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A Multi-level Trend-Renewal Process for Modeling Systems with Recurrence Data

Zhibing Xu
Virginia Tech

Yili Hong
Virginia Tech

William Q. Meeker
Iowa State University, wqmeeker@iastate.edu

Brock E. Osborn
GE Global Research Center

Kati Illouz
GE Global Research Center

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A repairable system is a system that can be restored to an operational state after a repair event. The system may experience multiple events over time, which are called recurrent events. To model the recurrent event data, the renewal process (RP), the nonhomogeneous Poisson process (NHPP), and the trend-renewal process (TRP) are often used. Compared to the RP and NHPP, the TRP is more flexible for modeling, because it includes both RP and NHPP as special cases. However, for a multi-level system (e.g., system, subsystem, and component levels), the original TRP model may not be adequate if the repair is effected by a subsystem replacement and if subsystem level replacement events affect the rate of occurrence of the component-level replacement events. In this paper, we propose a general class of models to describe replacement events in a multilevel repairable system by extending the TRP model. We also develop procedures for estimation of model parameters and prediction of future events based on historical data. The proposed model and method are validated by simulation studies and are illustrated by an industrial application.

Keywords

multi-level repairable systems, Nonhomogeneous Poisson process, renewal process, random effect, time-dependent covariate, trend-renewal process

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Brock E. Osborn[3], and Kati Illouz[3]

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[1]Zhibing Xu and Yili Hong are with the Department of Statistics, Virginia Tech, Blacksburg, VA, 24061, USA (e-mail: xzb8382@vt.edu; yilihong@vt.edu)

[2]William Q. Meeker is with the Department of Statistics, Iowa State University, Ames, IA, 50011, USA (e-mail: wqmeeker@iastate.edu)

[3]Brock E. Osborn is formerly with and Kati Illouz is with the Applied Statistics Laboratory, GE Global Research Center, Niskayuna, NY, 12309, USA (e-mail: brockosborn@gmail.com; illouz@ge.com)

1 Introduction

1.1 Background

A repairable system is defined as a system that will be restored to an operational state after a repair. In practice, a repairable system may experience multiple replacement events at different levels over time. For example, we consider a repairable vehicle with three levels: system (e.g., a truck), subsystem (e.g., the truck engine), and component (e.g., the oil pump). The replacement events can be the replacement of the oil pump (may be new or refurbished) or the replacement of the entire engine (may be new or refurbished). For some other examples, the failure of a computer motherboard can be repaired by replacing the whole motherboard or just by replacing the failed capacitors on the motherboard. The failure of a gearbox can be repaired by replacing the whole gearbox or just by replacing the failed gear.

In this paper, we consider a two-level repairable system where repair events can occur at the subsystem level, or the component (within a subsystem) level. We focus on a specific subsystem (e.g., the engine) in a vehicle and a particular component within that subsystem (e.g., the oil pump), although a subsystem may have many components. In particular,

- The replacement of a subsystem is called a subsystem event. In this case, the system can only be fixed by a subsystem replacement.
- The replacement of a component is called a component event. In this case, the system is fixed by a component replacement.

Often, the failed subsystems or components are replaced with refurbished units that are not as good as new units. When the subsystem is replaced, of course, the components inside the subsystem will be replaced at the same time leading to a change of the risk of having a failure at the component level. This repair information at multiple levels is available through maintenance records. In addition to replacement event times, dynamic covariates, such as system usage information, loading, and shocks may also be available.

One important goal of the modeling of the replacement events is to do field failure prediction, which is useful for purposes such as prognostics, maintenance scheduling, and spare parts provisioning. Prediction of component replacement events is difficult when there are also subsystem replacements. The objective of this paper is to use replacement information at multiple levels and system usage information to make field failure predictions for a critical component in a subsystem. We need a model that can incorporate the effects of system usage information and other possibly unobservable factors, and the effects that replacements at different levels have on the component failure process. The model can also handle situations in which replacements may not be perfect and there are possible system-to-system differences.

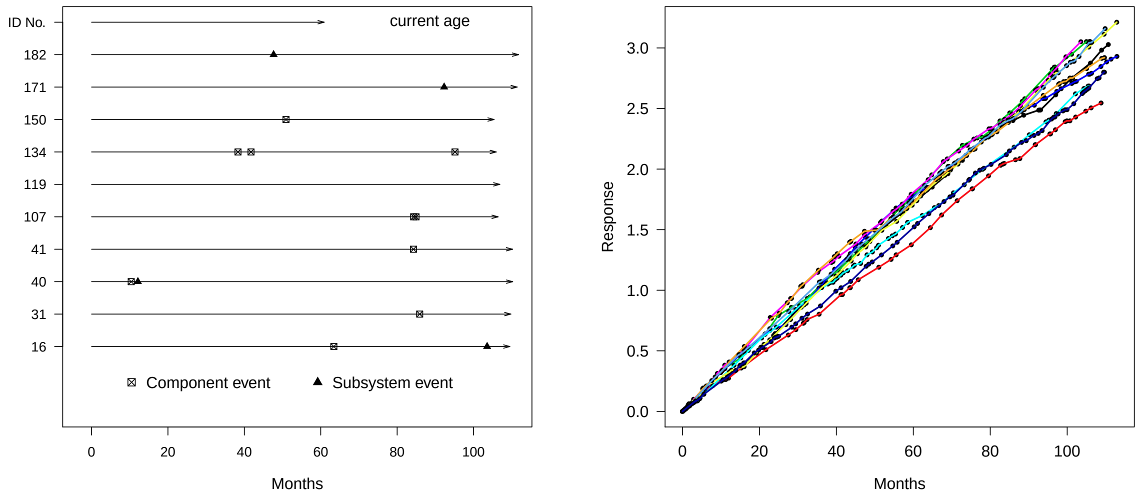
1.2 Motivating Application

This paper is motivated by the need to model recurrent events from a fleet of industrial systems, which we call Vehicle B. Vehicle B is a two-level repairable system, and it may experience subsystem (engine) and component (oil pump) events over time. To protect sensitive and proprietary information, names, the scales of time and the covariates have been changed in the following analysis. There are $n = 203$ systems being tracked in the field, and the data freeze date (DFD) is around 110 months after the first installation. The total number of component events and subsystem events are 219 and 44, respectively. The event histories of ten randomly selected units from the Vehicle B fleet are shown in Figure 1(a).

In addition to the event histories, several time-dependent covariates were also recorded. All the covariates, however, have a strong linear relationship to each other indicating that after using one covariate, there is little or no additional information in the others. Thus we used the cumulative usage covariate. Figure 1(b) shows the cumulative usage for ten randomly selected units from the Vehicle B fleet. Compared to the length of running times of the systems, the time needed to effect a repair is ignorable and is assumed to be zero. A prediction of the total number of component events in a future time period is needed in this application.

1.3 Related Literature and This Work

The nonhomogeneous Poisson process (NHPP) and the renewal process (RP) are the two most commonly used models in the analysis of recurrent event data (e.g., Zhao and Liu 2003, Leemis 2004, and Hong et al. 2013) with the assumption that the effect of repair is perfect or minimal, respectively. For general repairs, Brown and Proschan (1983) proposed an imperfect repair model. Kijima (1989) introduced two types of virtual age models by reducing the age of the system after each repair. Lawless and Thiagarajah (1996) used a proportional intensity model to incorporate renewals and time trends. Wang and Pham (1996) proposed a quasi renewal process with the consideration of maintenance cost. Doyen and Gaudoin (2004) proposed two new classes of imperfect models.



(a) Recurrent event processes for a subset of system units

(b) Cumulative usage processes for a subset of system units

Figure 1: Plots of event processes and cumulative usage processes for ten randomly selected units in the Vehicle B fleet.

Lindqvist et al. (2003) and Lindqvist (2006) introduced a trend-renewal process (TRP) which includes the NHPP and RP as special cases. The TRP model has been widely used in the literature (e.g., Yang et al. 2012, and Pietzner and Wienke 2013). Heggland and Lindqvist (2007) derived the non-parametric maximum likelihood estimator of the intensity function for the TRP. Franz et al. (2013) proposed methods for point prediction and interval prediction for the first time to failure using simulation. For virtual age models, Yańez et al. (2002) and Yu et al. (2013) proposed methods to estimate the expected number of failures using Monte Carlo simulation and an analytic approach, respectively.

For multi-level repairable system analysis, some papers have focused on the reliability analysis of a system by combining information from different levels using Bayesian methods. Examples include Johnson et al. (2005), Wilson et al. (2006), and Liu et al. (2011). We know of no previous work that has been done for reliability estimation and prediction for multi-level repairable systems with the consideration of the effect of subsystem events on component events.

Motivated by the Vehicle B application, we propose a multi-level trend renewal process (MTRP) with time-dependent covariates in the modeling of component events. Based on the MTRP model, we also develop procedures for obtaining point predictions and prediction intervals for the number of component events in a future time. To incorporate system-to-system variability, random effects are introduced in the MTRP model and the parameters are estimated by the Metropolis-within-Gibbs algorithm. Finite-sample properties of the estimators and prediction method are validated by simulation studies and illustrated with the Vehicle B application.

1.4 Overview

The rest of the paper is organized as follows. Section 2 introduces existing models and the proposed MTRP model. Section 3 develops estimation methods for the unknown parameters in the proposed MTRP model. Section 4 develops procedures for point predictions and prediction intervals (PI) based on Monte Carlo simulation. Section 5 validates the proposed methods by simulations. Section 6 illustrates the methods of modeling and predictions based on the Vehicle B application. Section 7 gives a summary and some related future research topics.

2 Repairable System Models

2.1 Existing Models

Let $0 < T_1 \cdots < T_i < \cdots$ be the event times from a repairable system. Let $N(t)$ denote the counting process for the number of events that occur in time interval $(0, t]$, and let $\mathcal{F}_t$ be the event history up to time t. The event intensity for the counting process is

$$\lambda(t|\mathcal{F}_{t^-}) = \lim_{\Delta t \to 0} \frac{\Pr\{N(t+\Delta t) - N(t) = 1|\mathcal{F}_{t^-}\}}{\Delta t},$$

where $\mathcal{F}_{t^-}$ is the event history immediately prior to time t. The cumulative event intensity function is defined as $\Lambda(t) = \int_0^t \lambda(u|\mathcal{F}_{u^-})\, du$.

The RP, denoted by $\text{RP}(F)$, corresponds to a perfect repair (i.e., replacement with a new unit), and the gaps between event times are independently and identically distributed (iid) with F. Here F is a cumulative distribution function (cdf). That is, $T_{i+1} - T_i \overset{\text{iid}}{\sim} F,\ i = 1, 2, \cdots$. Let $h(z)$ be the hazard function corresponding to F. The event intensity function of the RP is $\lambda(t|\mathcal{F}_{t^-}) = h[t - T_{N(t^-)}]$, where $T_{N(t^-)}$ is the last event time before time t. The NHPP corresponds to a minimal repair (e.g., adjustment or replacement of a small part of a large unit). The intensity function is $\lambda(t|\mathcal{F}_{t^-}) = \lambda(t)$, which does not depend on the event history. For the NHPP, the transformed event times $\Lambda(T_i)$ can be described by a homogeneous Poisson process (HPP) with a mean of one. The gaps between the transformed times are iid with an exponential distribution with mean one. That is, $\Lambda(T_{i+1}) - \Lambda(T_i) \overset{\text{iid}}{\sim} \text{Exp}(1),\ i = 1, 2, \cdots$.

The TRP model describes situations that are in-between NHPP and RP, and contains the NHPP and RP models as special cases. The gaps between the transformed event times are iid with RP(F). That is, $\Lambda(T_{i+1}) - \Lambda(T_i) \overset{\text{iid}}{\sim} F,\ i = 1, 2, \cdots$. We denote the TRP by TRP(F, λ), where $\lambda(t) = d\Lambda(t)/dt$ is called the trend function and F is called the renewal distribution function. The event intensity function is $\lambda(t|\mathcal{F}_{t^-}) = h\{\Lambda(t) - \Lambda[T_{N(t^-)}]\}\lambda(t)$. The trend function $\lambda(t)$ reflects system deterioration (or improvement) overtime, independent of replacement events or other repair-related events. The factor $h\{\Lambda(t) - \Lambda[T_{N(t^-)}]\}$ reflects the effect of the most recent repair at time $T_{N(t^-)}$. After each repair, there is a change in the event intensity function. The behavior of the change is determined by the hazard function of the renewal distribution function F.

2.2 Notation for Data

We consider a fleet of n multi-level repairable systems, which are under observation over the time interval $(0, \tau_i]$, where $i = 1, \cdots, n$. The subsystem consists of many components and we focus on the replacement of one particular critical component that had been carefully tracked. Let $N_i(t) = N_{is}(t) + N_{ic}(t)$ be the total number of replacement events up to time t, where $N_{is}(t)$ and $N_{ic}(t)$ are the number of subsystem events and the number of component events up to time t for system i, respectively. Let $0 < t^s_{i1} < \cdots < t^s_{i,N_{is}(\tau_i)} < \tau_i$ be the times for subsystem events, and let $0 < t^c_{i1} < \cdots < t^c_{i,N_{ic}(\tau_i)} < \tau_i$ be the times for component events. The replacement event times, regardless of the types, are denoted by $0 < t_{i1} < \cdots < t_{i,N(\tau_i)} < \tau_i$.

In the Vehicle B data, the time-dependent covariate (i.e., cumulative usage) at time t is denoted by $X_i(t)$ for system i, where $i = 1, \cdots, n$. The time-dependent covariate process for system i is denoted by $\boldsymbol{X}_i(t)$, where $\boldsymbol{X}_i(t) = \{X_i(u) : 0 < u \leq t\}$. The covariate process $\boldsymbol{X}_i(t)$ is recorded at time t_{ik}, where $k = 1, \cdots, m_i$, and m_i is the number of time points where the covariate information is available for system i before the end of observation τ_i.

With the consideration of the time-dependent covariate, the replacement events history up to time t is $\mathcal{F}_t = \{N_{ic}(u), N_{is}(u), X_i(u) : 0 < u \leq t\}$, and the history of subsystem events up to time t is $\mathcal{F}^s_t = \{N_{is}(u), X_i(u) : 0 < u \leq t\}$.

2.3 The Proposed Multi-level Trend-renewal Process

In a two-level repairable system i, the intensity functions for the subsystem and component level events are modeled as follows:

$$\text{Subsystem level: } \lambda_i^{s\star}(t|\mathcal{F}_{t^-}^s;\boldsymbol{\theta}^s) = h^{s\star}\{\Lambda_i^\star(t) - \Lambda_i^\star[t_{i,N_{is}(t^-)}^s];\boldsymbol{\theta}^s\}\lambda_i^\star(t;\boldsymbol{\theta}^s), \tag{1}$$

$$\text{Component level: } \lambda_i^c(t|\mathcal{F}_{t^-};\boldsymbol{\theta}^c) = h^c\left\{\Lambda_i^s(t|\mathcal{F}_{t^-}^s) - \Lambda_i^s\left[t_{i,N_i(t^-)}\,|\,\mathcal{F}^s_{t^-_{i,N_i(t^-)}}\right];\boldsymbol{\theta}^c\right\}\lambda_i^s(t|\mathcal{F}_{t^-}^s;\boldsymbol{\theta}^c). \tag{2}$$

In (1), we use a TRP model, TRP($F^{s\star}, \lambda_i^\star$), without random effects, to describe the subsystem-level events. The unknown model parameters in (1) are denoted by $\boldsymbol{\theta}^s$. Here, we use "$\star$" to denote the functions used in the model for subsystem replacement events. Let $F^{s\star}$ denote the renewal distribution for the subsystem-event process, $h^{s\star}(\cdot)$ denote the corresponding hazard function, and $\Lambda_i^\star(t) = \int_0^t \lambda_i^\star(u;\boldsymbol{\theta}^s)du$ denote the cumulative intensity function. The function $\lambda_i^\star(t;\boldsymbol{\theta}^s) = \lambda_b^\star(t)\exp\{\kappa g[X_i(t)]\}$ is the intensity trend function for system i with $\lambda_b^\star(t)$ as the baseline intensity function and κ as the coefficient of a transformed function of the time-dependent covariate (i.e., $g[X_i(t)]$). In the rest of this paper, we use $g[X_i(t)] = \log[X_i(t)]$ as the function of the time-dependent covariate.

The proposed MTRP model for component events in (2) is an extension of the TRP in the sense that we use an additional trend function $\lambda_i^s(t|\mathcal{F}_{t^-}^s;\boldsymbol{\theta}^c)$ for the component-event process that can incorporate the effect of subsystem events on the component events, because the intensity of component events may be affected by the subsystem events. In particular,

$$\lambda_i^s(t|\mathcal{F}_{t^-}^s;\boldsymbol{\theta}^c) = h^s\{\Lambda_i(t) - \Lambda_i[t_{i,N_{is}(t^-)}^s]\}\lambda_i(t;\boldsymbol{\theta}^c), \tag{3}$$

where $\boldsymbol{\theta}^c = (\theta_1^c,\cdots,\theta_p^c)'$ denotes a vector of unknown parameters with length of p in (2). The function $h^s(\cdot)$ in (3) describes the effect that subsystem events have on the intensity of component events. As in (1), the form of $h^s(\cdot)$ can be taken to be a hazard function, and its corresponding cdf form is $F^s(\cdot)$. The function $\lambda_i(t;\boldsymbol{\theta}^c)$ describes the effect of covariate and other unknown factors on the component intensity function. Here, $\Lambda_i(t) = \int_0^t \lambda_i(u;\boldsymbol{\theta}^c)\,du$.

The renewal distribution function of the component-event model in (2) is denoted by $F^c(\cdot)$. We use $f^c(t)$, $S^c(t) = 1 - F^c(t)$, and $h^c(t)$ to denote, respectively, the probability density function (pdf), survival function, and hazard function corresponding to F^c. Also, let $\Lambda_i^s(t|\mathcal{F}_{t^-}^s) = \int_0^t \lambda_i^s(u|\mathcal{F}_{u^-}^s;\boldsymbol{\theta}^c)\,du$, and $\Lambda_i(t) = \int_0^t \lambda_i(u;\boldsymbol{\theta}^c)\,du$.

Note that $\lambda_i^s(t|\mathcal{F}_{t^-}^s;\boldsymbol{\theta}^c)$ in (3) has the same parametric form of the TRP intensity model for subsystem events [i.e., $\lambda_i^{s\star}(t|\mathcal{F}_{t^-}^s;\boldsymbol{\theta}^s)$ in (1)], but with a different set of parameters $\boldsymbol{\theta}^c$, instead of $\boldsymbol{\theta}^s$. The trend function $\lambda_i^s(t|\mathcal{F}_{t^-}^s;\boldsymbol{\theta}^c)$ in (3) reflects the effect that subsystem events have on the component-event intensity. For example, when $h^s(\cdot)$ is a constant function, subsystem events have no effect on components [e.g., Figure 2(a)]; when $\lambda_i(t;\boldsymbol{\theta}^c)$ in (3) is a constant function over time and $h^s(\cdot)$ is not a constant function, a subsystem event corresponds to a perfect repair and results in an immediate reduction of the intensity function [e.g., Figure 2(b)]; when neither $\lambda_i(t;\boldsymbol{\theta}^c)$ nor $h^s(\cdot)$ is a constant function, the subsystem events are imperfect repairs [e.g., Figure 2(c)]. Thus, (3) is a flexible trend function for describing the effect that subsystem events have on the component-event intensity. Figure 3 illustrates the intensity function of component events in the MTRP model (2) for a simulated event history. In Figure 3, the intensity function of component events is affected by both the subsystem and component events.

Similar to the subsystem-event process, the incorporation of the time-dependent covariate can be achieved by

$$\lambda_i(t;\boldsymbol{\theta}^c) = \lambda_b(t)\exp\{\gamma\log[X_i(t)]\} \quad i = 1,\cdots,n. \tag{4}$$

Here, $\lambda_b(t)$ denotes the baseline intensity trend function if no component/subsystem replacement events occur, and γ is the coefficient of the function of the time-dependent covariate. The MTRP with a time-dependent covariate can be denoted by MTRP(F^c, F^s, λ_i). We want to point out that model (4) provides a flexible way of including time-dependent covariates into the event intensity function. Also, model (4) can be generalized, without difficulty, to include multiple time-dependent covariates.

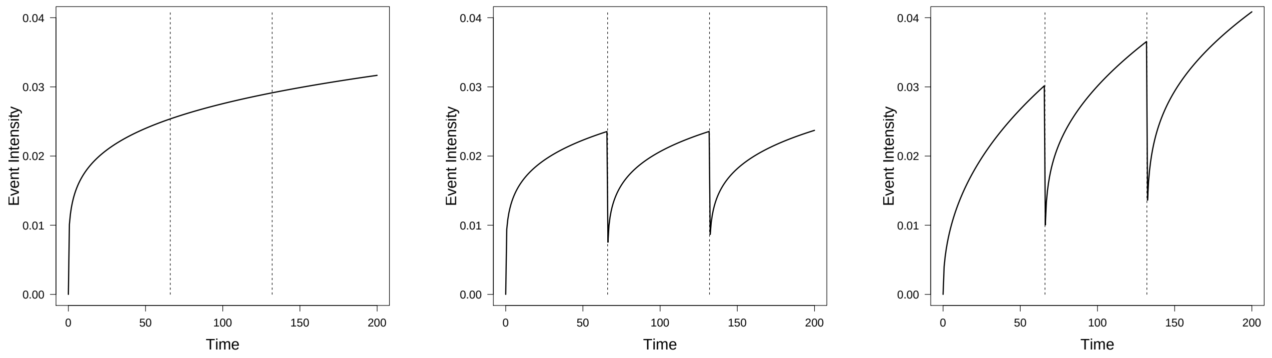
To incorporate unit-to-unit variability in a component-event process, we use random effects in the intensity function (4) as follows,

$$\lambda_i(t;\boldsymbol{\theta}^c) = \lambda_b(t)\exp\{\gamma\log[X_i(t)] + w_i\} \quad i = 1,\cdots,n, \tag{5}$$

where the random effect for system i, w_i, is iid with $\mathrm{N}(0,\sigma_r^2)$. Here, σ_r^2 is the variance of the normal distribution, and is not contained in $\boldsymbol{\theta}^c$. Define $\boldsymbol{w} = (w_1, w_2, \cdots, w_n)'$. The heterogeneous MTRP for component events in system i is denoted by HMTRP(F^c, F^s, λ_i).

2.4 Properties and Special Cases of MTRP

From another perspective, the MTRP model in (2) includes two TRP models in a hierarchical structure. The higher-level TRP model is used to describe the effect of subsystem events on



(a) Minimal repair effects (b) Perfect repair effects (c) Imperfect repair effects

Figure 2: Different cases of the trend function in (3). The vertical dotted lines indicate the occurrence of subsystem event.

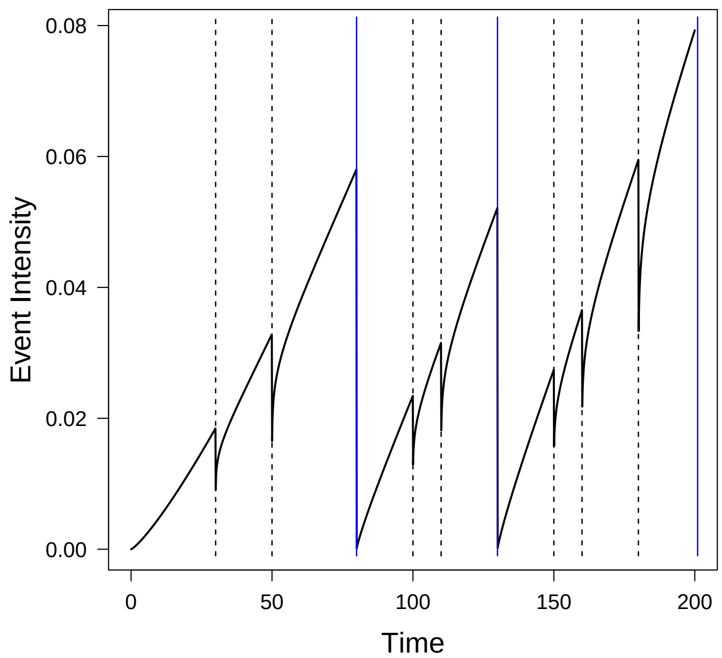


Figure 3: Illustration of the intensity function of component events in the MTRP model (2) for a simulated event history. The vertical solid (dashed) lines indicate the subsystem (component) event times.

the intensity of component events. The lower-level TRP model is used to model the component events with the higher-level TRP intensity function as its trend function. Thus, the MTRP can be defined as,

$$\Lambda_i(T^s_{i,j+1}) - \Lambda_i(T^s_{ij}) \overset{\text{iid}}{\sim} F^s(\cdot),$$

$$\Lambda^s_i(T^c_{ij}|\mathcal{F}^s_{T^c_{ij}-}) - \Lambda^s_i\left[T_{i,N_i(T^c_{ij}-)}\,|\,\mathcal{F}^s_{T^-_{i,N_i(T^c_{ij}-)}}\right] \overset{\text{iid}}{\sim} F^c(\cdot), \quad i = 1, \cdots, n, \quad j = 1, 2, \cdots .$$

For system i, T^c_{ij} is component event time j, and $T_{N_i(T^c_{ij}-)}$ denotes the most recent event time before T^c_{ij}. The cumulative intensity function for component events is computed as

$$\begin{aligned}\Lambda^c_i(t|\mathcal{F}_{t^-}) =& \int_0^t \lambda^c_i(u|\mathcal{F}_{u^-};\boldsymbol{\theta}^c)\, du\\ =& H^c\left\{\Lambda^s_i(t|\mathcal{F}^s_{t^-}) - \Lambda^s_i[t_{i,N_i(t^-)}|\mathcal{F}^s_{t^-_{i,N(t^-)}}]\right\} + \sum_{j=1}^{N_i(t^-)} H^c\left[\Lambda^s_i(t_{ij}|\mathcal{F}^s_{t^-_{ij}}) - \Lambda^s_i(t_{i,j-1}|\mathcal{F}^s_{t^-_{i,j-1}})\right],\end{aligned}$$

where $H^c(t) = \int_0^t h^c(u)\, du$ is the cumulative hazard function corresponding to F^c.

The MTRP model is a general model that includes the TRP, RP, NHPP and HPP models as special cases. Here, the application of the special cases, TRP, RP, NHPP, and HPP models, are slightly different from the usual application of these models because subsystem events induce censoring during system operation. For example, if a subsystem is replaced, then observation is terminated on the components in that subsystem. The intensity function (2) of the MTRP model reduces to the TRP model if $h^s(\cdot)$ is a constant function indicating no subsystem repair effect (i.e., only minimal repair effects from subsystem events). When both $h^s(\cdot)$ and $\lambda_i(t;\boldsymbol{\theta}^c)$ are constant functions, the MTRP model reduces to the RP model, indicating that subsystem events have no effect on component events, and component events are perfect repairs. When $h^c(\cdot)$, and $h^s(\cdot)$ are constant functions and $\lambda_i(t;\boldsymbol{\theta}^c)$ is a function of t, the MTRP model reduces to an NHPP model, indicating component events are minimal repairs and there are no subsystem-event effects. When $F^c(\cdot)$ is an Exp(1) distribution and both $h^s(\cdot)$ and $\lambda_i(t;\boldsymbol{\theta}^c)$ are constant functions, the MTRP model reduces to the HPP model. The likelihood ratio test can be used in the comparison and selection of these nested models. More details on the subject of model selection in the context of the TRP model can be found in Lindqvist et al. (2003).

3 Parameter Estimation

The estimate of parameter $\boldsymbol{\theta}^s$ in (1) can be obtained by using the method of Lindqvist et al. (2003). So we focus on the estimation of the parameters in the component-event model in (2).

3.1 The Likelihood Function

Note that component events for system i follow MTRP(F^c, F^s, λ_i). For convenience in the expression of the likelihood function, the component events are denoted by $\{t_{ij}, \delta^c_{ij}\}$, where $i = 1, \cdots, n$, and $j = 1, \cdots, N_i(\tau_i)$. Here, t_{ij} is the event time for system i, and δ^c_{ij} is the component-event indicator. If the replacement is for a component, then the indicator is equal to one, and zero otherwise. Let $t_{i0} = 0$, $t_{i,N_i(\tau_i)+1} = \tau_i$, $\Lambda^c_i(0|\mathcal{F}_0) = \Lambda^s_i(0|\mathcal{F}^s_0) = 0$, $\delta^c_{i,N_i(\tau_i)+1} = 0$, and $\mathcal{F} = \{N_{ic}(u), N_{is}(u), X_i(u) : 0 < u \le \tau_i,\ i = 1, \cdots, n\}$. The likelihood function for component events in the MTRP model can be expressed as

$$
\begin{aligned}
L(\boldsymbol{\theta}^c; \mathcal{F}) =& \prod_{i=1}^{n} \left(\left\{ \prod_{j=1}^{N_i(\tau_i)+1} [\lambda^c_i(t_{ij}|\mathcal{F}_{t^-_{ij}}; \boldsymbol{\theta}^c)]^{\delta^c_{ij}} \right\} \times \exp[-\Lambda^c_i(\tau_i|\mathcal{F}_{\tau^-_i})] \right) \\
=& \prod_{i=1}^{n} \prod_{j=1}^{N_i(\tau_i)+1} \left(\left\{ f^c[\Lambda^s_i(t_{ij}|\mathcal{F}^s_{t^-_{ij}}) - \Lambda^s_i(t_{i,j-1}|\mathcal{F}^s_{t^-_{i,j-1}})]\lambda^s_i(t_{ij}|\mathcal{F}^s_{t^-_{ij}}; \boldsymbol{\theta}^c) \right\}^{\delta^c_{ij}} \right. \\
& \left. \times \left\{ S^c[\Lambda^s_i(t_{ij}|\mathcal{F}^s_{t^-_{ij}}) - \Lambda^s_i(t_{i,j-1}|\mathcal{F}^s_{t^-_{i,j-1}})] \right\}^{1-\delta^c_{ij}} \right). \qquad (6)
\end{aligned}
$$

3.2 Estimation Procedure

We first discuss the estimation procedure for the unknown parameters in the model with random effects (i.e., HMTRP). Bayesian methods using diffuse prior distributions provide a convenient method to obtain the estimates of the unknown parameters. We suggest a Metropolis-within-Gibbs algorithm because some conditional distributions do not have closed forms. Define $\upsilon = 1/\sigma^2_r$ as the precision of the random effects distribution, and let $\upsilon \sim$ Gamma(a_1, a_2) be the conjugate prior distribution for the random effects. Here, a_1 is the shape parameter and a_2 is the rate parameter of a gamma distribution. Gelman (2006) and DePalma (2013) suggested using 0.001 for both a_1 and a_2. Then the mean and variance of the prior distribution of υ are 1 and 1000, respectively. We use a uniform distribution to describe the prior information on $\boldsymbol{\theta}^c$. Define $L_i(\boldsymbol{\theta}^c|\mathcal{F}_{\tau_i}, w_i)$ as the conditional likelihood function of system i given random effect w_i, and $L(\boldsymbol{\theta}^c|\mathcal{F}, \boldsymbol{w}) = \prod_{i=1}^{n} L_i(\boldsymbol{\theta}^c|\mathcal{F}_{\tau_i}, w_i)$ as the conditional likelihood function for all n systems. Then, the pdf of the full joint distribution of parameters in the MTRP model is

$$P(\boldsymbol{\theta}^c, \boldsymbol{w}, \upsilon|\mathcal{F}) \propto L(\boldsymbol{\theta}^c|\mathcal{F}, \boldsymbol{w})P(\boldsymbol{w}|\upsilon)P(\upsilon), \tag{7}$$

where $P(\boldsymbol{w}|\upsilon)$ is the pdf of a multivariate normal distribution with mean $\mathbf{0}$ and variance-covariance matrix $\boldsymbol{\Sigma}_r$. That is $\boldsymbol{w} \sim \mathrm{N}(\mathbf{0}, \boldsymbol{\Sigma}_r = \boldsymbol{I}/\upsilon)$, where $\boldsymbol{I}$ is an $n \times n$ identity matrix. The pdf of Gamma(a_1, a_2) is denoted by $P(\upsilon)$. Based on the full joint distribution, we can obtain the joint posterior distribution of the parameters in the model as follows,

$$w_i|\boldsymbol{\theta}^c, \upsilon \propto L_i(\boldsymbol{\theta}^c|\mathcal{F}_{\tau_i}, w_i)\upsilon^{1/2}\exp\left(-\frac{\upsilon w_i^2}{2}\right) \tag{8}$$

$$\upsilon|\boldsymbol{w}, \boldsymbol{\theta}^c \propto \mathrm{Gamma}\left(\frac{n}{2} + a_1, \frac{\boldsymbol{w}'\boldsymbol{w}}{2} + a_2\right) \tag{9}$$

$$\boldsymbol{\theta}^c|\boldsymbol{w}, \upsilon \propto L(\boldsymbol{\theta}^c|\mathcal{F}, \boldsymbol{w})\,. \tag{10}$$

Algorithm 1:

1. Initialize all the parameters $\boldsymbol{w}^{(0)}$, $\upsilon^{(0)}$ and $\boldsymbol{\theta}^{c(0)}$;

2. Update $w_i^{(j)}$, $i = 1, \cdots, n$ using Metropolis algorithm at step j:

   a) Sample $w_i^* \sim \mathrm{N}(w_i^{(j-1)}, \sigma_{w_i}^2)$, where $\sigma_{w_i}^2$ is the variance of the proposed distribution for unit i;

   b) Accept w_i^* as $w_i^{(j)}$ with the probability
   $\min\left\{\frac{L_i\left(\boldsymbol{\theta}^{c(j-1)}|\mathcal{F}_{\tau_i}, w_i^*\right)\exp\left(-\frac{\upsilon^{(j-1)}w_i^{*2}}{2}\right)}{L_i\left(\boldsymbol{\theta}^{c(j-1)}|\mathcal{F}_{\tau_i}, w_i^{(j-1)}\right)\exp\left(-\frac{\upsilon^{(j-1)}\left[w_i^{(j-1)}\right]^2}{2}\right)}, 1\right\}$, otherwise, set $w_i^{(j)} = w_i^{(j-1)}$.

3. Sample $\upsilon^{(j)} \sim \mathrm{Gamma}\left(n/2 + a_1, \boldsymbol{w}^{(j)\prime}\boldsymbol{w}^{(j)}/2 + a_2\right)$;

4. Update values of elements of $\boldsymbol{\theta}^{c(j)} = (\theta_1^{c(j)}, \cdots, \theta_i^{c(j)}, \cdots, \theta_p^{c(j)})'$ successively at step j. Let $\boldsymbol{\theta}_{i-1}^{c(j)} = (\theta_1^{c(j)}, \cdots, \theta_{i-1}^{c(j)}, \theta_i^{c(j-1)}, \cdots, \theta_p^{c(j-1)})'$. Note that $\boldsymbol{\theta}_0^{c(j)} = \boldsymbol{\theta}^{c(j-1)}$.

   a) Sample $\theta_i^{c*} \sim \mathrm{N}(\theta_i^{c(j-1)}, \sigma_{\theta_i^c}^2)$, where $\sigma_{\theta_i^c}^2$ is the variance of the proposed distribution. Let $\boldsymbol{\theta}_i^{c*} = (\theta_1^{c(j)}, \cdots, \theta_{i-1}^{c(j)}, \theta_i^{c*}, \theta_{i+1}^{c(j-1)}, \cdots, \theta_p^{c(j-1)})'$.

   b) Accept θ_i^{c*} as $\theta_i^{c(j)}$ in $\boldsymbol{\theta}_i^{c(j)}$ with the probability
   $\min\left\{\frac{L[\boldsymbol{\theta}_i^{c*}|\mathcal{F}, \boldsymbol{w}^{(j)}]}{L[\boldsymbol{\theta}_{i-1}^{c(j)}|\mathcal{F}, \boldsymbol{w}^{(j)}]}, 1\right\}$, otherwise, set $\theta_i^{c(j)} = \theta_i^{c(j-1)}$ in $\boldsymbol{\theta}_i^{c(j)}$.

c) Repeat steps a) and b) for $i = 1, \cdots, p$ at the given j. Let $\boldsymbol{\theta}^{c(j)} = \boldsymbol{\theta}_p^{c(j)}$.

5. Repeat steps 2-4 until a large number (e.g., 10,000) draws from the joint posterior distribution have been obtained.

To achieve optimal acceptance rates (around .44 according to Gelman et al. 1997 and Roberts and Rosenthal 2001), the tuning parameters σ_{w_i} and $\sigma_{\theta_i^c}$, $i = 1, \cdots, p$ can be adjusted by applying the method given in Roberts and Rosenthal (2009).

For a model without random effects (MTRP), **Algorithm 1** can still be used by omitting steps 1-3. With no random effect parameters, however, it is straightforward to estimate the unknown parameters by using the maximum likelihood (ML) method based on the likelihood function (6).

Once the parameters estimates in the MTRP/HMTRP model are obtained, the residuals of the model can be estimated by using the cumulative hazard function. Specifically, the residuals can be estimated by evaluating $R_{ij} = H^c[\Lambda_i^s(t_{ij}|\mathcal{F}^s_{t_{ij}^-}) - \Lambda_i^s(t_{i,j-1}|\mathcal{F}^s_{t_{i,j-1}^-})]$ using the values of the parameter estimates. The residuals $(R_{ij}, \delta_{ij}^c = 1)$ are expected to behave as samples from the Exp(1) distribution, which can be used to evaluate the goodness of fit of the model.

4 Prediction for Component Events

4.1 Point Prediction

Accurate prediction of future events is important to product manufacturers who provide service contracts, or to the operators of fleets of systems, for purposes of controlling operating costs, optimizing the number of spare components, and assessing the risk of excessive repair expenses or warranty returns. Here, we focus on the prediction of events at the component level.

Let $\boldsymbol{\theta}^x$ denote the parameters in the model for the time-dependent covariate, and let $\boldsymbol{X}_i(t_1, t_2) = \{X_i(t);\ t_1 < t \leq t_2\}$. The predicted cumulative number of component events in the future time period t^* for a fleet of n units can be obtained by:

$$\begin{aligned} N_c(t^*; \boldsymbol{\theta}^c, \boldsymbol{\theta}^s, \boldsymbol{\theta}^x) &= \sum_{i=1}^{n} N_{ic}(t^*; \boldsymbol{\theta}^c, \boldsymbol{\theta}^s, \boldsymbol{\theta}^x) \\ &= \sum_{i=1}^{n} \mathbf{E}_{\boldsymbol{X}_i(\tau_i, \tau_i+t^*)|\boldsymbol{X}(\tau_i)} \mathbf{E}_{w_i} \big\{ N_{ic}[t^*, \boldsymbol{X}_i(\tau_i, \tau_i + t^*), w_i; \boldsymbol{\theta}^c, \boldsymbol{\theta}^s, \boldsymbol{\theta}^x] \big\}, \end{aligned} \quad (11)$$

where $N_{ic}(t^*; \boldsymbol{\theta}^c, \boldsymbol{\theta}^s, \boldsymbol{\theta}^x)$ is the predicted cumulative number of component events in system i in the future time period t^*. Because the trend function of the component-event intensity depends on the history of subsystem events, the component-event model will also depend on the subsystem-event model. Hence, the prediction of component events will depend on the parameter vector of the subsystem-event model, $\boldsymbol{\theta}^s$. Because a closed form for (11) is not available, numerical methods or Monte Carlo simulation must be used. For prediction in the general recurrent process, Monte Carlo simulation is more common and easier for computation as illustrated in Yańez et al. (2002) and Franz et al. (2013).

The simulation of $N_c(t^*; \boldsymbol{\theta}^c, \boldsymbol{\theta}^s, \boldsymbol{\theta}^x)$ can be achieved by using the following steps.

- Simulate the time-dependent covariates;
- Simulate the subsystem events with the TRP model;
- Simulate the component events with the MTRP/HMTRP model.

4.2 Prediction for the Time-Dependent Covariate

To predict future recurrent events for a system with a time-dependent covariate, it is necessary to have a parametric model for the covariate process. Based on the covariate pattern shown in Figure 1(b), we use a linear mixed effects model to describe the dynamic covariate data. In particular,

$$X_i(t_{ik}) = t_{ik}(\beta_x + \nu_i) + \epsilon_i(t_{ik}) \quad i = 1, \cdots, n, \quad k = 1, \cdots, m_i, \quad (12)$$

where β_x is the coefficient of time, ν_i is the random effect, and $\epsilon_i(t_{ik})$ is the error term. We assume that $\nu_i \overset{\text{iid}}{\sim} \text{N}(0, \sigma_\nu^2)$, and $\epsilon_i(t_{ik}) \overset{\text{iid}}{\sim} \text{N}(0, \sigma_x^2)$ is independent of ν_i. The parameters in (12) are denoted by $\boldsymbol{\theta}^x = (\beta_x, \sigma_\nu, \sigma_x)'$. The estimation of $\boldsymbol{\theta}^x$ in the covariate model can be accomplished by using existing software packages (e.g., using the R function `lme`).

We use an approach that is similar to that used by Hong and Meeker (2013) for the covariate prediction for a different kind of failure-time model. Let $\boldsymbol{t}_i = (t_{i1}, \cdots, t_{im_i})'$,

$\boldsymbol{t}_{it^*} = (t_{i,m_{i+1}}, \cdots, t_{i,m_i+z_i})'$ be the observed time points before τ_i and the predicted time points during $(\tau_i, \tau_i + t^*]$, respectively. Let $\boldsymbol{X}_i(\boldsymbol{t}_i) = [X_i(t_{i1}), \cdots, X_i(t_{im_i})]'$, $\boldsymbol{X}_i(\boldsymbol{t}_{it^*}) = [X_i(t_{i,m_{i+1}}), \cdots, X_i(t_{i,m_i+z_i})]'$ be the corresponding time-dependent covariate processes. Here, z_i is the number of predicted time points for system i. The joint distribution of $\boldsymbol{X}_i(\boldsymbol{t}_i)$ and $\boldsymbol{X}_i(\boldsymbol{t}_{it^*})$ can be expressed as

$$\begin{bmatrix} \boldsymbol{X}_i(\boldsymbol{t}_i) \\ \boldsymbol{X}_i(\boldsymbol{t}_{it^*}) \end{bmatrix} \sim \mathrm{N}\left[\begin{pmatrix} \boldsymbol{t}_i \\ \boldsymbol{t}_{it^*} \end{pmatrix}\beta_x, \begin{pmatrix} \boldsymbol{\Sigma}_{i11} & \boldsymbol{\Sigma}_{i12} \\ \boldsymbol{\Sigma}_{i21} & \boldsymbol{\Sigma}_{i22} \end{pmatrix}\right],$$

where $\boldsymbol{\Sigma}_{i11} = \sigma_\nu^2 \boldsymbol{t}_i \boldsymbol{t}_i' + \sigma_x^2 \boldsymbol{I}_{m_i}$, $\boldsymbol{\Sigma}_{i22} = \sigma_\nu^2 \boldsymbol{t}_{it^*} \boldsymbol{t}_{it^*}' + \sigma_x^2 \boldsymbol{I}_{z_i}$, and $\boldsymbol{\Sigma}_{i12} = \sigma_\nu^2 \boldsymbol{t}_i \boldsymbol{t}_{it^*}'$. Here, $\boldsymbol{I}_{m_i}$ and $\boldsymbol{I}_{z_i}$ are $m_i \times m_i$ and $z_i \times z_i$ identity matrices. The conditional distribution of $\boldsymbol{X}_i(\boldsymbol{t}_i)|\boldsymbol{X}_i(\boldsymbol{t}_{it^*})$ is

$$\mathrm{N}\left(\boldsymbol{t}_{it^*}\beta_x + \boldsymbol{\Sigma}_{i21}\boldsymbol{\Sigma}_{i11}^{-1}[\boldsymbol{X}_i(\boldsymbol{t}_i) - \boldsymbol{t}_i\beta_x],\ \boldsymbol{\Sigma}_{i22} - \boldsymbol{\Sigma}_{i21}\boldsymbol{\Sigma}_{i11}^{-1}\boldsymbol{\Sigma}_{i12}\right). \tag{13}$$

The derivation of (13) is given in Appendix A. Based on (13), the time-dependent covariate processes can be predicted.

4.3 Subsystem Event Simulations

Because the model for component events depends on the history of subsystem events, the simulation of subsystem events is needed in the prediction of component events. Let $\varsigma_i = \tau_i + t^*$ be the prediction ending time of system i, $\widehat{F}^{s\star}$ be the estimate of renewal distribution function $F^{s\star}$, $\widehat{\Lambda}_i^\star$ be the estimate of $\Lambda_i^\star$, and $\widehat{\Lambda}_i^{\star-1}(\cdot)$ be the corresponding inverse function given $\widehat{\boldsymbol{\theta}}^s$ and $\widehat{\boldsymbol{\theta}}^x$. Here, $\widehat{\boldsymbol{\theta}}^s$ and $\widehat{\boldsymbol{\theta}}^x$ are ML estimates of $\boldsymbol{\theta}^s$ and $\boldsymbol{\theta}^x$, respectively. Based on the definition of the TRP model, the gaps between two consecutive transformed subsystem event times follow distribution $F^{s\star}$. That is, $\Lambda_i^\star(t_{i,j+1}^s) - \Lambda_i^\star(t_{ij}^s) \overset{\text{iid}}{\sim} F^{s\star}$, where $i = 1, \cdots, n$ and $j = 1, 2, \cdots$. The subsystem events can be simulated as follows.

Algorithm 2

1. Simulate a realization of $\boldsymbol{X}_i(\boldsymbol{t}_{it^*})$, the ith time-dependent covariate process, based on $\widehat{\boldsymbol{\theta}}^x$ using the conditional distribution (13).

2. Compute $\widehat{\Lambda}_i^\star(\varsigma_i)$ as the prediction ending time for unit i.

3. Generate a sequence of random variables U_{ij} from distribution $\widehat{F}^{s\star}$ and obtain the sequence of simulated event times in a transformed time scale, $T_{ij}^* = \widehat{\Lambda}_i^\star[t_{i,N_{is}(\tau_i)}^s] + \sum_{k=1}^j U_{ik}$, $j = 1, \cdots, C_i^s$, until $T_{i,C_i^s+1}^* > \widehat{\Lambda}_i^\star(\varsigma_i)$. Here, $T_{ij}^*, j = 1, \cdots, C_i^s$ are the event times in

the transformed time scale according to the RP($F^{s\star}$) model. Then, C_i^s is the random number of simulated subsystem events for unit i.

4. Compute the simulated subsystem event times $T_{ij}^s = \widehat{\Lambda}_i^{\star -1}(T_{ij}^*),\ j = 1, \cdots, C_i^s$.

5. Repeat steps 1-4 for each system i, where $i = 1, \cdots, n$.

Note that in step 3, the time of the first simulated subsystem event T_{i1}^s should be larger than τ_i, because the simulation is conditioned on the history. Otherwise it needs to be re-simulated.

4.4 Computation of Point Predictions

According to the definition of the MTRP model, the gaps (i.e., $d_1, d_2, \cdots$) between the component event and the most recent event (either component event or subsystem event) in the transformed time scale [i.e., $\Lambda^s(t)$] follow the F^c distribution as shown in Figure 4. Because the component-event process is censored by the subsystem events, the component events can be simulated in the intervals (i.e., $I_1, I_2, ...$ in Figure 4) of the subsystem events under the transformed time [i.e., $\Lambda^s(t)$] scale. In particular, for each system,

$$\Lambda_i^s(T_{ij}^c|\mathcal{F}_{T_{ij}^c-}^s) - \Lambda_i^s\left[T_{i,N_i(T_{ij}^c-)}\,|\,\mathcal{F}_{T^-_{i,N_i(T_{ij}^c-)}}^s\right] \overset{\text{iid}}{\sim} F^c(\cdot),$$

where $i = 1, \cdots, n$ and $j = 1, 2, \cdots$. Let $\widehat{\boldsymbol{\theta}}^c$ and $\widehat{\upsilon}$ be the mean of posterior distributions of $\boldsymbol{\theta}^c$ and υ, respectively. The prediction of the cumulative number of component events, $\widehat{N}_c(t^*; \widehat{\boldsymbol{\theta}}^c, \widehat{\boldsymbol{\theta}}^s, \widehat{\boldsymbol{\theta}}^x)$, can be computed by using the following algorithm.

Algorithm 3

1. Repeat **Algorithm 2** steps 1-4. Then the predicted covariate process and simulated times of subsystem events for system i are obtained. The simulated subsystem events times are denoted by $t_{i,N_{is}(\tau_i)+1}^s, \cdots, t_{i,N_{is}(\tau_i)+C_i^s}^s$. Here C_i^s is the simulated number of subsystem events. Set $t_{i,N_{is}(\tau_i)+0}^s = t_{i,N_{is}(\tau_i)}^s$ and $t_{i,N_{is}(\tau_i)+C_i^s+1}^s = \varsigma_i$.

2. The random effect w_i can be obtained by the Metropolis algorithm. The conditional pdf of w_i is proportional to the product of $L_i\left(\widehat{\boldsymbol{\theta}}^c|\mathcal{F}_t, w_i\right)$ and $P(w_i|\widehat{\boldsymbol{\theta}}^c, \widehat{\upsilon})$. For the MTRP model without random effects, this step can be skipped.

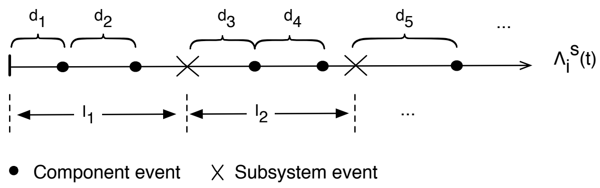


Figure 4: Illustration of the component event simulation.

3. Calculate the simulated time intervals which are separated by the simulated subsystem events for system i in the transformed time scale. That is $I_{ik} = \widehat{\Lambda}_i^s\big[\widehat{\Lambda}_i(t^s_{i,N_{is}(\tau_i)+k})\big] - \widehat{\Lambda}_i^s\big[\widehat{\Lambda}_i(t^s_{i,N_{is}(\tau_i)+k-1})\big],\ k = 1, \cdots, C_i^s + 1$, where $\widehat{\Lambda}_i^s(\cdot)$ and $\widehat{\Lambda}_i(\cdot)$ are the estimates of $\Lambda_i^s(\cdot)$ and $\Lambda_i(\cdot)$ based on the estimates $\widehat{\boldsymbol{\theta}}^c$ and $\widehat{\boldsymbol{\theta}}^x$, respectively.

4. For each simulated time interval, generate random variables U_{ikl} from distribution $\widehat{F}^c$, while $\sum_l U_{ikl} \leq I_{ik}$. The number of generated U_{ikl} values is recorded as C_{ik}^c.

5. Calculate the number of simulated component events in system i. That is $C_i^c = \sum_{k=1}^{C_i+1} C_{ik}^c$.

6. Repeat steps 1-5 for each system i, where $i = 1, \cdots, n$.

7. Repeat steps 1-6 B times and a series of $C_i^{c(b)},\ b = 1, \cdots, B,\ i = 1, \cdots, n$ are obtained. Then, the point prediction for the number of events between τ_i and $\tau_i + t^*$ is $\widehat{N}_c(t^*; \widehat{\boldsymbol{\theta}}^c, \widehat{\boldsymbol{\theta}}^s, \widehat{\boldsymbol{\theta}}^x) = \sum_{i=1}^n \sum_{b=1}^B C_i^{c(b)}/B$.

Note that in step 4 the time of the first simulated component event should be larger than τ_i. Otherwise it needs to be re-simulated.

4.5 Prediction Interval Computing

In order to obtain PIs for the cumulative number of events, we also need to take into account the distribution of estimated parameters as well as the uncertainty in the future values of the time-dependent covariate. Instead of sampling the posterior distributions, we use a multivariate normal distribution to approximate the distributions of the parameter estimators for fast computing. We focus on the PI for the cumulative number of component events. The algorithm is described as follows.

Algorithm 4

1. Simulate $\widehat{\boldsymbol{\theta}}^{x*}$, $\widehat{\boldsymbol{\theta}}^{s*}$, $\widehat{\boldsymbol{\theta}}^{c*}$, and $\widehat{\upsilon}^*$ from $\mathrm{N}(\widehat{\boldsymbol{\theta}}^{x}, \widehat{\boldsymbol{\Sigma}}_{\widehat{\boldsymbol{\theta}}^x})$, $\mathrm{N}(\widehat{\boldsymbol{\theta}}^{s}, \widehat{\boldsymbol{\Sigma}}_{\widehat{\boldsymbol{\theta}}^s})$, $\mathrm{N}(\widehat{\boldsymbol{\theta}}^{c}, \widehat{\boldsymbol{\Sigma}}_{\widehat{\boldsymbol{\theta}}^c})$ and $\mathrm{N}(\widehat{\upsilon}, \widehat{\sigma}^2_{\widehat{\upsilon}})$, respectively.

2. Replace $\widehat{\boldsymbol{\theta}}^{x}$ by $\widehat{\boldsymbol{\theta}}^{x*}$, $\widehat{\boldsymbol{\theta}}^{s}$ by $\widehat{\boldsymbol{\theta}}^{s*}$, $\widehat{\boldsymbol{\theta}}^{c}$ by $\widehat{\boldsymbol{\theta}}^{c*}$, and $\widehat{\upsilon}$ by $\widehat{\upsilon}^*$, and repeat steps 1-7 in **Algorithm 3** to obtain $\widehat{N}_c^*(t^*; \widehat{\boldsymbol{\theta}}^{c*}, \widehat{\boldsymbol{\theta}}^{s*}, \widehat{\boldsymbol{\theta}}^{x*})$.

3. Repeat steps 1-2 B times to obtain $\widehat{N}_c^{*(b)}(t^*; \widehat{\boldsymbol{\theta}}^{c*}, \widehat{\boldsymbol{\theta}}^{s*}, \widehat{\boldsymbol{\theta}}^{x*})$, where $b = 1, \cdots, B$.

4. The $100(1-\alpha)\%$ PI for N_c is the $(\alpha/2, 1-\alpha/2)$ quantile of the B ordered values of $\widehat{N}_c^{*(b)}(t^*; \widehat{\boldsymbol{\theta}}^{c*}, \widehat{\boldsymbol{\theta}}^{s*}, \widehat{\boldsymbol{\theta}}^{x*})$.

5 Finite-Sample Performance of Estimation Methods

In this section, we use simulation to study the effect that sample size and number of events has on the performance of the estimation methods.

5.1 Design of Simulations

In the simulation, time is defined as the calendar time of system, and the time for repair is ignored. Only one time-dependent covariate is considered with the form of (12) and 30 time points per system. The parameter settings are $\boldsymbol{\theta}^x = (\beta_x, \sigma_\nu, \sigma_x)' = (0.02, 0.004, 0.05)'$ which are similar to the Vehicle B application.

The subsystem events follow a TRP model with trend function $\lambda_i^\star(t; \boldsymbol{\theta}^s) = at^{a-1}\exp\{\kappa \log[X_i(t)]\}$ and the renewal distribution function $F^{s\star}$. We set $F^{s\star}$ to be a Weibull distribution and the corresponding hazard function is $h^\star(t) = (\beta/\eta)(t/\eta)^{\beta-1}$, where η is the scale parameter (also the approximate 0.63 quantile) and β is the shape parameter. Because the mean of renewal function is restricted to one, the corresponding hazard function can be expressed as

$$h^\star(t) = \Gamma(1+\sigma)^{\frac{1}{\sigma}} t^{\frac{1}{\sigma}-1} \left(\frac{1}{\sigma}\right), \tag{14}$$

where $\sigma = 1/\beta$. The parameters for the subsystem are $\boldsymbol{\theta}^s = (a, \sigma, \kappa)' = (0.3, 0.8, 0.8)'$.

Given the above simulated subsystem events, the simulation of component events is based on the HMTRP model with the consideration of time-dependent covariate and random effects.

Let $\lambda_b = (\alpha/\varphi)(t/\varphi)^{\alpha-1}$. The trend function of system i in (5) can be denoted by $\lambda_i(t;\boldsymbol{\theta}^c) = (\alpha/\varphi)(t/\varphi)^{\alpha-1}\exp\{\gamma\log[X_i(t)] + w_i\}$, where $w_i \sim \mathrm{N}(0,\sigma_r^2)$. Here, φ is set to be one in the HMTRP model. The renewal distributions for the subsystem F^s and the component F^c are both Weibull distributions with mean one. The mean-one restriction is used so that all model parameters are identifiable (Lindqvist et al. 2003). Similar to (14), the hazard functions can be expressed as $h^c(t) = \Gamma(1+\sigma_0)^{1/\sigma_0}t^{1/\sigma_0-1}(1/\sigma_0)$, and $h^s(t) = \Gamma(1+\sigma_1)^{1/\sigma_1}t^{1/\sigma_1-1}(1/\sigma_1)$, respectively. Then the intensity function of HMTRP can be expressed as

$$\lambda_i^c(t|\mathcal{F}_{t^-}) = h^c\left\{\Lambda_i^s(t|\mathcal{F}_{t^-}^s) - \Lambda_i^s\left[t_{i,N_i(t^-)}\,|\,\mathcal{F}^s_{t^-_{i,N_i(t^-)}}\right]\right\}h^s\left\{\Lambda_i(t) - \Lambda_i[t^s_{i,N_{is}(t^-)}]\right\}\lambda_i(t;\boldsymbol{\theta}^c), \quad (15)$$

where

$$\lambda_i(t;\boldsymbol{\theta}^c) = (\alpha/\varphi)(t/\varphi)^{\alpha-1}\exp\{\gamma\log[X_i(t)] + w_i\}.$$

The parameters for the component-event intensity are $\boldsymbol{\theta}^c = (\alpha,\sigma_0,\sigma_1,\gamma)' = (0.4, 0.6, 0.75, 0.9)'$ and $\sigma_r = 0.5$.

The number of systems n was selected to be 50, 100, and 200. For each value of n, we simulated data 1000 times based on the parameter settings. By selecting different DFDs, the expected number of subsystem events (q_1) and component events (q_2) in each sample size (i.e., $n = 50, 100$, and 200) were controlled to the following three combinations (q_1, q_2): (0.7, 1.3), (1.7, 4.3), and (3.9, 12.7).

5.2 Simulation Results

Based on the **Algorithm 1**, the estimated parameters and corresponding standard errors (SE) for the component-event model were obtained and are shown in Tables 1-3. In the results, the estimates are close to the true values of the parameters, and they are approximately equal to the true settings as the sample size (n) and the length of study time (DFD) increase. Also, the coverage probability (CP) of the confidence interval procedure for each parameter is close to the nominal value of 0.95. The results show that our proposed estimation procedure can estimate the parameters well.

6 Application for the Vehicle B Data

In this section, we use the Vehicle B data to illustrate our proposed method for estimation and prediction of component events. Considering that the difference of the number of com-

Table 1: Summary of the simulation studies of the HMTRP given average number of subsystem events $q_1 = 0.7$ and component events $q_2 = 1.3$. Here "SE" stands for "standard error", and "CP" stands for "coverage probability".

Sample size n	Parameter	Value	Mean	Bias	SE	$\sqrt{\text{MSE}}$	CP
50			.3999	.0001	.0285	.0285	.940
100	α	.40	.4008	.0008	.0187	.0187	.937
200			.4003	.0003	.0134	.0134	.930
50			.6274	.0274	.0953	.0992	.938
100	σ_0	.60	.6136	.0136	.0627	.0642	.925
200			.6068	.0068	.0418	.0424	.938
50			.7969	.0469	.1475	.1548	.927
100	σ_1	.75	.7697	.0197	.0855	.0877	.934
200			.7572	.0072	.0561	.0566	.942
50			1.003	.1030	.2267	.2490	.930
100	γ	.90	.9462	.0462	.1181	.1268	.942
200			.9167	.0167	.0795	.0813	.933
50			.5142	.0142	.1709	.1716	.911
100	σ_r	.50	.5103	.0103	.1015	.1021	.926
200			.5041	.0041	.0594	.0596	.951

Table 2: Summary of the simulation studies of the HMTRP given average number of subsystem events $q_1 = 1.7$ and component events $q_2 = 4.3$. Here "SE" stands for "standard error", and "CP" stands for "coverage probability".

Sample size n	Parameter	Value	Mean	Bias	SE	$\sqrt{\text{MSE}}$	CP
50			.3989	.0011	.0142	.0142	.935
100	α	.40	.3995	.0005	.0097	.0097	.942
200			.3995	.0005	.0066	.0066	.951
50			.6023	.0023	.0380	.0380	.940
100	σ_0	.60	.6013	.0013	.0258	.0258	.947
200			.6001	.0001	.0181	.0181	.943
50			.7547	.0047	.0429	.0432	.953
100	σ_1	.75	.7523	.0023	.0295	.0296	.945
200			.7510	.0010	.0212	.0212	.938
50			.9102	.0102	.0701	.0708	.920
100	γ	.90	.9059	.0059	.0443	.0447	.946
200			.9025	.0025	.0319	.0320	.951
50			.5105	.0105	.0774	.0781	.930
100	σ_r	.50	.5042	.0042	.0544	.0546	.922
200			.5030	.0030	.0361	.0362	.953

Table 3: Summary of the simulation studies of the HMTRP given average number of subsystem events $q_1 = 3.9$ and component events $q_2 = 12.7$. Here "SE" stands for "standard error", and "CP" stands for "coverage probability".

Sample size n	Parameter	Value	Mean	Bias	SE	$\sqrt{\text{MSE}}$	CP
50			.3996	.0004	.0125	.0125	.931
100	α	.40	.3998	.0002	.0087	.0087	.933
200			.3999	.0001	.0060	.0060	.939
50			.6008	.0008	.0202	.0202	.940
100	σ_0	.60	.6003	.0003	.0140	.0140	.935
200			.6003	.0003	.0096	.0096	.942
50			.7514	.0014	.0198	.0199	.926
100	σ_1	.75	.7503	.0003	.0139	.0139	.934
200			.7503	.0003	.0092	.0092	.951
50			.9033	.0033	.0350	.0352	.931
100	γ	.90	.9015	.0015	.0250	.0251	.942
200			.9008	.0008	.0176	.0176	.932
50			.5073	.0073	.0615	.0619	.934
100	σ_r	.50	.5041	.0041	.0421	.0423	.944
200			.5010	.0010	.0296	.0296	.940

ponent events among different units is small, we assume that the prior distribution $P(\upsilon)$ is Gamma(a_1, a_2) with $a_1 = 10$ and $a_2 = 0.06$ in the HMTRP model indicating a small mean of $\sigma_r = 1/\sqrt{\upsilon}$. The intensity function of the component events in the HMTRP model is similar to (15) using a Weibull distribution as renewal functions for F^c and F^s. We fit the subsystem events with the TRP model using $\lambda_i^{\star}(t; \boldsymbol{\theta}^s) = at^{a-1}\exp\{\kappa \log[X_i(t)]\}$ as trend function and a Weibull distribution for the renewal function ($F^{s\star}$).

6.1 Parameter Estimation

Table 4 lists the estimates and standard errors of parameters in the covariate model (12), subsystem-event model and component-event models. The Vehicle B data is also fitted by the HMTRP sub-models (i.e., HTRP, HRP and HNHPP). In the TRP model of subsystem events, the value of the Weibull shape parameter $1/\sigma$ is greater than one indicating the trend of the hazard function corresponding to the renewal function is increasing for subsystem events.

For the component events, several sub-models are compared to the HMTRP model. We use the deviance information criterion (DIC) as a criterion for Bayesian model selection. Similar to the Akaike information criterion (AIC), it considers both model adequacy and model complexity. Define the deviance as $D(\boldsymbol{\theta}) = -2\log[f(y|\boldsymbol{\theta})] + 2\log[g(y)]$, where $\boldsymbol{\theta}$ denotes the vector of unknown parameters, $f(y|\boldsymbol{\theta})$ is the likelihood function and $g(y)$ is a standardizing term. Then, the DIC can be expressed as DIC $= \bar{D} + p_D$, where $\bar{D}$ indicates the goodness of fit with the form of $\bar{D} = \mathbf{E}_{\boldsymbol{\theta}|y}[D(\boldsymbol{\theta})] = \mathbf{E}_{\boldsymbol{\theta}|y}[-2\ln f(y|\boldsymbol{\theta})]$ by setting $g(y) = 1$, and p_D indicates the penalty for model complexity with the form $p_D = \bar{D} - D(\bar{\boldsymbol{\theta}})$. Here, $\bar{\boldsymbol{\theta}}$ is the posterior mean of $\boldsymbol{\theta}$. By simple transformation, DIC can be re-expressed as DIC $= D(\bar{\boldsymbol{\theta}}) + 2p_D$, which is similar to the form of AIC. More information about DIC can be found in Spiegelhalter et al. (2002), and Berg et al. (2004).

The DIC is easy to compute via the MCMC method. The estimate of $\bar{D}$ and p_D can be computed by $\bar{D} = \sum_{b=1}^{B}[-2L(\boldsymbol{\theta}^{c(b)}|\mathcal{F}_t, \boldsymbol{w}^{(b)})]/B$ and $p_D = \bar{D} - [-2L(\widehat{\boldsymbol{\theta}}^c|\mathcal{F}_t, \widehat{\boldsymbol{w}})]$ with $\widehat{\boldsymbol{\theta}}^c = \sum_{b=1}^{B}\boldsymbol{\theta}^{c(b)}/B$ and $\widehat{\boldsymbol{w}} = \sum_{b=1}^{B}\boldsymbol{w}^{(b)}/B$, respectively. Here, $\boldsymbol{\theta}^{c(b)}$ and $\boldsymbol{w}^{(b)}$ are the simulated posterior estimates after burn in. The results of DIC in Table 4 show that the HMTRP model fits the data better than other sub-models because of the smallest value of DIC. However, the fit of the HMTRP model is just slightly better than the HTRP model due to the small number of subsystem events and component events. The value of $1/\sigma_1$ is larger than one, indicating that the $h^s(\cdot)$ is an increasing function of time and that subsystem events have effect on the

Table 4: Parameter estimates and standard errors for the component-event, subsystem-event, and covariate models, based on the Vehicle B data.

Component Event Models								
Models		α	φ	γ	σ_0	σ_1	σ_r	DIC
HMTRP	est.	.173	1	1.21	1.03	.795	.081	2412.2
	SE	.016	–	.241	.086	.129	.013	
HTRP	est.	.157	1	1.50	.982	1	.083	2413.9
	SE	.014	–	.152	.073	–	.016	
HRP	est.	1.0	88.3	0	.824	1	.083	2450.5
	SE	–	5.39	–	.046	–	.015	
HNHPP	est.	1.61	104.3	0	1	1	.080	2418.8
	SE	.115	4.163	–	–	–	.013	

Subsystem Event Model		
a	σ	κ
.118	.494	.706
.011	.069	.045

Covariate Model		
β_x	σ_ν	σ_x
.0281	.0017	.0556
.0001	.0001	.0004

intensity trend of component events. The value of $1/\sigma_0$ is close to one indicating that the replacement of a component does not change the intensity trend of component events (i.e., minimal repairs).

To check the goodness of fit of the model, we can use the Cox-Snell residual plot. The estimated residuals $\widehat{R}_{ij}$ (with δ_{ij}^c as the censoring indicator) are expected to behave like a censored sample from an Exp(1) distribution. Figure 5 is a Cox-Snell residual plot which shows that the HMTRP model provides a good fit to the data, as most of the points align well with the diagonal line.

6.2 Prediction Results

The cumulative number of component events is shown in Figure 6. From Figure 6, we observe that the trend of the increase of component events drops down in the last 10 months. This is due to the occurrence of the scheduled subsystem replacements which can be requested by the customers after usage covariate has reached a critical value according to the system service plan. There were 132 scheduled subsystem events during the last 15 months before the DFD.

The model in (1) only describes the unscheduled (randomly occurred) subsystem events. For prediction, it is important to incorporate the effect of scheduled subsystem events because it also affects the component-event intensity. After some exploratory analysis, we found that the scheduled subsystem event times T can be modeled adequately by a lognormal distribution. Specifically, $\log(T) \sim \mathrm{N}(\mu^*, \sigma^{*2})$, where μ^* and σ^* are the location parameter and scale

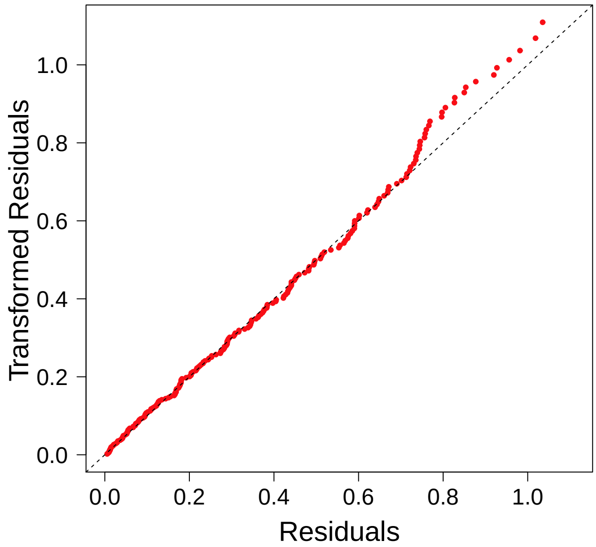


Figure 5: Residual plot for HMTRP model and the Vehicle B data.

parameter for the distribution of $\log(T)$, respectively. The ML estimates are $\widehat{\mu}^* = 1.026$ and $\widehat{\sigma}^* = 0.115$.

However, in the validation of the prediction algorithm, we set τ_i back 15 months to obtain the subset of the Vehicle B data that does not include the information of scheduled subsystem events. Thus, in the simulation of the component events, we need to simulate the scheduled subsystem events based on the lognormal distribution. Then, the subsystem and component events are simulated by treating the scheduled subsystem events as subsystem events, based on **Algorithms 2** and **3**.

The prediction of the component events are shown in Figure 7(a). The actual cumulative numbers of component events are close to the predicted values and inside the 95% PIs, indicating a good prediction performance. Using the Vehicle B data and prediction procedures described in **Algorithms 3** and **4**, we predict the cumulative number of component events in the next 30 months after the DFD which is shown in Figure 7(b). The results show that the expected total number of component events in the following 30 months will not exceed 80 with a 95% confidence level.

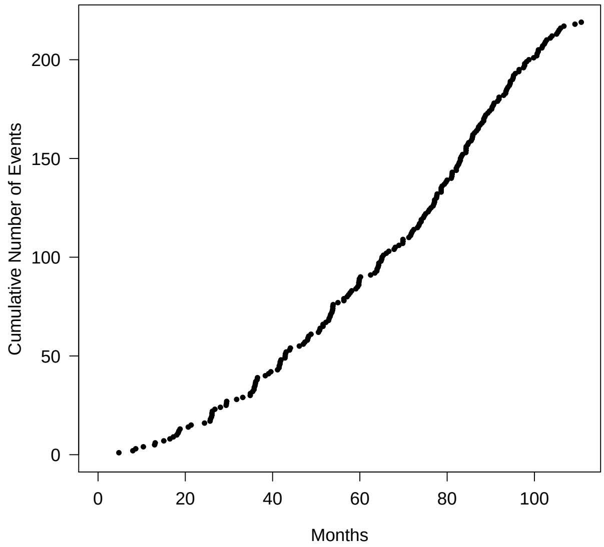
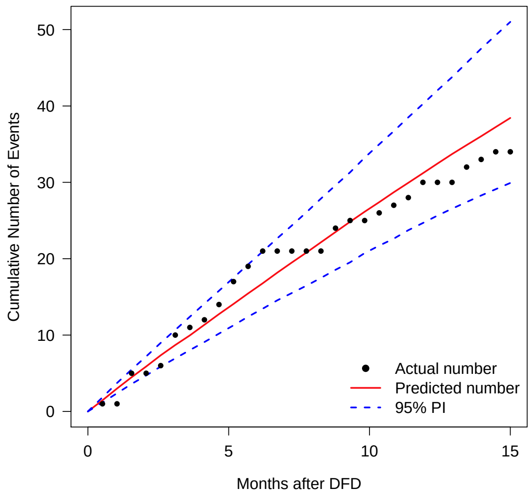
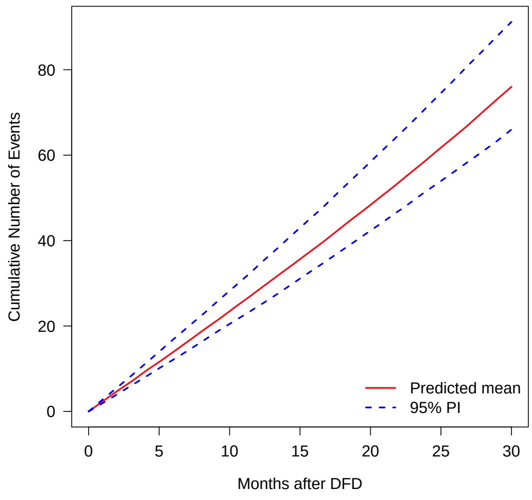


Figure 6: Plots of cumulative number of component events in Vehicle B dataset.



(a) Back test based on an early subset of the data



(b) Prediction of future events

Figure 7: Plots of predicted cumulative number of component events for Vehicle B.

7 Conclusion and Discussion

In this paper, we propose an MTRP model to describe component events in a multi-level repairable systems, extending the TRP model. Based on the MTRP, we also give Monte Carlo based procedures to provide point predictions and PIs for the cumulative number of future replacement events. The proposed MTRP model is a general recurrence process which includes the TRP, RP, and NHPP models as special cases. Using likelihood ratio tests or other criteria (e.g., AIC and DIC), an analyst can select the appropriate sub-model and determine the existence of effects from subsystem replacement events and component replacement events as well as the shape of the respective intensity functions. In order to explain more system-to-system variability, time-dependent covariates as well as random effects are introduced into the heterogeneous MTRP model (i.e., HMTRP). A Metropolis-within-Gibbs algorithm is suggested to estimate the unknown parameters in the HMTRP model. Performance of the estimation and prediction methods were checked with simulation studies. The Vehicle B industrial application is also used to illustrate the proposed method. Although only one time-dependent covariate was used in our application, the extension to multiple covariates is straightforward.

In the future related research, several possible areas can be continued.

- The proposed model and methods can apply to the system with more than two levels.
- A more complex system with multiple types of events (e.g., different failure modes) at the component level and at the subsystem level can be considered.
- The current model could also be extended to consider events that occur in many subsystems, with the possibility of interaction among subsystems.
- In some applications, there will be an opportunity to relate physical models for failure to the empirical replacement data and models for cumulative damage.

A Appendix

Let $X_i(t_{ij})$ and $X_i(t_{ik})$ denote two random variables of the time-dependent covariate. Based on (12) and the corresponding assumptions, it is easy to show that $\mathbf{E}[X_i(t_{ij})] = t_{ij}\beta_x$, $\mathbf{E}[X_i(t_{ik})] =$

$t_{ik}\beta_x$, $\text{Var}[X_i(t_{ij})] = \sigma_\nu^2 t_{ij}^2$, $\text{Var}[X_i(t_{ik})] = \sigma_\nu^2 t_{ik}^2$, and

$$\begin{aligned}\text{Cov}[X_i(t_{ij}), X_i(t_{ij})] =& \text{Cov}[t_{ij}(\beta_x + \nu_i) + \epsilon_i(t_{ij}), t_{ik}(\beta_x + \nu_i) + \epsilon_i(t_{ik})] \\ =& \text{Cov}[t_{ij}\nu_i, t_{ik}\nu_i] \\ =& \sigma_\nu^2 t_{ij} t_{ik}.\end{aligned}$$

Then, we can easily obtain the variance and covariance expressions for $\boldsymbol{X}_i(\boldsymbol{t}_i)$ and $\boldsymbol{X}_i(\boldsymbol{t}_{it^*})$: $\boldsymbol{\Sigma}_{i11} = \sigma_\nu^2 \boldsymbol{t}_i \boldsymbol{t}_i' + \sigma_x^2 \boldsymbol{I}_{m_i}$, $\boldsymbol{\Sigma}_{i22} = \sigma_\nu^2 \boldsymbol{t}_{it^*} \boldsymbol{t}_{it^*}' + \sigma_x^2 \boldsymbol{I}_{z_i}$, and $\boldsymbol{\Sigma}_{i12} = \sigma_\nu^2 \boldsymbol{t}_i \boldsymbol{t}_{it^*}'$. Based on the joint distribution of $\boldsymbol{X}_i(\boldsymbol{t}_i)$ and $\boldsymbol{X}_i(\boldsymbol{t}_{it^*})$, we can obtain the conditional distribution of $\boldsymbol{X}_i(\boldsymbol{t}_i)|\boldsymbol{X}_i(\boldsymbol{t}_{it^*})$:

$$\text{N}\left(\boldsymbol{t}_{it^*}\beta_x + \boldsymbol{\Sigma}_{i21}\boldsymbol{\Sigma}_{i11}^{-1}[\boldsymbol{X}_i(\boldsymbol{t}_i) - \boldsymbol{t}_i\beta_x],\ \boldsymbol{\Sigma}_{i22} - \boldsymbol{\Sigma}_{i21}\boldsymbol{\Sigma}_{i11}^{-1}\boldsymbol{\Sigma}_{i12}\right).$$

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