderate), [8] the area between Labrador and Greenland (weak), and [9] west of northern Africa. The Southern Hemisphere did not have any large TMS0(τ) maxima. Examination of the TMS0(τ) component plots, Figures 7.28 to 7.31, helped in assigning a cause to each of the maxima. Figure 7.28 shows very little σ(μ) component contribution to TMS0(τ). northern portion (having largest δσ/δμ).

The large σ(λ,t) contributions, shown in Figure 7.29, were in to the southern Tibet and Korea-Japan maxima; most of the southern Tibet maximum was attributable to the σ(λ,t) component, representing a lack of advection and time variation of σ in the model. Roughly half of the maximum north of Tibet is accounted for by the σ(λ,t) component.

The remaining TMS0(τ) maxima, shown in Figure 7.30, are artifacts of the nonlinear bias, although there are portions of the southern Tibet, Korea-Japan, and east U.S. coast maxima which are left as a residual error component (Figure 7.31)

240 Hour sequential F ratios Figure 7.32 shows, by 240 hours, much of the linear balance model improvement occurred in the tropics and NH mid latitudes. There was a relatively large area just upwind from Tibet where the baroclinic model produced a better forecast. Figure 7.33 shows, while many of the MCLBX biased forecasts occurred in the Northern Hemisphere, the Southern Hemisphere appeared to have some sort of propagation phenomenon occurring around South America and Africa.
Figure 7.27  240 Hour Forecast TMS0(τ)

Figure 7.28  240 Hour Forecast TMS0(τ) Component: σ(μ)
Figure 7.29 240 Hour Forecast TMS0(τ) Component: $\sigma(\lambda, t)$

Figure 7.30 240 Hour Forecast TMS0(τ) Component: Nonlinear Bias
Figure 7.31 240 Hour Forecast TMS0(\tau) Component: Residual Error
(a) MCLBX inferior to MCBCI \( (F < F_L) \)

(b) MCLBX superior to MCBCI \( (F > F_U) \)

Figure 7.32 240 Hour Sequential \( F[TMSO(\tau) \text{ Component}] \): \( \sigma(\lambda, t) \)
Figure 7.33 240 Hour Sequential FTMSO(τ) Component: Nonlinear Bias
Summary

Global and zonal evaluations indicate the \( \sigma(\mu) \) component was negligible throughout the forecast period. This same component was also negligible in the TMS0(\( \psi \)) partitioning. Therefore, a baroclinic model forecast is not improved by assuming a zonally constant \( \sigma \) (MCBC1) instead of a globally constant \( \sigma \) (MCBC0).

The primary MCLBX model improvements (over the baroclinic models), as indicated by the \( \sigma(\lambda, t) \) component F ratios, occurred after only 15 hours in the NH and after 2 1/2 days in the SH subtropics. The improvement was greatest in the NH mid latitudes and SH subtropics. There was no improvement in the SH polar regions.

The growth of the nonlinear bias component, tied to the error, was mainly in the active NH mid latitudes where the MCLBX forecast became biased after 4 days.

The lat-long plots of TMS0(\( \tau \)) components showed the \( \sigma(\lambda, t) \) component accounted for a major portion of a TMS0(\( \tau \)) maximum south of Tibet; nonlinear bias accounted for the predominant portion of remaining TMS0(\( \tau \)) maxima. The nonlinear bias F ratio lat-long plot seems to show a propagation phenomenon occurring in South America and around southern Africa.

Partitioning: TMS0(\( \varphi \))

Global evaluation

The global mean of TMS0(\( \varphi \)) partition components, Figure 7.34, shows the largest portion was due to the residual error component. The initial
decrease in the residual error component is probably a model "spin up"
effect due to adjustments in going from a initializing model having fixed
zonal coefficients to a forecast model having variable zonal coefficients.
The nonlinear bias component appears to increase exponentially over time.

Nonlinear bias component  The nonlinear bias component accounted
for an increasing portion of the TMS0(σ). By the end of the forecast
period, Figure 7.35 shows this error-related component accounted for about
14 percent of the total TMS0(σ). The relative frequency plot, Figure
7.36, indicates the MCLBX model bias was negligible out past the end of
the forecast period.

Zonal evaluation

Nonlinear bias component  The nonlinear bias component, shown in
Figure 7.37, accounted for an increasing portion of the MCLBX TMS0(σ). By
240 hours, the nonlinear bias accounted for about 15 percent in the NH mid
latitudes. The 0.8 line in Figure 7.38 shows the NH mid latitude MCLBX
model was not "effectively" biased until after the eighth day into the
forecast and the SH forecast bias was negligible throughout the forecast
period.

Lat-long evaluation

240 hour forecast  The 240 hour forecast TMS0(σ), Figure 7.39,
shows [1] a large complex of maxima extending from subtropical northeast
Africa, through the Middle East, around the southern part of Tibet, and
east through China; [2] a small maximum south east of Japan; [3] a line
Figure 7.34  Global Mean TMS0(σ) Components

Figure 7.35  Global Mean TMS0(σ) Sequential Composition

Figure 7.36  Global Relative Frequency \( [F_L \leq F \leq F_U] \): TMS0(σ)
Figure 7.37  Zonal Mean TMS0(σ) Sequential Composition: Nonlinear Bias

Figure 7.38  Zonal Relative Frequency [F_L ≤ F ≤ F_U]: TMS0(σ)
Nonlinear Bias Component

Figure 7.40 shows the nonlinear bias component accounted for most of the TMSO(σ) of maxima [2] and [5]. Somewhat less than half of the TMSO(σ) maximum over the eastern U.S. was accounted for by the nonlinear bias. The remaining TMSO(σ) in these and the other maxima was "accounted" for by the residual error component, Figure 7.41. Figure 7.41 shows that, although the nonlinear bias term did not account most of the TMSO(σ) in many of the maxima, its contribution was not negligible.

**Summary**

Global and zonal evaluations indicate the nonlinear bias was negligible until the final 1 to 2 days in the forecast period. Most of the areas having bias were in the NH mid latitudes.

The residual error remained the dominant component of the TMSO(σ).

**Summary**

This chapter dealt with the problem of partitioning model forecast errors by source. The approach taken was to partition the TMSO, total mean squared error relative to the true forecast. A simple operational partitioning technique, based upon taking the sequential differences of the calculated TMSO statistics from models in a hierarchy.
Figure 7.39  240 Hour Forecast TMSO(σ)

Figure 7.40  240 Hour Forecast TMSO(σ) Component: Nonlinear Bias
Figure 7.41  240 Hour Forecast TMS0(ω) Component: Residual Error

Figure 7.42  240 Hour Sequential F[TMS0(ω) Component]: Nonlinear Bias
As important as the TMSO partitioning, F ratios were developed to help determine if each of the partition components was negligible (i.e. "effectively" zero). An approach to evaluate the overall zonal and global contribution, using the relative frequency distribution of the gridded F ratios, was developed. The evaluation procedures, while somewhat crude, for the first time form a framework under which models and model improvements may be compared while accounting for the effects of errors in the initial conditions.

The partitioning and F-ratio evaluation procedure was applied to the model hierarchy consisting of three model types (barotropic, baroclinic, and linear balance) and one incremental model improvement (allowing the baroclinic $\sigma$ to be a zonally varying constant instead of the classical global constant). The results showed when, during the forecast period, the mean model forecasts were "effectively" different. The analyses also showed that, even though the incrementally "improved" baroclinic model improvement seemed to improve the forecasts slightly, over the classic baroclinic model, any actual improvement was masked by the forecast variability due to initial errors; therefore, with the current size of data errors, the incremental improvement was negligible.

Partitioning was also used as an early diagnostic tool to help in determining the source of forecast errors - whether due to a missing model component or due to residual error of the error-related nonlinear bias effects.
CHAPTER 8
SUMMARY AND TOPICS FOR FUTURE INVESTIGATION

Summary

In their paper, *The Reliability of Improvements in Deterministic Short-Range Forecasts in the Presence of Initial State and Modeling Deficiencies*, Tribbia and Baumhefner partitioned model forecast errors into a model and a random error component. This study partitioned each of their two components into subcomponents. Additionally, a procedure was developed to determine if the resulting partition components were negligible or if they represented "significant" model differences or biases.

Model hierarchy description

The partitioning procedure involved first selecting a hierarchy of models whose differences involved the desired partitions. The hierarchy, used in this study, consisted of a barotropic model, a classic baroclinic model, an incrementally-"improved" baroclinic model, and a linear balance model. The models were arranged from the simplest to the most complex. The barotropic model (MCBVX) has one layer and forecasts $\psi$. This model advects total vorticity and does not allow for any baroclinic development of weather systems. The two-layer classic baroclinic model (MCBCO) forecasts barotropic model variable, $\psi$, and a variable related to the mean potential temperature, $\tau$. This model assumes a global constant static stability, $\sigma$, and is the simplest model allowing for baroclinic development. The "improved" baroclinic model (MCBC1) relaxes the global constant
σ assumption by allowing σ to be a zonally varying constant. The linear balance model (MCLBX) forecasts ψ, τ, and, σ. Relative to the baroclinic models, this model promotes the growth of new disturbances and slows of growth in mature weather systems.

Error partitioning

The model forecast error was measured in terms of the total mean squared error, relative to the "true" forecast or TMSO. This statistic incorporated both the forecast variance and bias. Based upon the model hierarchy, the gridded TMSO could be partitioned into components. The general form of the available model deficiency partition components were those due to τ, σ(μ), and σ(λ,t) components,

\[ \delta \text{TMSO}(\tau) = \text{TMSO}_{\text{MCLBX}}(\tau) - \text{TMSO}_{\text{MBCO}}(\tau) \]
\[ \delta \text{TMSO}(\sigma(\mu)) = \text{TMSO}_{\text{MBCO}}(\sigma(\mu)) - \text{TMSO}_{\text{MCBC1}}(\sigma(\mu)) \]
\[ \delta \text{TMSO}(\sigma(\lambda,t)) = \text{TMSO}_{\text{MCBC1}}(\sigma(\lambda,t)) - \text{TMSO}_{\text{MCLBX}}(\sigma(\lambda,t)) \]

the initial error-related components were nonlinear bias component and the residual error component,

\[ \delta \text{TMSO}(\text{NLB}) = \text{TMSO}_{\text{MCLBX}}(\text{NLB}) - \hat{\sigma}_{\text{MCLBX}}^2(\text{NLB}) \]
\[ \delta \text{TMSO}(\text{ResErr}) = \hat{\sigma}_{\text{MCLBX}}^2(\text{ResErr}) \]
Not all of the TMSO of the three primary variables ($\psi, \tau, \sigma$) could use all partitions; the TMSO($\psi$) was partitioned with all components, the TMSO($\tau$) could be partitioned with only the $\sigma(\mu)$, $\sigma(\lambda, t)$ and initial error-related components, and the TMSO($\sigma$) could be partitioned with only the initial error-related components. Even though this seemed to be a limitation, it was not, since all meaningful model differences could be evaluated.

**Partition evaluation**

This study showed the following F ratios,

$$F\left[\delta\text{TMSO}(\tau)\right] = \frac{\text{TMSO}_{\text{MCBVX}}(\tau)}{\text{TMSO}_{\text{MCBCO}}(\tau)}$$

$$F\left[\delta\text{TMSO}(\sigma(\mu))\right] = \frac{\text{TMSO}_{\text{MCBCO}}(\sigma(\mu))}{\text{TMSO}_{\text{MCBC1}}(\sigma(\mu))}$$

$$F\left[\delta\text{TMSO}(\sigma(\lambda, t))\right] = \frac{\text{TMSO}_{\text{MCBC1}}(\sigma(\lambda, t))}{\text{TMSO}_{\text{MCLBX}}(\sigma(\lambda, t))}$$

$$F\left[\delta\text{TMSO}_{\text{NLB}}\right] = \frac{\text{TMSO}_{\text{MCLBX}}}{\sigma^2_{\text{MCLBX}}}$$

can be used to evaluate if a partition component is negligible. This is equivalent to determining if model differences are negligible.

For overall zonal and global evaluations, the observed distribution of gridded F ratios was examined. The partition component was considered
to be negligible if at least a minimum percentage fell within a certain interval; for this study, a partition was considered negligible if at least 80 percent of the gridded F ratios were within 20 percent of 1 (0.8 to 1.2).

Time and zonal plots show how the partitions changed over time in the forecast.

Application to model hierarchy

These partitioning and evaluation techniques were applied to a sample of forecasts, based on the same set of initial conditions. The initial conditions were generated using Monte-Carlo techniques.

ψ partition Only the σ(μ) component of the TMSO(ψ) was negligible during the whole forecast period indicating the incrementally "improved" baroclinic model was negligibly different from the classic baroclinic model. After 12 hours classic baroclinic model (τ component) forecasts were better than barotropic forecasts in the NH mid latitude and in the SH after 3 1/2 days. Most linear balance model [σ(λ,t)] improvements, over the baroclinic models, occurred in the NH nonpolar regions after 2 to 3 days and SH tropics after 4 to 5 days. The nonlinear bias was negligible through the first half of the forecast period.

τ partition As with the ψ partition, the σ(μ) component of the TMSO(ψ) was negligible during the whole forecast period. The differences between the baroclinic and linear balance model became apparent after only 15 hours in the NH mid latitudes. Nonlinear bias was negligible during the first 4 days in the NH mid latitudes.
The only component evaluated, nonlinear bias, was negligible during the first 8 to 9 days of the forecast period.

Application summary

The partitioning and evaluation techniques, when applied to a sample of forecasts using the selected model hierarchy, indicated that the use of the incrementally "improved" baroclinic model gave negligible forecast improvements. The other partitions were "significant". The results of the partitioning and evaluations are condensed into the following basic observations and suggestions for model applications (for members of the model hierarchy and within the 10 day forecast period of the study):

(1) Nonlinear bias is negligible for the first 100-120 hours

(2) Forecasting $\psi$: out to 12 hours: Use the barotropic model 
out to 60 hours: Use the baroclinic model 
beyond 60 hours: Use the linear balance model

(3) Forecasting $\tau$: out to 15 hours: Use the baroclinic model 
beyond 15 hours: Use the linear balanced model

(4) Forecasting $\sigma$: default to: linear balanced model

Conclusions

The most important results of this study were the partitioning and partition evaluation techniques developed. These allow the numerical modeler to systematically partition model forecast error into components
due to particular model differences and evaluate the partition components as to which are negligible, relative to model solution variability induced by errors in the initial conditions. The partitioning and evaluation combination puts numerical weather model development on a better footing.

**Topics for Future Investigation**

As with many studies, there were a number of interesting items which could not be investigated fully. The following is a brief list of topics and questions for future investigation.

**Initialization**

The problem of determining the best technique of Monte-Carlo model initialization is an extremely important one. In fact, this problem alone conceivably could be another dissertation topic. The solution to this problem would not only benefit simulations such as those performed for this study, but also operational Monte-Carlo forecast models.

**Alternatives**

Can Monte-Carlo initialization be simplified? Do all variances and covariances need to be specified or do patterns in the correlation matrix allowing simplifications to be made?

**Sensitivity to initial conditions**

How sensitive are the forecast results to the correlation pattern of the initial conditions. What is a reasonable correlation matrix? How sensitive are the partitioning results to the size of the initial errors?
**Improved "truth" model**

One weakness of any study of this type is the definition of the "true" forecast model which was assumed to be the linear balance model in this study. A better "true" model would be a multilayer primitive equation model having a full set of physics, such as the National Center for Atmospheric Research (NCAR) Community Climate Model (CCM). In the absence of the NCAR CCM, a possible "true" model could be a PE model having better physics and more layers than the models being studied.

**Improvements to current models**

The models in the hierarchy are just basic "bare-bones" models. Numerous improvements may be made and the partitioning-testing techniques applied to evaluated their "significance".

**Increased horizontal resolution**  
This is perhaps the easiest improvement to make, at least within the memory limitations of the computer and mass storage. Only array size parameters need to be changed.

**Increased vertical resolution**  
Increasing the vertical resolution would involve a complete rewrite of the numerical models. Because of this, increasing the vertical resolution should be combined with implementing a different vertical coordinate system, presumably the \( \sigma \)-coordinate system.

How sensitive are the partitioning results to increasing the vertical resolution; is there a "significant" forecast improvement?

**Different vertical coordinates**  
A rewrite of the models, to increase the vertical resolution, should include a conversion from the use
of pressure coordinates to the use of \( \sigma \) coordinates. This system would allow better terrain forcing and be one step closer to the main-stream operational numerical models.

**Improved physical parameterizations** The only physical parameterizations included in the models used by this study were simple terrain forcing and horizontal diffusion. The terrain forcing could be improved by going to \( \sigma \) coordinates and estimating the actual surface conditions. Moisture processes and radiational heating could be added to the models.

**Improved numerics** The iterative solution for the velocity potential component, \( \delta \), could be improved. Calculating \( \delta \) was by far the most time consuming portion of the whole forecast; so, any improvement in this step would speed up the model.

The current models used second-order time stepping. A fourth-order time stepping routine would be desirable - especially if implementation could be coupled with improvements in the speed of the \( \delta \) solution.

**Change forecast output frequency** Depending upon the specific statistic of interest, the frequency of forecast storage may be changed. For instance, to study the early forecast evolution, only the first 5 days of forecasts might be needed, but at 6 hour intervals. Another example would be in studying a statistic which varies exponentially with time, forecasts stored for \( \Delta t, 2\Delta t, 4\Delta t, \) etc. would be useful.

**Additional hierarchy models**

The model hierarchy stopped with the linear balance model. It could be extended to include the full nonlinear balance model and the primitive
equation model. This would be desirable not only from the standpoint of comparing more models, but also redefining the "true" model.

**Nonlinear balance model**  The nonlinear balance model is a balance model having the least number of restrictions. Implementing this model would be a major undertaking, as can be seen by comparing the balance equations, (5.15) to (5.17), with the linear balance equations, (5.18) to (5.20). The resulting nonlinear balance "omega" equation would be very difficult and time consuming to solve.

**Primitive equation model**  The primitive equation (PE) model, expressed in its vorticity-divergence form, equations (5.8) to 5.11), would be the most general model type in a new hierarchy.

**Analysis and evaluation**

This study presents a basis for evaluation of model differences and residual error partitions. This is another area with a rich potential for improvement.

The critical F values were empirically selected in this study. More research is needed to determine the appropriate critical F values.

Multivariate tests on the differences of mean spectral coefficients are alternative candidates for partition testing. The problem with this approach is that, unless the sample size is increased dramatically, only a subset of the coefficients, such as the short-wave coefficients, may be evaluated.

Other statistics, such as the mean squared differences between models, may be yield some valuable information.
REFERENCES CITED


Epstein, E.S. "Stochastic dynamic prediction." Tellus, 21, no. 6 (1969), 739-757.


Thompson, P.D. "Uncertainty of initial state as a factor in the predictability of large-scale atmospheric flow patterns." *Tellus*, 9, no. 3 (1957), 275-295.


Introduction

Chapter 4 presents a relatively simple application of error partitioning to the inertial oscillation equations. The analysis of the Monte-Carlo simulation results combined the two horizontal wind velocity components, \( u \) and \( v \), into a single complex quantity, \( V_c = u + iv \).

This appendix describes a simple approach to dealing with the statistics of complex quantities. The techniques and descriptive statistics are based upon the idea that the cartesian expression of a complex quantity, \( X_c \), is just a linear combination of two real quantities; \( x_r \), the real component, and \( x_i \), the imaginary component of \( X_c \).

Basic Statistics of Complex Random Variables

A complex random variable, \( X_c \), may be expressed in standard cartesian form as

\[
X_c = x_r + i x_i , \quad (A.1)
\]

where \( i = \sqrt{-1} \). The complex random variable may be viewed as a linear combination of the real and imaginary components.

Complex mean

The complex mean of \( X_c \) is the expected value of the linear combination, Equation (A.1),
\[ \mu_c(X_r) = \mu(x_r) + i \mu(x_i) \] \hspace{1cm} (A.2)

Defining \( (*) \) to be the complex conjugate of the quantity \( () \), then

\[ \mu_c(X^*) = \overline{\mu_c(X_c)} \] \hspace{1cm} (A.3)

**Complex bias**

The complex bias of \( X_c \) is the difference between the complex mean and the true complex value

\[ B_c(X_c) = \mu_c(X_c) - X_c^{\text{true}} \] \hspace{1cm} (A.4)

**Complex variance**

The complex variance of \( X_c \) is the variance of the linear combination, Equation (A.1),

\[ \sigma_c^2(X_r) = \sigma_{x_r}^2 + i^2 \sigma_{x_i}^2 + 2i \sigma_{x_r x_i} \]

\[ = \sigma_{x_r}^2 - \sigma_{x_i}^2 + 2i \sigma_{x_r x_i} \] \hspace{1cm} (A.5)

This definition of the variance of \( X_c \), or complex variance, may be zero under the following two conditions:

1. \( x_r \) and \( x_i \) are constants
2. \( \sigma_{x_r}^2 = \sigma_{x_i}^2 \) and \( \sigma_{x_r x_i} = 0 \).
The first condition is the familiar one encountered when working with real-valued random variables; the second condition results from the information lost (one degree of freedom) when using the two real values in the complex variance to describe the three real values used to calculate it.

As with the complex mean,

$$\sigma_c^2(X_c^*) = \sigma_c^2(X_c^*).$$  \hfill (A.6)

**Complex covariance**

The complex covariance between two complex random variables, $X_c$ and $Y_c$, is defined as

$$\sigma_c^2(X_c, Y_c) = \mathbb{E} \left[ (X_c - \mu_c(X_c))(Y_c - \mu_c(Y_c)) \right]$$

$$= \left[ \sigma_{x_r y_r} - \sigma_{x_i y_1} \right] + i \left[ \sigma_{x_r y_1} - \sigma_{x_i y_r} \right]$$  \hfill (A.7)

The complex covariance of complex conjugates of the random variables are

$$\sigma_c^2(X_c^*, Y_c^*) = \left[ \sigma_{x_r y_r} + \sigma_{x_i y_1} \right] + i \left[ \sigma_{x_r y_1} - \sigma_{x_i y_r} \right]$$  \hfill (A.8)

$$\sigma_c^2(X_c, Y_c^*) = \sigma_c^2(X_c^*, Y_c)^*$$  \hfill (A.9)

$$\sigma_c^2(X_c^*, Y_c^*) = \sigma_c^2(X_c, Y_c)^*$$  \hfill (A.10)
Complex mean squared error

The complex mean squared error (CMSE), as with the case of real-valued random variables, is defined as

\[
\text{MSE}_c(X_c) = E \left[ (X_c - X_c(\text{true}))^2 \right], \tag{A.11}
\]

where \(X_c(\text{true})\) is the true value. The CMSE may be expressed in terms of the complex variance and complex bias

\[
\text{MSE}_c(X_c) = \left[ \left( \sigma_{x_r}^2 + B_{x_r}^2 \right) - \left( \sigma_{x_1}^2 + B_{x_1}^2 \right) \right] + i \left[ 2 \left( \sigma_{x_r} x_1 + B_{x_r} B_{x_1} \right) \right]
\]

\[
\text{MSE}_c(X_c) = \text{MSE}_r(X_c) + i \text{MSE}_i(X_c) \tag{A.12}
\]

\[
\text{MSE}_c(X_c) = \left[ \sigma_{x_r}^2 - \sigma_{x_1}^2 + 2i \sigma_{x_r} x_1 \right] + \left[ B_{x_r}^2 - B_{x_1}^2 + 2i B_{x_r} B_{x_1} \right]
\]

\[
\text{MSE}_c(X_c) = \sigma_c^2(X_c) + B_c^2(X_c) \tag{A.13}
\]

where \(\sigma_c^2(X_c)\) and \(B_c^2(X_c)\) are the complex variance and the square of the complex bias, respectively.

Linear Combinations of Complex Random Variables

The previously defined statistics of complex random variables are based upon the idea of treating the complex random variable as a linear combination of two real components. This simplifies the definition of the statistics of linear combinations of complex random variables. Let the
complex random variable \( Z_c \) be the following linear combination of the complex random variables, \( X_c \) and \( Y_c \),

\[
Z_c = \alpha X_c + \beta Y_c + \gamma
\]

where \( \alpha, \beta, \) and \( \gamma \) are complex constants. The various statistics of \( Z_c \) are

\[
\mu_c(Z_c) = \alpha \mu_c(X_c) + \beta \mu_c(Y_c) + \gamma
\]

\[
B_c(Z_c) = \alpha B_c(X_c) + \beta B_c(Y_c)
\]

\[
\sigma_c^2(Z_c) = \alpha^2 \sigma_c^2(X_c) + \beta^2 \sigma_c^2(Y_c) + 2\alpha\beta \sigma_c(X_c, Y_c)
\]

\[
MSE_c(Z_c) = \alpha^2 MSE_c(X_c) + \beta^2 MSE_c(Y_c) + 
\]

\[
+ 2\alpha\beta \left[ \sigma_c^2(X_c, Y_c) + B_c(X_c) B_c(Y_c) \right]
\]

**Application: Spectral Transformations**

The preceding definitions of the statistics and linear combinations of complex random variables aid in the calculation of the variance and covariance of the spatial scalar field point values from spectral coefficients.

A scalar spatial field on a sphere may be represented as the linear combination of spherical harmonics, \( V_n^m(\mu, \lambda) \),
where $\mu$, $\lambda$, and $P_n^m(\mu)$ are the sine of latitude, longitude, and the associated Legendre polynomial of order $n$ and degree $m$, respectively. If the spectral coefficients, $A_n^m$, are a set of random variables, the scalar field values, $f(\phi, \lambda)$, are also random variables resulting from a linear combination of $A_n^m$.

**Complex variance**

From the preceding discussion on the linear combination of complex random variables, the "complex" variance of the scalar field is

$$\sigma_C^2(f(\mu, \lambda)) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[ \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} \sigma_C A_n^m A_n^{m'} \ Y_n^m(\mu, \lambda) \right] Y_n^m(\mu, \lambda) \ . \ (A.15)$$

The complex variance of $f(\mu, \lambda)$ is a double inverse spectral transformation of the variance-covariance matrix of the spectral coefficients.

**Complex covariance**

The covariance between two points in the random scalar field may be calculated by performing one of the inverse transformations at one point, $(\mu^*, \lambda^*)$, and the second inverse transformation at the other point, $(\mu, \lambda)$.
\[ \sigma_c \left( f(\mu, \lambda), f(\mu^*, \lambda^*) \right) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} \sigma_c \left( A_n^{n'}, A_n^{n'} \right) Y_n^{n'}(\mu^*, \lambda^*) \right) Y_n^m(\mu, \lambda) \] (A.19)

**Summary**

The Euler expansion of a complex random variable may be used to treat the variable as a linear combination. The basic statistics are then easily obtained. The complex variance of a random variable may be zero under two conditions (1) the real and imaginary components are constant or (2) the real and imaginary components are equivariant and uncorrelated.

The statistics of linear combinations of complex random variables are easily obtained using the euler expansion of each random variable and the usual linear theory.

The variance or covariance of a scalar function, \( f(\mu, \lambda) \), could be calculated by performing a double inverse spectral transformation on the complex variance-covariance matrix of the spectral coefficients.
APPENDIX B

INERTIAL OSCILLATION: DERIVATIONS AND PROOFS

Stochastic-dynamic Predictive Equations

The derivation of the stochastic-dynamic (SD) inertial oscillation predictive equations involves the application of the Delta-method technique to the simplified differential equations of inertial oscillation motion.

The simplified inertial oscillation equations of motion on a beta-plane, accounting for centrifugal acceleration, are

\[ \frac{du}{dt} = a_0 v + a_1 yv + a_2 uv + a_3 yuv \]  \hspace{1cm} (B.1)

\[ \frac{dv}{dt} = b_0 u + b_1 yu + b_2 u^2 + b_3 yu^2 \] \hspace{1cm} (B.2)

\[ \frac{dy}{dt} = v \] \hspace{1cm} (B.3)

where \((a_0, a_1, a_2, a_3)\) and \((b_0, b_1, b_2, b_3)\) are model constants. In this model, \(b_1 = -a_1\). If the three predictive quantities, \((u, v, y)\), are assumed to have some initial random measurement error, they may be represented as being composed of a mean value, \(\mu(t)\), and a deviation from the mean, \(\varepsilon(t)\)

\[ u(t) = \mu_u(t) + \varepsilon_u(t) \] \hspace{1cm} (B.4)

\[ v(t) = \mu_v(t) + \varepsilon_v(t) \] \hspace{1cm} (B.5)

\[ y(t) = \mu_y(t) + \varepsilon_y(t) \] \hspace{1cm} (B.6)

Substituting these representations for \((u, v, y)\) into the simplified equations of motion gives
\[
\frac{d\mu_u}{dt} + \frac{d\varepsilon}{dt} = a_0 \left( \mu + \varepsilon \right) + \\
+ a_1 \left( \mu \mu_y + \mu \varepsilon_y + \mu \varepsilon \varepsilon_y + \varepsilon \varepsilon_y \right) + \\
+ a_2 \left( \mu \mu_v + \mu \varepsilon_v + \mu \varepsilon \varepsilon_v + \varepsilon \varepsilon_v \right) + \\
+ a_3 \left( \mu \mu_{uv} + \mu \mu \varepsilon_y + \mu \mu \varepsilon \varepsilon_y + \mu \mu \varepsilon \varepsilon v \right) + \\
+ a_3 \left( \mu \mu \varepsilon \varepsilon y + \mu \mu \varepsilon \varepsilon v + \mu \mu \varepsilon \varepsilon u \varepsilon y \right) \quad \text{(B.7)}
\]

\[
\frac{d\mu_v}{dt} + \frac{d\varepsilon}{dt} = b_0 \left( \mu + \varepsilon \right) + \\
+ b_1 \left( \mu \mu_y + \mu \varepsilon_y + \mu \varepsilon \varepsilon_y + \varepsilon \varepsilon_y \right) + \\
+ b_2 \left( \mu \mu_u + \mu \varepsilon_u + \mu \varepsilon \varepsilon_u + \varepsilon \varepsilon_u \right) + \\
+ b_3 \left( \mu \mu \mu_u y + \mu \mu \mu_u \varepsilon_y + \mu \mu \mu \varepsilon \varepsilon_y + \mu \mu \mu \varepsilon \varepsilon v \right) + \\
+ b_3 \left( \mu \mu \mu_u \varepsilon y + \mu \mu \mu \varepsilon \varepsilon y + \mu \mu \mu \varepsilon \varepsilon u \varepsilon y \right) \quad \text{(B.8)}
\]

\[
\frac{d\mu_y}{dt} + \frac{d\varepsilon}{dt} = \left( \mu + \varepsilon \right) \quad \text{(B.9)}
\]

The stochastic-dynamic equations are expected values of functions of equations (B.7), (B.8), and (B.9). The expected values of the errors and product of errors are handled in the following manner.
Stochastic-dynamic equations for the mean

The SD equations for the mean are obtained by taking the expected value of equations (B.7) to (B.9). The resulting equations are

\[
\frac{d\mu_u}{dt} = a_0(\mu_v) + \nonumber
\]

\[
+ a_1(\mu_u \mu + \sigma_{uv}) + \nonumber
\]

\[
+ a_2(\mu_u \sigma_u + \sigma_{uv}) + \nonumber
\]

\[
+ a_3(\mu_u \mu \mu_y + \mu_u \sigma_{uy} + \mu \sigma_{uy} + \mu \sigma_{uy} + \sigma_{uy}) \quad (B.10)
\]

\[
\frac{d\mu_v}{dt} = b_0(\mu_u) + \nonumber
\]

\[
+ b_1(\mu_u \mu_y + \sigma_{uy}) + \nonumber
\]

\[
+ b_2(\mu_u \mu_u + \sigma_{uu}) + \nonumber
\]

\[
+ b_3(\mu_u \mu \mu_y + \mu_u \sigma_{uy} + \mu \sigma_{uy} + \mu \sigma_{uy} + \sigma_{uy}) \quad (B.11)
\]
Stochastic-dynamic equations for variances and covariances

Before deriving the SD variance and covariance equations, equations (B.10) through (B.12) are subtracted from equations (B.7) through (B.9), respectively. The results, equations (B.13) through (B.15), are the predictive error equations.

\[
\frac{d\mu_y}{dt} = \mu_v \tag{B.12}
\]

\[
\frac{de_u}{dt} = a_0(e_v) + \\
+ a_1\left(\mu_v \varepsilon_y + \mu_y \varepsilon_v + (e_v e_y - \sigma_{vy}) \right) + \\
+ a_2\left(\mu_u \varepsilon_v + \mu_v \varepsilon_u + (e_v e_u - \sigma_{uv}) \right) + \\
+ a_3\left(\mu_u \varepsilon_y + \mu_y \varepsilon_u + \mu_v \varepsilon_u \right) + \\
+ a_3\left(\mu_u (e_v e_y - \sigma_{vy}) + \mu_v (e_v e_u - \sigma_{uv}) \right) + \\
+ a_3\left(\mu_y (e_u e_v - \sigma_{uv}) + (e_u e_v e_y - \sigma_{uvy}) \right) \tag{B.13}
\]

\[
\frac{de_v}{dt} = b_0(e_u) + 
\]
\[ + b_1 \left( \mu_{y} \varepsilon_y + \mu_{y} \varepsilon_u + \left( \varepsilon_{u} \varepsilon_y - \sigma_{uy} \right) \right) + \]

\[ + b_2 \left( \mu_{u} \varepsilon_u + \mu_{u} \varepsilon_u + \left( \varepsilon_{u} \varepsilon_u - \sigma_{uu} \right) \right) + \]

\[ + b_3 \left( \mu_{u} \varepsilon_u \varepsilon_y + \mu_{u} \varepsilon_y \varepsilon_u + \mu_{u} \varepsilon_y \varepsilon_y \right) + \]

\[ + b_3 \left( \mu_{u} \varepsilon_u \varepsilon_y - \sigma_{uy} \right) + \mu_{u} \left( \varepsilon_{u} \varepsilon_y - \sigma_{uy} \right) + \]

\[ + b_3 \left( \mu_{y} \varepsilon_u \varepsilon_u - \sigma_{uu} \right) + \left( \varepsilon_{u} \varepsilon_y - \sigma_{uy} \right) \]  \hfill (B.14)

\[ \frac{d \varepsilon_y}{dt} = \begin{pmatrix} \varepsilon_y \end{pmatrix} \]  \hfill (B.15)

Since \( \sigma_{\alpha\beta} = E[\varepsilon_\alpha \varepsilon_\beta] \), the time derivative of \( \sigma_{\alpha\beta} \) is

\[ \frac{d}{dt} \sigma_{\alpha\beta} = \frac{d}{dt} E[\varepsilon_\alpha \varepsilon_\beta] \]

\[ = E \left[ \frac{d}{dt} (\varepsilon_\alpha \varepsilon_\beta) \right] \]

\[ = E \left[ \varepsilon_\alpha \frac{d}{dt} (\varepsilon_\beta) + \varepsilon_\beta \frac{d}{dt} (\varepsilon_\alpha) \right] . \]

This relation, along with the predictive error equations, are used to obtain the predictive equations for all of the variances and covariances.
Stochastic-dynamic variance equations

The resulting SD predictive equations for the variances are

\[
\frac{d}{dt} \sigma^2_u = 2a_0 \left( \sigma'_{uv} \right) + \\
+ 2a_1 \left( \mu \sigma'_{uv} + \mu_y \sigma'_{uv} + \sigma'_{uvy} \right) + \\
+ 2a_2 \left( \mu \sigma'_{uv} + \sigma'_{uvy} + \sigma_{uv} \right) + \\
+ 2a_3 \left( \mu \sigma'_{uv} + \mu \sigma'_{uvy} + \mu \sigma_y \right) + \\
+ 2b_2 \left( \mu \sigma_{uuv} + \mu \sigma_{uvy} + \sigma_{uvy} \right) + \\
+ 2b_3 \left( \mu \sigma_{uvy} + \mu \sigma_{uv} + \sigma_{uv} \right) + \\
+ 2 \sigma_{vy} 
\]  
(B.16)

\[
\frac{d}{dt} \sigma^2_v = 2b_0 \left( \sigma'_{uv} \right) + \\
+ 2b_1 \left( \mu \sigma'_{uv} + \mu \sigma'_{uv} + \sigma'_{uvy} \right) + \\
+ 2b_2 \left( \mu \sigma'_{uv} + \mu \sigma_{uvy} + \sigma_{uvy} \right) + \\
+ 2b_3 \left( \mu \sigma'_{uv} + \mu \sigma_{uvy} + \mu \sigma_{uv} \right) + \\
+ 2b_3 \left( \mu \sigma_{uvy} + \mu \sigma_{uv} + \sigma_{uv} \right) + \\
+ 2 \sigma_{vy} 
\]  
(B.17)

\[
\frac{d}{dt} \sigma^2_y = 2 \sigma_{vy} 
\]  
(B.18)
Stochastic-dynamic covariance equations

The resulting SD predictive equations for the covariances are

\[
\frac{d}{dt} \sigma_{uv} = a_0 \sigma_{vv} + b_0 \sigma_{uu} + \\
+ a_1 \left( \mu_y \sigma_{vy} + \mu_v \sigma_{uv} + \sigma_{vvy} \right) + b_1 \left( \mu_y \sigma_{uy} + \mu_y \sigma_{uu} + \sigma_{uyu} \right) + \\
+ a_2 \left( \mu_u \sigma_{uv} + \mu_v \sigma_{uv} + \sigma_{uvv} \right) + b_2 \left( \mu_u \sigma_{uu} + \mu_u \sigma_{uu} + \sigma_{uuv} \right) + \\
+ a_3 \left( \mu \mu_y \sigma_{vv} + \mu \mu_v \sigma_{uv} + \mu \mu_v \sigma_{vvy} \right) + \\
+ a_3 \left( \mu \mu_u \sigma_{uv} + \mu \mu_u \sigma_{uu} + \mu \mu_u \sigma_{uuv} \right) + \\
+ b_3 \left( \mu \mu_y \sigma_{uy} + \mu \mu_y \sigma_{uu} + \mu \mu_y \sigma_{uuv} \right) + \\
+ b_3 \left( \mu \mu_u \sigma_{uy} + \mu \mu_u \sigma_{uu} + \mu \mu_u \sigma_{uuv} \right) \\
\text{(B.19)}
\]

\[
\frac{d}{dt} \sigma_{uy} = \sigma_{uv} + a_0 \sigma_{vy} + \\
+ a_1 \left( \mu_y \sigma_{yy} + \mu_v \sigma_{vy} + \sigma_{vyy} \right) + \\
+ a_2 \left( \mu_u \sigma_{vy} + \mu_v \sigma_{uy} + \sigma_{uvy} \right) + \\
+ a_3 \left( \mu \mu_y \sigma_{vy} + \mu \mu_v \sigma_{uy} + \mu \mu_v \sigma_{yy} \right) + \\
+ a_3 \left( \mu \mu_u \sigma_{vvy} + \mu \mu_v \sigma_{uuy} + \mu \sigma_{uyu} + \sigma_{uuvy} \right) \\
\text{(B.20)}
\]
\[ \frac{d\sigma_{vy}}{dt} = \sigma_{vv} + b_0 \sigma_{uy} + b_1 \left( \mu \sigma_{yy} + \mu \sigma_{uy} + \sigma_{uyy} \right) + b_2 \left( \mu \sigma_{uy} + \mu \sigma_{uy} + \sigma_{uyy} \right) + b_3 \left( \mu \mu \sigma_{uu} + \mu \mu \sigma_{uy} + \mu \mu \sigma_{uy} \right) + b_3 \left( \mu \sigma_{uyy} + \mu \sigma_{uyy} + \mu \sigma_{uyy} + \sigma_{uyyy} \right) \] (B.21)

Summary

Examination of the resulting SD predictive equations for the means, variances, and covariances shows the inclusion of nonlinear terms into the model equations, \((a_i,b_i,a'_i,b'_i,a'_i,b'_i)\) terms, result in the inclusion of higher order moments in the SD equations; thus, with nonlinear terms in the model, second and third moments are needed to predict the means exactly, third and fourth moments are needed to predict the second moments exactly, etc. This, and the increasing complexity and number of predictive equations for the higher order moments, leads to the necessity of truncating the equations after the second or third moments. Fleming (1971) examines using moment closure along with the truncation.

Conservation of Kinetic Energy

Defining \(E_k\) to be the kinetic energy,

\[ E_k = \frac{u^2 + v^2}{2} \]
the time derivative of kinetic energy is
\[
\frac{dE_k}{dt} = 2\left(u\frac{du}{dt} + v\frac{dv}{dt}\right).
\]
Substituting the full equations of motion for the inertial oscillation model, equations (4.6) and (4.7), into the above derivative gives
\[
\frac{dE_k}{dt} = 2\left(2\Omega + \frac{u}{a\cos\alpha}\right) \sin\lambda \; uv - \left(2\Omega + \frac{u}{a\cos\alpha}\right) \sin\lambda \; uv
\]
\[
= 0.
\]
Since the time derivative is zero, the kinetic energy is conserved in the system described by the inertial oscillation equations. This result is useful in checking the stability of the numerical solution.

**Conservation Property of First Two Statistical Moments**

Based upon the conservation of particle kinetic energy in a system described by the inertial oscillation equations, another conservation property may be derived for the means and variances of the velocity components.

Suppose there is a population of N particles in an inertial oscillation system. If the velocity components of the ith particle are \((u_i(t), v_i(t))\), the kinetic energy of the ith particle is
\[
E_{k1}(t) = \frac{u_i^2(t) + v_i^2(t)}{2}.
\]
The average kinetic energy for the population of particles is

\[ E_k(t) = \frac{1}{2} \left( E[u^2(t)] + E[v^2(t)] \right) \]

Since the kinetic energy for each individual particle is conserved, the mean kinetic energy for the population is conserved. Working with only the sum of the expected values of the squares of the velocity components and the identity

\[ E[x^2] = \sigma_x^2 + \mu_x^2, \]

it is simple to show the following quantity is conserved in the inertial oscillation system

\[ \left[ \sigma_u^2(t) + \sigma_v^2(t) \right] + \left[ \mu_u^2(t) + \mu_v^2(t) \right] = \text{constant}. \]

**f-Plane Solution by Complex Variables**

The solution to the inertial oscillation equations in an f-plane, while simple, is tedious to obtain. A shorter derivation may be done by combining the velocity components into a single complex quantity, \( V_c = u + iv \). The two equations of motion

\[ \frac{du}{dt} = fv \]

\[ \frac{dv}{dt} = -fu \]
become the single complex differential equation

\[ \frac{dV_c}{dt} = -i f V_c \]

which may be solved by inspection to give

\[ V_c(t) = V_c(0) e^{-i ft} . \]

If we assume

\[ V_c(t) = \mu_c(t) + \varepsilon_c(t) , \]

then the complex expected value of \( V_c \) is

\[ \mu_c(t) = \mu_c(0) e^{-i ft} \]

and the complex variance of \( V_c \) is

\[ \sigma_c^2(t) = \sigma_c^2(0) e^{-i 2 ft} . \]

Notice that the magnitudes of the mean complex velocity and the complex variance are constants in the case of inertial oscillation in an f-plane. This is another useful property to check the numerical solution schemes.
APPENDIX C
ALTERNATIVE MODEL VARIABLES

Introduction

The traditional approach, that of Lorenz (1960), to the definition of model variables is to use the sum and difference of the upper and lower layer streamfunction $\psi$, divergence $\chi = \nabla \cdot \mathbf{V}$, and potential temperature $\theta$ as variables in a two layer model, as in equations (5.19) to (5.24). This appendix presents an alternative formulation of the potential temperature variables, $\theta$ and $\sigma$. These alternate variables have some attractive properties, especially for a baroclinic model.

Definition

Only the definition of the potential temperature-related variables change, the vorticity-related variables remain the same. The new thermodynamic variables are

$$\theta^* = \left[ \theta_1 \theta_3 \right]^{1/2}$$  \hspace{1cm} (C.1)

$$\sigma^* = \left[ \frac{\theta_1}{\theta_3} \right]^{1/2}$$  \hspace{1cm} (C.2)

The logarithm of the stability variable, $\sigma^*$, is proportional to the temperature lapse rate. To show this, let the temperature lapse rate be defined as

$$\Gamma = -\frac{\partial T}{\partial z}$$  \hspace{1cm} (C.3)
The temperature profile, with respect to pressure, for this lapse rate is

\[ T = T_R \left( \frac{P}{P_*} \right)^{\frac{\Gamma_R}{g}} = \theta_R \left( \frac{P}{P_*} \right)^{\frac{\Gamma_R}{g}} \]  

(C.4)

where \( \theta_R \) is the temperature at \( P_* = 1000\text{mb} \). Solving for the potential temperature in terms of \( \theta_R \) gives

\[ \theta = \theta_R \left( \frac{P}{P_*} \right)^{\frac{\Gamma_R}{g}(\Gamma - \Gamma_d)} \]  

(C.5)

where

\[ \Gamma_d = \frac{g}{C_p} \]

is the dry adiabatic lapse rate. Finally, defining

\[ \gamma = \frac{R}{g}(\Gamma - \Gamma_d) \]  

(C.6)

the potential temperature at any point is

\[ \theta = \theta_R \left( \frac{P}{P_*} \right)^{\gamma} \]  

(C.7)

Substitution of this result into the definitions of \( \theta^* \) and \( \sigma^* \) gives

\[ \theta^* = \left( \theta_1 \theta_3 \right)^{1/2} = \theta_R \left( \frac{P_1 P_3}{P_*^2} \right)^{\gamma/2} \]  

(C.8)
\[ \sigma^* = \left( \frac{\theta_1}{\theta_3} \right)^{1/2} = \left( \frac{P_1}{P_3} \right)^{\gamma/2} \]  
(C.9)

and

\[ \theta_1 = \theta^* \sigma^* \]  
(C.10)

\[ \theta_3 = \theta^*/\sigma^* \]  
(C.11)

Notice that \( \sigma^* \) is a function of \( \gamma \) (thus \( \Gamma \)) only and an assumption that \( \sigma^* \) is constant, i.e. the assumption made for a baroclinic model, would imply the temperature lapse rate is also constant. A variation of \( \ln(\sigma^*) \) in time, as allowed in a linear balance model, is proportional to a variation of the local lapse rate in time. It can be easily shown that

\[ \frac{\partial \ln(\sigma^*)}{\partial t} = \left( \frac{R}{2g} \ln \frac{P_1}{P_3} \right) \frac{\partial \Gamma}{\partial t} . \]  
(C.12)

Finally, using the relations between \( \theta_1, \theta_3 \) and \( (\theta^*, \sigma^*) \), the following conversion equations between the traditional variables \( (\theta, \sigma) \) and \( (\theta^*, \sigma^*) \) are derived

\[ \theta^* = \left( \theta^2 - \sigma^2 \right)^{1/2} , \]  
(C.13)

\[ \sigma^* = \left( \frac{\theta + \sigma}{\theta - \sigma} \right)^{1/2} , \]  
(C.14)
\[ \theta = \frac{\theta^*}{2\sigma^*} \left( \sigma^{*2} + 1 \right) \hspace{1cm} (C.15) \]
\[ \sigma = \frac{\theta^*}{2\sigma^*} \left( \sigma^{*2} - 1 \right) \hspace{1cm} (C.16) \]

**Linear Balance Model Equations**

While the vorticity equations for \( \psi \) and \( \tau \) remain unchanged, the alternative thermodynamic variables, \( \theta^* \) and \( \sigma^* \), give rise to slightly modified thermodynamic equations.

**Vorticity equations**

The vorticity equations are the same as those for the traditional model

\[
\nabla^2 \psi = - J(\psi, \nabla^2 \psi + f) - J(\tau, \nabla^2 \tau) - \nabla \chi \cdot \nabla f - f \nabla^2 \chi \hspace{1cm} (5.36)
\]
\[
\nabla^2 \tau = - J(\tau, \nabla^2 \psi + f) - J(\psi, \nabla^2 \tau) - \nabla \delta \cdot \nabla f - f \nabla^2 \delta \hspace{1cm} (5.37)
\]

**Balance equation**

Except for the substitution of \( \theta^* \) for \( \theta \), the linear balance equation remains unchanged from (5.33).

\[
\frac{R \Delta P}{2} \left( \frac{P_2}{P^*} \right)^{R/C_p} \nabla^2 \theta^* = \nabla f \cdot \nabla \tau + f \nabla^2 \tau \hspace{1cm} (C.17)
\]
The use of \( \theta^* = \exp\left(\frac{\ln \theta_1 + \ln \theta_3}{2}\right) \), instead of \( \theta = \frac{\theta_1 + \theta_3}{2} \), in the balance equation is supported by calculating, for a given lapse rate, the pressure at which the actual potential temperature is equal to \( \theta \) and \( \theta^* \), as defined in this appendix. The results are

for \( \theta \):
\[
P = \left(\frac{P_1 P_3}{P_1 + P_3}\right)^{1/y}
\]

for \( \theta^* \):
\[
P = \left(P_1 P_3\right)^{1/2}
\]

In the two-layer model used in this study, the resulting pressures, for various lapse rates are given below in Table C.1

<table>
<thead>
<tr>
<th>( \Gamma )</th>
<th>( P(\theta) )</th>
<th>( P(\theta^*) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 K/km</td>
<td>446.68 mb</td>
<td>458.26 mb</td>
</tr>
<tr>
<td>3</td>
<td>450.19</td>
<td>458.26</td>
</tr>
<tr>
<td>6</td>
<td>453.75</td>
<td>458.26</td>
</tr>
<tr>
<td>9</td>
<td>457.33</td>
<td>458.26</td>
</tr>
</tbody>
</table>

Using the series expansion of \( a^x \) and the limit

\[
\lim_{x \to 0} \left(1 + bx\right)^{b} = e^b
\]

it can be shown that

\[
\lim_{\Gamma \to \Gamma_d} P(\theta) = P(\theta^*)
\]
Thermodynamic equations

The model thermodynamic equations are based upon the total derivative of \( \ln(\theta) \) instead of the total derivative of \( \theta \). Beginning with the model level \( i \) total derivative of \( \ln(\theta) \),

\[
\frac{d\ln(\theta_i)}{dt} = \frac{1}{C_p} \frac{\dot{q}_i}{T_i} = \frac{s_i}{C_p}
\]

define \( s_i = \frac{s_i}{C_p} \) and expand the total derivative to get

\[
\frac{\partial \ln(\theta_i)}{\partial t} + J(\psi_i, \ln(\theta_i)) + \nabla \chi_i \cdot \nabla \ln(\theta_i) + \omega_i \frac{\partial \ln(\theta)}{\partial P} \bigg|_1 = S_i,
\]

where \( J(A,B) \) represents the Jacobian of \( A \) and \( B \).

At this point, based upon how the potential temperature is assumed to vary with pressure, a number of models may be developed. The two versions considered here are (1) \( \frac{\partial \ln(\theta)}{\partial \ln(P)} \) is constant (\( \Gamma \) is constant) and (2) \( \frac{\partial \ln(\theta)}{\partial P} \) is constant.

**Assumption: \( \frac{\partial \ln(\theta)}{\partial \ln(P)} \) is constant**

The assumption of \( \frac{\partial \ln(\theta)}{\partial \ln(P)} \) being constant is equivalent to assuming the temperature lapse rate (and \( \gamma \)) is constant. Making this assumption, the \( \frac{\partial \ln(\theta)}{\partial P} \) term equation \((C.20)\) may be replaced by

\[
\frac{\partial \ln(\theta)}{\partial P} = \frac{1}{P} \frac{\partial \ln(\theta)}{\partial \ln(P)} = \frac{\gamma}{P}
\]

\[(C.19)\]

to give
\[
\frac{\partial \ln \theta_1}{\partial t} + J(\psi_1, \ln \theta_1) + \nabla \chi_1 \cdot \nabla \ln \theta_1 + \omega_1 \frac{\gamma_1}{\frac{\rho_1}{\rho}} = S_1 . \tag{C.20}
\]

For a two-layer model, assuming a constant temperature lapse rate \((\gamma_1 = \gamma_3 = \gamma)\),

\[
\frac{\partial \ln \theta}{\partial \rho} = \frac{2 \ln \sigma^*}{\rho \ln (\frac{\rho_1}{\rho_3})} . \tag{C.21}
\]

Substituting (C.23) into (C.20) and applying the resulting equation to model levels 1 and 3 gives

\[
\frac{\partial \ln \theta_1}{\partial t} = - J(\psi_1, \ln \theta_1) - \nabla \chi_1 \cdot \nabla \ln \theta_1 - \omega_1 \frac{2 \ln \sigma^*}{\rho_1 \ln (\frac{\rho_1}{\rho_3})} + S_1 \tag{C.22}
\]

\[
\frac{\partial \ln \theta_3}{\partial t} = - J(\psi_3, \ln \theta_3) - \nabla \chi_3 \cdot \nabla \ln \theta_3 - \omega_3 \frac{2 \ln \sigma^*}{\rho_3 \ln (\frac{\rho_1}{\rho_3})} + S_3 \tag{C.23}
\]

Applying the continuity equation to the two-layer model,

\[
V^2 \chi = - \frac{\omega_4}{2 \Delta \rho} \quad \rightarrow \quad \frac{\omega_4}{2} = - \Delta \rho V^2 \chi \tag{5.28}
\]

\[
V^2 \delta = - \frac{\omega_2}{\Delta \rho} - V^2 \chi \quad \rightarrow \quad \frac{\omega_2}{2} = - \frac{\Delta \rho}{2} \left(V^2 \chi + V^2 \delta \right) \tag{5.29}
\]

the resulting estimates of \(\omega_1\) and \(\omega_3\) are

\[
\omega_1 = \frac{\omega_4 + \omega_0}{2} = \frac{\omega_2}{2} = - \frac{\Delta \rho}{2} \left(V^2 \chi + V^2 \delta \right) \tag{C.24}
\]

\[
\omega_3 = \frac{\omega_4 + \omega_2}{2} = - \frac{\Delta \rho}{2} \left(3V^2 \chi + V^2 \delta \right) . \tag{C.25}
\]
Adding (C.22) to (C.23), dividing by 2, and substituting for $\omega_1$ and $\omega_3$, gives the predictive equation for $\ln\theta^*$

$$
\frac{\partial}{\partial t} \ln\theta^* = -J(\psi, \ln\theta^*) - J(\tau, \ln\theta^*) - V\chi \cdot V \ln\theta^* - V\delta \cdot V \ln\theta^* + \\
+ A_\theta \cdot \ln\sigma^* \psi^2 \chi + B_\theta \cdot \ln\sigma^* \psi^2 \delta + S_{\ln\theta^*},
$$

(C.26)

where

$$
A_\theta^* = \frac{\Delta P}{2 \ln(P_1/P_3)} \left( \frac{1}{P_1} + \frac{3}{P_3} \right) = -1.798434287
$$

$$
B_\theta^* = \frac{\Delta P}{2 \ln(P_1/P_3)} \left( \frac{1}{P_1} + \frac{1}{P_3} \right) = -1.124021430
$$

$$
S_{\ln\theta^*} = \frac{S_{\ln\theta_1} + S_{\ln\theta_3}}{2}.
$$

The numerical values are for a two layer model with $(P_1, P_3, \Delta P) = (300mb, 700mb, 400mb)$.

The predictive equation for $\ln(\sigma^*)$ is obtained by subtracting (C.23) from (C.22), dividing by 2, and substituting for $\omega_1$ and $\omega_3$. The result is

$$
\frac{\partial}{\partial t} \ln\sigma^* = -J(\psi, \ln\sigma^*) - J(\tau, \ln\sigma^*) - V\chi \cdot V \ln\sigma^* - V\delta \cdot V \ln\sigma^* + \\
+ A_\sigma \cdot \ln\sigma^* \psi^2 \chi + B_\sigma \cdot \ln\sigma^* \psi^2 \delta + S_{\ln\sigma^*},
$$

(C.27)

where
\[ A^* = \frac{\Delta P}{2 \ln(P_1/P_3)} \left( \frac{1}{P_1} - \frac{3}{P_3} \right) = 0.2248042861 \]

\[ B^* = \frac{\Delta P}{2 \ln(P_1/P_3)} \left( \frac{1}{P_1} - \frac{1}{P_3} \right) = -0.4496085716 \]

\[ S_1^{\ln \theta_1} = \frac{S_1^{\ln \theta_3} - S_1^{\ln \theta_3}}{2} \]

The numerical values are for a two layer model with \((P_1, P_3, \Delta P) = (300\text{mb}, 700\text{mb}, 400\text{mb})\).

Assumption: \(\partial \ln \theta / \partial P\) is constant Making the assumption of constant \(\partial \ln \theta / \partial P\), the \(\partial \ln \theta / \partial P\) term in the thermodynamic equation (C.18) may be replaced by

\[ \frac{\partial \ln \theta}{\partial P} = -\left( \ln \theta_1 - \ln \theta_3 \right) = -\frac{2}{\Delta P} \ln \sigma^* \]

to give

\[ \frac{\partial \ln \theta_1}{\partial t} + J(\psi, \ln \theta_1) + \nabla \chi_1 \cdot \nabla \ln \theta_1 - 2 \frac{\omega_1}{\Delta P} \ln \sigma^* = S_1 \quad \text{(C.28)} \]

Application of (C.30) to model levels 1 and 3 gives

\[ \frac{\partial \ln \theta_1}{\partial t} = -J(\psi, \ln \theta_1) - \nabla \chi_1 \cdot \nabla \ln \theta_1 + 2 \frac{\omega_1}{\Delta P} \ln \sigma^* + S_1 \quad \text{(C.29)} \]

\[ \frac{\partial \ln \theta_3}{\partial t} = -J(\psi, \ln \theta_3) - \nabla \chi_3 \cdot \nabla \ln \theta_3 + 2 \frac{\omega_3}{\Delta P} \ln \sigma^* + S_3 \quad \text{(C.30)} \]
Adding (C.31) to (C.32), dividing by 2, and substituting for $\omega_1$ and $\omega_3$ gives the predictive equation for $\ln\theta^*$

$$
\frac{\partial}{\partial t} \ln\theta^* = -J(\psi, \ln\theta^* ) - J(\tau, \ln\sigma^* ) - \nabla \chi \cdot \nabla \ln\sigma^* - \nabla \delta \cdot \nabla \ln\sigma^* + 
+A_{\theta^*} \ln\sigma^* \nabla^2 \chi + B_{\theta^*} \ln\sigma^* \nabla^2 \delta + S_{\ln\theta^*}
$$

(C.31)

where

$$
A_{\theta^*} = -2.000
$$

$$
B_{\theta^*} = -1.000
$$

$$
S_{\ln\theta^*} = \frac{S_{\ln\theta_1} + S_{\ln\theta_3}}{2}
$$

The predictive equation for $\ln(\sigma^*)$ is obtained by subtracting (C.30) from (C.29), dividing by 2, and substituting for $\omega_1$ and $\omega_3$. The result is

$$
\frac{\partial}{\partial t} \ln\sigma^* = -J(\psi, \ln\sigma^* ) - J(\tau, \ln\theta^* ) - \nabla \chi \cdot \nabla \ln\sigma^* - \nabla \delta \cdot \nabla \ln\theta^* + 
+A_{\sigma^*} \ln\sigma^* \nabla^2 \chi + B_{\sigma^*} \ln\sigma^* \nabla^2 \delta + S_{\ln\sigma^*}
$$

(C.32)

where

$$
A_{\sigma^*} = 1.000
$$

$$
B_{\sigma^*} = 0.000
$$

$$
S_{\ln\sigma^*} = \frac{S_{\ln\theta_1} - S_{\ln\theta_3}}{2}
$$
Final thermodynamic equations. Multiplying equation (C.31) by \( \theta^* \) gives the predictive equation for \( \theta^* \)

\[
\frac{\partial \theta^*}{\partial t} = -J(\psi, \theta^*) - \theta^* J(\tau, \ln \sigma^*) - \nabla \chi \cdot \nabla \theta^* - \theta^* \nabla \delta \cdot \nabla \ln \sigma^* + \\
+ A_\theta (\theta^* \ln \sigma^*) \nabla^2 \chi + B_\theta (\theta^* \ln \sigma^*) \nabla^2 \delta + S_{\theta^*}^* . \tag{C.33}
\]

The heating terms, \( S_{\theta^*} \) and \( S_{\ln \sigma^*} \), may be rewritten in terms of the traditional model level heating terms, \( H_1 \), as

\[
H_1 = \left( \frac{P_1}{P} \right) \frac{R}{C_p} \frac{q}{q_i} \frac{\dot{q}_i}{C_p} \tag{5.25}
\]

\[
S_{\theta^*} = \frac{H_1 + \sigma^2 H_3}{2 \sigma^*} \tag{C.34}
\]

\[
S_{\ln \sigma^*} = \frac{H_1 - \sigma^2 H_3}{2 \sigma^*} \tag{C.35}
\]

Expanding \( S_{\theta^*} \) and \( S_{\ln \sigma^*} \) in the thermodynamic equations gives the final form

\[
\frac{\partial \theta^*}{\partial t} = -J(\psi, \theta^*) - \theta^* J(\tau, \ln \sigma^*) - \nabla \chi \cdot \nabla \theta^* - \theta^* \nabla \delta \cdot \nabla \ln \sigma^* + \\
+ A_\theta (\theta^* \ln \sigma^*) \nabla^2 \chi + B_\theta (\theta^* \ln \sigma^*) \nabla^2 \delta + \frac{H_1 + \sigma^2 H_3}{2 \sigma^*} . \tag{C.36}
\]

\[
\frac{\partial \ln \sigma^*}{\partial t} = -J(\psi, \ln \sigma^*) - J(\tau, \ln \sigma^*) - \nabla \chi \cdot \nabla \ln \sigma^* - \nabla \delta \cdot \nabla \ln \sigma^* + \\
+ A_\sigma (\ln \sigma^*) \nabla^2 \chi + B_\sigma (\ln \sigma^*) \nabla^2 \delta + \frac{H_1 - \sigma^2 H_3}{2 \sigma^*} . \tag{C.37}
\]
The different assumptions ($\partial \ln \theta / \partial \ln P$ or $\partial \ln \theta / \partial P$ is constant) give the same overall predictive equations for $\theta^*$ and $\sigma^*$, but different constants ($A_{\theta^*}$, $B_{\theta^*}$) and ($A_{\sigma^*}$, $B_{\sigma^*}$). The following derivation of the "omega" equation is general and the results are valid for both assumptions.

Omega equation

The "omega" equation is the diagnostic equation used to determine $\delta$. The derivation proceeds in the same manner as with the traditional model equations, using the balance, predictive $\tau$, and predictive $\theta^*$ equations. These equations, written in abbreviated form, are

\[ C_{\text{bal}} \nabla^2 \theta^* = \nabla f \cdot \nabla \tau + f \nabla^2 \tau \quad (C.38) \]

\[ \nabla^2 \frac{\partial \tau}{\partial t} = \nabla f + D_{\tau} - f \nabla^2 \delta \quad (C.39) \]

\[ \frac{\partial}{\partial t} \theta^* = A F_{\theta^*} + D_{\theta^*} + B_{\theta^*} (\theta^* \ln \sigma^* \nabla^2 \delta) \quad (C.40) \]

The various abbreviated components are

\[ C_{\text{bal}} = \frac{R}{2} \frac{\Delta P}{P_2} \left( \frac{P_2}{P_\tau} \right) \]

\[ A_{\tau} = - J(\tau, \nabla^2 \psi + f) - J(\psi, \nabla^2 \tau) \]

\[ D_{\tau} = - \nabla \nabla \cdot \nabla f \]

\[ A F_{\theta^*} = - J(\psi, \theta^*) - \theta^* J(\tau, \ln \sigma^*) - \nabla \chi \cdot \nabla \theta^* + A_{\theta^*} (\theta^* \ln \sigma^* \nabla^2 \chi) + S_{\theta^*} \]

\[ D_{\theta^*} = - \theta^* \nabla \nabla \cdot \nabla \ln \sigma^* \]
Taking the time derivative of the balance equation (C.38) and substituting in the product of the Coriolis term, \( f \), and the \( \tau \) equation (C.39) gives, after some rearranging of terms

\[
C_{ba1} \frac{\partial^2 \theta^*}{\partial t^2} = f \left( A + D \right) + \nabla f \cdot \frac{\partial \theta}{\partial t} - f^2 \nabla^2 \theta^* .
\]  
\[(C.41)\]

Taking the product of \( C_{ba1} \) and the Laplacian of the \( \theta^* \) equation (C.40) gives, after rearranging terms

\[
C_{ba1} B_{\theta^*} \nabla^2 (\theta^* \nabla^2 \phi^*) = C_{ba1} \nabla^2 \theta^* - C_{ba1} \nabla^2 \left( A \phi^* + D \phi^* \right) .
\]  
\[(C.42)\]

Combining the two equations and solving for the \( \nabla^2 \phi^* \) terms give the "omega" equation

\[
\nabla^2 (\theta^* \nabla^2 \phi^*) + \frac{f^2}{C_{ba1} B_{\theta^*}} \nabla^2 \phi^* = - \frac{1}{B_{\theta^*}} \nabla^2 \left( A \phi^* + D \phi^* \right) + \frac{f}{C_{ba1} B_{\theta^*}} \left( A + D \right) + \]
\[
+ \frac{1}{C_{ba1} B_{\theta^*}} \nabla f \cdot \frac{\partial \theta}{\partial t} .
\]
\[(C.43)\]

Operational equations

Since this study used utility subroutines which performed operations on a unit radius sphere, to ease the task of model implementation, the following equations have been converted to equations on a unit sphere, where () denotes a unit sphere operation and \( a \) is the radius of the actual sphere (earth).
Vorticity equations

\[ \nabla_s^2 \theta^* = \frac{1}{a^2} \left\{ J_s(\psi, \nabla_s^2 \psi) + J_s(\tau, \nabla_s^2 \tau) \right\} + \]

\[ - 2 \Omega \left\{ \frac{\partial \psi}{\partial \lambda} + (1-\mu^2) \frac{\partial \chi}{\partial \mu} + \mu \nabla_s^2 \chi \right\} \]  
(C.44)

\[ \nabla_s^2 \tau^* = \frac{1}{a^2} \left\{ J_s(\psi, \nabla_s^2 \tau) + J_s(\tau, \nabla_s^2 \psi) \right\} \]

\[ - 2 \Omega \left\{ \frac{\partial \tau}{\partial \lambda} + (1-\mu^2) \frac{\partial \delta}{\partial \mu} + \mu \nabla_s^2 \delta \right\} \]  
(C.45)

Balance equation

\[ \frac{R}{2} \left( \frac{P_2}{P_*} \right)^{R/C_p} \nabla_s^2 \theta^* = 2 \Omega \left\{ (1-\mu^2) \frac{\partial \tau}{\partial \mu} + \mu \nabla_s^2 \tau \right\} \]  
(C.46)

Thermodynamic equations

\[ \frac{\partial \theta^*}{\partial t^*} = - \frac{1}{a^2} \left\{ J_s(\psi, \theta^*) - \theta^* J_s(\tau, \ln \sigma^*) - \nabla_s \chi \cdot \nabla_s \theta^* + \Lambda_{\theta^*}(\theta^* \ln \sigma^*) \nabla_s^2 \chi \right\} + \]

\[ - \frac{1}{a^2} \left\{ \theta^* \nabla_s \delta \cdot \nabla_s \ln \sigma^* - B_{\theta^*}(\theta^* \ln \sigma^*) \nabla_s^2 \delta \right\} + \frac{H_1 + \sigma^* H_3}{2 \sigma^*} \]  
(C.47)

\[ \frac{\partial \ln \sigma^*}{\partial t^*} = - \frac{1}{a^2} \left\{ J_s(\psi, \sigma^*) + J_s(\tau, \ln \theta^*) + \nabla_s \chi \cdot \nabla_s \ln \sigma^* - \Lambda_{\sigma^*} \ln \sigma^* \nabla_s^2 \chi \right\} + \]

\[ - \frac{1}{a^2} \left\{ \nabla_s \delta \cdot \nabla_s \ln \theta^* - B_{\sigma^*} \ln \sigma^* \nabla_s^2 \delta \right\} + \frac{H_1 - \sigma^* H_3}{2 \sigma^* \sigma^*} \]  
(C.48)
Omega equation

\[ V^2_a(\theta^* \ln \sigma^* V^2 \delta) + \frac{4\Omega^2 a^2}{C_{Ba1} B_\theta^*} \mu^2 V^2 \delta = - \frac{a^2}{B_\theta^* V^2_a} \left( \mu \left( A F_{\theta^*} D_{\theta^*} \right) + \frac{1-\mu^2}{\Omega} \frac{\delta \tau}{\delta t} \right) \]  

(C.49)

Model modifications

The final model equations, (C.44) through (C.49), are for a linear balance model. In a baroclinic-type model, the stability variable, \( \sigma^* \), is assumed to be constant. Two variations of this may be considered:

1. The "classic" condition: \( \sigma^*(\lambda, \mu, t) = \sigma^*_0 \).

2. The modified "classic" condition: \( \sigma^*(\lambda, \mu, t) = \sigma^*_0(\mu) \).

The classic \( \sigma^*_0 \) may be initialized from the initial global mean value of \( \ln \sigma^*(\lambda, \mu) \). In a similar manner, the modified classic \( \sigma^*_0(\mu) \) may be initialized from the initial zonal mean values of \( \ln \sigma^*(\lambda, \mu) \).
APPENDIX D
INITIALIZATION OF MONTE-CARLO GLOBAL SPECTRAL MODELS

Introduction

One of the problems with using a Monte-Carlo forecast technique is the generation of an initial sample. This is especially difficult when dealing with spectral models which have complex coefficients. For example, a simple T15 barotropic vorticity model will have 136, 16 real and 120 complex, spectral coefficients. Using this model and assuming initial multivariate normality, the simplest Monte-Carlo forecast must have the means, variances, and covariances between all 256 (16 + 2*120) real random variables specified. To individually specify the vector of means and variance-covariance matrix for all but the simplest low resolution models is not realistic.

The approach taken in this study was to

(1) Use generalized linear regression to estimate the vector of means and variance-covariance matrix of the subset of the zonal (real) spectral coefficients.

(2) Generate a multivariate normally distributed sample based upon the estimated mean vector and variance-covariance matrix.

(3) Use the sampled zonal coefficients as input for a "super" model. Keeping the zonal coefficients constant, this model was allowed to run for x model days.

(4) The output from the "super" model comprise the actual initial starting sample values of the Monte-Carlo simulation.
Spectral representation of zonal means

A two dimensional physical field, \( f(\mu, \lambda, t) \), may be expressed as the sum of waves, each wave represented by \( F_m^n(t)P_n^m(\mu)e^{im\lambda} \),

\[
f(\lambda, \mu, t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} F_m^n(t)P_n^m(\mu)e^{im\lambda}.
\]

The complex spectral coefficients, \( F_m^n(t) \), may be functions of time. If the field is averaged over longitude, \( \lambda \), the resulting zonal means, \( f_{\lambda}(\mu, t) \), are linear combinations of the zonal spectral coefficients, \( F_0^n(t) \), and the associated Legendre polynomials, \( P_0^0(\mu) \):

\[
<f(\lambda, \mu, t)>_\lambda = f_{\lambda}(\mu, t) = \sum_{n=0}^{\infty} F_0^n(t)P_0^0(\mu).
\]

The values of \( P_0^0(\mu) \) may be calculated for any latitude \( \phi \) (\( \mu = \sin(\phi) \)); so, if the zonal means are known for various latitudes, linear regression may be used to estimate the zonal spectral coefficients.

Linear regression

The spectral representation, in matrix notation, of the zonal means of \( f(\lambda, \mu, t) \) at various latitudes is:
From statistical linear model theory (Searle, 1971), given the mean vector and variance-covariance matrix of \( \mathbf{Y} \), \( \mu_Y \) and \( \Sigma_Y \) respectively, the mean vector and variance-covariance matrix of \( \beta \) are simply

\[
P = (X'\Sigma_Y^{-1}X)^{-1}X'\Sigma_Y^{-1} \tag{D.4}
\]

\[
\mu_\beta = P \mu_Y \tag{D.5}
\]

\[
\Sigma_\beta = (X'\Sigma_Y^{-1}X)^{-1} \tag{D.6}
\]

In this case, \( \mathbf{Y} \) represents the known zonal means and \( \beta \) represents the unknown zonal spectral coefficients. One of the benefits of obtaining the spectral coefficients using linear regression is that an estimated variance-covariance matrix for the estimated zonal spectral coefficients may be calculated. This variance-covariance matrix is exactly what is needed to initialize the Monte-Carlo models in this study.
Application to Initialization of Monte-Carlo Models

The zonal coefficients of some variables, such as temperature (or potential temperature, \( \theta \)), may be obtained directly from the zonal means using the regression approach. With some extra effort, the zonal spectral coefficients of variables which are not observed directly, such as the streamfunction (\( \psi \)) and velocity potential (\( \chi \)) may also be obtained with linear regression techniques.

Streamfunction and velocity potential

Zonal spectral coefficients of the streamfunction (\( \psi \)) and velocity potential (\( \chi \)) may be derived from the zonal average of the \( u \) and \( v \) wind components,

\[
u(\lambda, \mu, t) = \frac{1}{a \cos(\phi)} \frac{\partial}{\partial \lambda} \psi(\lambda, \mu, t) + \frac{1}{a} \frac{\partial}{\partial \phi} \psi(\lambda, \mu, t)
\]

\[
v(\lambda, \mu, t) = \frac{1}{a \cos(\phi)} \frac{\partial}{\partial \lambda} \chi(\lambda, \mu, t) + \frac{1}{a} \frac{\partial}{\partial \phi} \chi(\lambda, \mu, t)
\]

to get

\[
u_\lambda(\mu, t) = -\frac{1}{a} \frac{\partial}{\partial \phi} \psi_\lambda(\mu, t) = -\frac{(1-\mu)^{1/2}}{a} \frac{\partial}{\partial \mu} \psi_\lambda(\mu, t)
\]

\[
v_\lambda(\mu, t) = -\frac{1}{a} \frac{\partial}{\partial \phi} \chi_\lambda(\mu, t) = -\frac{(1-\mu)^{1/2}}{a} \frac{\partial}{\partial \mu} \chi_\lambda(\mu, t)
\]

Solving for the derivatives of \( \psi \) and \( \chi \), gives
These expansions are linear combinations of \( \frac{d\varphi^0}{d\mu_n}(\mu) \) and zonal spectral coefficients of \( \psi \) and \( \chi \).

Table D.1 summarizes the regression \( X \), \( Y \), and \( \beta \) values associated with the estimation of the zonal spectral coefficients of \( \psi \), \( \xi \), and \( \theta \); \( \psi_n^0 \), \( \chi_n^0 \), and \( \theta_n^0 \) respectively.

**Generation of random samples**

Once estimates of the zonal spectral coefficients and their variance-covariance matrix, \( \Sigma_F \), have been obtained, a random sample of zonal coefficients may be generated by using the Cholesky factorization of \( \Sigma_F \), \( \Sigma_F = A_F A_F^T \), a vector of independent standard normal random numbers, \( Z \), and the equation, \( F = \mu_F + A_F Z \). This random sampling is performed for each of the independent physical variables in the model. The two-layer linear balance model, as defined in this study, was the initializing "super" model, so independent samples of zonal spectral coefficients of the primary variables \( \psi = (\psi_1 + \psi_3)/2 \), \( \tau = (\psi_1 - \psi_3)/2 \), and \( \sigma = (\theta_1 - \theta_3)/2 \) were generated.
Model initialization

The sample of zonal coefficients was used as the initial conditions in the initializing "super" model. This model allowed the non-zonal spectral coefficients to evolve over time.

Zonal heating coefficients may be initialized by defining them as being the heating needed to keep the zonal coefficients constant.

The output from the initialization model comprises a intercorrelated sample of model variables which may be used as starting values for Monte-Carlo models. Further processing may be done to obtain spectral coefficients with a uniform variance in the real-space field. In this study, this last processing step, discussed in the next section, is left as purely theory and a topic for future method improvement.

Uniform-variance Spatial Field

Modification of the initialized sample spectral coefficients, $F^m_n(t)$, giving uniform grid variance, may be possible; however the size of the intermediate variance-covariance matrices make this impractical for all but the most limited resolution models.

The spectral space to grid space conversion,

$$f(\lambda, \mu, t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} F^m_n(t) \mathcal{P}^{lm}_n(\mu) e^{ilm\lambda}$$  \hspace{1cm} (D.13)

is a linear combination of spectral coefficients, $F^m_n(\mu)$, and spherical harmonics, $\mathcal{P}^{lm}_n(\mu) e^{ilm\lambda}$. Rewritten as
\[ f(\lambda_1, \mu_1, t) = \sum_k \left( \frac{p^m_k(\mu_j)}{n_k} e^{i \lambda_1 n_k} \right) F^m_k(t) \quad (D.14) \]

\[ f_{ij} = X'_{ij} \quad F \]

or, for all \( \mu \) and \( \lambda \) coordinates in the grid, forming the vector \( f \) and matrix \( X \),

\[
\begin{pmatrix}
  f_{11} \\
  f_{12} \\
  \vdots \\
  f_{1J}
\end{pmatrix}
\begin{pmatrix}
  x'_{11} \\
  x'_{12} \\
  \vdots \\
  x'_{1J}
\end{pmatrix}
= \begin{pmatrix}
  F^0(t) \\
  F^1(t) \\
  \vdots \\
  F^J(t)
\end{pmatrix}
\]

\[ f = X F \quad (D.15) \]

Based upon the linear relation between \( f \) and \( F \), the variance-covariance matrices are

\[ \text{VAR}(F) = \Sigma_f \quad (D.16) \]

\[ \text{VAR}(f) = \Sigma_f = X \Sigma_f X' \quad (D.17) \]

\( \Sigma_f \) is composed of two parts, a standard deviation part and a correlation part:

\[ \Sigma_f = \begin{pmatrix} \sigma_{ij} \end{pmatrix} R_f \begin{pmatrix} \sigma_{ij} \end{pmatrix} \quad (D.18) \]

where
\[ \{ \sigma_{ij} \} = \begin{pmatrix} \sigma_{11} & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \sigma_{21} & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sigma_{11} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \sigma_{12} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & \sigma_{IJ} \end{pmatrix} \]  
(D.19)

and

\[ R = \begin{pmatrix} 1 & \rho_{11,21} & \ldots & \rho_{11,11} & \rho_{11,12} & \ldots & \rho_{11,IJ} \\ \rho_{11,21} & 1 & \ldots & \rho_{21,11} & \rho_{21,12} & \ldots & \rho_{21,1J} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \rho_{11,11} & \rho_{21,11} & \ldots & 1 & \rho_{11,12} & \ldots & \rho_{11,1J} \\ \rho_{11,12} & \rho_{21,12} & \ldots & \rho_{11,12} & 1 & \ldots & \rho_{12,1J} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \rho_{11,1J} & \rho_{21,1J} & \ldots & \rho_{11,1J} & \rho_{12,1J} & \ldots & 1 \end{pmatrix}. \]  
(D.20)

\( R \) is the correlation matrix and \( \{ \sigma_{ij} \} \) is a diagonal matrix consisting of the standard deviations of each grid point \((\lambda_i, \mu_j)\). This factorization is the key to being able to adjust, at least in theory, the initialized sample spectral coefficients to give a uniform variance gridded field.

From linear regression, letting \( P = (X'\Sigma^{-1}X)^{-1} X'\Sigma^{-1} \),

\[ \Sigma_T = (X'\Sigma^{-1}X)^{-1} \]
\[
\Sigma_f = \left( X' \left( \sigma_{1j} R_f \sigma_{1j} \right)^{-1} X \right)^{-1}
\]

letting \( \sigma_{1j} = \sigma I \),

\[
\Sigma_f = \sigma^2 \left( X' R_f^{-1} X \right)^{-1}
\]  \( \text{(D.21)} \)

The last relation occurs when the variance of \( f(\lambda_i, \mu_j, t) \) are uniform and is needed to adjust the initialized spectral coefficients.

The basic adjustment procedure is

1. calculate \( \Sigma_f \) from the sample of initialized spectral coefficients.
2. calculate the Cholesky factorization of \( \Sigma_f \)
   \[
   \Sigma_f = \Lambda_f \Lambda_f'
   \]
3. calculate the resulting \( \Sigma_f \) of the grid points, \( f(\lambda_i, \mu_j, t) \),
   \[
   \Sigma_f = X \Sigma_f X'
   \]
4. calculate the correlation matrix \( R_f \)
   \[
   R_f = \left( \frac{1}{\sigma_{1j}} \right) \sum_r \left( \frac{1}{\sigma_{1j}} \right)
   \]
5. specify the uniform grid point variance, \( \sigma^2 \)
   \[
   \sigma^2 = \sigma_{1j}^2
   \] (possibly)
(6) calculate the revised $\Sigma_f$

$$\Sigma_f^r = \sigma^2 \left( X'R_f^{-1}X \right)^{-1}$$

(7) calculate the revised Cholesky factorization of $\Sigma_f^r$

$$\Sigma_f^r = A_f^r A_f'^r$$

(8) revise the initialized spectral coefficients sample

$$\Gamma^F = \left[ I - A_f^r A_f'^{-1} \right] \mu_F + A_f^r A_f'^{-1}F$$

The Cholesky inverses are generalized inverses.

A number of potential problems need to be addressed before using this procedure. One problem is the correlations between the coefficients of different variables and their effect on the correction. Another problem, array size, was alluded to earlier. The size of $\Sigma_f$ is $(I_{\text{Max}} \cdot J_{\text{Max}})^2$ and the size of $\Sigma_f^r$ is the square of the number of non-zero spectral coefficient components, $(2K_{\text{Max}} - N_{\text{Max}} - 1)^2$. Table 2 tabulates the sizes of each of these arrays for various resolutions and typical values of $I_{\text{Max}}$, $J_{\text{Max}}$, $K_{\text{Max}}$, and $N_{\text{Max}}$. Examination of the table shows that the technique is impractical for all but the most limited resolution models.
## Table D.1: Regression variables and coefficients

<table>
<thead>
<tr>
<th>Model Variable</th>
<th>Regression Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi$</td>
<td>$u_\lambda(\mu, t)\frac{d\varphi_0(\mu)}{d\mu_n(t)}$</td>
</tr>
<tr>
<td>$\chi$</td>
<td>$v_\lambda(\mu, t)\frac{d\varphi_0(\mu)}{d\mu_n(t)}$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$\theta_\lambda(\mu, t)\frac{\varphi_0(\mu)}{n_0(t)}$</td>
</tr>
</tbody>
</table>

## Table D.2: Size of Variance-Covariance Arrays

<table>
<thead>
<tr>
<th>Resolution</th>
<th>$I_{\text{Max}}$</th>
<th>$J_{\text{Max}}$</th>
<th>$K_{\text{Max}}$</th>
<th>$N_{\text{Max}}$</th>
<th>$\Sigma_f$</th>
<th>$\Sigma_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T01</td>
<td>6</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>144</td>
<td>4</td>
</tr>
<tr>
<td>T02</td>
<td>9</td>
<td>4</td>
<td>6</td>
<td>2</td>
<td>1296</td>
<td>9</td>
</tr>
<tr>
<td>T03</td>
<td>12</td>
<td>6</td>
<td>10</td>
<td>3</td>
<td>5184</td>
<td>16</td>
</tr>
<tr>
<td>T04</td>
<td>16</td>
<td>8</td>
<td>15</td>
<td>4</td>
<td>16384</td>
<td>25</td>
</tr>
<tr>
<td>T05</td>
<td>24</td>
<td>10</td>
<td>21</td>
<td>5</td>
<td>57600</td>
<td>36</td>
</tr>
<tr>
<td>T06</td>
<td>24</td>
<td>10</td>
<td>28</td>
<td>6</td>
<td>57600</td>
<td>49</td>
</tr>
<tr>
<td>T07</td>
<td>32</td>
<td>12</td>
<td>36</td>
<td>7</td>
<td>147456</td>
<td>64</td>
</tr>
<tr>
<td>T08</td>
<td>32</td>
<td>14</td>
<td>45</td>
<td>8</td>
<td>200704</td>
<td>81</td>
</tr>
<tr>
<td>T09</td>
<td>32</td>
<td>14</td>
<td>55</td>
<td>9</td>
<td>200704</td>
<td>100</td>
</tr>
<tr>
<td>T10</td>
<td>36</td>
<td>16</td>
<td>66</td>
<td>10</td>
<td>331776</td>
<td>121</td>
</tr>
<tr>
<td>T11</td>
<td>40</td>
<td>18</td>
<td>78</td>
<td>11</td>
<td>518400</td>
<td>144</td>
</tr>
<tr>
<td>T12</td>
<td>48</td>
<td>20</td>
<td>91</td>
<td>12</td>
<td>921600</td>
<td>169</td>
</tr>
<tr>
<td>T13</td>
<td>48</td>
<td>20</td>
<td>105</td>
<td>13</td>
<td>921600</td>
<td>196</td>
</tr>
<tr>
<td>T14</td>
<td>48</td>
<td>22</td>
<td>120</td>
<td>14</td>
<td>1115136</td>
<td>225</td>
</tr>
<tr>
<td>T15</td>
<td>54</td>
<td>24</td>
<td>136</td>
<td>15</td>
<td>1679616</td>
<td>256</td>
</tr>
<tr>
<td>T18</td>
<td>64</td>
<td>28</td>
<td>190</td>
<td>18</td>
<td>3211264</td>
<td>361</td>
</tr>
<tr>
<td>T21</td>
<td>72</td>
<td>32</td>
<td>253</td>
<td>21</td>
<td>5308416</td>
<td>484</td>
</tr>
</tbody>
</table>
APPENDIX E
SUMMARY OF MODEL EQUATIONS

This appendix summarizes the model equations used in this study. The equations are presented in the following order:

Raw Equations:
- Raw Vorticity Equations
- Raw Divergence/Balance Equations
- Miscellaneous Raw Equations

Model Equations:
- Model Vorticity Equations
- Model Thermodynamic Equations
- Model Balance Equations
- Miscellaneous Model Equations
- Model "Omega" Equation

Unit Sphere Model Equations
- Linear Balance Equations using Unit Sphere Operations

Spectral Operations
**RAW VORTICITY EQUATIONS**

**Primitive equation model**

\[
\frac{\partial \psi}{\partial t} = -\mathbf{J}(\psi, \nabla^2 \psi + f) - \mathbf{\nabla} \cdot \nabla f - f\nabla^2 \chi - \mathbf{\nabla} \chi \cdot \nabla (\nabla^2 \psi) - \nabla^2 \chi \cdot \nabla^2 \psi - \omega \nabla^2 \left( \frac{\partial \psi}{\partial \mathbf{p}} \right) - \nabla \omega \cdot \nabla \left( \frac{\partial \psi}{\partial \mathbf{p}} \right) - \mathbf{J} \left( \omega, \frac{\partial \chi}{\partial \mathbf{p}} \right)
\]

**Balance model**

\[
\frac{\partial \psi}{\partial t} = -\mathbf{J}(\psi, \nabla^2 \psi + f) - \mathbf{\nabla} \cdot \nabla f - f\nabla^2 \chi - \mathbf{\nabla} \chi \cdot \nabla (\nabla^2 \psi) - \nabla^2 \chi \cdot \nabla^2 \psi - \omega \nabla^2 \left( \frac{\partial \psi}{\partial \mathbf{p}} \right) - \nabla \omega \cdot \nabla \left( \frac{\partial \psi}{\partial \mathbf{p}} \right)
\]

**Linear balance model**

\[
\frac{\partial \psi}{\partial t} = -\mathbf{J}(\psi, \nabla^2 \psi + f) - \mathbf{\nabla} \cdot \nabla f - f\nabla^2 \chi
\]

**Baroclinic model**

\[
\frac{\partial \psi}{\partial t} = -\mathbf{J}(\psi, \nabla^2 \psi + f) - \mathbf{\nabla} \cdot \nabla f - f\nabla^2 \chi
\]

**Barotropic model**

\[
\frac{\partial \psi}{\partial t} = -\mathbf{J}(\psi, \nabla^2 \psi + f) - \mathbf{\nabla} \cdot \nabla f - f\nabla^2 \chi
\]
RAW DIVERGENCE/BALANCE EQUATIONS

Primitive equations model
\[
\frac{\partial^2 \chi}{\partial t^2} = -\nabla^2 \phi + \nabla f \cdot \nabla \psi + f \nabla^2 \psi - \nabla \cdot \left( \nabla \psi \cdot \nabla \psi \right) - J(f, \chi) - \nabla \cdot \left( \nabla \psi \cdot \nabla \chi \right) - \nabla \cdot \left( \nabla \chi \cdot \nabla \psi \right) - \\
- \nabla \cdot \left( \nabla \chi \cdot \nabla \chi \right) - \nabla \cdot \frac{\partial}{\partial \rho} \left( \nabla \psi + \nabla \chi \right)
\]

Balance model
\[
0 = -\nabla^2 \phi + \nabla f \cdot \nabla \psi + f \nabla^2 \psi - \nabla \cdot \left( \nabla \psi \cdot \nabla \psi \right)
\]

Linear balance model
\[
0 = -\nabla^2 \phi + \nabla f \cdot \nabla \psi + f \nabla^2 \psi
\]

Baroclinic model
\[
0 = -\nabla^2 \phi + \nabla f \cdot \nabla \psi + f \nabla^2 \psi
\]

Barotropic model
\[
0 = -\nabla^2 \phi + \nabla f \cdot \nabla \psi + f \nabla^2 \psi
\]
MISCELLANEOUS RAW EQUATIONS

Thermodynamic equation

\[
\frac{\partial \theta}{\partial t} = -J(\psi, \theta) - \nabla \chi \cdot \nabla \theta - \omega \frac{\partial \theta}{\partial P} + H_1
\]

\[
\kappa = \frac{R}{C_p}
\]

\[
H_1 = \left\{ \frac{p}{p^*} \right\}^\kappa \cdot \frac{q}{C_p}
\]

Hydrostatic equation

\[
\frac{\partial \phi}{\partial P} = -\left\{ \frac{p}{p^*} \right\}^\kappa \theta
\]

Continuity equation

\[
\nabla \chi = -\frac{\partial \omega}{\partial P}
\]
MODEL VORTICITY EQUATIONS

**PSI (ψ)**

**MCBVX:** \[ \frac{\partial \nabla^2 \psi}{\partial t} = -J(\psi, \nabla^2 \psi + f) \quad - \nabla^2 \psi + f^2 \]

**MCBC0:** \[ \frac{\partial \nabla^2 \psi}{\partial t} = -J(\psi, \nabla^2 \psi + f) - J(\tau, \nabla^2 \tau) - \nabla^2 \psi + f^2 \]

**MCBC1:** \[ \frac{\partial \nabla^2 \psi}{\partial t} = -J(\psi, \nabla^2 \psi + f) - J(\tau, \nabla^2 \tau) - \nabla^2 \psi + f^2 \]

**MCLBX:** \[ \frac{\partial \nabla^2 \psi}{\partial t} = -J(\psi, \nabla^2 \psi + f) - J(\tau, \nabla^2 \tau) - \nabla^2 \psi + f^2 \]

**TAU (τ)**

**MCBVX:** One layer model - Tau is not forecast

**MCBC0:** \[ \frac{\partial \nabla^2 \tau}{\partial t} = -J(\tau, \nabla^2 \tau + f) - J(\psi, \nabla^2 \tau) - \nabla^2 \tau + f^2 \]

**MCBC1:** \[ \frac{\partial \nabla^2 \tau}{\partial t} = -J(\tau, \nabla^2 \tau + f) - J(\psi, \nabla^2 \tau) - \nabla^2 \tau + f^2 \]

**MCLBX:** \[ \frac{\partial \nabla^2 \tau}{\partial t} = -J(\tau, \nabla^2 \tau + f) - J(\psi, \nabla^2 \tau) - \nabla^2 \tau + f^2 \]
MODEL THERMODYNAMIC EQUATIONS

THETA (θ)

MCBVX: \[
\frac{\partial \theta}{\partial t} = -J(\psi, \theta) - \nabla \theta \cdot \nabla \theta - 2\sigma \nabla^2 \theta + H_\theta
\]

MCBC0: \[
\frac{\partial \theta}{\partial t} = -J(\psi, \theta) - \nabla \theta \cdot \nabla \theta - 2\sigma \nabla^2 \theta + H_\theta
\]

MCBC1: \[
\frac{\partial \theta}{\partial t} = -J(\psi, \theta) - \nabla \theta \cdot \nabla \theta - 2\sigma \nabla^2 \theta + H_\theta
\]

MCLBX: \[
\frac{\partial \theta}{\partial t} = -J(\psi, \theta) - J(\tau, \sigma) - \nabla \theta \cdot \nabla \theta - \nabla \delta \cdot \nabla \theta - 2\sigma \nabla^2 \theta + H_\theta
\]

SIGMA (σ)

MCBVX: \[
\sigma = \sigma_0 \quad \text{initial global mean}
\]

MCBC0: \[
\sigma = \sigma_0 \quad \text{initial global mean}
\]

MCBC1: \[
\sigma = \sigma_0(\mu) \quad \text{initial zonal means}
\]

MCLBX: \[
\frac{\partial \sigma}{\partial t} = -J(\psi, \sigma) - J(\tau, \theta) - \nabla \sigma \cdot \nabla \theta - \nabla \delta \cdot \nabla \theta + \sigma \nabla^2 \theta + H_\sigma
\]
MODEL BALANCE EQUATION

\[ C_{B}\nabla^2 \theta = f \cdot \nabla \tau + f \nabla^2 \tau \]

\[ C_{B} = \frac{R \Delta P}{\frac{2}{\kappa} \left( \frac{p_2}{\kappa} \right)} \]

MISCELLANEOUS MODEL EQUATIONS

\[ \nabla^2 \chi = -\frac{\omega_1}{2\Delta P} \]

\[ \nabla^2 \delta = -\frac{\omega_2}{\Delta P} - \nabla^2 \chi \]

\[ \kappa = \frac{R}{C_p} \]
MODEL "OMEGA" EQUATION

\[\nabla^2 (\psi) - \frac{f^2}{c_B} (\nabla^2 \delta) = \nabla^2 \left( A_F + D_\theta + H_\theta \right) - \frac{f}{c_B} \left( A_F + D_\tau \right) - \frac{1}{c_B} v_f \nabla \cdot \nabla \delta \]

where

\[\nabla^2 \frac{\partial \psi}{\partial t} = \left( -J(\psi, \nabla^2 \psi + f) - J(\tau, \nabla^2 \tau) - \nabla \delta \cdot \nabla f - f \nabla^2 \chi \right) \]
\[= A_F \psi \]

\[\nabla^2 \frac{\partial \tau}{\partial t} = \left( -J(\tau, \nabla^2 \psi + f) - J(\psi, \nabla^2 \tau) \right) + \left( -\nabla \delta \cdot \nabla f \right) - f \nabla^2 \delta \]
\[= A_F \tau + D_\delta - f \nabla^2 \delta \]

\[\frac{\partial \theta}{\partial t} = \left( -J(\psi, \theta) - J(\tau, \sigma) - \nabla \delta \cdot \nabla \theta - 2\sigma \nabla^2 \chi \right) + H_\theta + \left( -\nabla \delta \cdot \nabla \sigma \right) - \sigma \nabla^2 \delta \]
\[= A_F \theta + H_\theta + D_\theta - \sigma \nabla^2 \delta \]

\[c_B \nabla^2 \theta = f \cdot \nabla \tau + f \nabla^2 \tau \]
LINEAR BALANCE EQUATIONS USING UNIT SPHERE OPERATIONS

\[ v_s^2 \frac{\partial \psi}{\partial t} = - \frac{1}{a^2} \left\{ J_s(\psi, v_s^2 \psi) + J_s(\tau, v_s^2 \tau) \right\} - 2\Omega \left\{ \frac{\partial \psi}{\partial \lambda} + \mu \nu_s^2 \chi + (1-\mu^2) \frac{\partial \chi}{\partial \mu} \right\} \]

\[ v_s^2 \frac{\partial \tau}{\partial t} = - \frac{1}{a^2} \left\{ J_s(\psi, v_s^2 \tau) + J_s(\tau, v_s^2 \psi) \right\} - 2\Omega \left\{ \frac{\partial \tau}{\partial \lambda} + \mu \nu_s^2 \delta + (1-\mu^2) \frac{\partial \delta}{\partial \mu} \right\} \]

\[ \frac{\partial \theta}{\partial t} = - \frac{1}{a^2} \left\{ J_s(\psi, \theta) + J_s(\tau, \sigma) + \nabla_s \chi \cdot \nabla_s \theta + 2\sigma \nu_s^2 \chi \right\} - \frac{1}{a^2} \left\{ \nabla_s \delta \cdot \nabla_s \sigma \right\} - \frac{1}{a^2} \left\{ \sigma \nu_s^2 \delta \right\} + H_{\theta} \]

\[ \frac{\partial \sigma}{\partial t} = - \frac{1}{a^2} \left\{ J_s(\psi, \sigma) + J_s(\tau, \theta) + \nabla_s \chi \cdot \nabla_s \sigma - \sigma \nu_s^2 \chi \right\} - \frac{1}{a^2} \left\{ \nabla_s \delta \cdot \nabla_s \theta \right\} + H_{\sigma} \]

\[ C_B v_s^2 \theta = 2\Omega \left\{ (1-\mu^2) \frac{\partial \tau}{\partial \mu} + \mu \nu_s^2 \tau \right\} \]
\[
\begin{align*}
\left( \frac{\nu_s}{s} - \frac{2\Omega^2 a^2}{C_s^2} \right) \nu_s^2 & = \frac{s^2}{s} \nu_s^2 \left( A_F^\theta + H_\theta + D_\theta \right) - \nu_s^2 \left( \sigma' \nu_s^2 \delta \right) \\
& - \frac{2\Omega a^2}{C_s^2} \left( \frac{\mu}{\Omega} \left( A_F^\tau + D_\tau \right) + (1-2\mu^2) \nu_s^2 \delta + \frac{1-\mu^2}{\Omega} \frac{\partial}{\partial \mu} \left( \frac{\partial \tau}{\partial t} \right) \right)
\end{align*}
\]

where

\[
A_F^\tau = - \frac{1}{a^2} \left( J_s(\psi, \nu_s^2 \tau) + J_s(\tau, \nu_s^2 \psi) \right) - 2\Omega \left( \frac{\partial \tau}{\partial \lambda} \right)
\]

\[
A_F^\theta = - \frac{1}{a^2} \left( J_s(\psi, \theta) + J_s(\tau, \sigma) + \nu_s \chi \cdot \nu_s \theta + 2\sigma \nu_s^2 \chi \right)
\]

\[
D_\tau = - 2\Omega (1-\mu^2) \frac{\partial \delta}{\partial \mu}
\]

\[
D_\theta = - \frac{1}{a^2} \left( \nu_s \delta \cdot \nu_s \sigma \right)
\]

\[
\sigma = \sigma' + \sigma' \]

\[
f^2 = f^2 + f'^2 = 2\Omega^2 - 2\Omega^2 (1-2\mu^2)
\]
SPECTRAL OPERATIONS

Dot Product of Gradients

\[
\nabla_A \cdot \nabla_B = \frac{1}{a^2} \left( \nabla_s A \cdot \nabla_s B \right) = \frac{1}{a^2} \left( \frac{1}{1-\mu^2} \frac{\partial A}{\partial \lambda} \frac{\partial B}{\partial \lambda} + (1-\mu^2) \frac{\partial A}{\partial \mu} \frac{\partial B}{\partial \mu} \right)
\]

Jacobian

\[
J(A,B) = \frac{1}{a^2} \left( J_s(A,B) \right) = \frac{1}{a^2} \left( \frac{\partial A}{\partial \lambda} \frac{\partial B}{\partial \mu} - \frac{\partial A}{\partial \mu} \frac{\partial B}{\partial \lambda} \right)
\]

Laplacian

\[
\left( \nabla^2_A \right)^n = \frac{1}{a^2} \left( \nabla^2_s A \right)^n = \frac{1}{a^2} \left( -n(n+1)A_n \right)
\]
SPECTRAL OPERATIONS (CONTINUED)

Note: for $f = 2\Omega \sin \phi = 2\Omega \mu$,

\[ J(A, f) = \frac{2\Omega}{a^2} \left( \frac{\partial A}{\partial \lambda} \right) \]

\[ \nabla f \cdot \nabla A = \frac{2\Omega}{a^2} \left[ (1-\mu^2) \frac{\partial A}{\partial \mu} \right] \]

\[ f \nabla^2 A = \frac{2\Omega}{a^2} \left( \mu \nabla^2 A \right) \]

\[ \nabla \cdot (f \nabla A) = \frac{2\Omega}{a^2} \left[ (1-\mu^2) \frac{\partial A}{\partial \mu} + \mu \nabla^2 A \right] \]
APPENDIX F

DATASETS

The following sets of data and terrain plots are presented in this appendix.

Sets of Data

Winter (DJF) Mean Zonal Data: 700mb
Winter (DJF) Mean Zonal Data: 300mb
T21 Terrain Spectral Coefficients

Terrain Plots

Raw 5° x 5° Gridded Terrain
Smoothed T15 Surface Terrain
Winter (DJF) Mean Zonal Data: 300mb

Source: *Global Atmospheric Circulation Statistics 1958-1973*
NOAA Professional Parer 14

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>-80</td>
<td>3.7</td>
<td>-0.2</td>
<td>-0.1</td>
<td>-52.4</td>
<td>-602</td>
<td>0.0</td>
<td>1.1</td>
<td>0.2</td>
<td>0.9</td>
<td>0.9</td>
<td>14</td>
<td>0.010</td>
</tr>
<tr>
<td>-70</td>
<td>4.8</td>
<td>-0.1</td>
<td>-0.1</td>
<td>-51.7</td>
<td>-560</td>
<td>0.1</td>
<td>1.5</td>
<td>0.2</td>
<td>0.9</td>
<td>0.8</td>
<td>14</td>
<td>0.010</td>
</tr>
<tr>
<td>-65</td>
<td>8.1</td>
<td>0.0</td>
<td>-0.2</td>
<td>-50.6</td>
<td>-495</td>
<td>0.1</td>
<td>1.7</td>
<td>0.2</td>
<td>0.7</td>
<td>0.7</td>
<td>13</td>
<td>0.010</td>
</tr>
<tr>
<td>-60</td>
<td>13.2</td>
<td>0.0</td>
<td>-0.2</td>
<td>-49.2</td>
<td>-394</td>
<td>0.1</td>
<td>2.1</td>
<td>0.3</td>
<td>0.7</td>
<td>0.5</td>
<td>14</td>
<td>0.010</td>
</tr>
<tr>
<td>-55</td>
<td>18.5</td>
<td>0.1</td>
<td>-0.3</td>
<td>-47.2</td>
<td>-251</td>
<td>0.1</td>
<td>2.4</td>
<td>0.3</td>
<td>0.6</td>
<td>0.5</td>
<td>14</td>
<td>0.010</td>
</tr>
<tr>
<td>-50</td>
<td>22.2</td>
<td>0.2</td>
<td>-0.4</td>
<td>-45.0</td>
<td>-88</td>
<td>0.1</td>
<td>2.3</td>
<td>0.3</td>
<td>0.4</td>
<td>0.6</td>
<td>14</td>
<td>0.010</td>
</tr>
<tr>
<td>-45</td>
<td>22.6</td>
<td>0.3</td>
<td>0.0</td>
<td>-42.7</td>
<td>75</td>
<td>0.1</td>
<td>1.9</td>
<td>0.3</td>
<td>0.6</td>
<td>0.6</td>
<td>14</td>
<td>0.010</td>
</tr>
<tr>
<td>-40</td>
<td>21.1</td>
<td>0.2</td>
<td>0.5</td>
<td>-40.2</td>
<td>223</td>
<td>0.2</td>
<td>1.5</td>
<td>0.3</td>
<td>0.8</td>
<td>0.7</td>
<td>13</td>
<td>0.020</td>
</tr>
<tr>
<td>-35</td>
<td>18.1</td>
<td>0.0</td>
<td>0.6</td>
<td>-37.7</td>
<td>336</td>
<td>0.2</td>
<td>1.3</td>
<td>0.3</td>
<td>0.6</td>
<td>0.7</td>
<td>12</td>
<td>0.020</td>
</tr>
<tr>
<td>-30</td>
<td>14.4</td>
<td>-0.1</td>
<td>0.2</td>
<td>-35.4</td>
<td>420</td>
<td>0.2</td>
<td>1.2</td>
<td>0.2</td>
<td>0.3</td>
<td>0.7</td>
<td>11</td>
<td>0.030</td>
</tr>
<tr>
<td>-25</td>
<td>10.0</td>
<td>-0.2</td>
<td>0.2</td>
<td>-33.4</td>
<td>474</td>
<td>0.3</td>
<td>1.3</td>
<td>0.2</td>
<td>0.5</td>
<td>0.7</td>
<td>11</td>
<td>0.030</td>
</tr>
<tr>
<td>-20</td>
<td>5.4</td>
<td>-0.2</td>
<td>-0.4</td>
<td>-32.0</td>
<td>505</td>
<td>0.3</td>
<td>1.3</td>
<td>0.2</td>
<td>0.6</td>
<td>0.6</td>
<td>11</td>
<td>0.030</td>
</tr>
<tr>
<td>-15</td>
<td>1.4</td>
<td>-0.2</td>
<td>-2.1</td>
<td>-31.5</td>
<td>515</td>
<td>0.3</td>
<td>1.1</td>
<td>0.2</td>
<td>0.5</td>
<td>0.6</td>
<td>12</td>
<td>0.030</td>
</tr>
<tr>
<td>-10</td>
<td>-1.5</td>
<td>0.3</td>
<td>-1.8</td>
<td>-31.3</td>
<td>516</td>
<td>0.3</td>
<td>0.9</td>
<td>0.2</td>
<td>0.4</td>
<td>0.6</td>
<td>12</td>
<td>0.030</td>
</tr>
<tr>
<td>-5</td>
<td>-2.6</td>
<td>0.7</td>
<td>-2.3</td>
<td>-31.3</td>
<td>518</td>
<td>0.3</td>
<td>0.7</td>
<td>0.1</td>
<td>0.5</td>
<td>0.6</td>
<td>12</td>
<td>0.030</td>
</tr>
<tr>
<td>0</td>
<td>-2.7</td>
<td>0.8</td>
<td>-1.7</td>
<td>-31.2</td>
<td>522</td>
<td>0.3</td>
<td>0.8</td>
<td>0.1</td>
<td>0.5</td>
<td>0.7</td>
<td>12</td>
<td>0.020</td>
</tr>
<tr>
<td>5</td>
<td>-0.1</td>
<td>0.8</td>
<td>-0.6</td>
<td>-31.2</td>
<td>525</td>
<td>0.3</td>
<td>1.0</td>
<td>0.1</td>
<td>0.6</td>
<td>0.7</td>
<td>12</td>
<td>0.020</td>
</tr>
<tr>
<td>10</td>
<td>4.0</td>
<td>0.7</td>
<td>1.9</td>
<td>-31.5</td>
<td>528</td>
<td>0.2</td>
<td>1.4</td>
<td>0.1</td>
<td>0.2</td>
<td>0.7</td>
<td>13</td>
<td>0.020</td>
</tr>
<tr>
<td>15</td>
<td>11.5</td>
<td>0.6</td>
<td>3.1</td>
<td>-32.5</td>
<td>503</td>
<td>0.2</td>
<td>1.6</td>
<td>0.1</td>
<td>0.3</td>
<td>0.7</td>
<td>12</td>
<td>0.020</td>
</tr>
<tr>
<td>20</td>
<td>20.7</td>
<td>0.3</td>
<td>-1.6</td>
<td>-34.1</td>
<td>451</td>
<td>0.2</td>
<td>1.7</td>
<td>0.1</td>
<td>0.3</td>
<td>0.6</td>
<td>10</td>
<td>0.020</td>
</tr>
<tr>
<td>25</td>
<td>28.6</td>
<td>0.1</td>
<td>1.3</td>
<td>-36.9</td>
<td>349</td>
<td>0.2</td>
<td>1.8</td>
<td>0.1</td>
<td>0.4</td>
<td>0.6</td>
<td>8</td>
<td>0.010</td>
</tr>
<tr>
<td>30</td>
<td>31.7</td>
<td>-0.1</td>
<td>1.6</td>
<td>-40.6</td>
<td>217</td>
<td>0.2</td>
<td>1.6</td>
<td>0.1</td>
<td>0.4</td>
<td>0.4</td>
<td>6</td>
<td>0.010</td>
</tr>
<tr>
<td>35</td>
<td>29.5</td>
<td>-0.3</td>
<td>0.2</td>
<td>-44.8</td>
<td>67</td>
<td>0.1</td>
<td>1.2</td>
<td>0.1</td>
<td>0.3</td>
<td>0.3</td>
<td>4</td>
<td>0.010</td>
</tr>
<tr>
<td>40</td>
<td>25.2</td>
<td>-0.3</td>
<td>-0.2</td>
<td>-48.1</td>
<td>-84</td>
<td>0.1</td>
<td>1.4</td>
<td>0.1</td>
<td>0.2</td>
<td>0.4</td>
<td>5</td>
<td>0.010</td>
</tr>
<tr>
<td>45</td>
<td>20.3</td>
<td>-0.3</td>
<td>-0.1</td>
<td>-50.4</td>
<td>-233</td>
<td>0.1</td>
<td>1.7</td>
<td>0.1</td>
<td>0.2</td>
<td>0.4</td>
<td>8</td>
<td>0.005</td>
</tr>
<tr>
<td>50</td>
<td>15.5</td>
<td>-0.2</td>
<td>-0.4</td>
<td>-52.2</td>
<td>-344</td>
<td>0.0</td>
<td>2.0</td>
<td>0.1</td>
<td>0.4</td>
<td>0.4</td>
<td>10</td>
<td>0.005</td>
</tr>
<tr>
<td>55</td>
<td>11.9</td>
<td>-0.1</td>
<td>-0.5</td>
<td>-53.5</td>
<td>-438</td>
<td>0.0</td>
<td>1.9</td>
<td>0.1</td>
<td>0.3</td>
<td>0.3</td>
<td>11</td>
<td>0.005</td>
</tr>
<tr>
<td>60</td>
<td>9.3</td>
<td>0.0</td>
<td>-0.4</td>
<td>-54.5</td>
<td>-514</td>
<td>0.0</td>
<td>1.8</td>
<td>0.1</td>
<td>0.5</td>
<td>0.4</td>
<td>11</td>
<td>0.005</td>
</tr>
<tr>
<td>65</td>
<td>7.8</td>
<td>0.1</td>
<td>-0.2</td>
<td>-55.6</td>
<td>-579</td>
<td>0.0</td>
<td>1.8</td>
<td>0.1</td>
<td>0.4</td>
<td>0.5</td>
<td>11</td>
<td>0.005</td>
</tr>
<tr>
<td>70</td>
<td>6.6</td>
<td>0.1</td>
<td>0.1</td>
<td>-56.5</td>
<td>-635</td>
<td>0.0</td>
<td>2.1</td>
<td>0.1</td>
<td>0.6</td>
<td>0.7</td>
<td>14</td>
<td>0.005</td>
</tr>
<tr>
<td>80</td>
<td>4.2</td>
<td>0.0</td>
<td>0.4</td>
<td>-58.1</td>
<td>-726</td>
<td>0.0</td>
<td>1.8</td>
<td>0.1</td>
<td>0.6</td>
<td>1.1</td>
<td>21</td>
<td>0.005</td>
</tr>
</tbody>
</table>
## Winter (DJF) Mean Zonal Data: 700mb

Source: *Global Atmospheric Circulation Statistics 1958-1973*
NOAA Professional Paper 14

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>-80</td>
<td>0.6</td>
<td>0.0</td>
<td>-0.2</td>
<td>-19.1</td>
<td>-310</td>
<td>0.7</td>
<td>0.3</td>
<td>0.5</td>
<td>0.9</td>
<td>0.8</td>
<td>11</td>
<td>0.05</td>
</tr>
<tr>
<td>-70</td>
<td>-1.2</td>
<td>0.0</td>
<td>-0.2</td>
<td>-17.1</td>
<td>-316</td>
<td>0.9</td>
<td>0.5</td>
<td>0.6</td>
<td>0.7</td>
<td>0.6</td>
<td>8</td>
<td>0.07</td>
</tr>
<tr>
<td>-65</td>
<td>1.2</td>
<td>0.0</td>
<td>-0.6</td>
<td>-15.1</td>
<td>-295</td>
<td>1.0</td>
<td>0.6</td>
<td>0.6</td>
<td>0.8</td>
<td>0.6</td>
<td>8</td>
<td>0.08</td>
</tr>
<tr>
<td>-60</td>
<td>5.3</td>
<td>0.0</td>
<td>-0.6</td>
<td>-12.6</td>
<td>-251</td>
<td>1.2</td>
<td>0.8</td>
<td>0.5</td>
<td>0.9</td>
<td>0.6</td>
<td>9</td>
<td>0.11</td>
</tr>
<tr>
<td>-55</td>
<td>9.3</td>
<td>0.0</td>
<td>-0.6</td>
<td>-9.4</td>
<td>-183</td>
<td>1.5</td>
<td>0.9</td>
<td>0.4</td>
<td>0.7</td>
<td>0.6</td>
<td>9</td>
<td>0.13</td>
</tr>
<tr>
<td>-50</td>
<td>11.7</td>
<td>0.0</td>
<td>-0.6</td>
<td>-5.8</td>
<td>-102</td>
<td>1.8</td>
<td>0.9</td>
<td>0.3</td>
<td>0.6</td>
<td>0.6</td>
<td>9</td>
<td>0.14</td>
</tr>
<tr>
<td>-45</td>
<td>11.4</td>
<td>0.1</td>
<td>0.0</td>
<td>-2.1</td>
<td>-20</td>
<td>2.2</td>
<td>0.7</td>
<td>0.2</td>
<td>0.9</td>
<td>0.6</td>
<td>8</td>
<td>0.14</td>
</tr>
<tr>
<td>-40</td>
<td>9.3</td>
<td>0.1</td>
<td>0.9</td>
<td>1.5</td>
<td>49</td>
<td>2.6</td>
<td>0.5</td>
<td>0.1</td>
<td>0.8</td>
<td>0.5</td>
<td>7</td>
<td>0.13</td>
</tr>
<tr>
<td>-35</td>
<td>6.3</td>
<td>0.0</td>
<td>1.1</td>
<td>4.5</td>
<td>96</td>
<td>3.2</td>
<td>0.5</td>
<td>0.1</td>
<td>0.7</td>
<td>0.5</td>
<td>5</td>
<td>0.13</td>
</tr>
<tr>
<td>-30</td>
<td>3.1</td>
<td>0.0</td>
<td>0.7</td>
<td>6.9</td>
<td>123</td>
<td>3.8</td>
<td>0.6</td>
<td>0.2</td>
<td>0.5</td>
<td>0.4</td>
<td>4</td>
<td>0.16</td>
</tr>
<tr>
<td>-25</td>
<td>0.6</td>
<td>0.0</td>
<td>0.1</td>
<td>8.3</td>
<td>133</td>
<td>4.6</td>
<td>0.6</td>
<td>0.2</td>
<td>0.7</td>
<td>0.4</td>
<td>3</td>
<td>0.22</td>
</tr>
<tr>
<td>-20</td>
<td>-1.0</td>
<td>0.0</td>
<td>-0.3</td>
<td>9.1</td>
<td>135</td>
<td>5.3</td>
<td>0.5</td>
<td>0.1</td>
<td>0.6</td>
<td>0.4</td>
<td>2</td>
<td>0.23</td>
</tr>
<tr>
<td>-15</td>
<td>-1.4</td>
<td>-0.3</td>
<td>-1.9</td>
<td>9.3</td>
<td>131</td>
<td>5.7</td>
<td>0.4</td>
<td>0.1</td>
<td>0.3</td>
<td>0.4</td>
<td>3</td>
<td>0.22</td>
</tr>
<tr>
<td>-10</td>
<td>-1.4</td>
<td>-0.5</td>
<td>-1.4</td>
<td>9.3</td>
<td>129</td>
<td>5.9</td>
<td>0.5</td>
<td>0.1</td>
<td>0.4</td>
<td>0.4</td>
<td>3</td>
<td>0.23</td>
</tr>
<tr>
<td>-5</td>
<td>-2.1</td>
<td>-0.7</td>
<td>-1.4</td>
<td>9.4</td>
<td>129</td>
<td>6.0</td>
<td>0.5</td>
<td>0.1</td>
<td>0.6</td>
<td>0.4</td>
<td>2</td>
<td>0.25</td>
</tr>
<tr>
<td>0</td>
<td>-3.1</td>
<td>-0.8</td>
<td>-1.6</td>
<td>9.6</td>
<td>129</td>
<td>5.9</td>
<td>0.4</td>
<td>0.1</td>
<td>0.5</td>
<td>0.4</td>
<td>2</td>
<td>0.25</td>
</tr>
<tr>
<td>5</td>
<td>-3.9</td>
<td>-0.7</td>
<td>-1.6</td>
<td>9.7</td>
<td>132</td>
<td>5.3</td>
<td>0.5</td>
<td>0.2</td>
<td>0.6</td>
<td>0.4</td>
<td>3</td>
<td>0.21</td>
</tr>
<tr>
<td>10</td>
<td>-3.9</td>
<td>-0.4</td>
<td>-0.8</td>
<td>10.0</td>
<td>139</td>
<td>4.5</td>
<td>0.6</td>
<td>0.2</td>
<td>0.3</td>
<td>0.5</td>
<td>3</td>
<td>0.19</td>
</tr>
<tr>
<td>15</td>
<td>-1.3</td>
<td>-0.2</td>
<td>2.7</td>
<td>9.2</td>
<td>140</td>
<td>3.7</td>
<td>0.7</td>
<td>0.2</td>
<td>0.3</td>
<td>0.5</td>
<td>3</td>
<td>0.16</td>
</tr>
<tr>
<td>20</td>
<td>2.6</td>
<td>0.0</td>
<td>1.8</td>
<td>7.5</td>
<td>133</td>
<td>3.1</td>
<td>0.8</td>
<td>0.2</td>
<td>0.4</td>
<td>0.4</td>
<td>2</td>
<td>0.12</td>
</tr>
<tr>
<td>25</td>
<td>6.2</td>
<td>-0.1</td>
<td>2.3</td>
<td>4.4</td>
<td>110</td>
<td>2.6</td>
<td>0.7</td>
<td>0.2</td>
<td>0.4</td>
<td>0.4</td>
<td>2</td>
<td>0.09</td>
</tr>
<tr>
<td>30</td>
<td>8.4</td>
<td>-0.1</td>
<td>1.7</td>
<td>0.5</td>
<td>77</td>
<td>2.2</td>
<td>0.7</td>
<td>0.1</td>
<td>0.4</td>
<td>0.4</td>
<td>2</td>
<td>0.08</td>
</tr>
<tr>
<td>35</td>
<td>9.4</td>
<td>0.0</td>
<td>0.0</td>
<td>-3.8</td>
<td>32</td>
<td>1.8</td>
<td>0.5</td>
<td>0.1</td>
<td>0.4</td>
<td>0.3</td>
<td>2</td>
<td>0.07</td>
</tr>
<tr>
<td>40</td>
<td>9.1</td>
<td>0.0</td>
<td>-0.2</td>
<td>-8.0</td>
<td>-18</td>
<td>1.5</td>
<td>0.6</td>
<td>0.1</td>
<td>0.2</td>
<td>0.4</td>
<td>3</td>
<td>0.05</td>
</tr>
<tr>
<td>45</td>
<td>8.4</td>
<td>0.0</td>
<td>-0.2</td>
<td>-11.8</td>
<td>-76</td>
<td>1.3</td>
<td>0.8</td>
<td>0.0</td>
<td>0.5</td>
<td>0.5</td>
<td>4</td>
<td>0.05</td>
</tr>
<tr>
<td>50</td>
<td>6.8</td>
<td>0.0</td>
<td>-0.8</td>
<td>-15.0</td>
<td>-129</td>
<td>1.1</td>
<td>1.0</td>
<td>0.1</td>
<td>0.7</td>
<td>0.7</td>
<td>5</td>
<td>0.05</td>
</tr>
<tr>
<td>55</td>
<td>5.3</td>
<td>0.0</td>
<td>-1.0</td>
<td>-17.5</td>
<td>-172</td>
<td>0.9</td>
<td>1.0</td>
<td>0.1</td>
<td>0.4</td>
<td>0.8</td>
<td>5</td>
<td>0.04</td>
</tr>
<tr>
<td>60</td>
<td>3.9</td>
<td>0.0</td>
<td>-0.8</td>
<td>-19.6</td>
<td>-205</td>
<td>0.8</td>
<td>1.0</td>
<td>0.1</td>
<td>0.6</td>
<td>0.7</td>
<td>6</td>
<td>0.04</td>
</tr>
<tr>
<td>65</td>
<td>3.0</td>
<td>0.0</td>
<td>-0.3</td>
<td>-21.5</td>
<td>-231</td>
<td>0.7</td>
<td>1.1</td>
<td>0.1</td>
<td>0.4</td>
<td>0.7</td>
<td>7</td>
<td>0.04</td>
</tr>
<tr>
<td>70</td>
<td>2.2</td>
<td>0.0</td>
<td>0.1</td>
<td>-23.4</td>
<td>-253</td>
<td>0.6</td>
<td>1.3</td>
<td>0.1</td>
<td>0.7</td>
<td>0.6</td>
<td>10</td>
<td>0.04</td>
</tr>
<tr>
<td>80</td>
<td>1.6</td>
<td>0.0</td>
<td>1.1</td>
<td>-26.3</td>
<td>-285</td>
<td>0.4</td>
<td>0.9</td>
<td>0.2</td>
<td>0.6</td>
<td>0.7</td>
<td>17</td>
<td>0.03</td>
</tr>
</tbody>
</table>
T21 Terrain Spectral Coefficients

Storage Format:  (215, 2(2X, F12.5))

253 Spectral Coefficients

<table>
<thead>
<tr>
<th>N</th>
<th>M</th>
<th>Real</th>
<th>Imaginary</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>370.45710</td>
<td>0.00000</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>20.60013</td>
<td>0.00000</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>-62.87037</td>
<td>75.94886</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>153.42200</td>
<td>0.00000</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>-2.92545</td>
<td>63.46180</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-74.85590</td>
<td>-27.91179</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>-212.83750</td>
<td>0.00000</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>24.87309</td>
<td>53.46894</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-95.06419</td>
<td>-50.74427</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-58546</td>
<td>29.48803</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>105.58820</td>
<td>0.00000</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>3.71422</td>
<td>-97.89577</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-97.20409</td>
<td>-8.94858</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-36.20285</td>
<td>-64.83897</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>19.29775</td>
<td>-57.79919</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>-214.06350</td>
<td>0.00000</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>-8.03865</td>
<td>9.38725</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-7.02126</td>
<td>21.83667</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-23.78462</td>
<td>-57.93907</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>70.45421</td>
<td>48.77405</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>12.50182</td>
<td>50.20034</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>192.02670</td>
<td>0.00000</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>-1.31899</td>
<td>-100.44220</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>38.05332</td>
<td>19.58101</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-11.14706</td>
<td>-9.40065</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>62.73124</td>
<td>28.32125</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>35.76534</td>
<td>17.08641</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>-28.49724</td>
<td>1.19033</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>-60.29871</td>
<td>0.00000</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>-32.60131</td>
<td>51.05579</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>83.64118</td>
<td>25.90179</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-19.73756</td>
<td>11.89591</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>15.25127</td>
<td>48.265</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>10.13309</td>
<td>42.46312</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>-21.38115</td>
<td>-1.72313</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>14.46208</td>
<td>-34.84799</td>
</tr>
</tbody>
</table>
### T21 Terrain Spectral Coefficients (continued)

<table>
<thead>
<tr>
<th>N</th>
<th>M</th>
<th>Real</th>
<th>Imaginary</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>0</td>
<td>103.34990</td>
<td>0.0000</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>-18.09871</td>
<td>-24.57148</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>6.14593</td>
<td>23.44492</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>-2.65446</td>
<td>29.07279</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>-22.23955</td>
<td>3.11393</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>18.68182</td>
<td>39.00995</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>-12.21140</td>
<td>-12.59873</td>
</tr>
<tr>
<td>8</td>
<td>7</td>
<td>-48.23433</td>
<td>-18.70435</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>-13.57024</td>
<td>3.52385</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>-99.44944</td>
<td>0.0000</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>-16.68157</td>
<td>77.88860</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>9.18631</td>
<td>12.77131</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>-7.68719</td>
<td>14.86717</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>-45.90007</td>
<td>-14.00867</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>18.07529</td>
<td>1.32267</td>
</tr>
<tr>
<td>9</td>
<td>6</td>
<td>-15.04644</td>
<td>-20.20338</td>
</tr>
<tr>
<td>9</td>
<td>7</td>
<td>-22.27954</td>
<td>-37.71495</td>
</tr>
<tr>
<td>9</td>
<td>8</td>
<td>40.74888</td>
<td>2.59655</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>9.09607</td>
<td>3.84628</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>-1.09477</td>
<td>0.0000</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>-12.42832</td>
<td>-51.12587</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>-61.46961</td>
<td>5.90860</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>16.19080</td>
<td>-33.28292</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>-40.83589</td>
<td>-21.75436</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>5.11659</td>
<td>-15.48902</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>.45582</td>
<td>.55278</td>
</tr>
<tr>
<td>10</td>
<td>7</td>
<td>-14.51223</td>
<td>-28.54709</td>
</tr>
<tr>
<td>10</td>
<td>8</td>
<td>4.05525</td>
<td>.14924</td>
</tr>
<tr>
<td>10</td>
<td>9</td>
<td>-9.72276</td>
<td>14.28326</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>22.86475</td>
<td>8.60163</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>-68.19119</td>
<td>0.0000</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>-17.58171</td>
<td>-8.94695</td>
</tr>
<tr>
<td>11</td>
<td>2</td>
<td>11.98617</td>
<td>9.82008</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
<td>3.06058</td>
<td>-40.41620</td>
</tr>
<tr>
<td>11</td>
<td>4</td>
<td>-9.24259</td>
<td>4.54858</td>
</tr>
<tr>
<td>11</td>
<td>5</td>
<td>3.05618</td>
<td>-13.51683</td>
</tr>
<tr>
<td>11</td>
<td>6</td>
<td>19.84404</td>
<td>4.22943</td>
</tr>
<tr>
<td>11</td>
<td>7</td>
<td>-4.20907</td>
<td>-19.88583</td>
</tr>
<tr>
<td>11</td>
<td>8</td>
<td>6.64773</td>
<td>-3.67265</td>
</tr>
<tr>
<td>11</td>
<td>9</td>
<td>14.58311</td>
<td>39.04762</td>
</tr>
<tr>
<td>11</td>
<td>10</td>
<td>-1.58434</td>
<td>-11.64400</td>
</tr>
<tr>
<td>11</td>
<td>11</td>
<td>-8.66647</td>
<td>-9.77351</td>
</tr>
<tr>
<td>N</td>
<td>M</td>
<td>Real</td>
<td>Imaginary</td>
</tr>
<tr>
<td>----</td>
<td>----</td>
<td>----------</td>
<td>-----------</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>14.48172</td>
<td>.00000</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
<td>6.64767</td>
<td>-43.29528</td>
</tr>
<tr>
<td>12</td>
<td>2</td>
<td>-6.18390</td>
<td>9.41072</td>
</tr>
<tr>
<td>12</td>
<td>3</td>
<td>32.55259</td>
<td>-33.63586</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
<td>1.67589</td>
<td>8.27958</td>
</tr>
<tr>
<td>12</td>
<td>5</td>
<td>2.56182</td>
<td>-1.74167</td>
</tr>
<tr>
<td>12</td>
<td>6</td>
<td>21.63368</td>
<td>1.88232</td>
</tr>
<tr>
<td>12</td>
<td>7</td>
<td>11.04680</td>
<td>-3.04488</td>
</tr>
<tr>
<td>12</td>
<td>8</td>
<td>-4.82356</td>
<td>-.06529</td>
</tr>
<tr>
<td>12</td>
<td>9</td>
<td>5.43135</td>
<td>21.85673</td>
</tr>
<tr>
<td>12</td>
<td>10</td>
<td>-.33911</td>
<td>-2.29289</td>
</tr>
<tr>
<td>12</td>
<td>11</td>
<td>-.34980</td>
<td>16.44980</td>
</tr>
<tr>
<td>12</td>
<td>12</td>
<td>-12.79563</td>
<td>9.49770</td>
</tr>
<tr>
<td>13</td>
<td>0</td>
<td>25.43242</td>
<td>.00000</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>2.61439</td>
<td>17.94163</td>
</tr>
<tr>
<td>13</td>
<td>2</td>
<td>51.93503</td>
<td>26.58115</td>
</tr>
<tr>
<td>13</td>
<td>3</td>
<td>1.40473</td>
<td>-.65868</td>
</tr>
<tr>
<td>13</td>
<td>4</td>
<td>8.21747</td>
<td>17.16706</td>
</tr>
<tr>
<td>13</td>
<td>5</td>
<td>-4.73691</td>
<td>10.60808</td>
</tr>
<tr>
<td>13</td>
<td>6</td>
<td>3.03364</td>
<td>8.27216</td>
</tr>
<tr>
<td>13</td>
<td>7</td>
<td>6.59457</td>
<td>19.47282</td>
</tr>
<tr>
<td>13</td>
<td>8</td>
<td>-18.41869</td>
<td>4.80850</td>
</tr>
<tr>
<td>13</td>
<td>9</td>
<td>-7.12570</td>
<td>14.76070</td>
</tr>
<tr>
<td>13</td>
<td>10</td>
<td>4.39519</td>
<td>14.82047</td>
</tr>
<tr>
<td>13</td>
<td>11</td>
<td>-9.46049</td>
<td>-27.45619</td>
</tr>
<tr>
<td>13</td>
<td>12</td>
<td>-2.75092</td>
<td>-22.21806</td>
</tr>
<tr>
<td>13</td>
<td>13</td>
<td>8.17551</td>
<td>7.48923</td>
</tr>
<tr>
<td>14</td>
<td>0</td>
<td>7.55592</td>
<td>.00000</td>
</tr>
<tr>
<td>14</td>
<td>1</td>
<td>-2.61855</td>
<td>21.24874</td>
</tr>
<tr>
<td>14</td>
<td>2</td>
<td>-5.00577</td>
<td>18.23068</td>
</tr>
<tr>
<td>14</td>
<td>3</td>
<td>6.47258</td>
<td>6.22125</td>
</tr>
<tr>
<td>14</td>
<td>4</td>
<td>-9.18240</td>
<td>14.77865</td>
</tr>
<tr>
<td>14</td>
<td>5</td>
<td>-9.88149</td>
<td>24.62053</td>
</tr>
<tr>
<td>14</td>
<td>6</td>
<td>-7.29871</td>
<td>1.86575</td>
</tr>
<tr>
<td>14</td>
<td>7</td>
<td>-1.50768</td>
<td>15.17421</td>
</tr>
<tr>
<td>14</td>
<td>8</td>
<td>-15.74679</td>
<td>-.33950</td>
</tr>
<tr>
<td>14</td>
<td>9</td>
<td>-10.02808</td>
<td>5.30635</td>
</tr>
<tr>
<td>14</td>
<td>10</td>
<td>19.25025</td>
<td>7.84568</td>
</tr>
<tr>
<td>14</td>
<td>11</td>
<td>.88773</td>
<td>-10.97177</td>
</tr>
<tr>
<td>14</td>
<td>12</td>
<td>-6.41231</td>
<td>8.50095</td>
</tr>
<tr>
<td>14</td>
<td>13</td>
<td>-1.90898</td>
<td>-.39608</td>
</tr>
<tr>
<td>14</td>
<td>14</td>
<td>-.83088</td>
<td>-9.33370</td>
</tr>
<tr>
<td>N</td>
<td>M</td>
<td>Real</td>
<td>Imaginary</td>
</tr>
<tr>
<td>----</td>
<td>----</td>
<td>----------</td>
<td>-----------</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
<td>2.27593</td>
<td>0.0000</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
<td>-1.23193</td>
<td>13.58924</td>
</tr>
<tr>
<td>15</td>
<td>2</td>
<td>3.21564</td>
<td>17.86961</td>
</tr>
<tr>
<td>15</td>
<td>3</td>
<td>-27.47334</td>
<td>12.87034</td>
</tr>
<tr>
<td>15</td>
<td>4</td>
<td>-21.28995</td>
<td>11.74356</td>
</tr>
<tr>
<td>15</td>
<td>5</td>
<td>-9.13585</td>
<td>4.20706</td>
</tr>
<tr>
<td>15</td>
<td>6</td>
<td>-10.86150</td>
<td>4.2767</td>
</tr>
<tr>
<td>15</td>
<td>7</td>
<td>-2.08087</td>
<td>5.14441</td>
</tr>
<tr>
<td>15</td>
<td>8</td>
<td>-7.07036</td>
<td>-9.32271</td>
</tr>
<tr>
<td>15</td>
<td>9</td>
<td>-10.75442</td>
<td>-1.72581</td>
</tr>
<tr>
<td>15</td>
<td>10</td>
<td>8.67291</td>
<td>.38916</td>
</tr>
<tr>
<td>15</td>
<td>11</td>
<td>-2.21129</td>
<td>-3.14111</td>
</tr>
<tr>
<td>15</td>
<td>12</td>
<td>-7.17367</td>
<td>-17.55185</td>
</tr>
<tr>
<td>15</td>
<td>13</td>
<td>13.80835</td>
<td>-3.17322</td>
</tr>
<tr>
<td>15</td>
<td>14</td>
<td>2.09809</td>
<td>16.32617</td>
</tr>
<tr>
<td>15</td>
<td>15</td>
<td>-5.81129</td>
<td>1.47045</td>
</tr>
<tr>
<td>16</td>
<td>0</td>
<td>-25.66617</td>
<td>0.0000</td>
</tr>
<tr>
<td>16</td>
<td>1</td>
<td>5.74303</td>
<td>24.75127</td>
</tr>
<tr>
<td>16</td>
<td>2</td>
<td>-33.52258</td>
<td>-11.70473</td>
</tr>
<tr>
<td>16</td>
<td>3</td>
<td>22.10673</td>
<td>-23.18609</td>
</tr>
<tr>
<td>16</td>
<td>4</td>
<td>-25.33197</td>
<td>-10.88033</td>
</tr>
<tr>
<td>16</td>
<td>5</td>
<td>11.84396</td>
<td>-4.38600</td>
</tr>
<tr>
<td>16</td>
<td>6</td>
<td>-10.61594</td>
<td>-2.09749</td>
</tr>
<tr>
<td>16</td>
<td>7</td>
<td>1.21757</td>
<td>-3.37996</td>
</tr>
<tr>
<td>16</td>
<td>8</td>
<td>-5.50590</td>
<td>-5.92221</td>
</tr>
<tr>
<td>16</td>
<td>9</td>
<td>-2.22590</td>
<td>-10.56994</td>
</tr>
<tr>
<td>16</td>
<td>10</td>
<td>11.51115</td>
<td>1.38827</td>
</tr>
<tr>
<td>16</td>
<td>11</td>
<td>7.44633</td>
<td>.93465</td>
</tr>
<tr>
<td>16</td>
<td>12</td>
<td>14.668</td>
<td>-16.17876</td>
</tr>
<tr>
<td>16</td>
<td>13</td>
<td>-8.77184</td>
<td>10.22345</td>
</tr>
<tr>
<td>16</td>
<td>14</td>
<td>10.01994</td>
<td>4.80024</td>
</tr>
<tr>
<td>16</td>
<td>15</td>
<td>13.4967</td>
<td>-9.71383</td>
</tr>
<tr>
<td>16</td>
<td>16</td>
<td>-5.30093</td>
<td>3.57517</td>
</tr>
<tr>
<td>17</td>
<td>0</td>
<td>-14.44382</td>
<td>0.0000</td>
</tr>
<tr>
<td>17</td>
<td>1</td>
<td>16.04951</td>
<td>-33.04664</td>
</tr>
<tr>
<td>17</td>
<td>2</td>
<td>-4.30907</td>
<td>-4.39605</td>
</tr>
<tr>
<td>17</td>
<td>3</td>
<td>-3.38462</td>
<td>-17.16386</td>
</tr>
<tr>
<td>17</td>
<td>4</td>
<td>-3.44039</td>
<td>-9.84289</td>
</tr>
<tr>
<td>17</td>
<td>5</td>
<td>18.13156</td>
<td>-13.68109</td>
</tr>
<tr>
<td>17</td>
<td>6</td>
<td>-13.30778</td>
<td>-11.09376</td>
</tr>
<tr>
<td>17</td>
<td>7</td>
<td>5.05256</td>
<td>-7.95237</td>
</tr>
<tr>
<td>17</td>
<td>8</td>
<td>9.02179</td>
<td>-7.42492</td>
</tr>
<tr>
<td>17</td>
<td>9</td>
<td>11.23686</td>
<td>-2.52330</td>
</tr>
<tr>
<td>17</td>
<td>10</td>
<td>11.04685</td>
<td>2.99831</td>
</tr>
<tr>
<td>17</td>
<td>11</td>
<td>11.39544</td>
<td>7.46192</td>
</tr>
<tr>
<td>17</td>
<td>12</td>
<td>-7.00297</td>
<td>-15.22996</td>
</tr>
</tbody>
</table>
### T21 Terrain Spectral Coefficients (continued)

<table>
<thead>
<tr>
<th>N</th>
<th>M</th>
<th>Real</th>
<th>Imaginary</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>13</td>
<td>-5.86596</td>
<td>8.75641</td>
</tr>
<tr>
<td>17</td>
<td>14</td>
<td>6.98391</td>
<td>8.62736</td>
</tr>
<tr>
<td>17</td>
<td>15</td>
<td>-17.34429</td>
<td>6.92801</td>
</tr>
<tr>
<td>17</td>
<td>16</td>
<td>-11.56339</td>
<td>-6.01295</td>
</tr>
<tr>
<td>17</td>
<td>17</td>
<td>8.51794</td>
<td>-12.76257</td>
</tr>
<tr>
<td>18</td>
<td>0</td>
<td>2.61090</td>
<td>0.00000</td>
</tr>
<tr>
<td>18</td>
<td>1</td>
<td>2.12414</td>
<td>-11.13119</td>
</tr>
<tr>
<td>18</td>
<td>2</td>
<td>-5.21866</td>
<td>-11.19256</td>
</tr>
<tr>
<td>18</td>
<td>3</td>
<td>22.8834</td>
<td>-31.71273</td>
</tr>
<tr>
<td>18</td>
<td>4</td>
<td>3.30338</td>
<td>-2.62541</td>
</tr>
<tr>
<td>18</td>
<td>5</td>
<td>14.65711</td>
<td>3.22562</td>
</tr>
<tr>
<td>18</td>
<td>6</td>
<td>6.11146</td>
<td>-8.21247</td>
</tr>
<tr>
<td>18</td>
<td>7</td>
<td>3.70799</td>
<td>3.38433</td>
</tr>
<tr>
<td>18</td>
<td>8</td>
<td>12.88207</td>
<td>-5.22996</td>
</tr>
<tr>
<td>18</td>
<td>9</td>
<td>5.27212</td>
<td>15.43384</td>
</tr>
<tr>
<td>18</td>
<td>10</td>
<td>-5.78369</td>
<td>2.38369</td>
</tr>
<tr>
<td>18</td>
<td>11</td>
<td>6.12226</td>
<td>10.82889</td>
</tr>
<tr>
<td>18</td>
<td>12</td>
<td>-16.61502</td>
<td>-6.60886</td>
</tr>
<tr>
<td>18</td>
<td>13</td>
<td>0.93931</td>
<td>-7.10857</td>
</tr>
<tr>
<td>18</td>
<td>14</td>
<td>-0.57526</td>
<td>7.80980</td>
</tr>
<tr>
<td>18</td>
<td>15</td>
<td>6.32436</td>
<td>-9.00295</td>
</tr>
<tr>
<td>18</td>
<td>16</td>
<td>11.73075</td>
<td>-1.16787</td>
</tr>
<tr>
<td>18</td>
<td>17</td>
<td>-3.84577</td>
<td>9.56343</td>
</tr>
<tr>
<td>18</td>
<td>18</td>
<td>-0.54960</td>
<td>4.03322</td>
</tr>
<tr>
<td>19</td>
<td>0</td>
<td>4.67746</td>
<td>0.00000</td>
</tr>
<tr>
<td>19</td>
<td>1</td>
<td>11.91468</td>
<td>-18.91092</td>
</tr>
<tr>
<td>19</td>
<td>2</td>
<td>15.19859</td>
<td>3.98425</td>
</tr>
<tr>
<td>19</td>
<td>3</td>
<td>-2.03938</td>
<td>-0.85095</td>
</tr>
<tr>
<td>19</td>
<td>4</td>
<td>5.74583</td>
<td>16.61705</td>
</tr>
<tr>
<td>19</td>
<td>5</td>
<td>8.73671</td>
<td>1.23956</td>
</tr>
<tr>
<td>19</td>
<td>6</td>
<td>2.63124</td>
<td>0.00272</td>
</tr>
<tr>
<td>19</td>
<td>7</td>
<td>-5.62588</td>
<td>12.50869</td>
</tr>
<tr>
<td>19</td>
<td>8</td>
<td>13.86180</td>
<td>4.99670</td>
</tr>
<tr>
<td>19</td>
<td>9</td>
<td>-0.93332</td>
<td>11.78786</td>
</tr>
<tr>
<td>19</td>
<td>10</td>
<td>-10.44805</td>
<td>-1.97002</td>
</tr>
<tr>
<td>19</td>
<td>11</td>
<td>-0.69909</td>
<td>5.68582</td>
</tr>
<tr>
<td>19</td>
<td>12</td>
<td>-8.28507</td>
<td>-2.02931</td>
</tr>
<tr>
<td>19</td>
<td>13</td>
<td>2.20906</td>
<td>-8.63383</td>
</tr>
<tr>
<td>19</td>
<td>14</td>
<td>-3.10365</td>
<td>12.88590</td>
</tr>
<tr>
<td>19</td>
<td>15</td>
<td>2.77162</td>
<td>4.40550</td>
</tr>
<tr>
<td>19</td>
<td>16</td>
<td>-11.77443</td>
<td>-9.52097</td>
</tr>
<tr>
<td>19</td>
<td>17</td>
<td>-10.03783</td>
<td>-6.57494</td>
</tr>
<tr>
<td>19</td>
<td>18</td>
<td>9.19670</td>
<td>-0.88449</td>
</tr>
<tr>
<td>19</td>
<td>19</td>
<td>2.81090</td>
<td>-0.32323</td>
</tr>
<tr>
<td>N</td>
<td>M</td>
<td>Real</td>
<td>Imaginary</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>--------</td>
<td>-----------</td>
</tr>
<tr>
<td>20</td>
<td>0</td>
<td>23.41143</td>
<td>0.00000</td>
</tr>
<tr>
<td>20</td>
<td>1</td>
<td>-1.14384</td>
<td>33.58818</td>
</tr>
<tr>
<td>20</td>
<td>2</td>
<td>18.59042</td>
<td>-6.95005</td>
</tr>
<tr>
<td>20</td>
<td>3</td>
<td>13.08077</td>
<td>7.66580</td>
</tr>
<tr>
<td>20</td>
<td>4</td>
<td>3.50634</td>
<td>9.15522</td>
</tr>
<tr>
<td>20</td>
<td>5</td>
<td>10.65559</td>
<td>11.32690</td>
</tr>
<tr>
<td>20</td>
<td>6</td>
<td>-6.32790</td>
<td>-4.04120</td>
</tr>
<tr>
<td>20</td>
<td>7</td>
<td>-1.84646</td>
<td>14.11740</td>
</tr>
<tr>
<td>20</td>
<td>8</td>
<td>0.39436</td>
<td>4.03769</td>
</tr>
<tr>
<td>20</td>
<td>9</td>
<td>-1.59006</td>
<td>-3.43335</td>
</tr>
<tr>
<td>20</td>
<td>10</td>
<td>-12.91937</td>
<td>4.45496</td>
</tr>
<tr>
<td>20</td>
<td>11</td>
<td>-4.14845</td>
<td>-9.42321</td>
</tr>
<tr>
<td>20</td>
<td>12</td>
<td>-4.06266</td>
<td>-4.9036</td>
</tr>
<tr>
<td>20</td>
<td>13</td>
<td>-6.73677</td>
<td>-12.19727</td>
</tr>
<tr>
<td>20</td>
<td>14</td>
<td>-0.93798</td>
<td>8.77416</td>
</tr>
<tr>
<td>20</td>
<td>15</td>
<td>5.38905</td>
<td>0.86409</td>
</tr>
<tr>
<td>20</td>
<td>16</td>
<td>-5.30586</td>
<td>-8.08226</td>
</tr>
<tr>
<td>20</td>
<td>17</td>
<td>0.45985</td>
<td>4.53780</td>
</tr>
<tr>
<td>20</td>
<td>18</td>
<td>-6.34769</td>
<td>1.25681</td>
</tr>
<tr>
<td>20</td>
<td>19</td>
<td>-0.97030</td>
<td>0.91640</td>
</tr>
<tr>
<td>20</td>
<td>20</td>
<td>3.67267</td>
<td>7.31968</td>
</tr>
<tr>
<td>21</td>
<td>0</td>
<td>-17.75040</td>
<td>0.00000</td>
</tr>
<tr>
<td>21</td>
<td>1</td>
<td>8.07696</td>
<td>6.62099</td>
</tr>
<tr>
<td>21</td>
<td>2</td>
<td>4.25771</td>
<td>5.23502</td>
</tr>
<tr>
<td>21</td>
<td>3</td>
<td>-19.44112</td>
<td>24.25772</td>
</tr>
<tr>
<td>21</td>
<td>4</td>
<td>-6.57823</td>
<td>5.28662</td>
</tr>
<tr>
<td>21</td>
<td>5</td>
<td>-1.54114</td>
<td>4.02308</td>
</tr>
<tr>
<td>21</td>
<td>6</td>
<td>-18.31841</td>
<td>-8.58748</td>
</tr>
<tr>
<td>21</td>
<td>7</td>
<td>-4.67500</td>
<td>11.46621</td>
</tr>
<tr>
<td>21</td>
<td>8</td>
<td>-11.62866</td>
<td>1.81894</td>
</tr>
<tr>
<td>21</td>
<td>9</td>
<td>-0.92509</td>
<td>-3.84352</td>
</tr>
<tr>
<td>21</td>
<td>10</td>
<td>-11.64352</td>
<td>7.19269</td>
</tr>
<tr>
<td>21</td>
<td>11</td>
<td>-1.87285</td>
<td>-15.35326</td>
</tr>
<tr>
<td>21</td>
<td>12</td>
<td>-0.81147</td>
<td>2.86330</td>
</tr>
<tr>
<td>21</td>
<td>13</td>
<td>-2.65123</td>
<td>-7.91888</td>
</tr>
<tr>
<td>21</td>
<td>14</td>
<td>4.44355</td>
<td>8.13154</td>
</tr>
<tr>
<td>21</td>
<td>15</td>
<td>-3.55521</td>
<td>1.70272</td>
</tr>
<tr>
<td>21</td>
<td>16</td>
<td>4.23969</td>
<td>-1.07845</td>
</tr>
<tr>
<td>21</td>
<td>17</td>
<td>-1.78965</td>
<td>-6.06763</td>
</tr>
<tr>
<td>21</td>
<td>18</td>
<td>12.88077</td>
<td>2.16735</td>
</tr>
<tr>
<td>21</td>
<td>19</td>
<td>13.14884</td>
<td>11.59535</td>
</tr>
<tr>
<td>21</td>
<td>20</td>
<td>-10.56306</td>
<td>-1.20822</td>
</tr>
<tr>
<td>21</td>
<td>21</td>
<td>-5.99492</td>
<td>-4.85019</td>
</tr>
</tbody>
</table>
Figure F.1  Raw 5° x 5° Gridded Surface Terrain
Figure F.2  Filtered T15 Surface Terrain