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Discordant Relaxations of Misspecified Models

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Abstract

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Keywords

Partial identification, identified/outer set, misspecification, nonconflicting hypothesis, robust identified set

Disciplines

Econometrics | Economic Theory | Statistical Methodology

DISCORDANT RELAXATIONS OF MISSPECIFIED MODELS

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ABSTRACT. In many set identified models, it is difficult to obtain a tractable characterization of the identified set. Therefore, empirical works often construct confidence region based on an outer set of the identified set. Because an outer set is always a superset of the identified set, this practice is often viewed as conservative yet valid. However, this paper shows that, when the model is refuted by the data, a nonempty outer set could deliver conflicting results with another outer set derived from the same underlying model structure, so that the results of outer sets could be misleading in the presence of misspecification. We provide a sufficient condition for the existence of discordant outer sets which covers models characterized by intersection bounds and the Artstein (1983) inequalities. We also derive sufficient conditions for the non-existence of discordant submodels, therefore providing a class of models for which constructing outer sets cannot lead to misleading interpretations. In the case of discordancy, we follow Masten and Poirier (2020) by developing a method to salvage misspecified models, but unlike them we focus on discrete relaxations. We consider all minimum relaxations of a refuted model which restore data-consistency. We find that the union of the identified sets of these minimum relaxations is misspecification-robust and has a new and intuitive empirical interpretation.

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1. INTRODUCTION

A central challenge in structural estimation of economic models is that the hypothesized structure often fails to identify a single generating process for the data, either because of multiple equilibria or data observability constraints. In such a context, the econometrics of partially identified models has been trying to obtain a tractable characterization of parameters compatible with the

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available data and maintained assumptions (hereafter *identified set*). A question of particular relevance in applied work is that it is often very difficult to find a tractable characterization of the identified set and then to obtain a valid confidence region for it. To avoid this difficulty, a large part of the literature has been trying to provide confidence region for an *outer set*, i.e., a collection of values for the parameter of interest that contains the identified set but may also contain additional values.¹ Because of its tractability, constructing confidence region for an outer set has been entertained in various topic of studies where the parameters of interest are only partially identified, see for instance Blundell et al. (2007), Ciliberto and Tamer (2009), Aucejo, Bugni, and Hotz (2017), Sheng (2020), de Paula, Richards-Shubik, and Tamer (2018), Dickstein and Morales (2018), Honoré and Hu (2020), and Chesher and Rosen (2020), among many others.

In most of the empirical studies, obtaining a tight outer set² is very often interpreted as an evidence for a small and then informative identified set; and this is because, under correct specification, any outer set contains the identified set. However, estimating the outer set could provide very misleading results when the initial model itself is misspecified.

The first main contribution of this paper is to characterize a class of partially identified models, for which when the initial model is misspecified we can always find at least two nonempty outer sets that fail to detect the violation of the initial model and are discordant to each other. We show that various models for which the identified sets are characterized by intersection bounds, conditional moment inequalities, or the Artstein (1983) inequalities share this property. This is a negative result for this class of models. It shows that in presence of misspecification — which often cannot be directly tested because of the non-tractability of the identified set characterization — the result provided by an outer set could entirely be driven by the set of restrictions the researcher chooses to construct this outer set, and that we could always consider an alternative choice of restrictions that provides a result that conflicts with the one initially provided by the researcher. However, discordant submodels do not exist in all refuted models. We then derive sufficient conditions for the non-existence of discordant submodels. This second result characterizes a class of models for which constructing outer sets can not lead to misleading interpretations. In this case, outer sets would be conservative, but are always robust.

Before us, various papers have been concerned about misspecification in partially identified models. An important focus has been dedicated on analyzing the impact of model misspecification on standard confidence regions used for set identified models. Bugni, Canay, and Guggenberger (2012) analyze the behavior of usual inferential methods for moment inequality models under local model misspecification. Ponomareva and Tamer (2011) and Kaido and White (2013) consider the impact of misspecification on semiparametric partially identified models, respectively, in the linear regression model with an interval-valued outcome and in a framework where some nonparametric moment

¹See Molinari (2020) for a detailed discussion.

²A tight outer set here refers to an outer set that is very small and informative.

inequalities are correctly specified and misspecification is due to a parametric functional form. See also Allen and Rehbeck (2020) who propose a method for statistical inference on the minimum approximation error needed to explain aggregate data in quasilinear utility models. It is worth noting that if one tries to find a confidence region for an outer set, then no inference methods, including those developed in the pre-cited papers, can fix the issue we are raising here. This is because two non-empty outer sets derived from the same underlying model structure can lead to discordant results. Adopting one of these outer sets without checking the validity of the underlying model could lead to misleading conclusions. Therefore, we need to suggest a more primitive approach to deal with these discordant results in this paper.

This objective leads to our second main contribution which consists of providing a method to salvage models that are possibly misspecified because of the existence of discordant misspecified submodels or discordant nonempty outer sets. The main intuition is to construct some minimum relaxation of the full model by removing discordant submodels until all remaining submodels are compatible. Because, there could be multiple ways to relax a model to restore data consistency, we take the union of the identified set of all these relaxed models. By doing so, we construct what we call the *misspecification robust bound*. We provide general sufficient conditions under which our misspecification robust bound exists, and also provide a new and intuitive empirical interpretation for it. Intuitively, we will say that a hypothesis is robust to misspecification if the hypothesis is compatible with all relaxed models that are data consistent.

The misspecification robust bound concept is related to the minimally-relaxed identified set introduced in Andrews and Kwon (2019), and can be viewed as a special case of the falsification adaptive set concept introduced in Masten and Poirier (2020). The main departure from Masten and Poirier (2020) is that we focus on discrete relaxations while Masten and Poirier (2020) focused exclusively on relaxing assumptions in a continuous way. In general, the use of discrete or continuous relaxation depends on the empirical application under scrutiny. In the following, we explore various features of discrete relaxations beyond its formal definition, especially, we will point out circumstances under which it would be more informative to consider only discrete relaxations.

It is worth noting that discrete relaxations of misspecified models have been entertained in various existing papers, see for instance, Manski and Pepper (2000, 2009), Blundell et al. (2007), Kreider et al. (2012), Chen, Flores, and Flores-Lagunes (2018), Kédagni (2021), Mourifié, Henry, and Méango (2020), among many others. In these papers, when the initial model is too stringent, they suggested alternative weaker assumptions that are believed to be more compatible with the empirical application under scrutiny and for which the identified/outer set is not empty. However some alternative reasonable relaxations may generate results that are discordant with what they suggested. To mitigate this issue, our misspecification robust bound approach suggests to collect information from all potentially discordant minimum relaxations of the initial model.

We organize the rest of the paper as follows. Section 2 introduces our main idea using a simple illustrative example. Section 3 presents our general setting and main results on the characterization of discordant submodels. Section 4 discusses a class of models for which constructing outer sets do not lead to misleading interpretations. Section 5 introduces the misspecification robust bound used to salvage misspecified models. Section 6 illustrates our misspecification robust bounds in a return to education example. The last section concludes, and additional results and proofs are relegated to the appendix.

2. INTRODUCTORY EXAMPLE: INTERSECTION BOUNDS

Although the main idea of this paper can be applied to general models, let us start with a simple introductory model, where our main idea can be illustrated in a straightforward way. Let us consider a special case of the intersection bounds in Chernozhukov, Lee, and Rosen (2013) in which parameter θ is bounded by the conditional mean of an upper and lower bounds,

$$E[\underline{Y}|Z = z] \leq \theta \leq E[\overline{Y}|Z = z] \quad \text{almost surely,} \quad (2.1)$$

where \underline{Y} and \overline{Y} are two observable random bounds and Z a vector of instrumental variables. Let \mathcal{Z} be the support of Z , and define

$$\underline{\gamma} := \sup_{z \in \mathcal{Z}} E[\underline{Y}|Z = z] \quad \text{and} \quad \overline{\gamma} := \inf_{z \in \mathcal{Z}} E[\overline{Y}|Z = z].$$

The identified set of θ is interval $[\underline{\gamma}, \overline{\gamma}]$ when $\underline{\gamma} \leq \overline{\gamma}$. We assume the following regularity condition holds in this example.

Assumption 1. *Assume $E[\underline{Y}] < \infty$ and $E[\overline{Y}] < \infty$. In addition, assume that the conditional expectations $E[\underline{Y}|Z]$, and $E[\overline{Y}|Z]$ exist and $E[\underline{Y}|Z] \leq E[\overline{Y}|Z]$ almost surely.*

This simple framework encompasses some important treatment effect models.

Example 1 (Discrete treatment model). Consider a setting where $\mathcal{X} := \{x_1, \dots, x_K\}$ is the set of all possible treatments. Let Y_k be the potential outcome when the treatment is externally set to x_k . The observed outcome Y is defined as follows: $Y = \sum_k \mathbb{1}(X = x_k)Y_k$. Let us define $\theta_k \equiv E[Y_k]$ and assume that Y_k has a bounded support $[\underline{y}_k, \overline{y}_k]$. The random bound for Y_k can be constructed as follows $\underline{Y}_k \equiv Y \mathbb{1}(X = x_k) + \underline{y}_k \mathbb{1}(X \neq x_k)$ and $\overline{Y}_k \equiv Y \mathbb{1}(X = x_k) + \overline{y}_k \mathbb{1}(X \neq x_k)$. If we assume the mean independence assumption $E[Y_k|Z] = E[Y_k]$ we obtain a special case of (2.1).

Discrete treatment models with bounded potential outcomes are usually considered in Manski's work. See for instance Manski (1990, 1994) among many others.

Example 2 (Smooth Treatment Model). Consider a smooth treatment model as in Kim et al. (2018). When the treatment is x , the potential outcome is $Y(x) = g(x, \epsilon)$ where g is an unknown function, and ϵ is individual heterogeneous characterization. Assume $g(x, \epsilon)$ is Lipschitz continuous

in x with Lipschitz constant equal to τ . Suppose we are interested in $\theta_x = E[Y(x)]$. The lower and upper bounds can be constructed as $\underline{Y}(x) = Y - \|X - x\| \tau$ and $\overline{Y}(x) = Y + \|X - x\| \tau$. As in the discrete treatment case, if we assume $E[Y(x)|Z] = E[Y(x)]$, we obtain model (2.1). As a special case, one can also consider a linear model with heterogeneous coefficient, $Y = X'\beta + \epsilon$ where β is a vector of an unobserved random coefficient. Suppose the coefficient space for β is $[\underline{\beta}, \overline{\beta}]$. Then, $\underline{Y}(x) = Y + \sum_i \min \left\{ (x_i - X_i) \underline{\beta}_i, (x_i - X_i) \overline{\beta}_i \right\}$ where the subscript i stands for the i^{th} dimension of the corresponding variables. Similarly, $\overline{Y}(x) = Y + \sum_i \max \left\{ (x_i - X_i) \underline{\beta}_i, (x_i - X_i) \overline{\beta}_i \right\}$.

In practice, model (2.1) is sometimes implemented by solving its unconditional version,

$$E[h(Z)(\theta - \underline{Y})] \geq 0 \text{ and } E[h(Z)(\overline{Y} - \theta)] \geq 0, \quad (2.2)$$

where h is some nonnegative function mapping its input to \mathbb{R}_+^m with $m < \infty$, and the inequalities in (2.2) are vector inequalities. The inference for (2.2) is typically much simpler than the inference for the original model (2.1), especially when Z is multi-dimensional. Let $\tilde{\Theta}(h)$ be the identified set for θ in model (2.2). As made explicit in the notation, $\tilde{\Theta}(h)$ depends on the choice of instrumental function h . However, since (2.1) implies (2.2), we know that for every choice of h , $\tilde{\Theta}(h)$ is always an outer set of the interval $[\underline{\gamma}, \overline{\gamma}]$, the identified set for θ in model (2.1), i.e. $[\underline{\gamma}, \overline{\gamma}] \subseteq \tilde{\Theta}(h)$. This inclusion relation is often used as a justification for using model (2.2), it may not be as informative as model (2.1), but its identification result $\tilde{\Theta}(h)$ is often viewed as a conservative bound for $[\underline{\gamma}, \overline{\gamma}]$, the identified set for model (2.1).

Our first observation is that the result based on $\tilde{\Theta}(h)$ is not always reliable. To see this, define \mathcal{H}_m^+ to be the space of all nonnegative instrumental functions with dimension m . More formally, let $\mathcal{H}_m^+ := \{h : \mathcal{Z} \mapsto \mathbb{R}_+^m \text{ such that } E\|h(Z)\| < \infty, E\|\underline{Y}h(Z)\| < \infty, E\|\overline{Y}h(Z)\| < \infty \text{ and } E[h_i(Z)] > 0, \forall i = 1, \dots, m\}$. Then, we have the following theorem.

Theorem 1. *Suppose Assumption 1 holds. If the restriction in (2.1) is refuted, i.e. $\underline{\gamma} > \overline{\gamma}$, then, for any θ in $(\overline{\gamma}, \underline{\gamma})$, there exists some $h \in \mathcal{H}_2^+$ such that $\tilde{\Theta}(h) = \{\theta\}$. Conversely, if there exists some integer m and some $h \in \mathcal{H}_m^+$ such that $\tilde{\Theta}(h) = \{\theta\}$, then $\theta \in [\overline{\gamma}, \underline{\gamma}]$.*

When (2.1) is refuted, Theorem 1 shows that the unconditional moment restrictions can point identify any element in the crossed bound $(\overline{\gamma}, \underline{\gamma})$ with a properly chosen instrumental function. The width of $(\overline{\gamma}, \underline{\gamma})$ depends on the extent of the model violation: the worse the violation is, the wider this interval would be. In the extreme case where the mean independence condition is so much violated that $[E\underline{Y}, E\overline{Y}] \subseteq (\overline{\gamma}, \underline{\gamma})$, this means that any point in the Manski worst-case bounds can be picked up as the point identification result by some choice of h .

Theorem 1 suggests a caveat for outer sets. If one does not have the knowledge of whether the identified set is empty or not, one should not use an outer set $\tilde{\Theta}(h)$ to infer the identified set. An outer set $\tilde{\Theta}(h)$ could be *misleading* for this purpose in the sense that there might not exist a θ in

the identified set such that $\theta \in \tilde{\Theta}(h)$. For example, if one observes $\tilde{\Theta}(h) \subseteq (0, +\infty)$ for some h , one should not jump directly to the conclusion that the sign of θ is positive without verifying the non-emptiness of the identified set. As shown in Theorem 1, if the identified set happens to be empty, the value of $\tilde{\Theta}(h)$ can be arbitrarily chosen by selecting different h functions. To put it differently, Theorem 1 implies that, when the identified set of (2.1) is empty, there exist two h and h' such that both $\tilde{\Theta}(h)$ and $\tilde{\Theta}(h')$ are nonempty but $\tilde{\Theta}(h) \cap \tilde{\Theta}(h')$ is empty. Thus, two researchers could apply the same model on the same data set and yet draw completely different conclusions from the outer sets by choosing different h functions.

Remark 1. This result complements the findings in Andrews and Shi (2013). In Andrews, Kim, and Shi (2017), they propose an inference procedure and a Stata code named “**cmi-interval**” which constructs a confidence region for parameters θ in a model like (2.1). Their inference approach transform (2.1) into (2.2) by selecting h in a sub-family of \mathcal{H}_m^+ and letting $m \rightarrow \infty$ as the sample size increases. Our result shows that increasing m to infinity is crucial to ensure the robustness of the result if (2.1) could be misspecified. If the dimension of h is fixed, then the empirical result for (2.2) could be misleading even if the inference controls the size uniformly. The “**cmi-interval**” command gives the possibility to the researcher to fix m . In such a context, if (2.1) is indeed misspecified, two different researchers can choose two different pairs $(m, h) \neq (m', h')$ where each of them will provide a very tight outer set but with discordant information. This then raises the question on how to interpret the results under the model misspecification.

Finally, Theorem 1 also shows that the informativeness of an outer set does not necessarily imply the informativeness of the identified set. An outer set can be as tight as a singleton set, but such precision is superfluous because its value might not have anything to do with the underlying parameter.

This caveat of outer sets is somewhat overlooked in the literature. As we listed some papers in the introduction, it is common for researchers to construct a confidence interval for an outer set and draw conclusions based solely on its result. If the model is refutable and one only studies an outer set in the empirical analysis without knowing whether the identified set is empty or not, Theorem 1 shows that, in this case, results based on an outer set could be misleading in the intersection bound model. In the next section, we will extend our analysis to more general models, and will show that this caveat is indeed a concern for some other partial identification models that have been widely used in the literature.

3. MISLEADING SUBMODELS IN A GENERAL SETTING

In this section, we show the existence of similar results in Theorem 1 in a more general setting. Let A be some nonempty collection of assumptions. For any nonempty subset $A' \subseteq A$, define $\Theta_I(A')$ to be the identified set of θ when all $a \in A'$ are assumed to hold given the true distribution of the

observable data. Let \emptyset denote the empty set, and define $\Theta_I(\emptyset) = \Theta$ where Θ denotes the parameter space. Throughout the paper, we assume Θ is some subset in a metric space. For each $a \in A$, we abbreviate $\Theta_I(\{a\})$ as $\Theta_I(a)$. We interpret A as the *full model*, and $A' \subsetneq A$ as a *submodel*. In the introductory example, the role of A is played by the set of all models (2.2) indexed by the instrumental function h . And, (2.1) holds if and only if all assumptions a in A hold. By its definition, $\Theta_I(\cdot)$ has the following two properties: for any two submodels A' and A'' , (i) $\Theta_I(A') \subseteq \Theta_I(A'')$ if $A' \subseteq A''$; (ii) $\Theta_I(A' \cup A'') \subseteq \Theta_I(A') \cap \Theta_I(A'')$.

We say model A' is *data-consistent* if $\Theta_I(A')$ is nonempty, and call it *refuted* if the reverse is true. By definition, $\Theta_I(A) \subseteq \Theta_I(A')$ for any $A' \subseteq A$. Therefore, if the full model A is data-consistent, we know that each $A' \subseteq A$ is also data consistent, and $\Theta_I(A') \cap \Theta_I(A'')$ is nonempty for any two submodels A' and A'' . In other words, when the full model is data-consistent, each $\Theta_I(A')$ can be viewed as a conservative bound for $\Theta_I(A)$, and all of these submodels are mutually compatible.

We are interested in what would happen when the full model A is refuted. We will present two types of results: (i) In this section, we want to find conditions under which there exists two data-consistent submodels A' and $A'' \in \mathcal{A}$ such that $\Theta_I(A') \cap \Theta_I(A'')$ is empty. When such sets A' and A'' exist, we call them *discordant submodels*, because both of these two submodels fail to detect the failure of the full model and lead to discordant empirical results. This is important since it shows for which class of models the use of outer sets would lead to misleading interpretations in empirical works. We will illustrate this situation with examples, showing some common practices in the literature that would lead to misleading interpretations. (ii) In the next section, however, we will show that discordant submodels may not always exist in a refuted model. We will derive sufficient conditions for the non-existence of discordant submodels, and will illustrate with examples on how this result could be used to construct data-consistent submodels free of discordancy. This second result characterizes a class of models for which constructing outer sets can not lead to misleading interpretations. For now, let us first discuss when and why discordant submodels exist.

Theorem 2. *Assume the two following conditions hold:*

(T2.C1) *Whenever $\Theta_I(A) = \emptyset$, there exists some nonempty $A^* \subseteq A$ such that $\Theta_I(A^*) = \emptyset$ and $\Theta_I(a) \neq \emptyset$ for any $a \in A^*$.*

(T2.C2) *For any nonempty subset $B \subseteq A$, $\Theta_I(B) = \bigcap_{a \in B} \Theta_I(a)$.*

And If A is not a finite set, assume, in addition, that the following condition holds

(T2.C3) *$\Theta_I(a)$ is compact for each $a \in A$.*

Then, the full model is refuted, i.e. $\Theta_I(A) = \emptyset$, if and only if there exist two finite subset $A', A'' \subseteq A$, such that both $\Theta_I(A)$ and $\Theta_I(A')$ are nonempty but $\Theta_I(A) \cap \Theta_I(A')$ is empty.

Moreover, when $\Theta_I(A) = \emptyset$, for any submodel $B \subseteq A$ with nonempty $\Theta_I(B)$, there exist two finite subset B', B'' of A , such that $\Theta_I(B \cup B') \neq \emptyset$, $\Theta_I(B'') \neq \emptyset$ and $\Theta_I(B \cup B') \cap \Theta_I(B'') = \emptyset$.

First, Theorem 2 shows that, under condition (T2.C1)-(T2.C3), whenever the full model is rejected there always exists a pair of submodels which are data-consistent and mutually discordant. Second, it suggests that, for every data-consistent submodel B , there exists another data-consistent submodel which is not compatible with some stronger version of B . In addition, it shows that even if A is not a finite set, we require only two finite subsets to reject A . In the partial identification literature, whenever the validity of a strong assumption is in question, some weaker assumption (or what we call a submodel here) is often considered as a better modelling choice, especially when this weak assumption still leads to informative results. Theorem 2 suggests that this top-to-down strategy could be misleading unless the researcher has a preference order on these mutually incompatible weaker assumptions ex-ante, because one could get different conclusions by picking different submodels.

Let us now briefly discuss the three conditions in Theorem 2. Condition (T2.C1) is the key condition which generates misleading submodels. In condition (T2.C1), A^* can be A itself or some subset of it. A^* could also depend on the underlying data generating process. It requires the existence of a collection of assumptions which are data-consistent separately but are refuted jointly. The intuition is that if some data-consistent assumptions are refuted jointly, then some of them must be mutually incompatible. However, this condition alone does not guarantee the result of Theorem 2. In condition (T2.C2), we assume that the identified set of a set of assumptions is equal to the intersection of the identified sets of each assumption. This condition holds if every assumption a in A is imposed on observables and parameters, and it may not hold when some of the submodels involve assumptions on the unobservables. However, note that condition (T2.C2) can still cover the case where the full model involves assumptions on the unobservables. In most partially identified models, there exists a collection of assumptions A on the observables and parameters so that the original assumption which involves the unobservables holds if and only if assumption a holds for each $a \in A$. As shown later in Example 3, Theorem 2 is useful in these cases. In addition, condition (T2.C2) can be further weakened when A is a finite set. See Theorem 2' relegated to the Appendix for the sake of simplicity. Finally, condition (T2.C3) is satisfied in most empirical models where θ is of finite dimension and it is only needed when A contains an infinite number of assumptions. In Appendix B.2.1, we show the result of Theorem 2 can fail if any of these conditions is violated. In the following, we will explore the implications of Theorem 2 for different classes of partially identified models.

3.1. Introductory example continued. Before looking at those more advanced cases, it is probably helpful for us to revisit the introductory example to illustrate how these conditions can be verified in a simple model. In the introductory example, recall that \mathcal{H}_1^+ stands for the set of all one dimensional nonnegative instrumental functions which satisfy some basic regularity conditions. The set A in the introductory example is the set of all assumptions indexed by $h \in \mathcal{H}_1^+$ with which (2.2) holds. And, A is indeed the full model because (2.1) holds if and only if (2.2) holds with each

$h \in \mathcal{H}_1^+$. In addition, for all $h \in \mathcal{H}_1^+$, the identified set $\tilde{\Theta}(h)$ for model (2.2) is equal to the following interval

$$\left[\frac{E[h(Z)\underline{Y}]}{E[h(Z)]}, \frac{E[h(Z)\overline{Y}]}{E[h(Z)]} \right],$$

which is nonempty whenever $E[\underline{Y}|Z] \leq E[\overline{Y}|Z]$ almost surely. Therefore, if we simply let $A^* = A$, then $\Theta_I(a) \neq \emptyset$ for all $a \in A^*$. Thus, condition (T2.C1) is satisfied. Since (2.2) are moment inequalities which only depend on observables and the parameter, condition (T2.C2) is also satisfied. Finally, since the identified set characterized by (2.2) is always a closed interval, condition (T2.C3) is satisfied under the regularity conditions imposed within the definition of \mathcal{H}_1^+ .

Note that, in this example, if $B \subseteq A$ and B consists of m assumptions, then B refers to the submodel that (2.2) holds for some $h \in \mathcal{H}_m^+$. As a result, Theorem 2 implies that when (2.1) is refuted, there must exist some $h_1 \in \mathcal{H}_{m_1}^+$ and $h_2 \in \mathcal{H}_{m_2}^+$, such that $\tilde{\Theta}(h_1) \neq \emptyset$ and $\tilde{\Theta}(h_2) \neq \emptyset$ but $\tilde{\Theta}(h_1) \cap \tilde{\Theta}(h_2)$ is empty. This is a weaker result compared to what we found in Theorem 1, which is not surprising as Theorem 1 utilizes some model specific structures whereas Theorem 2 is built on more general properties. In Appendix A.1, we provide sufficient conditions under which Theorem 2 applies to the following conditional moment inequalities:

$$E[m(X, Z; \theta)|Z] \leq 0 \text{ almost surely,} \quad (3.1)$$

where $X \in \mathbb{R}^{k_1}$ and $Z \in \mathbb{R}^{k_2}$ are observable random variables and $m(\cdot, \cdot; \theta) \in \mathbb{R}$ is some known integrable function for each θ .

3.2. Random Sets and Choquet Capacity. In this subsection, we consider a model with a vector of random variables Y and a vector of exogenous observable covariates X . Let $\mathcal{Y}(\theta)$ be some random closed set and assume $P(Y \in \mathcal{Y}(\theta)) = 1$. Then, Artstein (1983) shows that the conditional distribution of Y given X equals F if and only if for any compact set K in the support of Y , the following inequality holds:

$$P_F(Y \in K|X) \leq L(K, X; \theta) := P(\mathcal{Y}(\theta) \cap K \neq \emptyset|X) \text{ almost surely,} \quad (3.2)$$

where P_F refers to the probability measure corresponding to the distribution F . The quantity $L(\cdot, X; \theta)$ is often known as the Choquet capacity function. These type of models are studied in Galichon and Henry (2011). Often in practice, either $P_F(Y \in K|X)$ or $L(K, X; \theta)$ can be identified from the data, and the other one can typically be derived or simulated from some additional assumptions. For the purpose of illustration, we consider that Y is observable and assume $L(K, X; \theta)$ is a known function of K and X given θ .

In general, one needs to check (3.2) for all compact sets in order to ensure the equivalence between the collection of inequalities in (3.2) and the distributional assumption on Y . In some circumstances, checking the inequalities for a collection of compact sets is equivalent to checking them for all compact sets, in which case, the collection is called the *core determining class* in the language

of Galichon and Henry (2011). However, in practice, researchers typically pre-select some finite collection \mathcal{K} of compact sets, which is often not a core determining class, and they only check (3.2) for $K \in \mathcal{K}$. For instance, in the treatment effect literature, the well-known Manski (1994) bounds on the potential outcome distributions implemented in various applications such as in Blundell et al. (2007), or Peterson (1976) bounds on competing risk, use only a finite and not sufficient collection of Artstein inequalities. See, respectively, Molinari (2020), and Mourifié, Henry, and Méango (2020) for a detailed discussion. In empirical games, auction and network applications we can also cite Ciliberto and Tamer (2009), Haile and Tamer (2003), Sheng (2020), Chesher and Rosen (2020), among many others who also focused on a finite and not sufficient collection of Artstein inequalities.³

We want to explore the consequences of this pre-selection procedure when the original model might be refuted. To fit this model into the general framework in Section 3, define \mathcal{X} to be the collection of all nonempty compact sets in the support of Y . Define A as the set of all assumptions indexed by $K \in \mathcal{X}$ for which (3.2) holds. Then, for any finite subset $A' \subseteq A$, $\Theta_I(A')$ refers to the identified set when (3.2) is checked for only a finite number of different K s. As in the previous section, condition (T2.C2) of Theorem 2 is satisfied by construction. We only need to verify conditions (T2.C1) and (T2.C3).

Condition (T2.C3) is often easy to check and is only needed when the support of Y is not finite. Condition (T2.C1) is more challenging. Without imposing more structures, the most natural choice for A^* is $A^* = A$. In general, we need more regularity conditions to ensure $\Theta_I(a) \neq \emptyset$ for all $a \in A$. When the support of Y is discrete and finite, the following proposition provides a sufficient condition for (T2.C1) to hold.

Lemma 1. *Let \mathcal{X} and \mathcal{Y} be the supports of X and Y , respectively. Suppose \mathcal{Y} is discrete and finite, and the parameter space $\Theta \subseteq \mathbb{R}^d$. In addition, suppose the following conditions hold for any $y \in \mathcal{Y}$:*

(L1.C1) $\inf_{x \in \mathcal{X}} P(Y = \{y\} | X = x) > 0$,

(L1.C2) *there exists a sequence $\theta_1, \theta_2, \dots$ in \mathbb{R}^d such that $\inf_{x \in \mathcal{X}} L(\{y\}, x; \theta_k) \rightarrow 1$ as $k \rightarrow \infty$,*

where the inf in the above two conditions denotes the essential infimum. Then, there exists some compact set $\tilde{\Theta}$ such that condition (T2.C1) of Theorem 2 holds whenever $\tilde{\Theta} \subseteq \Theta$.

Lemma 1 implies that the results in Theorem 2 hold in this example if (L1.C1) and (L1.C2) hold and if Θ is large enough to include the compact set $\tilde{\Theta}$. To see why we need Θ to be large enough, note that, if Θ is as small as a singleton set, then any data-consistent submodels will automatically have the same identified set, because the size of Θ makes it very restrictive. To prevent this kind of trivial cases, we need the parameter space to be large enough. The requirement that $\tilde{\Theta} \subseteq \Theta$ means that the size of Θ would not be a binding restriction in the analysis. The simplest way to ensure this condition is to let $\Theta = \mathbb{R}^d$. Lemma 1 and Theorem 2 then imply the following corollary.

³See Molinari (2020) for a detailed discussion.

Corollary 1. *Suppose the support of Y is discrete and finite and the parameter space $\Theta \subseteq \mathbb{R}^d$. Suppose (L1.C1) and (L1.C2) hold and Θ is large enough to include the $\tilde{\Theta}$ in Lemma 1. For any collection \mathcal{K} of compact sets, define $\Theta_I(\mathcal{K})$ as the identified set of the submodel satisfying (3.2) with each $K \in \mathcal{K}$. Then, the full model is refuted if and only if there exist two finite collections of compact sets \mathcal{K}_1 and \mathcal{K}_2 such that both $\Theta_I(\mathcal{K}_1)$ and $\Theta_I(\mathcal{K}_2)$ are nonempty, but these two identified sets have empty intersection.*

Moreover, whenever the full model is refuted, for any finite collection of compact sets $\tilde{\mathcal{K}}$ with nonempty $\Theta_I(\tilde{\mathcal{K}})$, there exist \mathcal{K}_1 and \mathcal{K}_2 such that $\Theta_I(\tilde{\mathcal{K}} \cup \mathcal{K}_1) \neq \emptyset$ and $\Theta_I(\mathcal{K}_2) \neq \emptyset$, but $\Theta_I(\tilde{\mathcal{K}} \cup \mathcal{K}_1) \cap \Theta_I(\mathcal{K}_2) = \emptyset$.

We conclude this subsection with an example of entry game model.

Example 3 (Entry game). Consider an n -player complete information entry game as in Ciliberto and Tamer (2009). Assume there are n players, where player i 's payoff function is specified as

$$\pi_i = \gamma_i + X_i' \beta - \sum_{j \neq i} \delta_j Y_j + \epsilon_i$$

where the X_i s are some covariates which might be player i specific, γ_i and β is the parameter coefficient, $Y_j \in \{0, 1\}$ stands for player j 's entry decision, and $\delta_j > 0$ is the interaction parameter of player j . For simplicity, we assume that $Y = (Y_i : i = 1, \dots, n)$ is always a pure strategy Nash equilibrium.

Assume $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ is independent of X and ϵ follows the normal distribution $N(0, \Sigma)$. Let $\gamma = (\gamma_1, \dots, \gamma_n)$ and $\theta = (\gamma, \beta, \delta, \Sigma)$ be the vector of all parameters. Let $\mathcal{Y} = \{y = (y_1, \dots, y_n) : y_i \in \{0, 1\}, i = 1, \dots, n\}$ be the set of all possible entry decisions, and let $2^{\mathcal{Y}}$ denote the set of all subsets of \mathcal{Y} . For any $K \in 2^{\mathcal{Y}}$, define $L(K, X, \theta)$ to be the probability that at least one $y \in K$ is a pure-strategy Nash equilibrium given X and θ . In practice, $L(K, X, \theta)$ can often be solved from numerical simulation.

In Galichon and Henry (2011), the identified set of this model is shown to be the set of all θ which satisfies (3.2) for all nonempty set K of \mathcal{Y} . The number of these inequalities increases with n very quickly in the order of 2^{2^n} . Galichon and Henry (2011) provide some methods to reduce the number of inequalities by removing redundant inequalities in (3.2), but, in general, sharp characterization of the identified set involves a large number of inequalities. In practice for the sake of computational feasibility, empirical researchers often pre-select a finite collection \mathcal{K} of subsets and only check (3.2) for each $K \in \mathcal{K}$. See, for example, Ciliberto and Tamer (2009), and Ciliberto, Murry, and Tamer (2020).

Let us now check conditions in Proposition 1. Condition (L1.C1) in Proposition 1 are low-level assumptions on the data. It can be implied by assuming that X has a compact support, $P(Y = y|X) > 0$ almost surely and $P(Y = y|X = x)$ is continuous in x for every y . Condition

(L1.C2) in Proposition 1 also holds, because for each $y \in \mathcal{Y}$, one can have $L(\{y\}, x; \theta_k) \rightarrow 1$ by simply fixing $\beta = 0$, $\delta = 0$ and let $\gamma \rightarrow \gamma^*$ where $\gamma_i^* = \infty$ if $y_i = 1$ and $\gamma_i^* = -\infty$ if $y_i = 0$.

4. COMPATIBLE SUBMODELS AND MINIMUM DATA-CONSISTENT RELAXATION

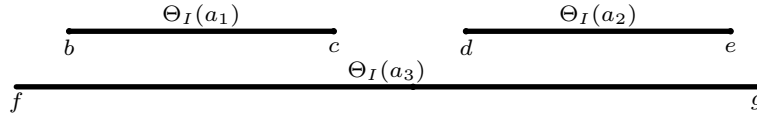
As discussed in the preceding section, there could exist discordant submodels when the full model is refuted. However, the falsification of the full model does not necessarily lead to discordancy of the submodels. In this section, we will present a sufficient condition which ensures that all data-consistent submodels are always compatible with each other.

To state our result, we need to introduce a new concept. When the full model is refuted, we can get some data-consistent submodel by dropping or relaxing some of the assumptions. We say that a data-consistent submodel is a minimum relaxation if we just relaxed the minimum amount of assumptions to restore the data consistency.

Definition 1. Let \tilde{A} be a subset of A . We say \tilde{A} is a *minimum data-consistent relaxation* of A if $\Theta_I(\tilde{A})$ is nonempty and for any $a \in A \setminus \tilde{A}$, $\Theta_I(\tilde{A} \cup \{a\})$ is empty.

To illustrate this concept, let us consider a simple example where $A = \{a_1, a_2, a_3\}$. The identified sets of each a_i are all closed intervals in \mathbb{R} as shown in Figure 1 with $\Theta_I(a_1) = [b, c]$, $\Theta_I(a_2) = [d, e]$, $\Theta_I(a_3) = [f, g]$ and $f \leq b \leq c < d \leq e \leq g$. Assume also that $\Theta_I(\{a, a'\}) = \Theta_I(a) \cap \Theta_I(a')$ for $a, a' \in \{a_1, a_2, a_3\}$.

FIGURE 1. The three-interval example



In this example, both $\{a_1, a_3\}$ and $\{a_2, a_3\}$ are minimum data-consistent relaxations. And, $\{a_3\}$ is not a minimum data-consistent relaxation, since it will remain data-consistent after including a_1 or a_2 . The following theorem establishes the existence of minimum data-consistent relaxations in a general setting.

Theorem 3. *Suppose one of the following two conditions is satisfied,*

(T3.C1) *A is a finite set.*

(T3.C2) *For any $a \in A$, $\Theta_I(a)$ is compact. Moreover, for any $B \subseteq A$, $\Theta_I(B) = \bigcap_{a \in B} \Theta_I(a)$.*

Then, there exists some minimum data-consistent relaxation of A . Moreover, for any data-consistent $A' \subseteq A$, there exists some minimum data-consistent relaxation \tilde{A} such that $A' \subseteq \tilde{A}$.

In general, the number of minimum data-consistent relaxations may or may not be unique. We will defer the discussion of multiple minimum data-consistent relaxations to the next section. In this section, we focus on the situation where there exists a unique minimum data-consistent relaxation. It turns out that the uniqueness of the minimum data-consistent relaxation is closely related to the compatibility of submodels.

Theorem 4. *Consider the following statements:*

(T4.C1) *There exists a unique data-consistent submodel A^* such that any data-consistent submodel A' is included within A^* . Hence, $\Theta_I(A^*) = \cap_{A': \Theta_I(A') \neq \emptyset} \Theta_I(A')$.*

(T4.C2) *for any $A' \subseteq A$, A' is data-consistent if and only if all $a \in A'$ are data-consistent.*

(T4.C3) *There exists a unique minimum data-consistent relaxation A^* .*

Then, (T4.C1) \Leftrightarrow (T4.C2) \Rightarrow (T4.C3) and the A^ in (T4.C1) and (T4.C3) is the same set. Moreover, if (T3.C1) or (T3.C2) hold, then (T4.C1) \Leftrightarrow (T4.C2) \Leftrightarrow (T4.C3).*

Theorem 4 involves three statements. (T4.C1) is a generalization of the following observation: when the full model A is data-consistent, we have $\Theta_I(A) \subseteq \Theta_I(A')$ for any submodel A' . In (T4.C1), when the full model A is refuted, the A^* plays the role of A : A^* is the most informative submodel and is compatible with any other data-consistent submodels. One implication of (T4.C1) is the non-existence of discordant submodels: for any two data-consistent submodels A_1 and A_2 , $\Theta_I(A_1) \cap \Theta_I(A_2)$ is nonempty, because $\Theta_I(A^*) \subseteq \Theta_I(A_1) \cap \Theta_I(A_2)$. More importantly, under (T4.C1), we no longer have the arbitrariness of $\Theta_I(A')$ of submodel A' that we saw in Theorems 1 and 2. No matter which data-consistent submodel one chooses to construct an outer set, the result always include $\Theta_I(A^*)$. The value of A^* depends on the distribution of the data. When the full model A is data-consistent, $A^* = A$. When A is refuted, A^* can be viewed as the model learned from the data by removing all refuted assumptions in A while keeping all the data-consistent ones. The interpretation of A^* and its role as the unique minimum data-consistent relaxation will be studied further in Section 5.1.

Theorem 4 provides a way to check whether (T4.C1) holds or not. Although the value of A^* in (T4.C1) depends on the underlying distribution of the data, we can use (T4.C2) to verify (T4.C1) without the knowledge of the data. Condition (T4.C2) means that all data-consistent submodels are compatible with each other. Equivalently, (T4.C2) also means that the set of data-consistent submodels, i.e., $\{A' \subseteq A : \Theta_I(A') \neq \emptyset\}$, is closed under the operation of taking unions: the union of data-consistent submodels remains data-consistent. One can verify (T4.C2) by deriving testable conditions that detect the violation of the assumptions. This is illustrated in the following models widely studied in the treatment effect literature. In Section 4.1.1, we consider a binary IV model, where the violation of the assumptions is known to be related to Pearl's instrumental inequalities. Condition (T4.C2) can then be verified by studying these inequalities. In Section 4.1.2, we present another way to achieve (T4.C2). In that example, all assumptions are nested, i.e., either a implies

a' or a' implies a for any two $a, a' \in A$. Therefore, the data-consistency of a set of assumptions is equal to the data-consistency of the strongest assumption in that set, which implies the validity of (T4.C2).

4.1. Some Illustrative Examples.

4.1.1. *Binary instrumental variable (IV) model.* Consider the following potential outcome model: $Y = [Y_{11}Z + Y_{10}(1-Z)]D + [Y_{01}Z + Y_{00}(1-Z)](1-D)$ where Y, D , and Z are all binary. Y_{dz} represents the potential outcome when D and Z are externally set to d and z , respectively. The parameters of interest are the average potential outcomes, i.e., $\theta_{dz} := E[Y_{dz}]$ with $\theta := (\theta_{11}, \theta_{10}, \theta_{01}, \theta_{00})$. Let's consider the following set of assumptions:⁴

- $a_1 := Y_{11} \geq Y_{10} \ \& \ Y_{dz} \perp Z, \ d, z \in \{0, 1\}$,
- $a_2 := Y_{11} \leq Y_{10} \ \& \ Y_{dz} \perp Z, \ d, z \in \{0, 1\}$,
- $a_3 := Y_{01} \geq Y_{00} \ \& \ Y_{dz} \perp Z, \ d, z \in \{0, 1\}$,
- $a_4 := Y_{01} \leq Y_{00} \ \& \ Y_{dz} \perp Z, \ d, z \in \{0, 1\}$.

Manski (1990) has studied the identification of the average potential outcomes under the IV independence and exclusion restrictions, i.e., $A = \{a_1, a_2, a_3, a_4\} = \{Y_{dz} \perp Z \ \& \ Y_{d1} = Y_{d0}, \ d \in \{0, 1\}\}$. Pearl (1994) has shown that model A is falsifiable by deriving the so called Pearl's instrumental inequalities, and Kitagawa (2021) proved that the instrumental inequalities are the most informative to detect observable violations of model A . See also the results in Kédagni and Mourifié (2020). Let us denote $q_{ij}(z) := P(Y = i, D = j | Z = z)$ for $i, j \in \{0, 1\}$. The Pearl instrumental inequalities are the following:

$$q_{11}(1) + q_{01}(0) \leq 1, \tag{4.1}$$

$$q_{11}(0) + q_{01}(1) \leq 1, \tag{4.2}$$

$$q_{10}(1) + q_{00}(0) \leq 1, \tag{4.3}$$

$$q_{10}(0) + q_{00}(1) \leq 1. \tag{4.4}$$

Model A will be rejected if at least one of these inequalities is violated. Notice that any assumption $a_i, i \in \{1, \dots, 4\}$ has a clear economic interpretation. For instance, a_1 implies that the average causal direct effect (ACDE) associated to the treatment $D = 1$ is non-negative, i.e., $ACDE(d) = \theta_{d1} - \theta_{d0} \geq 0$ for $d = 1$.⁵ In other terms, the instrument Z is not necessarily excluded from the outcome equation, but has a direct non-negative causal effect on the outcome when the treatment

⁴All the inequality restrictions on the random variables in assumptions a_1 - a_4 and other assumptions later in this paper should be understood as restrictions that hold almost surely.

⁵Please, refer to Pearl (2001) and Cai et al. (2008) for a detailed discussion on the ACDE.

is externally set to $D = 1$. One can easily show that:

$$\begin{aligned}
\Theta_I(\{a_2\}) = \emptyset &\iff (4.1) \text{ is violated,} \\
\Theta_I(\{a_1\}) = \emptyset &\iff (4.2) \text{ is violated,} \\
\Theta_I(\{a_4\}) = \emptyset &\iff (4.3) \text{ is violated,} \\
\Theta_I(\{a_3\}) = \emptyset &\iff (4.4) \text{ is violated.}
\end{aligned} \tag{4.5}$$

and (T4.C2) can be verified.⁶ By Theorem 4, there exists a unique minimum data-consistent relaxation A^* . The exact form of A^* is solved as follows:

$$A^* = \begin{cases} A & \text{if (4.1)-(4.4) hold,} \\ \{a_1, a_3, a_4\} & \text{if (4.2)-(4.4) hold, but (4.1) is violated,} \\ \{a_2, a_3, a_4\} & \text{if (4.1), (4.3), (4.4) hold, but (4.2) is violated,} \\ \{a_1, a_2, a_3\} & \text{if (4.1), (4.2), (4.4) hold, but (4.3) is violated,} \\ \{a_1, a_2, a_4\} & \text{if (4.1), (4.2), (4.3) hold, but (4.4) is violated,} \\ \{a_1, a_4\} & \text{if (4.2),(4.4) hold, but (4.1) and (4.3) are violated,} \\ \{a_1, a_3\} & \text{if (4.2),(4.3) hold, but (4.1) and (4.4) are violated,} \\ \{a_2, a_4\} & \text{if (4.2),(4.4) hold, but (4.1) and (4.3) are violated,} \\ \{a_2, a_3\} & \text{if (4.2),(4.3) hold, but (4.1) and (4.4) are violated,} \end{cases} \tag{4.6}$$

Then, we have:

$$\Theta_I(A^*) = \begin{cases} \Theta_I(A) & \text{if (4.1)-(4.4) hold,} \\ \Theta_I(\{a_1, a_3, a_4\}) & \text{if (4.2)-(4.4) hold, but (4.1) is violated,} \\ \Theta_I(\{a_2, a_3, a_4\}) & \text{if (4.1), (4.3), (4.4) hold, but (4.2) is violated,} \\ \Theta_I(\{a_1, a_2, a_3\}) & \text{if (4.1), (4.2), (4.4) hold, but (4.3) is violated,} \\ \Theta_I(\{a_1, a_2, a_4\}) & \text{if (4.1), (4.2), (4.3) hold, but (4.4) is violated,} \\ \Theta_I(\{a_1, a_4\}) & \text{if (4.2),(4.4) hold, but (4.1) and (4.3) are violated,} \\ \Theta_I(\{a_1, a_3\}) & \text{if (4.2),(4.3) hold, but (4.1) and (4.4) are violated,} \\ \Theta_I(\{a_2, a_4\}) & \text{if (4.2),(4.4) hold, but (4.1) and (4.3) are violated,} \\ \Theta_I(\{a_2, a_3\}) & \text{if (4.2),(4.3) hold, but (4.1) and (4.4) are violated,} \end{cases} \tag{4.7}$$

where

$$\Theta_I(A) = \left\{ \theta : \begin{cases} \sup_z q_{11}(z) \leq \theta_{10} = \theta_{11} \leq 1 - \sup_z q_{01}(z) \\ \sup_z q_{10}(z) \leq \theta_{01} = \theta_{00} \leq 1 - \sup_z q_{00}(z). \end{cases} \right\}$$

$$\Theta_I(\{a_1, a_3, a_4\}) = \left\{ \theta : \begin{cases} q_{11}(0) \leq \theta_{10} \leq 1 - q_{01}(0), \\ q_{11}(1) \leq \theta_{11} \leq 1 - q_{01}(1), \\ \theta_{11} - \theta_{10} \geq \max\{0, q_{11}(1) + q_{01}(0) - 1\}, \\ \sup_z q_{10}(z) \leq \theta_{01} = \theta_{00} \leq 1 - \sup_z q_{00}(z). \end{cases} \right\}$$

For the sake of conciseness, the remaining sets from $\Theta_I(\{a_2, a_3, a_4\})$ to $\Theta_I(\{a_2, a_3\})$ are relegated to Appendix A.2.

⁶Please, see Appendix A.2 for a detailed discussion.

4.1.2. *Adaptive Monotone IV (AMIV)*. Consider the following potential outcome model: $Y = \sum_{z \in \mathcal{Z}} \mathbb{1}(Z = z)[Y_{1z}D + Y_{0z}(1 - D)]$, where the treatment D is binary and the support \mathcal{Z} of instrument Z is discrete and finite. Y_{dz} is the potential outcome when the treatment and the instrument are externally set to d and z , respectively. Assume Z is one dimensional and assume, without loss of generality that $\mathcal{Z} = \{1, \dots, k\}$. We are interested in the average potential outcome $\theta_d = \sum_z P(Z = z)EY_{dz}$ for $d \in \{0, 1\}$. For any $z \in \{1, \dots, k\}$, define assumption a_z to be the collection of the following assumptions:

E.1 for each $d \in \{0, 1\}$ and any $t \in \{1, \dots, k\}$, $P(Y_{dt} \in [\underline{y}_d, \bar{y}_d]) = 1$.

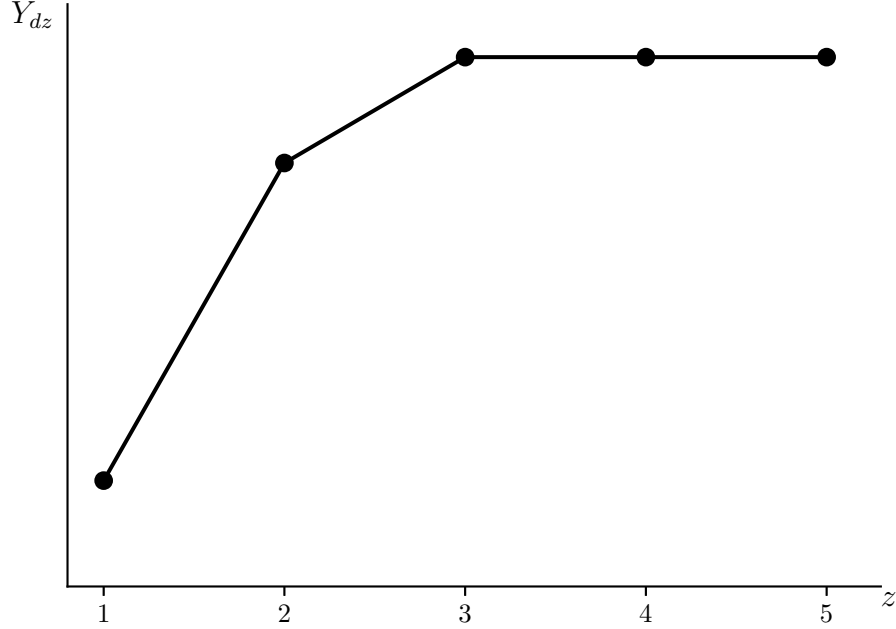
E.2 for each $d \in \{0, 1\}$ and any $t \in \{1, \dots, k\}$, $E[Y_{dt}|Z] = E[Y_{dt}]$ almost surely.

E.3 for each $d \in \{0, 1\}$, $Y_{dt} \leq Y_{dt'}$ for all $t \leq t'$, and $Y_{dt} = Y_{dz}$ for all $t \geq z$.

Each a_z has three parts. *E.1* requires the potential outcomes to have a bounded support. *E.2* is a mean independence assumption associated to the potential outcome Y_{dz} . The novelty here is *E.3* which is an adaptive relaxation of the exclusion restriction. Indeed, on one extreme case when $z = 1$, *E.3* is equivalent to the full exclusion restriction, that is, $Y_{dz} = Y_{dz'}$ for all d, z and z' , then *E.2* and *E.3* are equivalent to the mean independence assumption invoked in Manski (1990), i.e., $E[Y_d|Z] = E[Y_d]$. On the other extreme, when $z = k$, *E.2* and *E.3* imply the MIV assumption introduced in Manski and Pepper (2000), i.e., $z_1 < z_2 \Rightarrow E[Y_d|Z = z_1] \leq E[Y_d|Z = z_2]$. However, when $1 < z < k$, we are in a middle ground situation where the exclusion restriction is relaxed in such a way that $Y_{dz'}$ is monotone in z' , but remains flat for $z' \geq z$. See Figure 2 for an illustration of how Y_{dz} depends on z under *E.3*. We call this assumption the AMIV assumption because we allow this cut-off point z to be determined by the data. The economic rationality of the AMIV is that, even if Z is not a valid IV because it could positively affect the potential outcome, in some empirical context it could be reasonable to consider that the marginal effect of the IV on the potential outcome becomes null after a certain cut-off point.

By construction, for all $z = 1, \dots, k - 1$, a_z implies a_{z+1} . In addition, define a^\dagger as the collection of *E.1* and *E.2*. Let $A = \{a_1, \dots, a_k, a^\dagger\}$ be the collection of all assumptions. Then, the full model A is the classic mean independence assumption considered in Manski (1990). In addition, because assumptions in A are nested, we know that for any nonempty $A' \subseteq A$, there exists some $a^* \in A'$ such that a^* implies all assumptions in A' . Therefore, (T4.C2) holds in this example. Theorem 4 then implies that all data-consistent submodels will be compatible with each other and there exists a unique minimum data-consistent relaxation A^* . To solve the identified set of A^* , define $\underline{Y}_d = Y\mathbb{1}(D = d) + \underline{y}_d\mathbb{1}(D \neq d)$, $\bar{Y}_d = Y\mathbb{1}(D = d) + \bar{y}_d\mathbb{1}(D \neq d)$, $\underline{q}_{dz} = E[\underline{Y}_d|Z = z]$, and $\bar{q}_{dz} = E[\bar{Y}_d|Z = z]$. Then, we have the following result:

Proposition 1. *Assume that $P(Y \in [\underline{y}_d, \bar{y}_d]|D = d) = 1$ for any $d \in \{0, 1\}$. Let $\theta = (\theta_1, \theta_0)$ be the parameter of interest. Then, model A always has a unique minimum data-consistent relaxation A^* , and A^* always contains a^\dagger . In addition, for any $z = 1, \dots, k$, $a_z \in A^*$ if and only if the following*

FIGURE 2. Illustration of restriction $E.3$ when $z = 3$ and $k = 5$.

two conditions hold for each $d \in \{0, 1\}$:

$$\forall z' < z, \quad \max(\underline{q}_{dt} : t \leq z') \leq \min(\bar{q}_{dt} : t \geq z')$$

and

$$\max(\underline{q}_{dt} : t = 1, \dots, k) \leq \min(\bar{q}_{dt} : t \geq z)$$

Hence, $a_z \in A^*$ implies that $a_{z'} \in A^*$ for all $z' > z$. Moreover, if $\{z : a_z \in A^*\}$ is nonempty, define $z^* = \min\{z : a_z \in A^*\}$ and

$$\Gamma_{d,z^*} = \left[\sum_{z < z^*} P(Z = z) \max(\underline{q}_{dt}, t \leq z) + \sum_{z \geq z^*} P(Z = z) \max(\underline{q}_{dt} : t = 1, \dots, k), \right. \\ \left. \sum_{z < z^*} P(Z = z) \min(\bar{q}_{dt} : t \geq z) + \sum_{z \geq z^*} P(Z = z) \min(\bar{q}_{dt} : t \geq z^*) \right].$$

Then, $\Theta_I(A^*) = \Gamma_{1,z^*} \times \Gamma_{0,z^*}$. If $\{z : a_z \in A^*\}$ is empty, then $\Theta_I(A^*) = [E[\underline{Y}_1], E[\bar{Y}_1]] \times [E[\underline{Y}_0], E[\bar{Y}_0]]$.

Remark 2. It is worth noting that while, for simplicity, we impose the cut-off z^* to be the same for all potential outcomes in $E.3$, we do not need to do so. We can let the data determine the cut-offs for each potential outcome separately.

5. MISSPECIFICATION ROBUST BOUNDS

In this section, we consider the cases where there are multiple data-consistent relaxations. By Theorem 4, the multiplicity of data-consistent relaxations is a necessary condition of the existence of discordant submodels. Indeed, whenever there exist two mutually incompatible data-consistent submodels, there exist at least two minimum data-consistent relaxations. If there is no reason to favor one submodel over another *ex ante*, it is reasonable to take all of these relaxations into consideration.

Definition 2. Let \mathcal{A}_R be the set of all minimum data-consistent relaxations. The *misspecification robust bound* Θ_I^* is defined as $\Theta_I^* := \cup_{\tilde{A} \in \mathcal{A}_R} \Theta_I(\tilde{A})$.

The misspecification robust bound can be viewed as a special case of the falsification adaptive set concept introduced in Masten and Poirier (2020). However, a distinctive feature of this section is that we focus on discrete relaxations where an assumption is either dropped or kept, while Masten and Poirier (2020) focuses exclusively on relaxing assumptions in a continuous way. In general, the type of relaxation depends on the empirical question under study. In the following, we explore various features of discrete relaxations beyond its formal definition. Especially, we will point out circumstances under which it would be more valuable to consider only discrete relaxations.

5.1. Nonconflicting Statements. Consider a hypothetical statement on θ that $\theta \in S$ for some subset S of Θ . Then, this hypothesis would be implied by some submodel A' if $\Theta_I(A') \subseteq S$. Reversely, it would be rejected by submodel A' if $\Theta_I(A') \cap S = \emptyset$. We say the hypothetical statement that $\theta \in S$ is *nonconflicting* if

- (C1) $\theta \in S$ is implied by some data-consistent submodel. That is, there exists some submodel $A' \subseteq A$ such that $\Theta_I(A') \subseteq S$ and $\Theta_I(A') \neq \emptyset$.
- (C2) $\theta \in S$ is not rejected by any data-consistent submodel. That is, there does not exist a submodel $A' \subseteq A$ with $\Theta_I(A') \neq \emptyset$ such that $\Theta_I(A') \cap S = \emptyset$.

We use Λ to denote the collection of all nonconflicting statements:

$$\Lambda \equiv \left\{ S \subseteq \Theta : (C1) \text{ and } (C2) \text{ hold for } S \right\}.$$

Nonconflicting statements are not unique. If $S \in \Lambda$, so are its supersets. When the full model is refuted, different data-consistent submodels can imply different and potentially discordant statements on θ . Among all possible statements on θ , we think that being nonconflicting is a minimum requirement for a statement to be robust to model misspecification. If a statement fails to be nonconflicting, then either it is not implied by any of the data-consistent submodels, or it is rejected by some data-consistent submodels. In the following two theorems, we explore the linkage between nonconflicting statements and the misspecification robust bound Θ_I^* .

Theorem 5. *Suppose either (T3.C1) or (T3.C2) holds. Suppose also that $\Theta_I(a) \neq \emptyset$ for at least one $a \in A$. Then $\Theta_I^* \in \Lambda$, so that for any arbitrary nonempty subset S of Θ , $\Theta_I^* \subseteq S$ implies $S \in \Lambda$.*

Theorem 5 states that $\theta \in \Theta_I^*$ is nonconflicting. It also provides a sufficient condition for being nonconflicting: any statement that is implied by $\theta \in \Theta_I^*$ is also nonconflicting. To see what this means, consider the simple case where θ is a scalar. Suppose we are interested in the sign of θ . And, suppose Θ_I^* turns out to be within the positive real line, i.e., $\Theta_I^* \subseteq \mathbb{R}_{++}$. Then, Theorem 5 implies that the statement that θ is positive, i.e., $\theta \in \mathbb{R}_{++}$, is nonconflicting. It means that some submodels identify the sign of θ to be positive, and whenever the sign of θ can be identified by a submodel, the sign of θ is always positive.

We say S^* is the *smallest* element in Λ if $S^* \in \Lambda$ and $S^* \subseteq S$ for any $S \in \Lambda$. When Θ_I^* is the smallest element in Λ , then a statement that $\theta \in S$ is nonconflicting if and only if $\Theta_I^* \subseteq S$. In this case, Θ_I^* could have richer interpretation beyond Theorem 5. Consider the previous simple example again. Suppose it turns out that $\Theta_I^* \cap \mathbb{R}_{++} \neq \emptyset$ and $\Theta_I^* \cap \mathbb{R}_{--} \neq \emptyset$ so that $\theta \in \Theta_I^*$ does not imply the sign of θ . If we know Θ_I^* is the smallest element in Λ , then we have the following conclusion: both θ is positive and θ is negative are conflicting statements. In other words, the value of Θ_I^* in this case implies that the data and the model cannot provide a clear statement on the sign of θ . In the next theorem, we study when Θ_I^* is the smallest element of Λ .

Theorem 6. *Suppose the same assumptions in Theorem 5 hold. For the following three statements:*

- (S1) Θ_I^* is the smallest element in Λ ,
- (S2) Λ has a smallest element,
- (S3) at least one of the following two conditions is satisfied:
 - (T6.C1) there exists a unique minimum data-consistent relaxation,
 - (T6.C2) for any minimum data-consistent relaxation \tilde{A} , $\Theta_I(\tilde{A})$ is a singleton,

we have (S3) \Rightarrow (S1) \Rightarrow (S2). Moreover, if the following condition holds:

(T6.C3) *for any A' and A'' with $\Theta_I(A') \subseteq \Theta_I(A'')$ and $\Theta_I(A') \neq \emptyset$, we have $\Theta_I(A' \cup A'') \neq \emptyset$,*

then we have (S3) \Leftrightarrow (S1) \Leftrightarrow (S2).

Theorem 6 provides (S3) as a sufficient condition for Θ_I^* being the smallest element in Λ . We have discussed the case of (T6.C1) in the previous section. To understand (T6.C2), note that when the full model is misspecified, the identified set of a data-consistent submodel would gradually shrink when imposing more and more assumptions in A , and eventually shrink to an empty set. Condition (T6.C2) means that, during this process, the identified set of a submodel will always become a singleton before it becomes empty. As we will see later, this condition holds in our introductory example. When condition (T6.C3) holds, Theorem 6 states that (S3) is not only sufficient but also

necessary for Θ_I^* being the smallest element in Λ . It also implies that whenever Λ has a smallest element, the smallest element must be Θ_I^* .

Theorem 6 also shows that the only case where Λ has a smallest element but it is not equal to Θ_I^* is when (T6.C3) is violated. We do not view this as a limitation of Θ_I^* . Instead, we think the smallest element in Λ is not very reliable when (T3.C2) fails to hold. To see this, consider an example with $A = \{a_1, a_2\}$. Let F be the distribution of the observable random variables. For $i \in \{1, 2\}$, define \mathcal{G}_i to be the set of distribution G on both unobservable and observable random variables such that G satisfies assumption a_i , and that G 's marginal distribution on the observables equals F . Then, (T6.C3) would be violated if $\Theta_I(a_1) \subseteq \Theta_I(a_2)$ while $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$. In this case, $\Theta_I(A) = \emptyset$ and both $\{a_1\}$ and $\{a_2\}$ are minimum data-consistent relaxations. As a result, $\Theta_I^* = \Theta_I(a_1) \cup \Theta_I(a_2) = \Theta_I(a_2)$. Meanwhile, one can show that $\Theta_I(a_1)$ is the smallest element in Λ . In this example, Θ_I^* is still in Λ but it is no longer the smallest element in Λ . This discrepancy is due to the fact that the smallest element in Λ implicitly chooses $G \in \mathcal{G}_1$ over $G \in \mathcal{G}_2$ for informativeness, while Θ_I^* treats both $G \in \mathcal{G}_1$ and $G \in \mathcal{G}_2$ equally for robustness.

5.1.1. *Introductory example continued.* For the model (2.1), the misspecification robust bound is given in the following result:

Proposition 2. *Suppose Assumption 1 holds, then:*

$$\Theta_I^* = \begin{cases} [\underline{\gamma}, \bar{\gamma}] & \text{if } \underline{\gamma} \leq \bar{\gamma}, \\ [\bar{\gamma}, \underline{\gamma}] & \text{if } \bar{\gamma} < \underline{\gamma}, \quad P(E[\underline{Y}|Z] \leq \bar{\gamma}) > 0 \text{ and } P(E[\bar{Y}|Z] \geq \underline{\gamma}) > 0, \\ (\bar{\gamma}, \underline{\gamma}] & \text{if } \bar{\gamma} < \underline{\gamma}, \quad P(E[\underline{Y}|Z] \leq \bar{\gamma}) = 0 \text{ and } P(E[\bar{Y}|Z] \geq \underline{\gamma}) > 0, \\ [\bar{\gamma}, \underline{\gamma}) & \text{if } \bar{\gamma} < \underline{\gamma}, \quad P(E[\underline{Y}|Z] \leq \bar{\gamma}) > 0 \text{ and } P(E[\bar{Y}|Z] \geq \underline{\gamma}) = 0, \\ (\bar{\gamma}, \underline{\gamma}) & \text{if } \bar{\gamma} < \underline{\gamma}, \quad P(E[\underline{Y}|Z] \leq \bar{\gamma}) = 0 \text{ and } P(E[\bar{Y}|Z] \geq \underline{\gamma}) = 0. \end{cases} \quad (5.1)$$

Moreover, Condition (T6.C2) holds so that Θ_I^* is the smallest element in Λ , i.e., $\Theta_I^* = \bigcap_{S \in \Lambda} S$.

A direct implication of Proposition 2 is that if $P(E[\underline{Y}|Z] \leq \bar{\gamma}) > 0$ and $P(E[\bar{Y}|Z] \geq \underline{\gamma}) > 0$ hold which are mild technical requirements, the misspecification robust bound simplifies to $\Theta_I^* = [\min(\underline{\gamma}, \bar{\gamma}), \max(\underline{\gamma}, \bar{\gamma})]$ whether or not the full model is refuted.

5.1.2. *Binary IV model continued.* In the Binary IV model discussed in Section 4.1.1, $\Theta_I^* = \Theta_I(A^*)$ defined in Equation (4.7) is the unique minimum data-consistent relaxation, then Theorem 6 implies that Θ_I^* is the smallest element in Λ . Moreover, the uniqueness of the minimum data-consistent relaxation implies that we can infer the reason why the initial model is violated from the minimum data-consistent relaxation and Θ_I^* . For instance, if $\{a_1, a_3, a_4\}$ is the minimum data-consistent relaxation, we know that A was violated because $ACDE(1) = 0$ is a conflicting statement while $ACDE(1) > 0$ is a non-conflicting statement. Similar interpretations hold for the Adaptive Monotone IV example in Section 4.1.2.

5.2. Discrete Relaxation versus Continuous Relaxation. The misspecification robust bound relaxes a refuted model in a discrete way: an assumption is either fully kept or dropped during the relaxation. There are many other ways to relax and salvage a refuted model. One can also relax assumptions continuously as in Masten and Poirier (2020). In general, different relaxations will lead to different results, and it is hard to compare all the possible approaches. However, there does exist a special case where discrete relaxation always leads to stronger results than any other ways of relaxations.

In order to state our next result, we need to introduce the terminology used in Masten and Poirier (2020). For any $\epsilon \in [0, 1]$ and any $a \in A$, let a_ϵ denote the assumption after relaxing assumption a . The degree of relaxation is measured by ϵ : when $\epsilon = 0$, $a_\epsilon = a$; when $\epsilon \in (0, 1)$, the assumption a is partially relaxed but the exact form of a_ϵ would depend on the specific way of relaxation chosen by the researcher; when $\epsilon = 1$, the assumption a is completely relaxed and a_ϵ is a null assumption which does not impose any restriction. Assume the relaxation is monotone: if $\epsilon_1 \leq \epsilon_2$, a_{ϵ_1} is stronger than a_{ϵ_2} , in the sense that a_{ϵ_1} implies a_{ϵ_2} . For any $\delta : A \rightarrow [0, 1]$, define $A(\delta) := \{a_{\delta(a)} : a \in A\}$ as the perturbed full model. For any two $\delta_1 : A \rightarrow [0, 1]$ and $\delta_2 : A \rightarrow [0, 1]$, we write $\delta_1 < \delta_2$ if $\delta_1(a) \leq \delta_2(a)$ for all $a \in A$ and $\delta_1(a) < \delta_2(a)$ for some $a \in A$. Then, the *falsification frontier* in Masten and Poirier (2020) can be defined as $FF = \{\delta : A \rightarrow [0, 1] : \Theta_I(A(\delta)) \neq \emptyset \text{ and there does not exist } \delta' \text{ such that } \Theta_I(A(\delta')) \neq \emptyset, \Theta_I(A(\delta')) \subsetneq \Theta_I(A(\delta)) \text{ and } \delta' < \delta\}$. We slightly modified the definition of the falsification frontier of Masten and Poirier (2020) to ensure the nonemptiness of FF in some special cases.⁷ Then, the *falsification adaptive set* Θ_I^\dagger is defined as $\Theta_I^\dagger = \cup_{\delta \in FF} \Theta_I(A(\delta))$.

Note that Θ_I^\dagger depends on the specific way that one chooses to relax the assumptions. If one chooses to relax them discretely, i.e., if $a_\epsilon = a_{\mathbb{1}(\epsilon > 0)}$ for any ϵ and a , then Θ_I^\dagger is equal to the minimum data-consistent relaxation Θ_I^* . If one chooses a different way of relaxation, the Θ_I^\dagger is generally different. In some special cases, however, Θ_I^* is always included in Θ_I^\dagger no matter which way of relaxation is chosen.

Theorem 7. *Suppose Condition (S3) in Theorem 6 hold. Then, $\Theta_I^* \subseteq \Theta_I^\dagger$ for any type of relaxation chosen by the researcher.*

This theorem implies that Θ_I^* provides the most informative results among all possible relaxations considered above when (S3) holds. In this case, Theorem 6 also shows that Θ_I^* has the interpretation of the smallest non-conflicting statement. These results suggest that Θ_I^* provides a good way to

⁷The original definition in Masten and Poirier (2020), written in our notation, is $FF = \{\delta : A \rightarrow [0, 1] : \Theta_I(A(\delta)) \neq \emptyset \text{ and there does not exist } \delta' \text{ such that } \Theta_I(A(\delta')) \neq \emptyset \text{ and } \delta' < \delta\}$. With our modified definition, we do not need to worry about the possibility that there is a sequence of $\{\delta_i : i \geq 1\}$ such that $\delta_n \rightarrow \delta^*$, $\Theta_I(A(\delta^*)) = \emptyset$, $\Theta_I(A(\delta_n)) = \Theta_I(A(\delta_1)) \neq \emptyset$ and $\delta_{n+1} < \delta_n$ for all $n \geq 1$.

salvage a refuted model in the case of (S3). In particular, an important subcase of (S3) is when there exists a unique minimum data-consistent relaxation.

6. AN EMPIRICAL ILLUSTRATION

6.1. Context and Data. Estimating the causal impact of college education on later earnings has always been troublesome for economists because of endogeneity of the level of education. To evaluate the returns to schooling, different approaches have been proposed, and most of them rely on the validity of instruments such as parental education, tuition fees, quarter of birth, distance to college, etc. The validity of all these IVs have been widely criticized because of their potential correlation with children unobserved skills. In order to accommodate potentially invalid instruments, Manski and Pepper (2000, 2009) introduced the monotone IV (MIV) that does not require the IV to be valid but only impose a positive dependence relationship between the IV and the potential earnings. For instance, parental education may not be independent of potential wages, but plausibly does not negatively affect future earnings. In such a context, bounds on the average return to education can be derived. In this application, we will consider the AMIV assumption introduced in Section 4.1.2. Therefore, we consider that parental education can have a positive effect on children future earnings, but this marginal positive effect could plausibly becomes null after some cut-off. The particularity of our method is to let this cut-off be determined by the data using our misspecification robust bounds.

We consider the data used in Heckman, Tobias, and Vytlačil (2001, HTV). The data consists of a sample of 1,230 white males taken from the National Longitudinal Survey of Youth of 1979 (NSLY). The data contains information on the log weekly wage, college education, father’s education, mother’s education, among many other variables. Following HTV, we consider the college enrolment indicator as the treatment: it is equal to 1 if the individual has completed at least 13 years of education and 0 otherwise. In this empirical exercise, we use the maximum of the parental education as the candidate instrumental variable. Some summary statistics are reported in Table 1.

TABLE 1. Summary Statistics

	Total
Observations	1,230
log wage	2.4138 (0.5937)
college	0.4325 (0.4956)
father’s education	12.44715 (3.2638)
mother’s education	12.1781 (2.2781)
max(father’s education, mother’s education)	13.1699 (2.7123)

Average and standard deviation (in the parentheses)

6.2. Methodology and results. We start by constructing the 95% confidence region for the identified sets of the average structural functions $E[Y_d]$, $d \in \{0, 1\}$ and the average treatment effect $E[Y_1 - Y_0]$ under the Manski (1990) mean independence assumption, denoted as $\Theta_I(MI)$, and under the MIV assumption, denoted as $\Theta_I(MIV)$. In addition we construct an estimate of our misspecification robust bounds under the AMIV assumption, denoted as $\Theta_I^*(AMIV)$. As in Proposition 1, let a_{z^*} denote the strongest assumption in the minimum data-consistent relaxation. Recall that when z^* is equal to the lowest value in the support of Z , $\Theta_I^*(AMIV)$ is equal to $\Theta_I(MI)$. And, when z^* is equal to the highest value in the support of Z , $\Theta_I^*(AMIV)$ is equal to $\Theta_I(MIV)$.

The results are summarized in Table 2. In column (1), we compute the 95% confidence region for the ATE under the mean independence assumption, which turns out to be empty, meaning that the data shows clear evidence against the use of parental education as a valid IV. In column (4), we consider the MIV assumption, we first test the validity of the MIV using the test proposed by Hsu, Liu, and Shi (2019), we do not reject the MIV assumption even at 10 % level, then estimates of $\Theta_I(MIV)$ are presented in column (4). As can be seen, when using the Manski and Pepper (2000) MIV relaxation, we move from an empty identification region to a wide and non-informative identification region. However, our misspecification robust bounds provide relatively smaller set estimate for the ATE. Column (3) shows estimates of our misspecification robust bounds when the same cut-off is imposed for both potential outcomes as in Proposition 1, while column (2) shows estimates when we allow the cut-off to differ from potential outcomes as discussed in Remark 2.⁸ In the latter case, we see that the proposed approach almost identifies the sign of the ATE.

TABLE 2. Results

Set estimates/ 95% Conf. Bounds	(1)	(2)	(3)	(4)
	$\Theta_I(MI)$	$\Theta_I^*(AMIV)$ $(z_1^*, z_0^*) = (0, 11)$	$\Theta_I^*(AMIV)$ $(z_1^*, z_0^*) = (11, 11)$	$\Theta_I(MIV)$
$\theta_1 \equiv \mathbb{E}[Y_1]$	[2.535, 2.815]	[2.535, 2.815]	[2.412, 2.816]	[0.933, 2.815]
$\theta_0 \equiv \mathbb{E}[Y_0]$	Empty	[2.547, 2.591]	[2.547, 2.591]	[2.548, 2.814]
$ATE \equiv \mathbb{E}[Y_1 - Y_0]$	Empty	[-0.056, 0.268]	[-0.179, 0.269]	[-1.881, 0.267]

¹ All values in column (1) are the 95% confidence intervals.

² All values in column (2)-(4) are set estimates based on the 95% confidence interval of $\Theta_I(a_z)$.

7. DISCUSSION

In this paper, we show that there could exist discordant submodels in a wide range of models in the presence of model misspecification. This provides another reason why one should use the sharp

⁸Because our primarily focus in this paper is about identification, we do not attempt to study the statistical issues related to the derivation of a valid confidence region for the misspecification robust bound. We leave this open question for future research.

characterization of the identified set whenever it is possible: the identified set not only exhausts all the identification restrictions in the model structure and assumptions, but also is immune to the possible misleading conclusions of discordant submodels. Unlike an outer set, the identified set will be empty when the model is refuted by the data. In empirical applications where sharp characterization of the identified set is not tractable, our results suggest that empirical researchers should be more careful when working with outer sets, especially when the bounds that they get are very tight. For example, as a robustness check, one could construct the outer sets in different ways and see if there is any discordance between these outer sets.

Salvaging a refuted model is usually a challenging task, as it often involves some arbitrariness in how the model gets relaxed, and it could sometimes be computationally intractable. However, things get much easier when the minimum data-consistent relaxation is unique. In this case, it is apparent which assumptions are consistent with the data and which assumptions not, because all the data-consistent assumptions are compatible (Theorem 4). This result has a very good interpretation (Theorem 6). Moreover, the identified set of any data-consistent submodel can be viewed as a conservative bound for the misspecification robust bound, making the computation a lot easier. In practice, there often exist multiple different ways to disassemble the same model. For example, the construction of assumptions considered in our binary IV example is just one way to represent the independence and exclusion restriction on IV. Among all these possibilities, we suggest choosing the one that leads to a unique minimum data-consistent relaxation when possible.

Even when the uniqueness of minimum data-consistent relaxation is beyond the reach, one can still choose to find the misspecification robust bound we proposed in this paper. It always leads to non-conflicting statements (Theorem 5), and it is sometimes the most informative non-conflicting statement (Theorem 6). We work out the misspecification robust bound in some simple examples, but its exact solution could be too complicated to solve when the underlying model involves many structures. In those challenging cases, it might be possible to construct an outer set that always covers the misspecification robust bound proposed in this paper. This type of outer sets will be immune to the issue raised in this paper. It remains unclear how to construct such outer sets, but this could be one reasonable step beyond the findings in this paper.

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APPENDIX A. ADDITIONAL RESULTS

A.1. Conditional Moment Inequalities. Let us now consider a more general setting than the introductory example. Assume the full model is a conditional moment inequality,

$$E[m(X, Z; \theta)|Z] \leq 0 \text{ almost surely} \quad (\text{A.1})$$

where $X \in \mathbb{R}^{k_1}$ and $Z \in \mathbb{R}^{k_2}$ are observable random variables and $m(\cdot, \cdot; \theta)$ is some known integrable function with $E\|m(X, Z; \theta)\| < \infty$ for each θ . In practice, empirical researchers sometimes use the following unconditional model instead:

$$E[w(Z)m(X, Z; \theta)] \leq 0, \quad (\text{A.2})$$

where $w(\cdot)$ is some nonnegative weighting function. We want to understand what would happen when one conduct empirical analysis based on (A.2) when (A.1) happens to be refuted.

To answer this question, define \mathcal{W}_m^+ to be the set of all m -dimensional nonnegative function w which satisfies $0 < E\|w(Z)\|^2 < \infty$ and $E\|w(Z)m(X, Z; \theta)\| < \infty$ for all $\theta \in \Theta$. Define A to be the set of all submodels indexed by $w \in \mathcal{W}_m^+$. Then, each $B \subseteq A$ with m elements corresponds to the submodel which (A.2) hold for some $w \in \mathcal{W}_m^+$. By construction, Condition (T2.C2) are satisfied. What is left to check is whether Condition (T2.C1) and (T2.C3) are satisfied.

To verify Condition (T2.C1), we need to construct a A^* so that $\Theta_I(A^*) = \emptyset$ when the full model is refuted and $\Theta_I(a) \neq \emptyset$ for all $a \in A^*$. Suppose for any z_0 in the support of Z , there exists some $\theta_0 \in \Theta$ and some $\delta(z_0) > 0$ such that $E[m(X, Z; \theta_0)|Z] \leq 0$ for almost every Z with $\|Z - z_0\| \leq \delta(z_0)$. Then, we know the following submodel:

$$E[\mathbb{1}(\|Z - z_0\| < \delta(z_0))m(X, Z; \theta)] \leq 0$$

is always data-consistent. Therefore, if we define function $\delta_{z_0, \epsilon}$ as $\delta_{z_0, \epsilon}(z) = \mathbb{1}(\|z - z_0\| < \epsilon)$ and the collection of functions \mathcal{W}^* as $\mathcal{W}^* := \{\delta_{z, \epsilon} : \epsilon \in [0, \delta(z)], z \in \mathcal{Z}\}$ where \mathcal{Z} is the support of Z , then the A^* can be constructed as the set of submodels indexed by $w \in \mathcal{W}^*$ with which (A.2) hold. Under the regularity conditions listed in the following proposition, we can also ensure $\Theta_I(A^*) = \emptyset$ whenever $\Theta_I(A) = \emptyset$.

Proposition 3. *Let \mathcal{Z} be the support of Z . Suppose*

- (a) $E\|m(X, Z; \theta)\| < \infty$ for each $\theta \in \Theta$.
- (b) either \mathcal{Z} is discrete, or $E[m(X, Z; \theta)|Z = z]$ is continuous⁹ in z .
- (c) for any z_0 in the support of Z , there exists some neighborhood Ω_0 of z_0 and some $\theta_0 \in \Theta$ such that $E[m(X, Z; \theta_0)|Z] \leq 0$ for all most every Z in Ω_0 .

Then, Condition (T2.C1) of Theorem 2 is satisfied.

Note that a sufficient condition for Condition (c) in the above proposition is that for any z_0 in the support, there exists some $\theta_0 \in \Theta$ satisfying $E[m(X, Z; \theta_0)|Z = z_0] < 0$.

Finally, let us verify Condition (T2.C3) of Theorem 2. Suppose Θ is compact. Then, for any $a \in A$, we can ensure $\Theta_I(a)$ is compact if $E[w(Z)m(X, Z; \theta)]$ is a continuous function of θ for every $w \in \mathcal{W}_1^+$. A simple sufficient condition for $E[w(Z)m(X, Z; \theta)]$ to be continuous is that $E[m(X, Z; \theta)|Z]$ is continuous in θ almost surely and it is dominated by some integrable function.

Proposition 4. *Suppose*

- (a) $E[m(X, Z; \theta)|Z = z]$ is continuous in θ almost surely.
- (b) there exists some function $g(\cdot)$ such that $\sup_{\theta \in \Theta} \|E[m(X, Z; \theta)|Z]\| \leq g(Z)$ almost surely and $E[g(Z)]^2 < \infty$.
- (c) Θ is compact.

Then, Condition (T2.C3) of Theorem 2 is satisfied.

Combine these two propositions, we have the following result as a corollary of Theorem 2.

Corollary 2. *Suppose*

- (a) either \mathcal{Z} is discrete, or $E[m(X, Z; \theta)|Z = z]$ is continuous in z .
- (b) $E[m(X, Z; \theta)|Z = z]$ is continuous in θ almost surely.

⁹A more precise statement is that there exists some $g(z; \theta)$ such that $g(z; \theta)$ is continuous in z and $E[m(X, Z; \theta)|Z = z] = g(z; \theta)$ almost surely.

- (c) for any z_0 in the support of Z , there exists some neighborhood Ω_0 of z_0 and some $\theta_0 \in \Theta$ such that $E[m(X, Z; \theta_0)|Z] \leq 0$ for all most every Z in Ω_0 .
- (d) there exists some function $g(\cdot)$ such that $\sup_{\theta \in \Theta} \|E[m(X, Z; \theta)|Z]\| \leq g(Z)$ almost surely and $E|g(Z)|^2 < \infty$.
- (e) Θ is compact.

Then, (A.1) is refuted if and only if there exists $w_1 \in \mathcal{W}_{m_1}^+$ and $w_2 \in \mathcal{W}_{m_2}^+$ such that both the submodel (A.2) with $w = w_1$ and the submodel (A.2) with $w = w_2$ are not refuted but the identified sets of these two submodels have empty intersection.

Moreover, whenever (A.1) is refuted, for any $\tilde{w} \in \mathcal{W}_m^+$ with which submodel (A.2) is refuted, there exists some $w_1 \in \mathcal{W}_{m_1}^+$ and $w_2 \in \mathcal{W}_{m_2}^+$ such that both the submodel (A.2) with $w = (\tilde{w}, w_1)$ and the submodel (A.2) with $w = w_2$ are data-consistent but the identified sets of these two submodels have empty intersection.

This result complements the findings in Andrews and Shi (2013). In Andrews and Shi (2013), they propose an inference procedure for models like (A.1). Their inference transform (A.1) into (A.2) by selecting w in a sub-family of $\cup_{m \geq 1} \mathcal{W}_m^+$ and letting $m \rightarrow \infty$ as the sample size increases. Our result shows that increasing m to infinity is crucial to ensure the robustness of the result if (A.1) could be misspecified. If the dimension of w is fixed, then the empirical result for (A.2) could be misleading even if the inference controls the size uniformly. Since a finite m is often used in the finite sample, our result also raise the question how to interpret the result of a test under possible model misspecification.

A.2. Binary IV example continued. First, notice that $Y_{dz} \perp Z$ and $Y = [Y_{11}Z + Y_{10}(1-Z)]D + [Y_{01}Z + Y_{00}(1-Z)](1-D)$ imply the following inequalities:

$$q_{11}(0) \leq \theta_{10} \leq 1 - q_{01}(0), \quad (\text{A.3})$$

$$q_{11}(1) \leq \theta_{11} \leq 1 - q_{01}(1), \quad (\text{A.4})$$

$$q_{10}(0) \leq \theta_{00} \leq 1 - q_{00}(0), \quad (\text{A.5})$$

$$q_{10}(1) \leq \theta_{01} \leq 1 - q_{00}(1). \quad (\text{A.6})$$

The identified set of each of our single assumptions a_i , for $i \in \{1, 2, 3, 4\}$ is as follows:

$$\Theta_I(\{a_1\}) = \{\theta : (\text{A.3}) - (\text{A.6}) \text{ hold and } \theta_{11} - \theta_{10} \geq \max\{0, q_{11}(1) + q_{01}(0) - 1\}\}$$

$$\Theta_I(\{a_2\}) = \{\theta : (\text{A.3}) - (\text{A.6}) \text{ hold and } \theta_{11} - \theta_{10} \leq \min\{0, 1 - q_{01}(1) - q_{11}(0)\}\}$$

$$\Theta_I(\{a_3\}) = \{\theta : (\text{A.3}) - (\text{A.6}) \text{ hold and } \max\{0, q_{10}(1) + q_{00}(0) - 1\} \leq \theta_{01} - \theta_{00}\}$$

$$\Theta_I(\{a_4\}) = \{\theta : (\text{A.3}) - (\text{A.6}) \text{ hold and } \theta_{01} - \theta_{00} \leq \min\{0, 1 - q_{00}(1) - q_{11}(0)\}\}$$

It can be easily seen that

$\Theta_I(\{a_1\}) \cap \Theta_I(\{a_2\}) = \{\theta : (\text{A.3}) - (\text{A.6}) \text{ hold and } \max\{0, q_{11}(1) + q_{01}(0) - 1\} \leq \theta_{11} - \theta_{10} \leq \min\{0, 1 - q_{01}(1) - q_{11}(0)\}\} = \{\theta : (\text{A.3}) - (\text{A.6}) \text{ hold and } \theta_{11} = \theta_{10}\} = \Theta_I(\{a_1, a_2\})$. In addition, remark that when Eqs (A.3)-(A.4) hold $\theta_{11} - \theta_{10} \leq 1 - q_{01}(1) - q_{11}(0)$ which implies that if $\Theta_I(\{a_1\}) = \emptyset \iff (4.2)$ is violated. Similarly, we can show that $\Theta_I(\{a_3, a_4\}) = \Theta_I(\{a_2\}) \cap \Theta_I(\{a_4\})$. Furthermore, notice that when $Y_{dz} \perp Z$, assumptions a_1, a_2 which concern only Y_{1z} are entirely orthogonal to a_3 , and a_4 which concern only Y_{0z} . Therefore, for any $A_1 \subseteq \{a_1, a_2\}$ and $A_2 \subseteq \{a_3, a_4\}$, if A_1 and A_2 are data-consistent, so is $A_1 \cup A_2$. The above results imply (T4.C2) holds, which completes the proof. Finally, we can easily derived the following identified sets:

$$\Theta_I(\{a_2, a_3, a_4\}) = \left\{ \theta : \begin{cases} q_{11}(0) \leq \theta_{10} \leq 1 - q_{01}(0), \\ q_{11}(1) \leq \theta_{11} \leq 1 - q_{01}(1), \\ \theta_{11} - \theta_{10} \leq \min\{0, 1 - q_{01}(1) - q_{11}(0)\}, \\ \sup_z q_{10}(z) \leq \theta_{01} = \theta_{00} \leq 1 - \sup_z q_{00}(z). \end{cases} \right\}$$

$$\Theta_I(\{a_2, a_3, a_4\}) = \left\{ \theta : \begin{cases} q_{11}(0) \leq \theta_{10} \leq 1 - q_{01}(0), \\ q_{11}(1) \leq \theta_{11} \leq 1 - q_{01}(1), \\ \theta_{11} - \theta_{10} \leq \min\{0, 1 - q_{01}(1) - q_{11}(0)\}, \\ \sup_z q_{10}(z) \leq \theta_{01} = \theta_{00} \leq 1 - \sup_z q_{00}(z). \end{cases} \right\}$$

$$\Theta_I(\{a_1, a_2, a_3\}) = \left\{ \theta : \begin{cases} q_{10}(0) \leq \theta_{00} \leq 1 - q_{00}(0), \\ q_{10}(1) \leq \theta_{01} \leq 1 - q_{00}(1), \\ \max\{0, q_{10}(1) + q_{00}(0) - 1\} \leq \theta_{01} - \theta_{00} \\ \sup_z q_{11}(z) \leq \theta_{10} = \theta_{11} \leq 1 - \sup_z q_{01}(z). \end{cases} \right\}$$

$$\Theta_I(\{a_1, a_2, a_4\}) = \left\{ \theta : \begin{cases} q_{10}(0) \leq \theta_{00} \leq 1 - q_{00}(0), \\ q_{10}(1) \leq \theta_{01} \leq 1 - q_{00}(1), \\ \theta_{01} - \theta_{00} \leq \min\{0, 1 - q_{00}(1) - q_{10}(0)\}, \\ \sup_z q_{11}(z) \leq \theta_{10} = \theta_{11} \leq 1 - \sup_z q_{01}(z). \end{cases} \right\}$$

$$\Theta_I(\{a_1, a_4\}) = \left\{ \theta : \begin{cases} q_{11}(0) \leq \theta_{10} \leq 1 - q_{01}(0), \\ q_{11}(1) \leq \theta_{11} \leq 1 - q_{01}(1), \\ \max\{0, q_{11}(1) + q_{01}(0) - 1\} \leq \theta_{11} - \theta_{10} \leq 1 - q_{01}(1) - q_{11}(0), \\ q_{10}(0) \leq \theta_{00} \leq 1 - q_{00}(0), \\ q_{10}(1) \leq \theta_{01} \leq 1 - q_{00}(1), \\ \theta_{01} - \theta_{00} \leq \min\{0, 1 - q_{00}(1) - q_{11}(0)\}. \end{cases} \right\}$$

$$\Theta_I(\{a_1, a_3\}) = \left\{ \theta : \begin{cases} q_{11}(0) \leq \theta_{10} \leq 1 - q_{01}(0), \\ q_{11}(1) \leq \theta_{11} \leq 1 - q_{01}(1), \\ \max\{q_{11}(1) + q_{01}(0) - 1, 0\} \leq \theta_{11} - \theta_{10}, \\ q_{10}(0) \leq \theta_{00} \leq 1 - q_{00}(0), \\ q_{10}(1) \leq \theta_{01} \leq 1 - q_{00}(1), \\ \max\{0, q_{10}(1) + q_{00}(0) - 1\} \leq \theta_{01} - \theta_{00}. \end{cases} \right\}$$

$$\Theta_I(\{a_2, a_4\}) = \left\{ \theta : \begin{cases} q_{11}(0) \leq \theta_{10} \leq 1 - q_{01}(0), \\ q_{11}(1) \leq \theta_{11} \leq 1 - q_{01}(1), \\ \theta_{11} - \theta_{10} \leq \min\{1 - q_{01}(1) - q_{11}(0), 0\}, \\ q_{10}(0) \leq \theta_{00} \leq 1 - q_{00}(0), \\ q_{10}(1) \leq \theta_{01} \leq 1 - q_{00}(1), \\ \theta_{01} - \theta_{00} \leq \min\{1 - q_{00}(1) - q_{11}(0), 0\}. \end{cases} \right\}, \Theta_I(\{a_2, a_3\}) = \left\{ \theta : \begin{cases} q_{11}(0) \leq \theta_{10} \leq 1 - q_{01}(0), \\ q_{11}(1) \leq \theta_{11} \leq 1 - q_{01}(1), \\ \theta_{11} - \theta_{10} \leq \min\{1 - q_{01}(1) - q_{11}(0), 0\}, \\ q_{10}(0) \leq \theta_{00} \leq 1 - q_{00}(0), \\ q_{10}(1) \leq \theta_{01} \leq 1 - q_{00}(1), \\ \max\{0, q_{10}(1) + q_{00}(0) - 1\} \leq \theta_{01} - \theta_{00}. \end{cases} \right\}$$

APPENDIX B. PROOF OF THE MAIN RESULTS

B.1. **Proof of Theorem 1.** Theorem 1 is an immediate result of the following two lemmas.

Lemma 2. Suppose Assumption 1 hold and $\bar{\gamma} < \underline{\gamma}$. Define the interval \mathcal{W} as the following:

$$\mathcal{W} := \begin{cases} [\bar{\gamma}, \underline{\gamma}] & \text{if } P(E[\bar{Y}|Z] = \bar{\gamma}) > 0 \text{ and } P(E[\underline{Y}|Z] = \underline{\gamma}) > 0 \\ [\bar{\gamma}, \underline{\gamma}] & \text{if } P(E[\bar{Y}|Z] = \bar{\gamma}) > 0 \text{ and } P(E[\underline{Y}|Z] = \underline{\gamma}) = 0 \\ (\bar{\gamma}, \underline{\gamma}] & \text{if } P(E[\bar{Y}|Z] = \bar{\gamma}) = 0 \text{ and } P(E[\underline{Y}|Z] = \underline{\gamma}) > 0 \\ (\bar{\gamma}, \underline{\gamma}) & \text{if } P(E[\bar{Y}|Z] = \bar{\gamma}) = 0 \text{ and } P(E[\underline{Y}|Z] = \underline{\gamma}) = 0 \end{cases} \quad (\text{B.1})$$

For any integer m and any $h \in \mathcal{H}_m^+$, if $\tilde{\Theta}(h)$ is nonempty, then $\tilde{\Theta}(h) \cap \mathcal{W}$ is nonempty.

Proof of Lemma 2. Since h has m dimensions, we can write $h = (h_1, \dots, h_m)$. Then, $\tilde{\Theta}(h)$ can be characterized as $\tilde{\Theta}(h) = [\underline{\theta}, \bar{\theta}]$, where

$$\underline{\theta} = \max_i \frac{E[h_i(Z)\underline{Y}]}{E[h_i(Z)]} \quad \text{and} \quad \bar{\theta} = \min_i \frac{E[h_i(Z)\bar{Y}]}{E[h_i(Z)]}.$$

Let us first prove $\underline{\theta} \leq \underline{\gamma}$ by contradiction. Suppose $\delta := \underline{\theta} - \underline{\gamma} > 0$. Let $i' \in \arg \max_i E[h_i(Z)\underline{Y}]/E[h_i(Z)]$. Then, we have

$$E[h_{i'}(Z)(E[\underline{Y}|Z] - \underline{\theta})] = 0$$

Since $E[\underline{Y}|Z] - \underline{\theta} \leq -\delta$ and $h_{i'}$ is nonnegative, we have $E[h_{i'}(Z)\delta] \leq 0$, which contradicts to the fact that $\delta > 0$ and $E[h_{i'}(Z)] > 0$. Moreover, if $P(E[\underline{Y}|Z] = \underline{\gamma}) = 0$, then $E[h_i(Z)\underline{Y}] < \underline{\gamma} \cdot E[h_i(Z)]$ for all i so that $\underline{\theta} < \underline{\gamma}$.

Similarly, we can show $\bar{\theta} \geq \bar{\gamma}$, and that $\bar{\theta} > \bar{\gamma}$ if $P(E[\bar{Y}|Z] = \bar{\gamma}) = 0$. These result then implies that $\tilde{\Theta}(h) \cap \mathcal{W} \neq \emptyset$ whenever $\tilde{\Theta}(h) \neq \emptyset$. \square

Lemma 3. Suppose Assumption 1 hold and $\bar{\gamma} < \underline{\gamma}$. Let \mathcal{W} be the interval defined as in (B.1). Then, for any $\theta \in \mathcal{W}$, there exists some $h \in \mathcal{H}_2^+$ such that $\tilde{\Theta}(h) = \{\theta\}$.

Proof of Lemma 3. Fix any $\theta \in \mathcal{W}$. Define $\underline{\mathcal{S}}^+ = \{z : E[\underline{Y}|Z = z] \geq \theta\}$, $\underline{\mathcal{S}}^- = \{z : E[\underline{Y}|Z = z] \leq \theta\}$, $\overline{\mathcal{S}}^+ = \{z : E[\overline{Y}|Z = z] \geq \theta\}$ and $\overline{\mathcal{S}}^- = \{z : E[\overline{Y}|Z = z] \leq \theta\}$. Note that, for any $\vartheta > \overline{\vartheta}$, the definition of $\overline{\vartheta}$ implies that $P(\vartheta \geq E[\overline{Y}|Z]) > 0$. When $P(E[\overline{Y}|Z] = \overline{\vartheta}) > 0$, for any $\vartheta \geq \overline{\vartheta}$, we also have $P(\vartheta \geq E[\overline{Y}|Z]) > 0$. Since $\theta \in \mathcal{W}$, we conclude that $P(Z \in \overline{\mathcal{S}}^-) > 0$. Similarly, that $\theta \in \mathcal{W}$ also implies that $P(Z \in \underline{\mathcal{S}}^+) > 0$. Moreover, since $E[\underline{Y}|Z] \leq E[\overline{Y}|Z]$ almost surely, we know $\underline{\mathcal{S}}^+ \subseteq \overline{\mathcal{S}}^+$ and $\overline{\mathcal{S}}^- \subseteq \underline{\mathcal{S}}^-$ almost surely. Therefore, $P(Z \in \underline{\mathcal{S}}^-) > 0$ and $P(Z \in \overline{\mathcal{S}}^+) > 0$.

Next, we show there exists some nonnegative function h_1 which satisfies $E[\underline{Y}h_1(Z)] = \theta$ and $E[h_1(Z)] = 1$. Define $h_1^+(z) = \mathbf{1}(z \in \underline{\mathcal{S}}^+)/P(Z \in \underline{\mathcal{S}}^+)$ and $h_1^-(z) = \mathbf{1}(z \in \underline{\mathcal{S}}^-)/P(Z \in \underline{\mathcal{S}}^-)$. By construction, h_1^+ and h_1^- are nonnegative, and $E[h_1^+(Z)] = 1$ and $E[h_1^-(Z)] = 1$. Moreover, $E[\underline{Y}h_1^+(Z)] \geq \theta \geq E[\underline{Y}h_1^-(Z)]$. Hence, there must exist some $q \in [0, 1]$ such that $E[\underline{Y}(qh_1^-(Z) + (1-q)h_1^+(Z))] = \theta$. Let $h_1 = qh_1^-(Z) + (1-q)h_1^+(Z)$. Then, such h_1 satisfies $E[\underline{Y}h_1(Z)] = \theta$ and $E[h_1(Z)] = 1$. Similarly, there exists some nonnegative function h_2 which satisfies $E[\overline{Y}h_2(Z)] = \theta$ and $E[h_2(Z)] = 1$.

Then, $E[h_1(Z)(\tilde{\theta} - \underline{Y})] \geq 0$ is equivalent to $\tilde{\theta} \geq \theta$. To see this, note that

$$\begin{aligned} E[h_1(Z)(\tilde{\theta} - \underline{Y})] &\geq 0 \\ \Leftrightarrow E[h_1(Z)]\tilde{\theta} &\geq E[h_1(Z)\underline{Y}] \\ \Leftrightarrow \tilde{\theta} &\geq \theta \end{aligned}$$

where the second equivalence follows from $E[\underline{Y}h_1(Z)] = \theta$ and $E[h_1(Z)] = 1$. Similarly, we can show $E[h_2(Z)(\overline{Y} - \tilde{\theta})] \geq 0$ is equivalent to $\tilde{\theta} \leq \theta$. Let $h = (h_1, h_2)$. These equivalence relation implies that if $\tilde{\theta} \in \tilde{\Theta}(h)$, then $\tilde{\theta} = \theta$.

Moreover, we have

$$\begin{aligned} E[h_2(Z)\theta] &= \theta \\ &= E[h_2(Z)\overline{Y}] \\ &\geq E[h_2(Z)\underline{Y}] \end{aligned}$$

where the first equality follows from $E[h_2(Z)] = 1$, and the second equality follows from $\theta = E[h_2(Z)\overline{Y}]$, and the last inequality comes from $E[\underline{Y}|Z] \leq E[\overline{Y}|Z]$ almost surely. Similarly, we can show $E[h_1(Z)\theta] \leq E[h_1(Z)\overline{Y}]$. Therefore, $\theta \in \tilde{\Theta}(h)$. As a result, $\tilde{\Theta}(h) = \{\theta\}$. \square

B.2. Proof of Theorem 2. To prove Theorem 2, we need the following lemma.

Lemma 4. *Suppose Θ is a subset in a metric space. Suppose also that at least one of the following conditions hold:*

- (1) *A is a finite set.*
- (2) *For any $a \in A$, $\Theta_I(a)$ is compact. Moreover, for any $B \subseteq A$, $\Theta_I(B) = \bigcap_{a \in B} \Theta_I(a)$.*

Then, for any submodel $A_0 \subseteq A$ with $\Theta_I(A_0) \neq \emptyset$, there exists some $\tilde{A} \subseteq A$ such that (i) $A_0 \subseteq \tilde{A}$, (ii) $\Theta_I(\tilde{A}) \neq \emptyset$, (iii) for any $a \in A \setminus \tilde{A}$, $\Theta_I(\tilde{A} \cup \{a\}) = \emptyset$.

Proof of Lemma 4. Let A_0 be an arbitrary subset of A with $\Theta_I(A_0) \neq \emptyset$. If $\Theta_I(A) \neq \emptyset$, then we can let $\tilde{A} = A$ and the results hold trivially. In the following, we focus on cases where $\Theta_I(A) = \emptyset$.

Suppose A is finite. Then, A_0 is also finite, enumerate it as $A_0 = \{a_1, \dots, a_k\}$. Moreover, $\{a \in A : \Theta_I(a) \neq \emptyset\}$ is also finite and enumerate it as $\{a_1, \dots, a_k, a_{k+1}, \dots, a_n\}$. Construct B_i , $i = k, k+1, \dots, n$ recursively as follows. Let $B_k = A_0$. For any $i \geq k$, let B_{i+1} equal B_i if $\Theta_I(B_i \cup \{a_{i+1}\}) = \emptyset$ and equal $B_i \cup \{a_{i+1}\}$ if otherwise. By construction, $A_0 \subseteq B_i$ and $\Theta_I(B_i) \neq \emptyset$ for each $i = k, \dots, n$. Moreover, for any $a \in A \setminus B_n$, either $\Theta_I(a) = \emptyset$ which implies that $\Theta_I(B_n \cup \{a\}) = \emptyset$, or $\Theta_I(a) \neq \emptyset$ which implies that there must exist some $i \in \{k, \dots, n\}$ with $\Theta_I(B_i \cup \{a\}) = \emptyset$ so that $\Theta_I(B_n \cup \{a\}) = \emptyset$. In other words, $\Theta_I(B_n \cup \{a\}) = \emptyset$ for any $a \in A \setminus B_n$. Hence, let $\tilde{A} = B_n$ and we get the desired result.

Suppose A is not finite. Define $\mathcal{A} = \{A' \subseteq A : A_0 \subseteq A' \text{ and } \Theta_I(A') \neq \emptyset\}$. \mathcal{A} is not empty because $A_0 \in \mathcal{A}$. By the construction of \mathcal{A} , there exists a submodel \tilde{A} which satisfies all three desired conditions in the lemma if and only if \mathcal{A} has a maximum element in terms of partial order \subseteq , i.e. there exists some $\tilde{A} \in \mathcal{A}$ such that there does not exist an $A' \in \mathcal{A}$ with $\tilde{A} \subsetneq A'$. To show \mathcal{A} does have such a maximum element, we are going to apply Zorn's lemma. Let \mathcal{Z} be an arbitrary nonempty chain in terms of \subseteq , i.e. let \mathcal{Z} be an arbitrary nonempty subset of \mathcal{A} such that for any A' and A'' in \mathcal{Z} , either $A'' \subseteq A'$ or $A' \subseteq A''$. Define $A^\dagger = \bigcup_{A' \in \mathcal{Z}} A'$. Then, $\Theta_I(A^\dagger) = \bigcap_{A' \in \mathcal{Z}} \Theta_I(A')$.

We claim $\Theta_I(A^\dagger)$ is a compact set. To see why this is so, note that $\Theta_I(A^\dagger) = \bigcap_{a \in A^\dagger} \Theta_I(a)$. Since for any $a \in A$, $\Theta_I(a)$ is compact and since Θ is a metric space, $\Theta_I(A^\dagger)$ is also compact. Similarly, we can prove that, for any $A' \in \mathcal{Z}$, $\Theta_I(A')$ is a compact set.

We further claim that $\Theta_I(A^\dagger)$ is nonempty. We prove this claim by contradiction. Suppose $\Theta_I(A^\dagger)$ is empty, then $\Theta = (\Theta_I(A^\dagger))^C = \cup_{A' \in \mathcal{Z}} \Theta_I(A')^C$ by De Morgan's laws. For any $A' \in \mathcal{Z}$, because $\Theta_I(A')$ is compact, $\Theta_I(A')^C$ is an open set. Fix an arbitrary element B in \mathcal{Z} . Then, $\Theta_I(B) \subseteq \cup_{A' \in \mathcal{Z}} \Theta_I(A')^C$. Because $\Theta_I(B)$ is compact and nonempty, there exists a finite number of A_1, \dots, A_K such that $\Theta_I(B) \subseteq \cup_{k \in \{1, \dots, K\}} \Theta_I(A_k)^C$. This implies that $\Theta_I(B) \cap \Theta_I(A_1) \cap \dots \cap \Theta_I(A_K)$ is empty, which contradicts to the fact that \mathcal{Z} is a chain in terms of \subseteq and that $\Theta_I(A')$ is nonempty for each $A' \in \mathcal{Z}$.

Because $\Theta_I(A^\dagger)$ is nonempty, $A^\dagger \in \mathcal{A}$. Moreover, by the construction of A^\dagger , $A' \subseteq A^\dagger$ for any $A' \in \mathcal{Z}$. By Zorn's lemma, we conclude that \mathcal{A} contains a maximum element with partial order \subseteq . This completes the proof. \square

Proof of Theorem 2. Whenever there exists two submodels A_1 and A_2 with $\Theta_I(A_1) \neq \emptyset$, $\Theta_I(A_2) \neq \emptyset$ and $\Theta_I(A_1) \cap \Theta_I(A_2) = \emptyset$, $\Theta_I(A_1 \cup A_2) \subseteq \Theta_I(A_1) \cap \Theta_I(A_2)$ implies that $\Theta_I(A_1 \cup A_2) = \emptyset$ so that $\Theta_I(A) = \emptyset$. Moreover, the existence of A^* when $\Theta_I(A) = \emptyset$ implies that there always exists some nonempty submodel $B \subseteq A$ with $\Theta_I(B) \neq \emptyset$. Therefore, to prove all the results in Theorem 2, we only need to prove that when $\Theta_I(A) = \emptyset$, for any nonempty $B \subseteq A$ with $\Theta_I(B) \neq \emptyset$, there exists two finite set B' and B'' such that $\Theta_I(B \cup B') \neq \emptyset$, $\Theta_I(B'') \neq \emptyset$ and $\Theta_I(B \cup B') \cap \Theta_I(B'') = \emptyset$.

Now, suppose $\Theta_I(A) = \emptyset$ and let B be an arbitrary submodel with $\emptyset \neq B \subseteq A$ and $\Theta_I(B) \neq \emptyset$. Lemma 4 implies that there exists some $\tilde{A} \subseteq A$ such that (i) $B \subseteq \tilde{A}$, (ii) $\Theta_I(\tilde{A}) \neq \emptyset$, (iii) for any $a \in A \setminus \tilde{A}$, $\Theta_I(\tilde{A} \cup \{a\}) = \emptyset$. Since $\cap_{a \in A^*} \Theta_I(a) = \emptyset$, there must exist some $a^* \in A^*$ such that $\Theta_I(a^*) \cap \Theta_I(\tilde{A}) \subsetneq \Theta_I(\tilde{A})$, because otherwise $\Theta_I(\tilde{A}) \subseteq \cap_{a \in A^*} \Theta_I(a)$ which conflicts to $\cap_{a \in A^*} \Theta_I(a) = \emptyset$. Because $\Theta_I(a^*) \cap \Theta_I(\tilde{A}) \subsetneq \Theta_I(\tilde{A})$ implies that $a^* \notin \tilde{A}$, the construction of \tilde{A} implies that $\Theta_I(\{a^*\} \cup \tilde{A}) = \emptyset$. Under Condition (T2.C2), we know $\Theta_I(a^*) \cap \Theta_I(\tilde{A}) = \emptyset$.

Suppose A is a finite set. Then, $\tilde{A} \setminus B$ is also a finite set. Therefore, we get the desired result with $B' = \tilde{A} \setminus B$, and $B'' = \{a^*\}$.

Suppose A is an infinite set. Then, Condition (T2.C3) hold in this case. Define $\mathcal{T} := \{\Theta_I(B \cup \{a, a^*\}) : a \in \tilde{A}\}$. Then, \mathcal{T} is not empty because $\Theta_I(B \cup \{a^*\}) \in \mathcal{T}$ and $B \subseteq \tilde{A}$. Because $\Theta_I(a^*) \cap \Theta_I(\tilde{A}) = \emptyset$, and because Condition (T2.C2) hold, $\cap_{S \in \mathcal{T}} S = \Theta_I(\tilde{A} \cup \{a^*\}) = \emptyset$. Hence, $\Theta_I(B) \subseteq (\cap_{S \in \mathcal{T}} S)^C$ so that $\Theta_I(B) \subseteq \cup_{S \in \mathcal{T}} S^C$. Because any $S \in \mathcal{T}$ is compact under Condition (T2.C3) and hence closed, $\{S^C : S \in \mathcal{T}\}$ is an open covering for $\Theta_I(B)$. Because $\Theta_I(B)$ is compact under Condition (T2.C3), there must exist a finite number of $S_1, \dots, S_n \in \mathcal{T}$ such that $\Theta_I(B) \subseteq S_1^C \cup \dots \cup S_n^C = (S_1 \cap \dots \cap S_n)^C$, or equivalently, $\Theta_I(B) \cap S_1 \cap \dots \cap S_n = \emptyset$. This implies that there exist $a_1, a_2, \dots, a_n \in \tilde{A}$ such that $\Theta_I(B) \cap \Theta_I(B \cup \{a^*, a_1\}) \cap \dots \cap \Theta_I(B \cup \{a^*, a_n\}) = \emptyset$. Under Condition (T2.C2) this is equivalent to $\Theta_I(B \cup \{a_1, \dots, a_n\}) \cap \Theta_I(a^*) = \emptyset$. Since $a_1, \dots, a_n \in \tilde{A}$, we know $\Theta_I(B \cup \{a_1, \dots, a_n\}) \neq \emptyset$. Therefore, we get the desired result with $B' = \{a_1, \dots, a_n\}$ and $B'' = \{a^*\}$. \square

B.2.1. *When Theorem 2 fails to hold.* In this section, we present four counterexamples to show the result in Theorem 2 could fail if any of its four conditions are violated.

- Suppose only Condition (T2.C1) is violated. Consider the following example, where $\mathcal{A} = \{a_1, a_2, a_3\}$ with $\Theta_I(a_1) \subseteq \Theta_I(a_2)$, $\Theta_I(a_1) \neq \emptyset$ and $\Theta_I(a_3) = \emptyset$. Then, the full model A is refuted, but A only has three data-consistent submodels, namely, $\{a_1\}$, $\{a_2\}$ and $\{a_1, a_2\}$. And, $\Theta_I(a_1) \cap \Theta_I(a_2) \cap \Theta_I(\{a_1, a_2\}) = \Theta_I(a_1) \neq \emptyset$.
- Suppose only Condition (T2.C2) is violated. Consider the following example. Let ϵ be some unobservable random variable. Assumption a_1 assumes $E[\epsilon^2] = 1$ and assumption a_2 assumes $E[\epsilon^2] = 2$. Let θ be the variance of ϵ . If $\mathcal{A} = \{a_1, a_2\}$, we know A will always be refuted, since a_1 and a_2 are never compatible, but $\Theta_I(a_1) \cap \Theta_I(a_2) = [0, 1]$.
- Suppose only Condition (T2.C3) is violated and A is infinite. Consider the following example, let $\mathcal{A} = \{a_1, a_2, \dots\}$ with $\Theta_I(a_i) = (0, 1/i)$. Then, $\Theta_I(A) = \cap_{i \geq 1} \Theta_I(a_i) = \emptyset$, but for any $i \neq j$, $\Theta_I(a_i) \cap \Theta_I(a_j) = (0, 1/\max(i, j)) \neq \emptyset$. Since the identified set for each a_i is nested, we conclude that for any A_1, A_2 with $\Theta_I(A_1) \neq \emptyset$, $\Theta_I(A_2) \neq \emptyset$, we would have $\Theta_I(A_1) \cap \Theta_I(A_2) \neq \emptyset$.

B.2.2. *An alternative version of Theorem 2.*

Theorem 2'. *Suppose A is a finite set. In addition, assume*

- (1) *whenever $\Theta_I(A) = \emptyset$, there exists a nonempty subset A^* of A such that $\Theta_I(A^*) = \emptyset$ and $\Theta_I(a) \neq \emptyset$ for all $a \in A^*$.*
- (2) *for any $A', A'' \subseteq A$, $\Theta_I(A') \cap \Theta_I(A'') \neq \emptyset$ implies $\Theta_I(A' \cup A'') \neq \emptyset$.*

Then,

- (a) *when $\Theta_I(A) = \emptyset$, for any $B \subseteq A$ with $\Theta_I(B) \neq \emptyset$, there exists some B' and $B'' \subseteq A$ such that $B \subseteq B'$, $\Theta_I(B') \neq \emptyset$, $\Theta_I(B'') \neq \emptyset$ and $\Theta_I(B') \cap \Theta_I(B'') = \emptyset$.*
- (b) *$\Theta_I(A) = \emptyset$ if and only if there exists A' and $A'' \subseteq A$ such that $\Theta_I(A') \neq \emptyset$, $\Theta_I(A'') \neq \emptyset$ and $\Theta_I(A') \cap \Theta_I(A'') = \emptyset$.*

Proof of Theorem 2'. Let us first prove the first part of the theorem. Suppose $\Theta_I(A) = \emptyset$. Fix an arbitrary $B \subseteq A$ with $\Theta_I(B) \neq \emptyset$. Suppose there exists some $a \in A^*$ such that $\Theta_I(B) \cap \Theta_I(a) = \emptyset$, then we are done. Suppose, instead, that $\Theta_I(B) \cap \Theta_I(a) \neq \emptyset$ for any $a \in A^*$, define $\mathcal{A} = \{B \cup \{a\} : a \in A^*\}$. Since $\Theta_I(B) \cap \Theta_I(a) \neq \emptyset$ for any $a \in A^*$, we know $\Theta_I(B') \neq \emptyset$ for any $B' \in \mathcal{A}$. Moreover, \mathcal{A} is also a finite set and $\Theta_I(\cup_{B' \in \mathcal{A}} B') = \emptyset$. Enumerate $\mathcal{A} = \{B_1, \dots, B_n\}$ and define $\tilde{B}_k = \cup_{i=1, \dots, k} B_i$. Then, $\Theta_I(\tilde{B}_1) \neq \emptyset$ and $\Theta_I(\tilde{B}_n) = \emptyset$. Define $k^* = \min\{k : \Theta_I(\tilde{B}_k) = \emptyset\}$. Note $1 < k^* \leq n$. and Define $B' = \tilde{B}_{k^*-1}$ and $B'' = B_{k^*}$. Then, $\Theta_I(B') \neq \emptyset$ and $\Theta_I(B'') = \emptyset$. Moreover, $\Theta_I(B' \cup B'') = \emptyset$ implies that $\Theta_I(B') \cap \Theta_I(B'') = \emptyset$. Finally, by construction, $B \subseteq B'$, which completes the proof for the first part.

Let us now prove the second part of the theorem. If $\Theta_I(A) = \emptyset$, we know from the first result that there must exist some A' and $A'' \subseteq A$ such that $\Theta_I(A') \neq \emptyset$, $\Theta_I(A'') \neq \emptyset$ and $\Theta_I(A') \cap \Theta_I(A'') = \emptyset$. Reversely, if there exists some A' and $A'' \subseteq A$ such that $\Theta_I(A') \neq \emptyset$, $\Theta_I(A'') \neq \emptyset$ and $\Theta_I(A') \cap \Theta_I(A'') = \emptyset$, then we must have $\Theta_I(A' \cup A'') = \emptyset$ which implies that $\Theta_I(A) = \emptyset$. \square

B.3. Proof for Theorem 3. Suppose for any nonempty $B \subseteq A$, $\Theta_I(B) = \emptyset$. Recall that $\Theta_I(\emptyset) = \Theta$. Then, the empty set is the only minimum data-consistent relaxation of A and we are done.

Suppose there exists some $B \subseteq A$ such that $\Theta_I(B) \neq \emptyset$. Then, the desired result follows from Lemma 4.

B.4. Proof of Theorem 4.

- **(T4.C1) \Rightarrow (T4.C2):** Suppose (T4.C1) is true. We want to prove (T4.C2). Let A' be an arbitrary subset of A . Because $\Theta_I(A') \subseteq \Theta_I(a)$ for any $a \in A'$, if A' is data-consistent, then all $a \in A'$ are data-consistent. Reversely, if all $a \in A'$ is data-consistent, then (T4.C1) implies that $a \in A^*$ for all $a \in A'$, i.e. $A' \subseteq A^*$. Hence, $\Theta_I(A^*) \subseteq \Theta_I(A')$. Because A^* is data-consistent, so is A' . Hence, (T4.C2) is true.
- **(T4.C2) \Rightarrow (T4.C1):** Suppose (T4.C2) is true. We want to prove (T4.C1). Define $A^* = \{a \in A : \Theta_I(a) \neq \emptyset\}$. Because (T4.C2) is true, $\Theta_I(A^*) \neq \emptyset$. Fix an arbitrary data-consistent A' . We know $\Theta_I(a) \neq \emptyset$ for any $a \in A'$, so that $A' \subseteq A^*$. As a result, (T4.C1) is true.
- **(T4.C1) \Rightarrow (T4.C3) and A^* in (T4.C1) and (T4.C3) is the same set:** Suppose (T4.C1) is true. We want to prove (T4.C3). Let A^* be a data-consistent submodel in (T4.C1) that includes all data-consistent submodels. First of all, note that such A^* must be unique. If there exists another data-consistent \tilde{A} which includes all data-consistent submodels, then we must have $\tilde{A} \subseteq A^*$ and $A^* \subseteq \tilde{A}$. Secondly, the A^* in (T4.C1) must be a minimum data-consistent relaxation, because $a \notin A^*$ implies $\Theta_I(a) = \emptyset$ and, hence, $\Theta_I(A^* \cup \{a\}) = \emptyset$. Finally, A^* is the unique data-consistent relaxation. Because $\Theta_I(a) = \emptyset$ for any $a \notin A^*$, there cannot exist any other data-consistent submodel other than the subsets of A^* .
- **(T4.C3) \Rightarrow (T4.C1) if (T3.C1) or (T3.C2) hold:** Suppose (T4.C3) and either (T3.C1) or (T3.C2) hold. Let A' be an arbitrary data-consistent submodel. By Theorem 3, there exists some minimum data-consistent relaxation that includes A' . Because A^* is the unique minimum data-consistent relaxation, we conclude that $A' \subseteq A^*$. Hence, (T4.C1) must be true.

B.5. Proof of Theorem 5. Let us first prove the first result. To prove $\Theta_I^* \in \Lambda$, we need to check (C1) and (C2) for Θ_I^* . To prove (C1) for Θ_I^* , note that there exists some nonempty minimum data-consistent relaxation A^* by Theorem 3. Then, $\Theta_I(A^*) \neq \emptyset$ and $\Theta_I(A^*) \subseteq \Theta_I^*$. To prove (C2) for Θ_I^* , note that by Theorem 3, for any $A' \subseteq A$ with $\Theta_I(A') \neq \emptyset$, there exists some minimum data-consistent relaxation A^* such that $A' \subseteq A^*$. Therefore, $\Theta_I(A^*) \subseteq \Theta_I(A') \cap \Theta_I^*$ so that $\Theta_I(A') \cap \Theta_I^* \neq \emptyset$. This proves that $\Theta_I^* \in \Lambda$. Finally, for any S with $\Theta_I^* \subseteq S$, the fact that (C1) and (C2) holds for Θ_I^* implies that (C1) and (C2) also hold for S .

B.6. Proof of Theorem 6. We first prove (S3) \Rightarrow (S1) \Rightarrow (S2). Then, we prove (S2) \Rightarrow (S3) under Condition (T6.C3).

Prove (S3) \Rightarrow (S1): To show Θ_I^* is the smallest element in Λ , note that $\Theta_I^* \in \Lambda$ by Theorem 5. We only need to show that for any $S \in \Lambda$, $\Theta_I^* \subseteq S$.

Suppose there exists a unique minimum data-consistent relaxation A^* . Then, $\Theta_I^* = \Theta_I(A^*)$ by the definition of Θ_I^* . For any $S \in \Lambda$, there must exist some A' such that $\Theta_I(A') \subseteq S$ and $\Theta_I(A') \neq \emptyset$. By Theorem 3, we know $A' \subseteq A^*$ so that $\Theta_I(A^*) \subseteq \Theta_I(A') \subseteq S$. Hence, $\Theta_I^* \subseteq S$ for any $S \in \Lambda$.

Suppose for any minimum data-consistent relaxation \tilde{A} , $\Theta_I(\tilde{A})$ is a singleton set. For any $S \in \Lambda$, because (C2) holds for S , for any minimal data-consistent relaxation \tilde{A} , we have $\Theta_I(\tilde{A}) \cap S \neq \emptyset$. Since $\Theta_I(\tilde{A})$ is a singleton set, this implies that $\Theta_I(\tilde{A}) \subseteq S$ for all minimal data-consistent relaxation \tilde{A} . Hence, $\Theta_I^* \subseteq S$.

Prove (S1) \Rightarrow (S2): This result is immediate. If Θ_I^* is the smallest element in Λ , Λ of course has a smallest element.

Prove (S2) \Rightarrow (S3) under Condition (T6.C3): To show (S2) \Rightarrow (S3), we only need to show that when there exists at least two minimum data-consistent relaxations, (S2) implies that for any minimum data-consistent relaxation \tilde{A} , $\Theta_I(\tilde{A})$ is a singleton set.

To show this result, we need the following two lemmas, the proof of which will be shown later in Section B.6.1 and B.6.2.

Lemma 5. *Let S_1, \dots, S_n be a finite number of sets which satisfies: (i) for any $i = 1, \dots, n$, $S_i \neq \emptyset$; and (ii) for any $i, j = 1, \dots, n$ with $i \neq j$, $S_i \not\subseteq S_j$ and $S_j \not\subseteq S_i$. Define $T = \cup_{i=1}^n S_i$ and define $\mathcal{W} = \{S : \text{the following conditions hold for } S, \text{ (i) } \exists i \in \{1, \dots, n\} \text{ such that } S_i \subseteq S; \text{ (ii) } \forall j \in \{1, \dots, n\}, S_j \cap S \neq \emptyset\}$. Let $W = \cap_{S \in \mathcal{W}} S$. Then, we have the following results:*

- (1) If $n = 1$, $W \in \mathcal{W}$.
- (2) If $n \geq 2$, $W \in \mathcal{W}$ if and only if S_i is singleton for all $i = 1, \dots, n$.
- (3) $W \in \mathcal{W}$ implies $W = T$.

Lemma 6. *Let \mathcal{T} be a nonempty collection of sets which satisfies (i) for any $S \in \mathcal{T}$, $S \neq \emptyset$; and (ii) for any two $S, S' \in \mathcal{T}$, $S \cap S' = \emptyset$. Define $T = \cup_{S \in \mathcal{T}} S$ and define $\mathcal{W} = \{S : \text{the following conditions hold for } S, \text{ (i) } \exists S' \in \mathcal{T} \text{ such that } S' \subseteq S; \text{ (ii) } \forall S' \in \mathcal{T}, S' \cap S \neq \emptyset\}$. Let $W = \cap_{S \in \mathcal{W}} S$. Then, we have the following results:*

- (1) If \mathcal{T} only contains one set, then $W \in \mathcal{W}$.
- (2) If \mathcal{T} contains at least two sets, then $W \in \mathcal{W}$ if and only if S is singleton for all $S \in \mathcal{T}$.
- (3) $W \in \mathcal{W}$ implies $W = T$.

Suppose (S2) hold, i.e. suppose $\cap_{S \in \Lambda} S \in \Lambda$. Suppose also that there exists at least two minimum data-consistent relaxations. We are going to use the above two lemmas to show that for any minimum data-consistent relaxation \tilde{A} , $\Theta_I(\tilde{A})$ is a singleton set.

- When A is a finite set, we can enumerate all the minimum data-consistent relaxation as A_1, \dots, A_n with $n \geq 2$. Define $S_i = \Theta_I(A_i)$ for $i = 1, \dots, n$. By the definition of minimum data-consistent relaxation, $S_i \neq \emptyset$ for each $i = 1, \dots, n$. Moreover, Condition (T6.C3) implies that there does not exist $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$ such that $S_i \subseteq S_j$, because, if otherwise, $\Theta_I(A_i \cup A_j) \neq \emptyset$ so that A_i and A_j will not be minimum data-consistent relaxations. Define $\mathcal{W} = \{S : \text{the following conditions hold for } S, \text{ (i) } \exists i \in \{1, \dots, n\} \text{ such that } S_i \subseteq S; \text{ (ii) } \forall j \in \{1, \dots, n\}, S_j \cap S \neq \emptyset\}$. Then, $\mathcal{W} = \Lambda$ due to the results in Theorem 3. Because $n \geq 2$, the second result in Lemma 5 implies that S_i is singleton for each $i = 1, \dots, n$.
- Suppose A is an infinite set. Because we assume all the conditions in Theorem 3, we know that for any $a \in A$, $\Theta_I(a)$ is compact. Moreover, for any $B \subseteq A$, $\Theta_I(B) = \cap_{a \in B} \Theta_I(a)$. Then, for any two minimum data-consistent relaxation A' and A'' , we must have $\Theta_I(A') \cap \Theta_I(A'') = \emptyset$, because, if otherwise, $\Theta_I(A' \cup A'') = \Theta_I(A') \cap \Theta_I(A'') \neq \emptyset$ so that A' and A'' will not be minimum data-consistent relaxations. Define $\mathcal{T} = \{\Theta_I(A') : A' \text{ is a minimum data-consistent relaxation.}\}$ and define $\mathcal{W} = \{S : \text{the following conditions hold for } S, \text{ (i) } \exists S' \in \mathcal{T} \text{ such that } S' \subseteq S; \text{ (ii) } \forall S' \in \mathcal{T}, S' \cap S \neq \emptyset\}$. Then, $\mathcal{W} = \Lambda$ due to the results in Theorem 3. Because \mathcal{T} has at least two elements, The second result in Lemma 6 implies that all sets in \mathcal{T} are singleton sets.

Therefore, in both cases, we have shown that for any minimum data-consistent relaxation \tilde{A} , $\Theta_I(\tilde{A})$ is a singleton set.

B.6.1. *Proof of Lemma 5.* The proof of Lemma 5 builds on the following lemma, whose proof will be provided later in this section.

Lemma 7. *Let S_1, \dots, S_n be a finite number of sets which satisfies: (i) for any $i = 1, \dots, n$, $S_i \neq \emptyset$; and (ii) for any $i, j = 1, \dots, n$ with $i \neq j$, $S_i \not\subseteq S_j$ and $S_j \not\subseteq S_i$, and (iii) there exists $i = 1, \dots, n$ such that S_i contains at least two elements. Define $\mathcal{B} = \{S : \forall i = 1, \dots, n, S \cap S_i \neq \emptyset\}$. We say S is a minimal element in \mathcal{B} if $S \in \mathcal{B}$ and for any $S' \subsetneq S$, $S' \notin \mathcal{B}$. Then, \mathcal{B} has at least two different minimal elements.*

Proof of Lemma 5. First of all, note that $T \in \mathcal{W}$ by the definition of \mathcal{W} .

Suppose $n = 1$. Then, $T = S_1$ by the definition of T . Moreover, $S_1 \in \mathcal{W}$ by the definition of \mathcal{W} . Finally, for any $S \in \mathcal{W}$, we must have $S_1 \subseteq S$ by the first condition in the definition of \mathcal{W} . Hence, $S_1 = W$ and $W = T$.

Suppose $n \geq 2$. We are going to show that, for the following statements:

- (L7.S1) for each $i = 1, \dots, n$, S_i is singleton.
- (L7.S2) $W \in \mathcal{W}$
- (L7.S3) $W = T$

the following relations hold: (L7.S1) \Rightarrow ((L7.S2) and (L7.S3)) and that (L7.S2) \Rightarrow (L7.S1).

To show (L7.S1) \Rightarrow ((L7.S2) and (L7.S3)). Suppose (L7.S1) is true. Note that if S is singleton, then for any other set S' , $S' \cap S \neq \emptyset$ if and only if $S \subseteq S'$. Now, for any $S \in \mathcal{W}$, the second condition in the definition of \mathcal{W} implies that $S_i \subseteq S$ for all $i = 1, \dots, n$ so that $T \subseteq S$. Since $T \in \mathcal{W}$, we know $T = W$ and $W \in \mathcal{W}$.

To show (L7.S2) \Rightarrow (L7.S1). Suppose (L7.S2) is true. We prove S_i is singleton for all $i = 1, \dots, n$ using the following steps:

Step 1: We show that if there exists some $i = 1, \dots, n$ such that S_i is singleton, then S_j is singleton for all $j = 1, \dots, n$. To see why this is so, let $S^* \in \{S_i : i = 1, \dots, n\}$ be singleton. Suppose, for the purpose of contradiction, there exists $j = 1, \dots, n$ such that S_j is not a singleton. Then, apply Lemma 7 and we know that there exists two different minimal elements S' and S'' of $\mathcal{B} = \{S : \forall i = 1, \dots, n, S \cap S_i \neq \emptyset\}$ so that (i) for any $i = 1, \dots, n$, $S' \cap S_i \neq \emptyset$ and $S'' \cap S_i \neq \emptyset$; and (ii) $S' \cap S'' \notin \mathcal{B}$. Because $S^* \in \{S_i : i = 1, \dots, n\}$ is singleton, we know $S^* \subseteq S'$ and $S^* \subseteq S''$ so that $S' \in \mathcal{W}$ and $S'' \in \mathcal{W}$. Moreover, since $S' \cap S'' \notin \mathcal{B}$, we know $S' \cap S'' \notin \mathcal{W}$. Because $W \subseteq (S' \cap S'')$ by the definition of W , we conclude that $W \notin \mathcal{W}$, which leads to the contradiction.

Step 2: We show that there must exist some $i = 1, \dots, n$ such that S_i is singleton. Suppose, for the purpose of contradiction, that S_i is not singleton for all $i = 1, \dots, n$. Since $W \in \mathcal{W}$, there must exist some $S^* \in \{S_1, \dots, S_n\}$ such that $S^* \subseteq W$. Define $\mathcal{P} = \{S_i : S_i \neq S^*, S_i \cap S^* \neq \emptyset\}$ and $\mathcal{Q} = \{S_i : S_i \cap S^* = \emptyset\}$.

- When $\mathcal{Q} \neq \emptyset$, apply Lemma 7 to the sets in \mathcal{Q} , then there exists two different minimal elements S' and S'' of $\mathcal{B} = \{S : \forall S' \in \mathcal{Q}, S \cap S' \neq \emptyset\}$ so that (i) for any $S \in \mathcal{Q}$, $S' \cap S \neq \emptyset$ and $S'' \cap S_i \neq \emptyset$; and (ii) $S' \cap S'' \notin \mathcal{B}$. By the definition of \mathcal{W} and the construction of \mathcal{P} and \mathcal{Q} , we know that $S^* \cup S' \in \mathcal{W}$ and $S^* \cup S'' \in \mathcal{W}$. However, since $S' \cap S'' \notin \mathcal{B}$, we know $(S^* \cup S') \cap (S^* \cup S'') = S^* \cup (S' \cap S'') \notin \mathcal{W}$ because the second condition in the definition of \mathcal{W} is violated for some set in \mathcal{Q} . Since $W \subseteq S^* \cup (S' \cap S'')$, we conclude that $W \notin \mathcal{W}$, which leads to contradiction.
- When $\mathcal{Q} = \emptyset$, we know $\mathcal{P} \neq \emptyset$ (if otherwise, $n = 1$) and $S^* \in \mathcal{W}$. By the definition of W , we know $W \subseteq S^*$. Because $S^* \subseteq W$ by the definition of S^* , we conclude that $W = S^*$. To proceed, we consider two sub-cases:
 - when $S^* \not\subseteq \cup_{S \in \mathcal{P}} S$: Construct W' as $W' = \cup_{S \in \mathcal{P}} S$. By the construction of W' , we know $W' \in \mathcal{W}$. This implies that $W \subseteq W'$ by the definition of W . However, $W = S^* \not\subseteq W'$, which leads to the contradiction.
 - when $S^* \subseteq \cup_{S \in \mathcal{P}} S$: Then, \mathcal{P} contains at least two sets. If otherwise, $S^* \subseteq \cup_{S \in \mathcal{P}} S$ implies that there exists some $S \in \{S_i : i = 1, \dots, n\}$ such that $S^* \subseteq S$ which is ruled out the assumption of this lemma. Fix an arbitrary $\tilde{S} \in \mathcal{P}$. For any $S \in \mathcal{P}$ with $S \neq \tilde{S}$, pick arbitrary $\theta \in S \setminus S^*$. Let S^\dagger be the collection of these θ s. By the construction of S^\dagger , we know $S^\dagger \cap S^* = \emptyset$ and that, for any $S \in \mathcal{P}$ with $S \neq \tilde{S}$, $S^\dagger \cap S \neq \emptyset$. Define $\widetilde{W} = \tilde{S} \cup S^\dagger$. Then, $\widetilde{W} \in \mathcal{W}$, because $S^* \cap \widetilde{W} = S^* \cap \tilde{S} \neq \emptyset$ (recall $\tilde{S} \in \mathcal{P}$), $\tilde{S} \subseteq \widetilde{W}$ and that for any $S \in \mathcal{P}$ with $S \neq \tilde{S}$, $\emptyset \neq S^\dagger \cap S \subseteq \widetilde{W} \cap S$. However,

$$\begin{aligned}
W \cap \widetilde{W} &= S^* \cap \widetilde{W} \\
&= S^* \cap (\tilde{S} \cup S^\dagger) \\
&= (S^* \cap \tilde{S}) \cup (S^* \cap S^\dagger) \\
&= S^* \cap \tilde{S} \quad (\text{because } S^* \cap S^\dagger = \emptyset) \\
&\subsetneq S^* \quad (\text{because } S^* \not\subseteq \tilde{S} \text{ and } \tilde{S} \not\subseteq S^*).
\end{aligned}$$

Because $W = S^*$, this implies that $W \cap \widetilde{W} \subsetneq W$. But, this means that $W \not\subseteq \widetilde{W}$, which contradicts to that $\widetilde{W} \in \mathcal{W}$ and that $W = \cap_{S \in \mathcal{W}} S$.

Therefore, there is always a contradiction when $\mathcal{Q} = \emptyset$.

Since we get contradiction in all the above cases, we conclude that there must exist some $i = 1, \dots, n$ such that S_i is singleton. This result combines the Step 1 shows that S_i is singleton for all $i = 1, \dots, n$. Combine all the above results, we have shown that (L7.S1) \Leftrightarrow (L7.S2) when $n \geq 2$.

Finally, we need to show $W \in \mathcal{W}$ implies that $W = T$. Suppose $W \in \mathcal{W}$. When $n = 1$, we have shown that $W = T$. When $n \geq 2$, we have shown that (L7.S2) implies (L7.S1) which further implies (L7.S3). Hence, $W = T$ in both cases. Therefore, $W \in \mathcal{W}$ implies $W = T$. \square

Proof of Lemma 7. Without loss of generality, let us assume that S_1 contains at least two elements. Pick two arbitrary points θ_1 and θ'_1 in S_1 with $\theta_1 \neq \theta'_1$. Since we have $S_i \not\subseteq S_1$ for any $i = 2, \dots, n$, we can pick $\theta_i \in S_i \setminus S_1$ for any $i = 2, \dots, n$. By construction, $S_1 \cap \{\theta_2, \dots, \theta_n\} = \emptyset$, $\{\theta_1, \theta_2, \dots, \theta_n\} \in \mathcal{B}$ and $\{\theta'_1, \theta_2, \dots, \theta_n\} \in \mathcal{B}$. It's possible that there exists some $\theta_i \in \{\theta_1, \dots, \theta_n\}$ such that $\{\theta_1, \dots, \theta_n\} \setminus \theta_i \in \mathcal{B}$. Keep removing these redundant elements until there does not exist any

redundant element. In this way, one can find a subset $S^* \subseteq \{\theta_1, \dots, \theta_n\}$ such that S^* is a minimal element of \mathcal{B} . Similarly, one can find a subset $S' \subseteq \{\theta'_1, \theta_2, \dots, \theta_n\}$ such that S' is a minimal element of \mathcal{B} . Since $\{\theta_2, \dots, \theta_n\} \cap S_1 = \emptyset$, we must have $\theta_1 \in S^*$, because $S^* \notin \mathcal{B}$ if otherwise. Similarly, we know $\theta'_1 \in S'$. Since $\theta_1 \neq \theta'_1$, we know that $S^* \neq S'$. Therefore, \mathcal{B} has at least two different minimal elements. \square

B.6.2. Proof of Lemma 6. Suppose \mathcal{T} only contains one set, say $\mathcal{T} = \{S\}$. Then, $T = S$ by the definition of T . Moreover, for any $S' \in \mathcal{W}$, we know $S \subseteq S'$ by the first condition in the definition of \mathcal{W} . In addition, $S \in \mathcal{W}$. Therefore, $W = S = T$ and $W \in \mathcal{W}$.

Suppose \mathcal{T} contains at least two sets. We are going to show that, for the following statements:

- (L6.S1) for each $S \in \mathcal{T}$, S is singleton.
- (L6.S2) $W \in \mathcal{W}$
- (L6.S3) $W = T$

the following relations hold: (L6.S1) \Rightarrow ((L6.S2) and (L6.S3)) and that (L6.S2) \Rightarrow (L6.S1).

Suppose (L6.S1) is true. For any $S \in \mathcal{W}$, the second condition in the definition of \mathcal{W} implies that $T \subseteq S$. Moreover, $T \in \mathcal{W}$ by the definition of \mathcal{W} . Therefore, $T = W$ by the definition of W and $W \in \mathcal{W}$. Hence, (L6.S1) implies ((L6.S2) and (L6.S3))

Suppose (L6.S2) is true. Suppose, for the purpose of contradiction, (L6.S1) is not true. Then, there exists some $S^* \in \mathcal{T}$ such that S^* contains at least two element. Pick θ and θ' in S^* with $\theta \neq \theta'$. Construct $W_1 = \{\theta\} \cup (\cup_{S \in \mathcal{T}: S \neq S^*} S)$ and $W_2 = \{\theta'\} \cup (\cup_{S \in \mathcal{T}: S \neq S^*} S)$. Then, $W_1 \in \mathcal{W}$ and $W_2 \in \mathcal{W}$. Moreover, $W_1 \cap W_2 = \cup_{S \in \mathcal{T}: S \neq S^*} S$. Because that for any $S \in \mathcal{T}$ with $S \neq S^*$, $S \cap S^* = \emptyset$, we know $S^* \cap (W_1 \cap W_2) = \emptyset$ which implies that $W_1 \cap W_2 \notin \mathcal{W}$. Because $W \subseteq W_1 \cap W_2$, we conclude that $W \notin \mathcal{W}$, which leads to the contradiction. Therefore, (L6.S2) implies that (L6.S1).

The above result implies that (L6.S1) \Leftrightarrow (L6.S2) when \mathcal{T} contains at least two sets.

Finally, we need to prove that $W \in \mathcal{W}$ implies that $W = T$. Suppose $W \in \mathcal{W}$ hold. When \mathcal{T} only contains one set, we have already shown that $W = T$. When \mathcal{T} contains at least two sets, (L6.S2) implies (L6.S1) which further implies that (L6.S3) i.e. $W = T$. Hence, $W = T$ whenever $W \in \mathcal{T}$.

B.7. Proof of Theorem 7. Let \tilde{A} be any minimal data-consistent relaxation. Define δ as $\delta(a) = 1 - \mathbb{1}(a \in \tilde{A})$. Since, for any $a \in \tilde{A}$, $a_{\delta(a)} = a$ and for any $a \notin \tilde{A}$, $a_{\delta(a)}$ is a null assumption which does not have any restriction, we know $\Theta_I(A(\delta)) = \Theta_I(\tilde{A})$. Moreover, for any $\delta' < \delta$ with $\Theta_I(A(\delta')) \neq \emptyset$, we must have that $\Theta_I(\tilde{A}) = \Theta_I(A(\delta'))$, because $\Theta_I(A(\delta')) \subseteq \Theta_I(A(\delta)) = \Theta_I(\tilde{A})$ and $\Theta_I(\tilde{A})$ is singleton. This implies that $\Theta_I(\tilde{A}) \subseteq \Theta_I^\dagger$. Since this result holds for any minimal data-consistent relaxation \tilde{A} , we conclude that $\Theta_I^* \subseteq \Theta_I^\dagger$.

B.8. Results in Section A.1.

Proof of Proposition 3. Recall \mathcal{Z} is the support of Z . Define $B(z, \epsilon) = \{z' : \|z - z'\| < \epsilon\}$. For any $\theta \in \Theta$, let $g(z; \theta)$ be the continuous version of $E[m(X, Z; \theta) | Z = z]$. That is, $g(z; \theta)$ is continuous in z and $g(z; \theta) = E[m(X, Z; \theta) | Z = z]$ almost surely.

By the second condition in the hypothesis of this proposition, for any z_0 in the support of Z , there exists some $\theta_0(z_0) \in \Theta$ and $\delta(z_0) > 0$ such that for almost every Z in $B(z_0, \delta(z_0))$, $E[m(X, Z; \theta_0) | Z] \leq 0$. Therefore, if we define function $\delta_{z_0, \epsilon}$ as $\delta_{z_0, \epsilon}(z) = \mathbb{1}(\|z - z_0\| < \epsilon)$ and the collection of functions \mathcal{W}' as $\mathcal{W}' := \{\delta_{z, \epsilon} : \epsilon \in [0, \delta(z)), z \in \mathcal{Z}\}$ where \mathcal{Z} is the support of Z , then the A^* can be constructed as the set of submodels indexed by $w \in \mathcal{W}'$ with which (A.2) hold. Then, for each $a \in A^*$, $\Theta_I(a) \neq \emptyset$.

Next, we need to show that $\Theta_I(A^*) = \emptyset$ whenever $\Theta_I(A) = \emptyset$. Suppose $\Theta_I(A) = \emptyset$. Fix an arbitrary $\theta \in \Theta$. Since $\theta \notin \Theta_I(A)$, there must exists some $\mathcal{Z}' \subseteq \mathcal{Z}$ with $P(Z \in \mathcal{Z}') > 0$ such that $E[m(X, Z; \theta) | Z] > 0$ for all $Z \in \mathcal{Z}'$.

We claim there exists some z_0 and $\epsilon_0 \geq 0$ such that $E[\mathbb{1}(z \in B(z_0, \epsilon_0))m(X, Z; \theta)] > 0$. If \mathcal{Z} is discrete, then there exists some $z^* \in \mathcal{Z}'$ such that $P(Z = z^*) > 0$. Hence, our claim can be verified for $z_0 = z^*$ and $\epsilon_0 = 0$. If $E[m(X, Z; \theta) | Z = z]$ is continuous in z , for all $z' \in \mathcal{Z}'$, there must exist some $\epsilon(z') > 0$ such that $E[m(X, Z; \theta) | Z] > 0$ for almost every Z in $B(z', \epsilon(z'))$. Since \mathcal{Z} is a subset of real space \mathbb{R}^{k_2} , \mathcal{Z}' must be separable. That is, there exists a countable subset \mathcal{D} of \mathcal{Z}' such that \mathcal{D} is dense in \mathcal{Z}' . Since $\mathcal{D} \subseteq \mathcal{Z}'$, $E[m(X, Z; \theta) | Z] > 0$ for almost every Z in $\cup_{z \in \mathcal{D}} B(z, \epsilon(z))$. Moreover, since \mathcal{D} is dense in \mathcal{Z}' , we know $\mathcal{Z}' \subseteq \cup_{z \in \mathcal{D}} B(z, \epsilon(z))$ so that $P(Z \in \cup_{z \in \mathcal{D}} B(z, \epsilon(z))) > 0$. Since \mathcal{D} is countable, we have $P(Z \in B(z^*, \epsilon(z^*))) > 0$ for some $z^* \in \mathcal{D}$. Hence, our claim is verified for $z_0 = z^*$ and $\epsilon_0 = \epsilon(z^*)$.

Finally, the claim in the previous paragraph implies that there exists some $a \in A^*$ such that $\theta \notin \Theta_I(a)$. Hence, $\theta \notin \Theta_I(A^*)$. This completes the proof that $\Theta_I(A^*) = \emptyset$ whenever $\Theta_I(A) = \emptyset$. \square

Proof of Proposition 4. Because Θ is compact, to show $\Theta_I(a)$ is compact for all $a \in A$, we only need to show that for any $w \in \mathcal{W}_1^+$, $E[w(Z)m(X, Z; \theta)]$ is continuous in θ . For any $\theta \in \Theta$, and for any θ_n converges to θ , we know $E[m(X, Z; \theta_n)|Z] \rightarrow E[m(X, Z; \theta)|Z]$ almost surely as $n \rightarrow \infty$. Since the norm of $w(Z)E[m(X, Z; \theta)|Z]$ is dominated by $w(Z)g(Z)$ and $E\|w(Z)g(Z)\| \leq \sqrt{E[\|w(Z)\|^2]} \sqrt{E[\|g(Z)\|^2]}$, the dominated convergence theorem implies that

$$E[w(Z)m(X, Z; \theta_n)] \rightarrow E[w(Z)m(X, Z; \theta)]$$

as $n \rightarrow \infty$. This completes the proof. \square

B.9. Results in Section 3.2.

Proof for Proposition 3.2. Let \mathcal{Y} be the support of Y . Define $\delta = \min_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} P(Y = y|X = x)$ where the inf stands for the essential infimum. Then, $\delta > 0$. For each $y \in \mathcal{Y}$, there exists a sequence $\theta_1(y), \dots, \theta_n(y), \dots$ in \mathbb{R}^d , such that $L(\{y\}, x; \theta_n(y)) \rightarrow 1$ as $n \rightarrow 1$. Therefore, there must exists some n such that $L(\{y\}, x; \theta_n(y)) > 1 - \delta$ for any $y \in \mathcal{Y}$. Then, for any nonempty subset K of \mathcal{Y} with $K \subsetneq \mathcal{Y}$, and any $y \in K$,

$$P(Y \in K|X) \leq 1 - \delta < L(\{y\}, x; \theta_n(y)) \leq L(K, X; \theta_n(X, y)) \text{ almost surely}$$

Therefore, define $\Theta_U = \{\theta_i(y) : i \leq n, y \in \mathcal{Y}\}$. Therefore, for any subset K of Y , there exists some $\theta \in \Theta_U$ such that

$$P(Y \in K|X) \leq L(K, X; \theta) \text{ almost surely}$$

This completes the proof. \square

B.10. Proof of Proposition 2. Recall the A in the introductory example is the set of all assumption indexed by $h \in \mathcal{H}_1^+$ with which (2.2) holds.

Let us first show Θ_I^* is equal the interval specified in (5.1). By Lemma 3, we know that for each $\theta \in (\underline{\gamma}, \underline{\gamma})$, there exists some $A' \subseteq A$ such that $\Theta_I(A') = \{\theta\}$. By Theorem 3, there exists some minimum data-consistent relaxation A^* such that $A' \subseteq A^*$. Since $\Theta_I(A')$ is singleton, we know $\Theta_I(A^*) = \Theta_I(A') = \{\theta\}$. Therefore, $(\underline{\gamma}, \underline{\gamma}) \subseteq \Theta_I^*$.

We claim that if $P(E[\underline{Y}|Z] \leq \underline{\gamma}) > 0$, then $\underline{\gamma} \in \Theta_I^*$ and there exists some $A' \subseteq A$ with $\Theta_I(A') = \{\underline{\gamma}\}$. To see why it is so, suppose $P(E[\underline{Y}|Z] \leq \underline{\gamma}) > 0$. Then, define $S_1 = \{z : E[\underline{Y}|Z = z] \leq \underline{\gamma}\}$ and $S_2 = \{z : E[\underline{Y}|Z = z] \geq \underline{\gamma}\}$. Since $P(E[\underline{Y}|Z] \leq \underline{\gamma}) > 0$, we know $P(Z \in S_1) > 0$. Since $\underline{\gamma} > \underline{\gamma}$, we know $P(Z \in S_2) > 0$. Now, define $h_1(z) = \mathbb{1}(z \in S_1)/P(Z \in S_1)$ and $h_2(z) = \mathbb{1}(z \in S_2)/P(Z \in S_2)$. Then, $Eh_1(Z) = 1$, $Eh_2(Z) = 1$, $E[h_1(Z)\underline{Y}] \leq \underline{\gamma}$ and $E[h_2(Z)\underline{Y}] \geq \underline{\gamma}$. Therefore, there must exists \underline{h} as a convex combination of h_1 and h_2 such that $E\underline{h}(Z) = 1$ and $E[\underline{h}(Z)\underline{Y}] = \underline{\gamma}$. Hence, $\underline{\gamma} \in \tilde{\Theta}(\underline{h})$ and $\tilde{\Theta}(\underline{h}) \cap (-\infty, \underline{\gamma}) = \emptyset$. Moreover, for each $i = 1, 2, \dots$, construct $\bar{h}_i(z)$ as $\bar{h}_i(z) = \mathbb{1}(E[\bar{Y}|Z = z] \in [\underline{\gamma}, \underline{\gamma} + 1/i])$. By the definition of $\underline{\gamma}$, we know $E\bar{h}_i(Z) > 0$ for each $i \geq 1$. Note that the identified set of (2.2) of \bar{h}_i , $\tilde{\Theta}(\bar{h}_i)$ is

$$\left[\frac{E[\bar{h}_i(Z)\underline{Y}]}{E\bar{h}_i(Z)}, \frac{E[\bar{h}_i(Z)\bar{Y}]}{E\bar{h}_i(Z)} \right].$$

Because $E[\underline{Y}|Z] \leq E[\bar{Y}|Z]$ almost surely, the law of iterated expectation implies that $\underline{\gamma} \in \tilde{\Theta}(\bar{h}_i)$. Moreover, by construction, for any $\theta > \underline{\gamma}$, $\theta \notin \cap_i \tilde{\Theta}(\bar{h}_i)$. Therefore, if we define $\mathcal{H}' = \{\bar{h}_i : i \geq 1\} \cup \{\underline{h}\}$, we have $\cap_{h \in \mathcal{H}'} \tilde{\Theta}(h) = \{\underline{\gamma}\}$. This implies that $\underline{\gamma} \in \Theta_I^*$ and there exists some $A' \subseteq A$ with $\Theta_I(A') = \{\underline{\gamma}\}$.

Next, we claim that if $P(E[\underline{Y}|Z] \leq \underline{\gamma}) > 0$, then $\Theta_I^* \cap (-\infty, \underline{\gamma}) = \emptyset$. To see this, note that Lemma 2 implies that for any $a \in A$, $\Theta_I(a) \cap [\underline{\gamma}, \underline{\gamma}] \neq \emptyset$. Let A' be an arbitrary minimum data-consistent relaxation A' of A . Because Condition (T2.C2) holds in this example, the preceding result implies that for any minimum data-consistent relaxation A' of A , we know $\Theta_I(A') \cap [\underline{\gamma}, \underline{\gamma}] \neq \emptyset$. Our claim will be verified if we can prove $\Theta_I(A') \cap (-\infty, \underline{\gamma}) = \emptyset$. Suppose not, i.e suppose $\Theta_I(A') \cap (-\infty, \underline{\gamma}) \neq \emptyset$. Because $\Theta_I(A')$ is a closed interval, the fact that $\Theta_I(A') \cap [\underline{\gamma}, \underline{\gamma}] \neq \emptyset$ implies that $\underline{\gamma} \in \Theta_I(A')$. Because we've proven that $\cap_{h \in \mathcal{H}'} \tilde{\Theta}(h) = \{\underline{\gamma}\}$, and because A' is a minimum data-consistent relaxation, we know $\Theta_I(A') = \{\underline{\gamma}\}$ which leads to contradiction.

Next, we claim that if $P(E[\underline{Y}|Z] \leq \bar{\gamma}) = 0$, then $\Theta_I^* \cap (-\infty, \bar{\gamma}] = \emptyset$. To see this, note that $P(E[\underline{Y}|Z] \leq \bar{\gamma}) = 0$ implies $P(E[\underline{Y}|Z] > \bar{\gamma}) = 1$. Therefore, for any $h \in \mathcal{H}_1^+$, $E[h(Z)(\theta - \underline{Y})] \geq 0$ implies that

$$\begin{aligned} & E[h(Z)(\theta - \underline{Y})] \geq 0 \\ \Rightarrow & E[h(Z)\underline{Y}] \leq E[h(Z)]\theta \\ \Leftrightarrow & E[h(Z)E[\underline{Y}|Z]] \leq E[h(Z)]\theta \\ \Rightarrow & E[h(Z)]\bar{\gamma} < E[h(Z)]\theta \end{aligned}$$

where the last inequality follows from the fact that $P(E[\underline{Y}|Z] > \bar{\gamma}) = 1$. Therefore, we know for any $h \in \mathcal{H}_1^+$, $\tilde{\Theta}(h) \cap (-\infty, \bar{\gamma}] = \emptyset$. This implies $\Theta_I^* \cap (-\infty, \bar{\gamma}] = \emptyset$.

Following similar steps as above, we can also prove the following results:

- If $P(E[\bar{Y}|Z] \geq \underline{\gamma}) > 0$, then $\underline{\gamma} \in \Theta_I^*$ and there exists some $A' \subseteq A$ with $\Theta_I(A') = \{\underline{\gamma}\}$.
- If $P(E[\bar{Y}|Z] \geq \underline{\gamma}) > 0$, then $\Theta_I^* \cap (\underline{\gamma}, +\infty) = \emptyset$.
- If $P(E[\bar{Y}|Z] \geq \underline{\gamma}) = 0$, then $\Theta_I^* \cap [\underline{\gamma}, +\infty) = \emptyset$.

Combining these results and that $(\bar{\gamma}, \underline{\gamma}) \subseteq \Theta_I^*$, we conclude that Θ_I^* is equals to the interval specified in (5.1). Moreover, we have also shown that for any $\theta \in \Theta_I^*$, there exists some $A' \subseteq A$ such that $\Theta_I(A') = \{\theta\}$. This implies that for any minimum data-consistent relaxation \tilde{A} , $\Theta_I(\tilde{A})$ is a singleton set. By Theorem 6, we know Θ_I^* is the smallest element in Λ .

B.11. Proof of Proposition 1. Recall the notation used in this example: $\underline{Y}_d := Y\mathbb{1}(D = d) + \underline{y}_d\mathbb{1}(D \neq d)$, $\bar{Y}_d := Y\mathbb{1}(D = d) + \bar{y}_d\mathbb{1}(D \neq d)$, $\underline{q}_{dt} := E[\underline{Y}_d|Z = t]$ and $\bar{q}_{dt} := E[\bar{Y}_d|Z = t]$. Proposition 1 is an immediate corollary of the following two lemmas.

Lemma 8. In model $Y = \sum_{z \in \mathcal{Z}} \mathbb{1}(Z = z)[Y_{1z}D + Y_{0z}(1 - D)]$ where $\mathcal{Z} = \{1, 2, \dots, k\}$. Fix an arbitrary $z^* = 1, \dots, k$. Let Θ_{I, z^*} be the identified set of a_{z^*} , i.e. the identified set of E.1, E.2 and E.3 for $z = z^*$. Then,

- (1) $\Theta_{I, z^*} \neq \emptyset$ if and only if the following two conditions hold for each $d \in \{0, 1\}$:

$$\forall z < z^*, \quad \max(\underline{q}_{dt} : t \leq z) \leq \min(\bar{q}_{dt} : t \geq z) \quad (\text{B.2})$$

and

$$\max(\underline{q}_{dt} : t = 1, \dots, k) \leq \min(\bar{q}_{dt} : t \geq z^*) \quad (\text{B.3})$$

- (2) if $\Theta_{I, z^*} \neq \emptyset$, then $\Theta_{I, z^*} = \Gamma_{1, z^*} \times \Gamma_{0, z^*}$.

Lemma 9. In model $Y = \sum_{z \in \mathcal{Z}} \mathbb{1}(Z = z)[Y_{1z}D + Y_{0z}(1 - D)]$ where $\mathcal{Z} = \{1, 2, \dots, k\}$. Let Θ_I be the identified set of a^\dagger , i.e. the identified set of E.1 and E.2. Then,

- (1) $\Theta_I \neq \emptyset$ if and only if $P(Y \in [\underline{y}_d, \bar{y}_d] | D = d) = 1$ for any $d \in \{0, 1\}$.
- (2) when $\Theta_I \neq \emptyset$, $\Theta_I = [E[\underline{Y}_1], E[\bar{Y}_1]] \times [E[\underline{Y}_0], E[\bar{Y}_0]]$.

Proof of Lemma 8. The results of this lemma can be divided into the following two parts:

- (1) For any $z^* = 1, \dots, k$, $\Theta_{I, z^*} \neq \emptyset$ only if that (B.2) and (B.3) hold for each $d = 0, 1$. Moreover, $\Theta_{I, z^*} \subseteq \Gamma_{1, z^*} \times \Gamma_{0, z^*}$.
- (2) if (B.2) and (B.3) hold, then $\Theta_{I, z^*} \neq \emptyset$ and $\Theta_{I, z^*} \supseteq \Gamma_{1, z^*} \times \Gamma_{0, z^*}$.

Let us now prove these two parts one by one.

Part 1. Fix any $d \in \{0, 1\}$. Suppose assumption a_{z^*} hold, i.e. assumptions E.1, E.2 and E.3 hold for $z = z^*$. Assumption E.3 implies that for any $z' < z^*$ and $t \leq z'$ we have $Y_{dt} \leq Y_{dz'}$ so that $E[Y_{dt}|Z = z'] \leq E[Y_{dz'}|Z = z']$. Due to E.2, we know $E[Y_{dt}|Z = z'] = E[Y_{dt}|Z = t]$, so that $E[Y_{dt}|Z = t] \leq E[Y_{dz'}|Z = z']$. Since $\underline{q}_{dt} \leq E[Y_{dt}|Z = t]$, we conclude that $\max_{t \leq z'} \underline{q}_{dt} \leq E[Y_{dz'}|Z = z']$. Similarly, E.3 implies that for any $z' < z^*$ and $t \geq z'$, we have $Y_{dz'} \leq Y_{dt}$ so that $E[Y_{dz'}|Z = z'] \leq E[Y_{dt}|Z = z']$. Because of E.2, and because $\bar{q}_{dt} \geq E[Y_{dt}|Z = t]$, we know that $E[Y_{dz'}|Z = z'] \leq \min_{t \geq z'} \bar{q}_{dt}$. Hence, for any $d \in \{0, 1\}$,

$$\forall z' < z^*, \quad \max(\underline{q}_{dt} : t \leq z') \leq E[Y_{dz'}|Z = z'] \leq \min(\bar{q}_{dt} : t \geq z') \quad (\text{B.4})$$

Now, for any $z' \geq z^*$, E.3 implies that $Y_{dt} \leq Y_{dz'}$ for any $t \in \{1, \dots, k\}$. Hence, $E[Y_{dt}|Z = z'] \leq E[Y_{dz'}|Z = z']$ for all t . Because E.2 implies that $E[Y_{dt}|Z = t] = E[Y_{dt}|Z = z']$, we have $E[Y_{dt}|Z = t] \leq E[Y_{dz'}|Z = z']$ for all t , so that $\max(\underline{q}_{dt} : t = 1, \dots, k) \leq E[Y_{dz'}|Z = z']$. For any $z' \geq z^*$, assumption E.3 implies that $Y_{dt} \geq Y_{dz'}$ for all $t \geq z^*$. Hence,

$E[Y_{dt}|Z = z] \geq E[Y_{dz}|Z = z]$ for all $t \geq z^*$. Assumption *E.2* then implies that $E[Y_{dt}|Z = t] \geq E[Y_{dz'}|Z = z']$ for all $t \geq z^*$, so that $\min(\bar{q}_{dt} : t \geq z^*) \geq E[Y_{dz}|Z = z]$. Hence, we conclude that for any $d \in \{0, 1\}$:

$$\forall z' \geq z^*, \quad \max(\underline{q}_{dt} : t = 1, \dots, k) \leq E[Y_{dz'}|Z = z'] \leq \min(\bar{q}_{dt} : t \geq z^*). \quad (\text{B.5})$$

Combine (B.4) and (B.5), we conclude that for any d , $\theta_d \in \Gamma_{d, z^*}$, so that $\Theta_{I, z^*} \subseteq \Gamma_{1, z^*} \times \Gamma_{0, z^*}$. Moreover, because Assumption *E.1*, *E.2* and *E.3* imply (B.4) and (B.5), the violation of (B.2) and (B.3) implies that $\Theta_{I, z^*} = \emptyset$. Equivalently, $\Theta_{I, z^*} \neq \emptyset$ only if (B.2) and (B.3) hold for any $d \in \{0, 1\}$.

Part 2. We want to prove that (B.2) and (B.3) implies that $\Theta_{I, z^*} \neq \emptyset$ and $\Theta_{I, z^*} \supseteq \Gamma_{1, z^*} \times \Gamma_{0, z^*}$. Fix an arbitrary $d \in \{0, 1\}$. First of all, we are going to prove that one can construct Y_{dz} which achieves the lower bound in Γ_{d, z^*} , satisfies assumptions *E.1-E.3*, and is compatible with the data at the same time.

Define γ_z for each $z = 1, \dots, k$ as follows:

- for $z < z^*$, let γ_z be the value which solves

$$\max(\underline{q}_{dt} : t \leq z) = E[\mathbb{1}(D = d)Y|Z = z] + E[\mathbb{1}(D \neq d)Y|Z = z]\gamma_z$$

Then, $\gamma_z \in [\underline{y}_d, \bar{y}_d]$ if $\bar{q}_{dz} \geq \max(\underline{q}_{dt} : t \leq z)$, which is implied by (B.2).

- for $z \geq z^*$, let γ_z be the value which solves

$$\max(\underline{q}_{dt} : t = 1, \dots, k) = E[\mathbb{1}(D = d)Y|Z = z] + E[\mathbb{1}(D \neq d)Y|Z = z]\gamma_z$$

Then, $\gamma_z \in [\underline{y}_d, \bar{y}_d]$ if $\max(\underline{q}_{dt} : \forall t) \leq \bar{q}_{dz}$ which is implied by (B.3).

Define $W_{dz} := \mathbb{1}(D = d, Z = z)Y + \mathbb{1}(D \neq d, Z = z)\gamma_z$. Then, by construction,

$$E[W_{dz}|Z = z] = \begin{cases} \max(\underline{q}_{dt} : t \leq z) & \text{if } z < z^* \\ \max(\underline{q}_{dt} : t = 1, \dots, k) & \text{if } z \geq z^* \end{cases} \quad (\text{B.6})$$

which implies that

$$\forall z \leq t, \quad E[W_{dz}|Z = z] \leq E[W_{dt}|Z = t] \quad (\text{B.7})$$

$$\forall z \geq z^*, \quad E[W_{dz}|Z = z] = \max(\underline{q}_{dt} : t = 1, \dots, k) \quad (\text{B.8})$$

Moreover, because $\gamma_z \in [\underline{y}_d, \bar{y}_d]$ for any $z \in \{1, \dots, k\}$, we know $P(W_{dz} \in [\underline{y}_z, \bar{y}_z]) = 1$ for all $z \in \{1, \dots, k\}$. And, $P(W_{dz} = Y|D = d, Z = z) = 1$ for any d and z .

Now, for any $t \in \{1, \dots, k\}$, define, $\phi_{dt}(\alpha) := (1 - \alpha)W_{dt} + \alpha\bar{y}_d$ and $\psi_{dt}(\alpha) := (1 - \alpha)\underline{y}_d + \alpha W_{dt}$. We claim that, for any $t \neq z$, there exists $\alpha_{tz} \in [0, 1]$ which solves the following equations:

$$\begin{aligned} \forall t < z, \quad E[W_{dz}|Z = z] &= E[\phi_{dt}(\alpha_{tz})|Z = t] \\ \forall t > z, \quad E[W_{dz}|Z = z] &= E[\psi_{dt}(\alpha_{tz})|Z = t]. \end{aligned} \quad (\text{B.9})$$

To see why it is so, note that

$$\begin{aligned} \forall t < z, \quad E[W_{dt}|Z = t] &= E[\phi_{dt}(0)|Z = t] \text{ and } E[\phi_{dt}(1)|Z = t] = \bar{y}_d, \\ \forall t > z, \quad \underline{y}_d &= E[\phi_{dt}(0)|Z = t] \text{ and } E[\phi_{dt}(1)|Z = t] = E[W_{dt}|Z = t]. \end{aligned} \quad (\text{B.10})$$

These results, combined with (B.7), imply that

$$\begin{aligned} \forall t < z, \quad E[\phi_{dt}(0)|Z = t] &\leq E[W_{dz}|Z = z] \leq E[\phi_{dt}(1)|Z = t], \\ \forall t > z, \quad E[\psi_{dt}(0)|Z = t] &\leq E[W_{dz}|Z = z] \leq E[\phi_{dt}(1)|Z = t]. \end{aligned}$$

which implies the existence of $\alpha_{tz} \in [0, 1]$ satisfying (B.9) for all $t \neq z$.

In addition, $(\alpha_{tz} : t \neq z)$ has some extra properties. Because (B.7) holds and $E[\phi_{dt}(\alpha)|Z = t]$ is an increasing function of α ,

$$\forall t < z < z', \quad \alpha_{tz} \leq \alpha_{tz'}. \quad (\text{B.11})$$

Because (B.7) holds and $E[\psi_{dt}(\alpha)|Z = t]$ is an increasing function of α ,

$$\forall z < z' < t, \quad \alpha_{tz} \leq \alpha_{tz'}. \quad (\text{B.12})$$

Construct $Y_{dz} := \sum_{t < z} \phi_{dt}(\alpha_{tz}) + W_{dz} + \sum_{t > z} \psi_{dt}(\alpha_{tz})$. Because $P(W_{dz} \in [\underline{y}_d, \bar{y}_d]) = 1$, assumption *E.1* holds for this Y_{dz} . Because of (B.9), assumption *E.2* holds for this Y_{dz} , i.e. $E[Y_{dz}|Z = t] = E[Y_{dz}|Z = z]$ for any t, z with $t \neq z$.

To show assumption *E.3* also holds for this Y_{dz} , note that, for any z_1, z_2 with $1 \leq z_1 < z_2 \leq k$,

- If $Z < z_1$, $Y_{dz_1} = \phi_{dZ}(\alpha_{Zz_1}) \leq \phi_{dZ}(\alpha_{Zz_2}) = Y_{dz_2}$ because of (B.11) and because $\phi_{dZ}(\alpha)$ is increasing in α .
- If $Z = z_1$, $Y_{dz_1} = W_{dz_1} \leq \phi_{dZ}(\alpha_{Zz_2}) = Y_{dz_2}$ because of the definition of $\phi_{dZ}(\alpha)$.
- If $z_1 < Z < z_2$, $Y_{dz_1} = \psi_{dZ}(\alpha_{Zz_1}) \leq W_{dZ} \leq \phi_{dZ}(\alpha_{Zz_2}) = Y_{dz_2}$ because of the definition of $\phi_{dZ}(\alpha)$ and $\psi_{dZ}(\alpha)$.
- If $Z = z_2$, $Y_{dz_1} = \psi_{dZ}(\alpha_{Zz_1}) \leq W_{dZ} = Y_{dz_2}$ because of the definition of $\psi_{dZ}(\alpha)$.
- If $z_2 < Z$, $Y_{dz_1} = \psi_{dZ}(\alpha_{Zz_1}) \leq \psi_{dZ}(\alpha_{Zz_2}) = Y_{dz_2}$ because of (B.12) and because $\psi_{dZ}(\alpha)$ is increasing in α .

As a result, $Y_{dz_1} \leq Y_{dz_2}$ almost surely for any $z_1 \leq z_2$. Moreover, because of (B.8), $\alpha_{tz} = \alpha_{tz'}$ for any t , z and z' with $t < \min(z, z')$ and $z^* \leq \min(z, z')$. Because of (B.8) and (B.10), $\alpha_{tz} = 0$ for any $z^* \leq t < z$, and $\alpha_{tz} = 1$ for any $t > z \geq z^*$. Given these results, one can show that for any $z' \geq z^*$, $Y_{dz'} = \sum_{t < z^*} \phi_{dt}(\alpha_{tz^*}) + \sum_{t=z^*}^k W_{dt}$. This implies that assumption E.3 also holds. So far, we have shown that Y_{dz} constructed above satisfies assumption a_{z^*} .

Finally, because $E[Y_{dz}] = E[Y_{dz}|Z = z] = E[W_{dz}|Z = z]$ and because of (B.6), we know $\sum_z P(Z = z)E[Y_{dz}]$ achieves the lower bound in $\Gamma_{d,z}$. Moreover, because $P[Y_{dz} = Y|D = d, Z = z] = 1$, this construction of Y_{dz} is consistent with the data. Combine all the above results, for an arbitrary $d \in \{0, 1\}$, we have constructed Y_{dz} which satisfies assumption a_{z^*} and, at the same time, $\sum_z P(Z = z)E[Y_{dz}]$ achieves the lower bound of $\Gamma_{d,z}$.

Similarly, one can construct Y_{dz} which satisfies assumption a_{z^*} and $\sum_z P(Z = z)E[Y_{dz}]$ achieves the upper bound of $\Gamma_{d,z}$, by defining γ'_z as follows:

- for $z < z^*$, let γ'_z be the value which solves

$$\min(\bar{q}_{dt} : t \geq z) = E[\mathbb{1}(D = d)Y|Z = z] + E[\mathbb{1}(D \neq d)Y|Z = z]\gamma_z$$

- for $z \geq z^*$, let γ'_z be the value which solves

$$\min(\bar{q}_{dt} : t \geq z^*) = E[\mathbb{1}(D = d)Y|Z = z] + E[\mathbb{1}(D \neq d)Y|Z = z]\gamma_z$$

Following the same steps as before except replacing γ_z with γ'_z , one can show that the constructed Y_{dz} satisfies E.1-E.3 and $\sum_z P(Z = z)E[Y_{dz}]$ achieves the upper bound of $\Gamma_{d,z}$.

Taking convex combinations of the constructions which achieve the upper and lower bound, every point in $\Gamma_{d,z}$ can be achieved under assumption E.1-E.3. This completes the proof. \square

Proof of Lemma 9. Suppose E.1 and E.2 hold. For any $z \in \{1, 2, \dots, k\}$ and any $d \in \{0, 1\}$, we have

$$\mathbb{1}(Z = z, D = d)Y + \mathbb{1}(Z \neq z \text{ or } D \neq d)\underline{y}_d \leq Y_{dz} \leq \mathbb{1}(Z = z, D = d)Y + \mathbb{1}(Z \neq z \text{ or } D \neq d)\bar{y}_d$$

Therefore, $\underline{q}_{dz} \leq E[Y_{dz}|Z = z] \leq \bar{q}_{dz}$. Because of E.2, this implies that $\underline{q}_{dz} \leq E[Y_{dz}] \leq \bar{q}_{dz}$. As a result, $E[\underline{Y}_d] \leq \sum_z P(Z = z)EY_{dz} \leq E[\bar{Y}_d]$, which proves that $\Theta_I \subseteq [E[\underline{Y}_1], E[\bar{Y}_1]] \times [E[\underline{Y}_0], E[\bar{Y}_0]]$. Moreover, when $P(Y \in [\underline{y}_d, \bar{y}_d]|D = d) = 1$ for any $d \in \{0, 1\}$ fails to hold, E.1 will fail to hold. Hence, $\Theta_I \neq \emptyset$ only if $P(Y \in [\underline{y}_d, \bar{y}_d]|D = d) = 1$ for any $d \in \{0, 1\}$.

Suppose that $P(Y \in [\underline{y}_d, \bar{y}_d]|D = d) = 1$ for any $d \in \{0, 1\}$ hold. Then, we know that for each $z = 1, \dots, k$ and each d , $\underline{y}_d \leq \underline{q}_{dz} \leq \bar{q}_{dz} \leq \bar{y}_d$. Construct Y_{dz} as the following for each z and d :

$$Y_{dz} = \mathbb{1}(Z = z, D = d)Y + \mathbb{1}(Z = z, D \neq d)\underline{y}_d + \mathbb{1}(Z \neq z)\underline{q}_{dz}.$$

By construction, $\theta_d = \sum_z P(Z = z)EY_{dz} = \sum_z P(Z = z)\underline{q}_{dz} = E[\underline{Y}_d]$. Moreover, one can check that this construction also satisfies assumptions E.1 and E.2. Similarly, for each d , we can construct Y'_{dz} as

$$Y'_{dz} = \mathbb{1}(Z = z, D = d)Y + \mathbb{1}(Z = z, D \neq d)\bar{y}_d + \mathbb{1}(Z \neq z)\bar{q}_{dz}.$$

Again, Y'_{dz} satisfies assumptions E.1 and E.2 by construction. In addition, $\theta_d = \sum_z P(Z = z)EY'_{dz} = E[\bar{Y}_d]$. By considering (Y_{1z}, Y'_{0z}) , (Y'_{1z}, Y_{0z}) , (Y_{1z}, Y_{0z}) and (Y'_{1z}, Y'_{0z}) , we conclude that Θ_I is nonempty and $\Theta_I = [E[\underline{Y}_1], E[\bar{Y}_1]] \times [E[\underline{Y}_0], E[\bar{Y}_0]]$. \square