2-2016

Maximum density of induced 5-cycle is achieved by an iterated blow-up of 5-cycle

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Abstract
Let \( C(n) \) denote the maximum number of induced copies of 5-cycles in graphs on \( n \) vertices. For \( n \) large enough, we show that
\[
C(n) = a \cdot b \cdot c \cdot d \cdot e + C(a) + C(b) + C(c) + C(d) + C(e) = a \cdot b \cdot c \cdot d \cdot e + C(a) + C(b) + C(c) + C(d) + C(e),
\]
where \( a + b + c + d + e = n \) and \( a, b, c, d, e \) are as equal as possible.

Moreover, for \( n \) a power of 5, we show that the unique graph on \( n \) vertices maximizing the number of induced 5-cycles is an iterated blow-up of a 5-cycle.

The proof uses flag algebra computations and stability methods.

Disciplines
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Maximum density of induced 5-cycle is achieved by an iterated blow-up of 5-cycle

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\textbf{A B S T R A C T}

Let $C(n)$ denote the maximum number of induced copies of 5-cycles in graphs on $n$ vertices. For $n$ large enough, we show that $C(n) = a \cdot b \cdot c \cdot d \cdot e + C(a) + C(b) + C(c) + C(d) + C(e)$, where $a + b + c + d + e = n$ and $a, b, c, d, e$ are as equal as possible.

Moreover, for $n$ a power of 5, we show that the unique graph on $n$ vertices maximizing the number of induced 5-cycles is an iterated blow-up of a 5-cycle.

The proof uses flag algebra computations and stability methods.

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\section{1. Introduction}

In 1975, Pippenger and Golumbic\textsuperscript{[20]} conjectured that in graphs the maximum induced density of a $k$-cycle is $k!/(k^k - k)$ when $k \geq 5$. In this paper we solve their conjecture for $k = 5$. In addition, we also show that the extremal limit object is unique. The problem of maximizing the induced density of $C_5$ is also posted on http://flagmatic.org as one of the problems where the plain flag algebra method was applied but failed to provide an exact result. It was also mentioned by Razborov\textsuperscript{[25]}. 
Fig. 1. The graph $C_5^{k \times 5}$ maximizes the number of induced $C_5$-s.

Problems of maximizing the number of induced copies of a fixed small graph $H$ have attracted a lot of attention recently [8,14,29]. For a list of other results on this so called inducibility of small graphs of order up to 5, see the work of Even-Zohar and Linial [8].

Denote the $(k-1)$-times iterated blow-up of $C_5$ by $C_5^{k \times 5}$, see Fig. 1. Let $\mathcal{G}_n$ be the set of all graphs on $n$ vertices, and denote by $C(G)$ the number of induced copies of $C_5$ in a graph $G$. Define

$$C(n) = \max_{G \in \mathcal{G}_n} C(G).$$

We say a graph $G \in \mathcal{G}_n$ is extremal if $C(G) = C(n)$. Notice that, since $C_5$ is a self-complementary graph, $G$ is extremal if and only if its complement is extremal. If $n$ is a power of 5, we can exactly determine the unique extremal graph and thus $C(n)$.

**Theorem 1.** For $k \geq 1$, the unique extremal graph in $\mathcal{G}_5^{k \times 5}$ is $C_5^{k \times 5}$.

To prove Theorem 1, we first prove the following theorem. Note that this theorem is sufficient to determine the unique limit object (the graphon) maximizing the density of induced copies of $C_5$.

**Theorem 2.** There exists $n_0$ such that for every $n \geq n_0$

$$C(n) = a \cdot b \cdot c \cdot d \cdot e + C(a) + C(b) + C(c) + C(d) + C(e),$$

where $a + b + c + d + e = n$ and $a, b, c, d, e$ are as equal as possible.

Moreover, if $G \in \mathcal{G}_n$ is an extremal graph, then $V(G)$ can be partitioned into five sets $X_1, X_2, X_3, X_4,$ and $X_5$ of sizes $a, b, c, d$ and $e$ respectively, such that for $1 \leq i < j \leq 5$ and $x_i \in X_i, x_j \in X_j$, we have $x_ix_j \in E(G)$ if and only if $j-i \in \{1, 4\}$.

In Section 2, we give a brief overview of our method, in Section 3 we prove Theorem 2, and in Section 4 we prove Theorem 1.

2. Method and flag algebras

Our method relies on the theory of flag algebras developed by Razborov [21]. Flag algebras can be used as a general tool to attack problems from extremal combinatorics. Flag algebras were used for a wide range of problems, for example the Caccetta–Häggkvist conjecture [15,24], Turán-type problems in graphs [7,11,13,19,22,26,27], 3-graphs [9,10] and hypercubes [1,3], extremal problems in a colored environment [2,4,6], and also to problems in geometry [17] or extremal theory of permutations [5]. For more details on these applications, see a recent survey of Razborov [23].

A typical application of the so-called plain flag algebra method provides a bound on densities of substructures. To get a good bound, true inequalities and equalities involving the densities of substructures are combined with the help of semidefinite programming. This step is by now largely automated, there is even an open source application called Flagmatic [29], which gives easy to check certificates for the validity of this step. In some cases the bound is asymptotically sharp. Obtaining
an exact result from the sharp bound usually consists of first bounding the densities of some small substructures by $o(1)$, which can be read off from the flag algebra computation. Forbidding these structures can yield a lot of information about the structures of the extremal structure. Finally, stability arguments are used to extract the precise extremal structure.

A similar approach can work in some cases where the bound on the desired density is not asymptotically sharp but merely very close to the extremal example. In this case, one may find bounds very close to 0 for a number of small substructures, and again these bounds may suffice for a stability argument.

Both of these ‘lucky’ cases happen most often when the extremal construction is ‘clean’, for example a simple blow-up of a small graph, replacing each vertex by a large independent set. Simple blow-ups of small graphs appear very often as extremal graphs, in fact there are large families of graphs whose extremal graphs for the inducibility are of this type, see Hatami, Hirst and Norine [12]. However, there are also many problems where the extremal construction is an iterated blow-up as shown by Pikhurko [18].

For our problem, the conjectured extremal graph has such an iterated structure, for which it is rare to obtain the precise density from plain flag algebra computations alone. One such rare example is the problem to determine the inducibility of small out-stars in oriented graphs [9] (note that the problem of inducibility of all out-stars was recently solved by Huang [16] using different techniques). Hladký, Král’ and Norine announced that they found the inducibility of the oriented path of length 2, which also has an iterated extremal construction, via a flag algebra method. In [4] we determined the iterated extremal construction maximizing the number of rainbow triangles in 3-edge-colored complete graphs. Other than these three examples, we are not aware of any applications of flag algebras which completely determined an iterative structure.

For our question, a direct application of the plain method gives an upper bound on the limit value and shows that $\lim_{n \to \infty} C(n) / \binom{n}{5} < 0.03846157$, which is slightly more than the density of $C_5$ in the conjectured extremal construction, which is $\frac{1}{26} \approx 0.03846154$. This difference may appear very small, but the bounds on densities of subgraphs not appearing in the extremal structure are too weak to allow the standard methods to work.

Instead, we use flag algebras to find bounds on densities of other subgraphs, which appear with fairly high density in the extremal graph. This enables us to better control the slight lack of performance of the flag algebra bounds as these small errors have a weaker relative effect on larger densities. In the remainder of this section we will give a short description of this new method which provides a proof of Theorem 2, the most critical part of the proof of Theorem 1. Theorem 1 is obtained from Theorem 2 by taking the minimum counterexample $G$ and blowing it up such that the top-level structure resembles $G$. This gives a contradiction that the top-level structure should resemble $C_5$.

In studying the conjectured extremal example, the iterated blow-up $C_5^{k \times 5}$, one observes that the vast majority of induced $C_5$s contain a vertex in each of the five top-level sets. Starting with such a typical $C_5$ and picking an extra vertex, the adjacencies of this vertex to the $C_5$ determine conclusively to which top-level set the vertex belongs. Picking two extra vertices, the induced graph will be in one of two general classes: either the two additional vertices are in the same top-level set (we call this class $C_31111$) or in different sets (we call this class $C22111$), see Fig. 2.

With this observation in mind, we use flag algebra calculations to bound the densities of these two 7-vertex graph classes. We use the fact that we are studying the extremal example, and thus the induced density of $C_5$ can be bounded from below by $\frac{1}{26}$, the density in $C_5^{k \times 5}$ for $k \to \infty$. Using an averaging argument, we compute bounds on the number of graphs of these two classes a typical $C_5$ will lie in. We cannot expect very sharp bounds agreeing with the densities of a top-level $C_5$ in the iterated blow-up, as even in the iterated blow-up the lower level copies of $C_5$ affect the averaging. But this effect is small enough that these bounds enable us to go on.

Using a linear combination of the bounds on the numbers of graphs in $C31111$ and $C22111$ our now fixed typical base $C_5$ lies in, we can define five top-level sets and a left-over set, and bound the sizes of these sets. Further, we can even conclude that most edges and non-edges between the top-level sets follow the pattern of the base $C_5$, as otherwise the density of $C22111$ would be too small.

Using these bounds, we can use a fairly standard stability argument to show that in fact all edges and non-edges between the top-level sets follow the pattern of the base $C_5$—if one of the pairs was out of pattern we could change it and increase the total number of $C_5$s.
In the next two steps, we show that the left-over set from above must be empty. First, we show that every vertex in the left-over set must look very different from the vertices in each of the top-level sets, again with a stability argument changing exactly one pair which is out of pattern. Then we show that this implies that this vertex lies in comparatively few $C_5$s to set up another standard stability argument: replacing this vertex by a copy of a vertex which is in at least an average number of $C_5$s would increase the total number of $C_5$s, a contradiction to the extremality. This last bound relies on the solution of a fairly well-behaved quadratic program, which can be relaxed to a program with only 5 variables. One could possibly solve this program with analytic means, but we doubt that this would give much added insight into the problem. Instead, we use a fairly simple brute-force discretization to approximate the solution in a rigorous way.

The final step of the proof of Theorem 2 is a convexity argument which shows that the top-level sets are balanced.

3. Proof of Theorem 2

In our proofs we consider densities of 7-vertex subgraphs. Guided by their prevalence in the conjectured extremal graph, the following two types of graphs will play an important role. We call a graph $C_{22111}$ if it can be obtained from $C_5$ by duplicating two vertices. We call a graph $C_{31111}$ if it can be obtained from $C_5$ by tripling one vertex. The edges between the original vertices and their copies are not specified, and there are two complementary types of $C_{22111}$, depending on the adjacency of the two doubled vertices in $C_5$. Technically, $C_{22111}$ and $C_{31111}$ denote collections of several graphs. Examples of $C_{22111}$ and $C_{31111}$ are depicted in Fig. 2. We slightly abuse notation by using $C_{22111}$ and $C_{31111}$ also to denote the densities of these graphs, i.e., the probability that randomly chosen 7 vertices induce the appropriate 7-vertex blow-up of $C_5$. Moreover, for a set of vertices $Z$ we denote by $C_{22111}(Z)$ and $C_{31111}(Z)$ the densities of $C_{22111}$ and $C_{31111}$ containing $Z$, i.e., for a graph $G$ on $n$ vertices, $C_{22111}(Z)$ ($C_{31111}(Z)$) is the number of $C_{22111}$ ($C_{31111}$) containing $Z$ divided by $\binom{n-|Z|}{7-|Z|}$.

We start with the following statement.

Proposition 3. There exists $n_0$ such that every extremal graph $G$ on at least $n_0$ vertices satisfies:

$$C_5 < 0.03846157;$$

$$4 \cdot C_{22111} - 11.94 \cdot C_{31111} \geq \frac{1349894760355389179787709186391}{420000000000000000000000000000000} + o(1)$$

$$> 0.003214. \quad (1)$$

Proof. This follows from a standard application of the plain flag algebra method. The first inequality was obtained by Flagmatic [29], which also provides the corresponding certificate. The computation by Flagmatic was done on 8 vertices. For the second inequality, we minimize the left side with the extra constraint that $C_5 \geq \frac{1}{26}$. We performed the computation on 7 vertices since the resulting bound was sufficient and rounding the solution is easier on 7 vertices than on 8. There are 6178 graphs to consider on 8 vertices while there are only 1044 on 7 vertices. It may be possible that we could use an upper bound on $C_5$ obtained on 7 vertices instead of 8 vertices. But since Flagmatic provides the result for 8 vertices, we used 8 vertices. For certificates, see http://orion.math.iastate.edu/lidicky/pub/c5/.
The expressions from Proposition 3 may be compared to the following limiting values in the iterated blow-up $C_5^k$, where $k \to \infty$:

$$C_5 = \frac{1}{26} \approx 0.03846154;$$

$$4 \cdot C221111 - 11.94 \cdot C311111 = 4 \cdot \frac{5}{31} - 11.94 \cdot \frac{5}{93} \approx 0.0032258.$$  

Notice that in the iterated blow-up of $C_5$, in the limit $4 \cdot C221111 - 12 \cdot C311111 = 0$. For our method to work, we need a lower bound greater than zero. On the other hand, computational experiments convinced us that the method works best if the bound is only slightly above zero, where a suitable factor is again determined by computations.

Let $G$ be an extremal graph on $n$ vertices, where $n$ is sufficiently large to apply Proposition 3. Denote the set of all induced $C_{5k}$ in $G$ by $Z$. We assume that $a \in \mathbb{R}$ and $Z = z_1z_2z_3z_4z_5$ is an induced $C_5$ maximizing $C221111(Z) - a \cdot C311111(Z)$. Then

$$(C221111(Z) - a \cdot C311111(Z)) \left(\frac{n-5}{2}\right) \geq \frac{1}{|Z|} \sum_{Y \in Z} (C221111(Y) - a \cdot C311111(Y)) \left(\frac{n-5}{2}\right) = \frac{(4 \cdot C221111 - 3a \cdot C311111) \left(\frac{n}{2}\right)}{C_5} \geq \frac{4}{21} C221111 - \frac{9}{2} C311111 \left(\frac{n-5}{2}\right).$$

As mentioned above, computations indicate that we get the most useful bounds if $C221111(Z) - a \cdot C311111(Z)$ is close but not too close to 0. Using (1) and setting $a = 3.98$, we get

$$C221111(Z) - 3.98 \cdot C311111(Z) > 0.0039792.$$  

For $1 \leq i \leq 5$, we define sets of vertices $Z_i$ which look like $z_i$ to the other vertices of $Z$. Formally,

$$Z_i := \{v \in V(G) : G[(Z \setminus z_i) \cup v] \cong C_5\} \quad \text{for } 1 \leq i \leq 5.$$  

Note that $Z_i \cap Z_j = \emptyset$ for $i \neq j$. We call a pair $v_iv_j$ funky, if $v_iv_j$ is an edge but $z_iz_j$ is not an edge or vice versa, where $v_i \in Z_i, v_j \in Z_j, 1 \leq i < j \leq 5$. In other words, $G[Z \cup \{v_i, v_j\}] \not\cong C221111$, i.e., every funky pair destroys a potential copy of $C221111(Z)$. Denote by $E_f$ the set of funky pairs. With this notation, (2) implies for large $n$ we have

$$\sum_{1 \leq i < j \leq 5} |Z_i||Z_j| - |E_f| - 3.98 \sum_{i=5} |Z_i|^2/2 > 0.003979 \left(\frac{n-5}{2}\right).$$

For any choice of sets $X_i \subseteq Z_i$, where $i \in [5]$, let $X_0 := V(G) \setminus \bigcup X_i$. Let $f$ be the number of funky pairs not incident to vertices in $X_0$, divided by $n^2$ for normalization, and denote $x_i = \frac{1}{n} |X_i|$ for $i \in \{0, \ldots, 5\}$. Choose the $X_i$ (possibly $X_i = Z_i$) such that the left hand side in

$$2 \sum_{1 \leq i < j \leq 5} x_i x_j - 2f - 3.98 \sum_{i=5} x_i^2 > 0.003979$$  

is maximized. In order to simplify notation, we use $X_{i+5} = X_i$ and $x_{i+5} = x_i$ for all $i \geq 1$.

**Claim 4.** The following inequalities are satisfied:

$$0.19816 < x_i < 0.20184 \quad \text{for } i \in [5];$$

$$x_0 < 0.00263;$$

$$f < 0.000011.$$  

**Proof.** To obtain (4)–(6), we need to solve four quadratic programs. The objectives are to minimize $x_1$, maximize $x_5$, maximize $x_0$, and to maximize $f$, respectively. The constraints are (3) and $\sum_{i=0}^5 x_i = 1$ in all four cases. By symmetry, bounds for $x_1$ apply also for $x_2, x_3, x_4,$ and $x_5$.  

Here we describe the process of obtaining the lower bound on $x_1$ in (4). We need to solve the following program $(P)$:

$$
(P) \begin{align*}
\text{minimize} & \quad x_1 \\
\text{subject to} & \quad \sum_{i=0}^{5} x_i = 1, \\
& \quad 2 \sum_{1 \leq i < j \leq 5} x_i x_j - 2f - 3.98 \sum_{i \in [5]} x_i^2 > 0.003979, \\
& \quad x_i \geq 0 \quad \text{for } i \in \{0, 1, \ldots, 5\}.
\end{align*}
$$

We claim that if $(P)$ has a feasible solution $S$, then there exists a feasible solution $S'$ of $(P)$ where

$$
S'(x_1) = S(x_1), \quad S'(f) = 0, \quad S'(0) = S(0), \\
S'(x_2) = S'(x_3) = S'(x_4) = S'(x_5) = \frac{1}{4}(1 - S(x_1) - S(x_0)).
$$

Since $x_2, x_3, x_4$ and $x_5$ appear only in constraints, we only need to check whether (3) is satisfied. The left hand side of (3) can be rewritten as

$$
2x_1 \sum_{2 \leq i < j \leq 5} x_i + 2 \sum_{2 \leq i < j \leq 5} x_i x_j - 3.98 \sum_{1 \leq i < j \leq 5} x_i^2 - 2f
= 2x_1 \sum_{2 \leq i < j \leq 5} x_i - \sum_{2 \leq i < j \leq 5} (x_i - x_j)^2 - 0.98 \sum_{2 \leq i < j \leq 5} x_i^2 - 3.98x_1^2 - 2f.
$$

Note that the term $\sum_{2 \leq i < j \leq 5} (x_i - x_j)^2$ is minimized if $x_i = x_j$ for all $i, j \in \{2, 3, 4, 5\}$. The term $x_2^2 + x_3^2 + x_4^2 + x_5^2$, subject to $x_2 + x_3 + x_4 + x_5$ being a constant, is also minimized if $x_i = x_j$ for all $i, j \in \{2, 3, 4, 5\}$. Since $f \geq 0$, the term $2f$ is minimized when $f = 0$. Hence (3) is satisfied by $S'$ and we can add the constraints $x_2 = x_3 = x_4 = x_5$ and $f = 0$ to bound $x_1$. The resulting program $(P')$ is

$$(P') \begin{align*}
\text{minimize} & \quad x_1 \\
\text{subject to} & \quad x_0 + x_1 + 4y = 1, \\
& \quad 8x_1 y - 0.98 \cdot 4y^2 - 3.98x_1^2 \geq 0.003979, \\
& \quad x_0, x_1, y \geq 0.
\end{align*}
$$

We solve $(P')$ using Lagrange multipliers. We delegate the work to Sage [28] and we provide the Sage script at http://orion.math.iastate.edu/lidicky/pub/c5/. Finding an upper bound on $x_1$ is done by changing the objective to maximization.

Similarly, we can set $x_1 = x_2 = x_3 = x_4 = x_5 = 1/5$ to get an upper bound on $f$. We can set $f = 0$ and $x_1 = x_2 = x_3 = x_4 = x_5 = (1 - x_0)/5$ to get an upper bound on $x_0$. We omit the details. Sage scripts for solving the resulting programs are provided at http://orion.math.iastate.edu/lidicky/pub/c5/. □

For any vertex $v \in X_i, i \in [5]$ we use $d_f(v)$ to denote the number of funky pairs from $v$ to $(X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5) \setminus X_i$ after normalizing by $n$. If we move $v$ from $X_1$ to $X_0$, then the left hand side of (3) will decrease by

$$
- \frac{1}{n} \left(2(x_2 + x_3 + x_4 + x_5) - 2d_f(v) - 2 \cdot 3.98 \cdot x_1 + o(1)\right).
$$

If this quantity was negative, then the left hand side of (3) could be increased by moving $v$ to $X_0$, contradicting our choice of $X_i$. This together with (4) implies that

$$
d_f(v) \leq x_2 + x_3 + x_4 + x_5 - 3.98 \cdot x_1 + o(1) \leq 1 - 4.98 \cdot x_1 + o(1) \leq 0.0132. \quad (7)
$$

Symmetric statements hold also for every vertex $v \in X_2 \cup X_3 \cup X_4 \cup X_5$.

**Claim 5.** There are no funky pairs in $X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5$. 

Proof. Assume that there is a funky pair $uv$. By symmetry, we only need to consider two cases, either $u \in X_1$, $v \in X_2$ or $u \in X_1$, $v \in X_3$. In fact, it is sufficient to check the case where $u \in X_1$ and $v \in X_2$, so $uv$ is not an edge. The other case then follows from considering the complement of $G$.

Let $G'$ be a graph obtained from $G$ by adding the edge $uv$, i.e., changing $uv$ to be not funky. We compare the number of induced $C_5$'s containing $\{u, v\}$ in $G$ and in $G'$. In $G'$, there are at least

$$\left[ x_3x_4x_5 - (d_f(u) + d_f(v)) \max\{x_3x_4, x_3x_5, x_4x_5\} - f \cdot \max\{x_3, x_4, x_5\} \right] n^3$$

induced $C_5$'s containing $uv$, since we can pick one vertex from each of $X_3, X_4, X_5$ to form an induced $C_5$ as long as none of the resulting nine pairs is funky.

Now we count the number of induced $C_5$'s in $G$ containing $\{u, v\}$. The number of such $C_5$'s which contain vertices from $X_0$ is upper bounded by $x_0n^3/2$. Next we count the number of such $C_5$'s avoiding $X_0$. Observe that there are no $C_5$'s avoiding $X_0$ in which $uv$ is the only funky pair.

The number of $C_5$'s containing another funky pair $u'v'$ with $\{u, v\} \cap \{u', v'\} = \emptyset$ can be upper bounded by $fn^2$. We are left to count $C_5$'s where the other funky pairs contain $u$ or $v$. The number of $C_5$'s containing at least two vertices other than $u$ and $v$ which are in funky pairs can be upper bounded by $(d_f(u)/2 + d_f(v)/2 + d_f(u)d_f(v))n^3$.

It remains to count only $C_5$'s containing exactly one vertex where $uw$ and $vw$ are the options for funky pairs. The number of choices for $w$ is at most $(d_f(u) + d_f(v))n$. As $\{u, v, w\}$ is in an induced $C_5$, the set $\{u, v, w\}$ induces a path in either $G$ or the complement of $G$. Let the middle vertex of that path be in $X_i$. If $G[\{u, v, w\}]$ is a path, then the remaining two vertices of a $C_5$ cannot be in $X_{i+1} \cup X_{i+3}$. If $G[\{u, v, w\}]$ is the complement of a path, then the remaining two vertices cannot be in $X_{i+2} \cup X_{i+3}$. Hence the remaining two vertices of a $C_5$ containing $\{u, v, w\}$ can be chosen from at most $3n \cdot \max\{x_i\}$ vertices. This gives an upper bound of $(d_f(u) + d_f(v))n(3n^{\max\{x_i\}})$ for the number of such $C_5$'s.

Now we compare the number of induced $C_5$'s containing $uv$ in $G$ and in $G'$. We use $x_{\max}$ and $x_{\min}$ to denote the upper and lower bound respectively from (4), use $d_f$ to denote the upper bound on $d_f(u)$ and $d_f(v)$ from (7), and also use bounds from (5) and (6). The number of $C_5$'s containing $uv$ divided by $n^3$ is

$$\text{in } G : \quad \leq x_0/2 + f + 2d_f^2 + 9d_fx_{\max}^2 \leq 0.0065;$$

$$\text{in } G' : \quad \geq (x_{\min} - 2d_f)x_{\min} - f x_{\max} \geq 0.0067.$$}

This contradicts the extremality of $G$. \hfill $\square$

Next, we want to show that $X_0 = \emptyset$. For this, suppose that there exists an $x \in X_0$. We will add $x$ to one of the $X_i$, $i \in \{5\}$ such that $d_f(x)$ is minimal. By symmetry, we may assume that $x$ is added to $X_1$. Note that adding a single vertex to $X_1$ does not change any of the density bounds we used above by more than $o(1)$.

**Claim 6.** For every $x \in X_0$, if $x$ is added to $X_1$ then $d_f(x) \geq 0.0808$.

Proof. Let $xw$ be a funky pair, where $w \in X_2$. The case where $w \in X_3$ can be argued the same way by considering the complement of $G$. Let $G'$ be obtained from $G$ by adding the edge $xw$. Since $G$ is extremal, we have $C(G') \leq C(G)$. The following analysis is similar to the proof of **Claim 5**, however, we can say a bit more since every funky pair contains $x$.

First we count induced $C_5$'s containing $xw$ in $G$. The number of induced $C_5$'s containing $xw$ and other vertices from $X_0$ is easily bounded from above by $x_0n^3/2$.

Let $F$ be an induced $C_5$ in $G$ containing $xw$ and avoiding $X_0 \setminus \{x\}$. Since all funky pairs contain $x$, $F - x$ is an induced path $p_0p_1p_2p_3$ without funky pairs. Either $p_j \in X_j$ for all $j \in \{0, 1, 2, 3\}$ or there is an $i \in \{1, 2, 3, 4, 5\}$ such that $p_j \in X_{i+j}$ for all $j \in \{0, 1, 2, 3\}$. The first case is depicted in Fig. 3(a). Consider now the second case. If $i \in \{2, 3, 4\}$, then $xp_0p_1p_2p_3$ does not satisfy the definition of $F$. Hence $i \in \{1, 5\}$ and the possible $C_5$'s are depicted in Fig. 3(b) and (c). In each of the three cases, $F$ contains
For every vertex $C$ of $C$ a Proof. Assumethat thereisan

Claim 8. Since

obtain constraints since it is possible to express $d$ $d$

exactly two funky pairs, $xw$ and $xy$. The location of $y$ entirely determines the location of $F - x$. Hence the number of induced $C_5$s containing $xw$ is at most $d_f(x) x_{\max}^2 n^2$.

In $G'$, there are at least \( (x y)_{x y x y} - d_f(x) \cdot \max(x y x y, x y, x y, x y) \) $n^3$ induced $C_5$s containing $xw$. We obtain

\[
C(G) / n^3 \leq d_f(x) x_{\max}^2 + x_0 / 2 \quad \text{and} \quad C(G') / n^3 \geq (x_{\min} - d_f(x)) x_{\min}^2.
\]

Since $C(G') \leq C(G)$, we have

\[(x_{\min} - d_f(x)) x_{\min}^2 \leq d_f(x) x_{\max}^2 + x_0 / 2,
\]

which together with (4) and (5) gives $d_f(x) \geq 0.0808$. \(\Box\)

Claim 7. Every vertex of the extremal graph $G$ is in at least \((1/26 + o(1)) \left( \binom{n}{4} \right) \approx 0.001602564 n^4\) induced $C_5$s.

Proof. For every vertex $u \in V(G)$, denote by $C_u^u$ the number of $C_5$s in $G$ containing $u$. For any two vertices $u, v \in V(G)$, we show that $C_u^u - C_v^v < n^3$, which implies Claim 7. Denote by $C_u^u$ the number of $C_5$s in $G$ containing both $u$ and $v$. A trivial bound is $C_u^u \leq \binom{n-2}{3}$.

Let $G'$ be obtained from $G$ by deleting $v$ and duplicating $u$ to $u'$, i.e., for every vertex $x$ we add the edge $x u'$ iff $x u$ is an edge. As $G$ is extremal we have

\[
0 \geq C(G') - C(G) \geq C_u^u - C_u^u - C_u^u \geq C_u^u - C_u^u - \left( \binom{n-2}{3} \right).
\]

Claim 8. The set $X_0$ is empty.

Proof. Assume that there is an $x \in X_0$. We count $C_x^x$, the number of induced $C_5$s containing $x$. Our goal is to show that $C_x^x$ is smaller than the value in Claim 7, which is a contradiction. Let $a_i/n$ be the number of neighbors of $x$ in $X_i$ and $b_i/n$ be the number of non-neighbors of $x$ in $X_i$ for $i \in \{0, 1, 2, 3, 4, 5\}$.

The number of $C_5$s where the other four vertices are in $X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5$ is upper bounded by

\[
\left( a_1 b_2 b_3 a_4 + a_2 b_3 b_4 a_5 + a_3 b_4 b_5 a_1 + a_4 b_5 b_1 a_2 + a_5 b_1 b_2 a_3 + \frac{1}{4} \sum_{i=1}^5 a_i^2 b_i^2 \right) n^4.
\]

Moreover, we also need to include the $C_5$s containing vertices from $X_0$ in our bound, which we do very generously by adding all variables by $a_0$ or $b_0$.

Since $x_i = a_i + b_i$, we can use (4) for every $i \in [5]$ as constraints. We also use Claim 6 to obtain constraints since it is possible to express $d_f(x)$ using $a_i$s and $b_i$s if $x$ is added to $X_j$ for all $i, j \in [5]$.

By combining the previous objective and constraints, we obtain the following program $(P)$, whose objective gives an upper bound on the number of $C_5$s containing $x$ divided by $n^4$.

\[
\begin{align*}
\text{maximize} \quad & \sum_{i=1}^5 (a_i + a_0)(b_{i+1} + b_0)(b_{i+2} + b_0)(a_{i+3} + a_0) + \frac{1}{4} \sum_{i=1}^5 a_i^2 b_i^2 \\
\text{subject to} \quad & \sum_{i=0}^5 (a_i + b_i) = 1, \\
& 0.19816 \leq a_i + b_i \leq 0.20184 \quad \text{for } i \in \{1, 2, 3, 4, 5\}, \\
& a_0 + b_0 \leq 0.00263, \\
& b_2 + a_3 + a_4 \geq 0.0808, \\
& b_1 + a_4 + a_5 \geq 0.0808, \\
& b_2 + b_4 + a_1 + a_5 \geq 0.0808, \\
& b_3 + b_5 + a_1 + a_2 \geq 0.0808, \\
& b_4 + b_1 + a_2 + a_3 \geq 0.0808, \\
& a_i, b_i \geq 0 \quad \text{for } i \in \{0, 1, 2, 3, 4, 5\}.
\end{align*}
\]
This can be checked by a computer program mesh-opt.cpp on each of the 5 coordinates. Hence in a box with sidelength \( t \), always as large as possible, which decreases the number of the degrees of freedom.

The resulting upper bound is

\[
\frac{\partial f}{\partial a_1} \leq 0.21^3 + \frac{2 \cdot 0.21^3}{27} < 0.001.
\]

Hence in a box with side length \( t \) the value of \( f \) cannot be bigger than the value at a corner plus \( 5t/2 \cdot 0.001 \). The factor \( 5t/2 \) comes from the fact that the closest corner is in distance at most \( t/2 \) in each of the 5 coordinates.

If we set \( s = 100 \), we compute that the maximum over all grid points of \( (P') \) is less than 0.00157. This can be checked by a computer program mesh–opt.cpp which computes the values at all grid

\[
\text{maximize } f = \sum_{i=1}^{5} a_i b_{i+1} b_{i+2} a_{i+3} + \frac{1}{4} \sum_{i=1}^{5} a_i^2 b_i^2
\]

subject to \( a_i + b_i = 0.21 \) for \( i \in \{1, 2, 3, 4, 5\} \),

\[
b_2 + b_5 + a_3 + a_4 \geq 0.0808,
b_1 + b_3 + a_4 + a_5 \geq 0.0808,
b_2 + b_4 + a_1 + a_5 \geq 0.0808,
b_3 + b_5 + a_1 + a_2 \geq 0.0808,
b_4 + b_1 + a_2 + a_3 \geq 0.0808,
a_i, b_i \geq 0 \text{ for } i \in \{1, 2, 3, 4, 5\}.
\]

Note that the resulting program \( (P') \) has only 5 degrees of freedom. We find an upper bound on the solution of \( (P') \) by a brute force method. We discretize the space of possible solutions, and bound the gradient of the target function to control the behavior between the grid points.

For solving \( (P') \), we fix a constant \( s \) which will correspond to the number of steps. For every \( a_i \), we check \( s + 1 \) equally spaced values between 0 and 0.21 that include the boundaries. By this we have a grid of \( s^5 \) boxes where every feasible solution of \( (P') \), and hence also of \( (P) \), is in one of the boxes.

Next we need to find the partial derivatives of \( f \). Since \( f \) is symmetric, we only check the partial derivative with respect to \( a_1 \).

\[
\frac{\partial f}{\partial a_1} = b_2 b_3 a_4 + a_3 b_4 b_5 + \frac{1}{2} a_1 b_1^2.
\]

We want to find an upper bound on \( \frac{\partial f}{\partial a_1} \). Hence we assume \( a_1 + b_1 = a_3 + b_3 = a_4 + b_4 = b_2 = b_5 = 0.21 \) and we maximize

\[
b_2 b_3 a_4 + a_3 b_4 b_5 = 0.21 ((0.21 - a_3) a_4 + a_3 (0.21 - a_4)) = 0.21 (0.21 a_4 + 0.21 a_3 - 2 a_3 a_4).
\]

This is maximized if \( a_3 = 0, a_4 = 0.21 \) or \( a_3 = 0.21, a_4 = 0 \) and gives the value 0.21\(^3\). Hence

\[
\frac{1}{2} a_1 b_1^2 = \frac{4}{2} a_1 \cdot \frac{b_1}{2} \cdot \frac{b_1}{2} \leq \frac{2(a_1 + b_1)^3}{3^3} = \frac{2 \cdot 0.21^3}{27}.
\]

The resulting upper bound is

\[
\frac{\partial f}{\partial a_1} \leq 0.21^3 + \frac{2 \cdot 0.21^3}{27} < 0.001.
\]

Fig. 3. Possible C\(_5\)s with funky pair \( xw \). They all have exactly one other funky pair \( xy \). The dotted lines represent non-edges.

Instead of solving \( (P) \) we solve a slight relaxation \( (P') \) with increased upper bounds on \( a_i + b_i \), which allows us to drop \( a_0 \) and \( b_0 \). Since the objective function is maximizing, we can claim that \( a_i + b_i \) is always as large as possible, which decreases the number of the degrees of freedom.
points. With \( t < 0.21/s = 0.0021 \), we have \( 5t/2 \cdot 0.001 < 0.00001 \). We conclude that \( x \) is in less than \( 0.00158n^4 \) induced \( C_5 \)'s which contradicts Claim 7.

Let us note that if we had chosen \( s = 200 \), we could have concluded that \( x \) is less than \( 0.00147n^4 \).

We have just established the “outside” structure of \( G \). Observe that in this outside structure, an induced \( C_5 \) can appear only if it either intersects each of the classes in exactly one vertex, or if it lies completely inside one of the classes. This implies that

\[
C(n) = (x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5)n^5 + C(x_1n) + C(x_2n) + C(x_3n) + C(x_4n) + C(x_5n).
\]

By averaging over all subgraphs of \( G \) of order \( n - 1 \), we can easily see that \( C(n) \leq \frac{n}{n-5} C(n-1) \) for all \( n \), so

\[
\ell := \lim_{n \to \infty} \frac{C(n)}{ \binom{n}{5} }
\]

exists. Therefore,

\[
\ell + o(1) = 5! \cdot x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 + \ell(x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5),
\]

which implies that \( x_i = \frac{1}{5} + o(1) \), and \( \ell = \frac{1}{26} \), given the constraints on the \( x_i \).

In order to prove Theorem (2), it remains to show that in fact \( |X| - |X| \leq 1 \) for all \( i, j \in \{1, \ldots, 5\} \).

**Claim 9.** For \( n \) large enough, we have \( |X_i| - |X_j| \leq 1 \) for all \( i, j \in \{1, \ldots, 5\} \).

**Proof.** By symmetry, assume for contradiction that \( |X_1| - |X_2| \geq 2 \). Let \( v \in X_1 \) where \( C_5^v \) is minimized over the vertices in \( X_1 \) and let \( w \in X_2 \) where \( C_5^w \) is maximized over the vertices in \( X_2 \). As \( G \) is extremal, \( C_5^v + C_5^w - C_5^w \geq 0 \); otherwise, we can increase the number of \( C_5 \)'s by replacing \( v \) by a copy of \( w \).

Let \( y_1 := |X_1| = x_i n \). By the monotonicity of \( \frac{C(x)}{ \binom{x}{5} } \), we have

\[
\frac{1}{26} + o(1) \geq \frac{C(y_1)}{ \binom{y_1}{5} } \geq \frac{C(y_2)}{ \binom{y_2}{5} } \geq \frac{1}{26} - o(1).
\]

Therefore, using \( y_1 - y_2 \geq 2 \), we have

\[
\begin{align*}
C_5^v + C_5^w - C_5^w &\leq \frac{C(y_1)}{y_1} + y_2 y_3 y_4 y_5 + y_3 y_4 y_5 - \frac{C(y_2)}{y_2} - y_1 y_3 y_4 y_5 \\
&= \frac{y_2 C(y_1) - y_1 C(y_2)}{y_1 y_2} + (y_2 - y_1 + 1)y_3 y_4 y_5 \\
&\leq \left( \frac{1}{26} + o(1) \right) \frac{1}{y_1 y_2} \left( y_2 \left( \frac{y_1}{5} \right) - y_1 \left( \frac{y_2}{5} \right) \right) + (y_2 - y_1 + 1)y_3 y_4 y_5 \\
&\leq \left( \frac{1}{26 \cdot 5!} + o(1) \right) (y_2^4 - y_2^4) + (y_2 - y_1 + 1)y_3 y_4 y_5 \\
&= \left( \frac{1}{26 \cdot 5!} + o(1) \right) (y_2^4 - y_2^4) + (y_2 - y_1 + 1)y_3 y_4 y_5 \\
&= (y_2 - y_1) \left( \left( \frac{1}{26 \cdot 5!} + o(1) \right) \frac{4n^3}{125} - \frac{n^3}{125} \right) + (1 + o(1)n^3) \\
&\leq \left( \frac{2}{26 \cdot 5!} + o(1) \right) \frac{4n^3}{125} - \frac{(1 + o(1)n^3)}{125} < 0,
\end{align*}
\]

a contradiction.”

With this claim, the proof of Theorem 2 is complete.  

4. Proof of Theorem 1

Theorem 1 is a consequence of Theorem 2. The main proof idea is to take a minimal counterexample \( G \) and show that some blow-up of \( G \) contradicts Theorem 2.

Proof of Theorem 1. Theorem 1 is easily seen to be true for \( k = 1 \). Suppose for a contradiction that there is a graph \( G \) on \( n = 5^6 \) vertices with \( C(G) \geq C(C_5^{k \times}) \) that is not isomorphic to \( C_5^{k \times} \), where \( k \geq 2 \) is minimal. Let \( n_0 \) be the \( n_0 \) from the statement of Theorem 2.

We say that a graph \( F \) of size \( 5m \) can be 5-partitioned, if \( V(F) \) can be partitioned into five sets \( X_1, X_2, X_3, X_4, X_5 \) with \( |X_i| = m \) for all \( i \in [5] \) and for every \( 1 \leq i < j \leq 5 \), every \( x_i \in X_i \) and \( x_j \in X_j \) are adjacent if and only if \( |i - j| \in \{1, 4\} \). Notice that this is the structure described by Theorem 2. Hence if \( 5m \geq n_0 \), and \( F \) is extremal then \( F \) can be 5-partitioned.

If \( G \) can be 5-partitioned, then \( G \) is isomorphic to \( C_5^{k \times} \) by the minimality of \( k \), a contradiction. Therefore, \( G \) cannot be 5-partitioned.

Let \( H \) be an extremal graph on \( 5^6 > n_0 \) vertices. Blowing up every vertex of \( C_5^{k \times} \) by a factor of \( 5^6 \), and inserting \( H \) in every part, gives an extremal graph \( G_1 \) on \( 5^6 + \ell \) vertices by \( \ell \) applications of Theorem 2. On the other hand, the graph \( G_2 \) obtained by blowing up every vertex of \( G \) by a factor of \( 5^6 \), and inserting \( H \) in every part, contains at least as many \( C_5 \)'s as \( G_1 \),

\[
C(G_1) = 5^k \cdot C(H) + C(C_5^{k \times}) \cdot (5^6)^5, \quad C(G_2) = 5^6 \cdot C(H) + C(G) \cdot (5^6)^5,
\]

so \( C(G_1) \geq C(G_2) \). Hence \( G_2 \) must also be extremal. Therefore \( G_2 \) can be 5-partitioned into five sets \( X_1, X_2, X_3, X_4, X_5 \) with \( |X_i| = 5^6 + \ell \). In particular, two vertices in \( G_2 \) are in the same set \( X_i \) if and only if their adjacency pattern agrees on more than half of the remaining vertices. But this implies that for every copy \( H' \) of \( H \) inserted into the blow-up of \( G \), all vertices of \( H' \) are in the same \( X_i \), and thus the 5-partition of \( V(G_2) \) gives a 5-partition of \( V(G) \), a contradiction. \( \square \)

Acknowledgments

We would like to thank Jan Volec for fruitful discussions and two anonymous referees for a careful reading of the manuscript and several suggestions improving the write-up. Research of first author was partially supported by Simons Fellowship (263583), NSF CAREER Grant DMS-0745185, Arnold O. Beckman Research Award (UIUC Campus Research Board 13039) and Marie Curie FP7-PEOPLE-2012-IF 327763. Research of the third author was supported in part by NSF grant DMS-1266016. Research of fourth author was partially supported by a collaboration grant from the Simons Foundation (276726).

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