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Glacial landscape evolution by subglacial quarrying: A multiscale computational approach

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Abstract

Quarrying of bedrock is a primary agent of subglacial erosion. Although the mechanical theory behind the process has been studied for decades, it has proven difficult to formulate the governing principles so that large-scale landscape evolution models can be used to integrate erosion over time. The existing mechanical theory thus stands largely untested in its ability to explain postglacial topography. In this study we relate the physics of quarrying to long-term landscape evolution with a multiscale approach that connects meter-scale cavities to kilometer-scale glacial landscapes. By averaging the quarrying rate across many small-scale bedrock steps, we quantify how regional trends in basal sliding speed, effective pressure, and bed slope affect the rate of erosion. A sensitivity test indicates that a power law formulated in terms of these three variables provides an acceptable basis for quantifying regional-scale rates of quarrying. Our results highlight the strong influence of effective pressure, which intensifies quarrying by increasing the volume of the bed that is stressed by the ice and thereby the probability of rock failure. The resulting pressure dependency points to subglacial hydrology as a primary factor for influencing rates of quarrying and hence for shaping the bedrock topography under warm-based glaciers. When applied in a landscape evolution model, the erosion law for quarrying produces recognizable large-scale glacial landforms: U-shaped valleys, hanging valleys, and overdeepenings. The landforms produced are very similar to those predicted by more standard sliding-based erosion laws, but overall quarrying is more focused in valleys, and less effective at higher elevations.

1. Introduction

The global cooling trend of the Cenozoic accelerated around 2.5–3 Myr ago, leading to several cycles of widespread glaciation in the Northern Hemisphere [Shackleton and Kennett, 1975; Savin, 1977; Shackleton et al., 1984; Zachos et al., 2001; Ehlers and Gibbard, 2007]. Mountainous areas that are now ice-free were buried under thick ice and exposed to intense subglacial erosion, as is now evident from the characteristic glacial landforms found in these regions: U-shaped valleys, hanging valleys, cirques, overdeepenings, horns, etc. [Sugden and John, 1976]. Especially impressive are the deeply incised fjord systems of continental margins, such as along the coasts of Norway, Alaska, Chile, New Zealand, and Greenland. Although the influence of subglacial erosion is important for the present topography of Earth’s surface, the mechanisms driving subglacial erosion are less clear. Glaciers erode bedrock primarily by abrasion and quarrying, and both processes have for a long time been the subjects of detailed studies [Hallet, 1979; Iverson, 1991; Hallet, 1996; Cohen et al., 2006]. However, both abrasion and quarrying are often represented by simplified rules when included in computational landscape evolution models that are capable of simulating long-term erosion by glaciers [MacGregor et al., 2000; Hildes et al., 2004; Anderson et al., 2006; Kessler et al., 2008; Egholm et al., 2009; Jamieson et al., 2008; Tomkin, 2009; Herman et al., 2011].

Most such landscape evolution models assume that glacial erosion rate, $\dot{E}$, is a function of basal sliding speed or ice flux, $u$:

$$\dot{E} = c_1u^2$$  \hspace{1cm} (1)

where $c_1$ and $c_2$ are constants. Often, $c_2 = 1$ is assumed [MacGregor et al., 2000; Tomkin, 2003; Kessler et al., 2008; Egholm et al., 2009; MacGregor et al., 2009; Herman et al., 2011]. Theoretical considerations described by Hallet [1979] suggest that $c_2 = 2$ is a reasonable approximation when abrasion dominates subglacial erosion and rock debris transported in the basal ice drives erosion. On the other hand, subglacial quarrying operates...
by plucking of bedrock from the lee sides of bumps on the bed, and this process is known to be influenced by factors other than sliding speed, such as the effective pressure (difference between ice pressure on the bed and water pressure in cavities) [Iverson, 1991; 2012; Cohen et al., 2006] and the presence of preexisting fractures [Dühnforth et al., 2010; Hooyer et al., 2012]. The constants $c_1$ and $c_2$ are poorly constrained, and the values often used ($c_1 = 10^{-6} \text{ to } 10^{-4}$ and $c_2 = 1$) are based on a single study from Variegated Glacier in Alaska, where data were gathered over a few years that included major surges and uncertain release of basal debris from storage by subglacial water [Humphrey and Raymond, 1994]. Empirical studies of subglacial erosion rates report values ranging from 0.01 to 100 mm yr$^{-1}$ [Hallet et al., 1996; Koppes et al., 2015]. Verification of the erosion law (equation (1)) has recently been attempted both through theoretical models [Iverson, 2012] and by relating measurements of basal sliding to sediment yields inferred to reflect sediment production by bedrock erosion [Herman et al., 2015; Koppes et al., 2015]. Koppes et al. [2015] reported ice flux, sliding speed, and sediment yield from 15 glaciers in a transect spanning approximately 20° latitude from Patagonia to Antarctica. Rates of erosion vary by several orders of magnitude along the north–south transect, even though the geological setting and the ice discharge are roughly the same. The study thus demonstrated dependence on latitude and to a lesser degree basal sliding speed or ice flux. The authors suggested that the correlation with latitude relates primarily to variations in seasonal meltwater production. Rapid fluctuations in the amount of meltwater reaching the glacier bed lead to fluctuations in basal water pressure, which are known to affect the rate of quarrying by periodic amplification of differential stress in bedrock close to subglacial cavities [Iverson, 1991; Hallet, 1996; Cohen et al., 2006; Anderson, 2014].

The influence of hydrology on erosion may vary between erosion processes, and it is therefore important to study the erosion processes separately. The simple power law erosion law (equation (1)) ignores the complexities of representing erosion mechanisms at the meter scales of individual bedrock steps or subglacial cavities in landscape evolution models where the spatial resolution is, at best, a few hundred meters. However, a process-oriented model for subglacial quarrying proposed by Iverson [2012] allows for bridging the spatial scales that separate individual bedrock steps from the regional topographical trends captured by the typical cell spacing of landscape evolution models. The model approach integrates quarrying over many meter-scale bed steps to estimate the first-order controls on quarrying when averaged over larger areas. Iverson [2012] used the model to quantify the dependence of erosion rate on sliding speed and effective pressure. Beaud et al. [2014] applied the quarrying model in a flow band model of coupled ice and water flow to predict rates of subglacial quarrying and abrasion. However, they did not track the evolution of the bed in response to the erosion. We stress that our purpose is to make possible the use of a new physics-based erosion law in a landscape evolution model, but not to test or calibrate the theoretical model. That more challenging goal should be addressed in a future study. The objectives of the present study are thus to (1) Extend the analysis of Iverson [2012] to more comprehensively evaluate the dependence of quarrying on effective pressure, to assess the effect of bed slope, and construct an erosion law for quarrying based on these results; (2) implement the resulting erosion law in a large-scale landscape evolution model and explore the topographic patterns that it produces; (3) address the importance of seasonal hydrological transients on the rates and patterns of subglacial quarrying.

First we provide a brief introduction to the mechanics of subglacial quarrying.

### 2. Quarrying Mechanics

Iverson [2012] model of quarrying is derived from the theory on adhesive wear. In material science ‘adhesive wear’ refers to erosion at a rough contact where slip causes transfer of material from one side of the contact to the other. As an upscaled analogue to that process, Iverson [2012] considered ice sliding at velocity, $u$, over rough bedrock. The roughness is modeled as a series of two-dimensional steps (Figure 1). Cavities form at the lee sides of the steps when sliding velocity is sufficiently high that ice moves away from lee surfaces faster than it creeps toward them. The contact area between the ice and the bed decreases as cavities grow, and this amplifies the pressure (bed normal stress) on the remaining contact surfaces, promoting growth of preexisting cracks in the rock. Earlier quarrying models [Iverson, 1991; Hallet, 1996; Hildes et al., 2004] considered isolated cracks in otherwise homogeneous bedrock, with the rate of quarrying depending on the rate of subcritical crack growth [Hallet, 1996; Hildes et al., 2004]. In contrast, the model presented by Iverson [2012] includes a statistical treatment of bedrock weakness: larger rock bodies are weak because they are more likely to contain a large flaw [Jaeger and Cook, 1979]. This implies that quarrying is influenced both by fracture distribution

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Figure 1. Ice sliding over fractured bedrock. Water-filled cavities form in the lee of steps. Ice exerts a normal stress on the bedrock over the length $L - S$. $u$ is basal sliding speed, $L$ is step length, $h$ is step height, $S$ is cavity length, $P_w$ is the water pressure in the cavities, and $\sigma_n$ is the ice normal stress on the bed. After Iverson [2012].

and geometry and by the volume of glacially stressed bedrock. We summarize in the following the quarrying model presented by Iverson [2012].

The rate of bedrock erosion by quarrying, $\dot{E}_q$, depends on the likelihood of rock fracturing at a topographic step of height $h$ and length $L$ (Figure 1):

$$\dot{E}_q = \frac{ukh}{2L} (1 - S/L),$$

where $k$ is the probability of step failure, and $S$ is the length of the cavity associated with the topographic step (Figure 1). This equation follows from basic assumptions of adhesive wear theory: that the volume of eroded fragments at the sliding interface depends linearly on sliding distance, on the area of contact along the sliding interface, and on the probability $k$ of bumps on the contact breaking off per unit distance slid [see Iverson, 2012] (GSA Data Repository 2012193 for the derivation). The same starting assumptions, which are not specific to particular materials on either side of the sliding interface, have been used to evaluate erosion along fault surfaces [Scholz, 1987].

The cavity length is computed under steady state conditions using the following relation derived and tested experimentally by Iverson and Petersen [2011]

$$S = 4 \left( \frac{uh}{2\pi} \right)^{1/2} \left( \frac{B}{N} \right)^{n/2},$$

where $B$ and $n$ are, respectively, the pre-exponential factor and the stress exponent of Glen’s flow rule for ice. $N$ is the effective pressure, which is the difference between the ice pressure, $P_i$, on the bed and the water pressure, $P_w$, within cavities.

The probability of step failure, $k$, is estimated using a Weibull probability distribution of rock strength, which builds on the assumption that a given volume of rock is only as strong as its weakest part [Jaeger and Cook, 1979]:

$$k = 1 - \exp \left[ -\frac{h(L - S)}{V_0} \left( \frac{\sigma_d}{k\sigma_0} \right)^m \right].$$

Here $h(L - S)$ is the glacially stressed rock volume, per unit width; $V_0$ is a characteristic rock volume, per unit width, large enough to contain the largest preexisting fracture; $\sigma_d$ is the deviatoric stress; $\sigma_0$ is the Weibull scale parameter, which is close to the mean strength of the rock; $k$ is a factor that reduces that strength to account for subcritical (slow) crack growth in subglacial settings; and $m$ is the Weibull modulus, which describes the homogeneity of rock strength within a given lithology, i.e., the narrowness of the range of stress over which multiple samples of the same lithology might fail. A high Weibull modulus implies a homogeneous fracture distribution. Values of $m$ and $\sigma_0$ are determined through laboratory testing [Iverson, 2012].
Table 1. Parameters Used in the Ice Model, the Quarrying Model, and the Hydrological Model

<table>
<thead>
<tr>
<th>Symbols</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>Viscosity constant</td>
<td>$7.3 \times 10^6$ Pa s$^{1/3}$</td>
</tr>
<tr>
<td>$n$</td>
<td>Ice stress exponent</td>
<td>3.0</td>
</tr>
<tr>
<td>$\rho_i$</td>
<td>Density of ice</td>
<td>910 kg m$^{-3}$</td>
</tr>
<tr>
<td>$C$</td>
<td>Sliding parameter</td>
<td>0.25</td>
</tr>
<tr>
<td>$\Lambda_0$</td>
<td>Sliding parameter</td>
<td>$2 \times 10^{-17}$ m Pa$^{-3}$ yr$^{-1}$</td>
</tr>
<tr>
<td>$T_{sl}$</td>
<td>Temperature at sea level</td>
<td>6°C</td>
</tr>
<tr>
<td>$\Delta T$</td>
<td>Lapse rate</td>
<td>6°C km$^{-1}$</td>
</tr>
<tr>
<td>$m_{acc}$</td>
<td>Accumulation gradient</td>
<td>0.5 m yr$^{-1}$ °C$^{-1}$</td>
</tr>
<tr>
<td>$m_{abl}$</td>
<td>Ablation gradient</td>
<td>1.5 m yr$^{-1}$ °C$^{-1}$</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Rate scaling factor</td>
<td>1/3</td>
</tr>
<tr>
<td>$V_0$</td>
<td>Characteristic rock volume per unit width</td>
<td>10 m$^3$</td>
</tr>
<tr>
<td>$\sigma_0$</td>
<td>Weibull scale parameter</td>
<td>$10^6$ Pa</td>
</tr>
<tr>
<td>$\sigma_n^*$</td>
<td>Ice yield stress</td>
<td>$10^6$ Pa</td>
</tr>
<tr>
<td>$c$</td>
<td>Stress scaling factor</td>
<td>0.1</td>
</tr>
<tr>
<td>$m$</td>
<td>Weibull modulus</td>
<td>3.0</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Topographic wavelength</td>
<td>50 m</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Hurst number</td>
<td>0.2</td>
</tr>
<tr>
<td>$\rho_w$</td>
<td>Density of water</td>
<td>1000 kg m$^{-3}$</td>
</tr>
<tr>
<td>$L_f$</td>
<td>Latent heat of fusion</td>
<td>$3.35 \times 10^3$ J kg$^{-1}$</td>
</tr>
<tr>
<td>$l_m$</td>
<td>Bed step length</td>
<td>2.0 m</td>
</tr>
<tr>
<td>$h_0$</td>
<td>Lower limit of bed step height</td>
<td>0.1 m</td>
</tr>
<tr>
<td>$s_0$</td>
<td>Step height scaling parameter</td>
<td>0.3</td>
</tr>
<tr>
<td>$d_s$</td>
<td>Step height scaling parameter</td>
<td>0.2</td>
</tr>
<tr>
<td>$k_0$</td>
<td>Transmisivity</td>
<td>$0.01 \text{–} 0.1$ kg$^{-1/2}$ m$^{1/2}$</td>
</tr>
</tbody>
</table>

Following Hallet [1996] and Iverson [2012], the deviatoric stress in the bedrock step is

$$
s_d = \begin{cases} 
    c (\sigma_n - \rho_w) = cN \left( \frac{1}{1 + \frac{3}{2}} \right) & \text{for } \sigma_n - \rho_w < \sigma_n^* \\
    c\sigma_n^* & \text{for } \sigma_n - \rho_w \geq \sigma_n^* 
\end{cases} 
$$

(5)

where $c$ is a constant that scales the bedrock boundary stress to internal deviatoric stress and $\rho_w$ is the water pressure in the cavities. As the contact area between ice and bed decreases ($S \rightarrow L$), $\sigma_n - \rho_w$ increases, but it cannot exceed the stress, $\sigma_n^*$, at which ice crushes. Parameter values used in the quarrying model are listed in Table 1 and are similar to the values used by Iverson [2012].

3. Constructing an Erosion Law for Landscape Simulation

The quarrying model by Iverson [2012] allows the standard erosion law of equation (1) to be evaluated [Iverson, 2012]. Using equations (2)−(5), Iverson [2012] calculated the average erosion rate of 5000 steps for sliding velocities ranging between 0 and 5000 m yr$^{-1}$ and then performed a power law fit of equation (1) to the calculated erosion rates. The resulting fit demonstrated that a power law is able to describe the dependence of erosion rates on basal sliding speed. However, the nonlinearity of this sliding-based erosion rule depends sensitively on bedrock strength heterogeneity and effective pressure [Iverson, 2012].

We follow Iverson [2012] in averaging erosion on many randomly sized steps for a range of sliding velocities, $u$. However, we extend the analysis by exploring more thoroughly the dependence on effective pressure, $N$, and by including the average bed slope in the direction of ice flow, $s_b$, as a third control variable. These three
parameters \((u, N, s_b)\) are chosen as primary variables because they represent first-order subglacial conditions that control erosion rate in the quarrying model: (1) basal sliding speed controls the rate at which ice encounters steps on the bed and the cavity size, which helps control both the stress that ice exerts on bedrock and the area over which that stress acts, (2) effective pressure scales the stress and affects the cavity size and hence also the glacially stressed area, (3) bed slope, by influencing step heights, also influences cavity size (equation (3)). The objective is thus to evaluate if an expanded power law relationship of the form

\[
E_q = c_i N^{s_b} u^{s_b} (s_b + s_{b0})^{s_c}
\]

provides a sound basis for predicting quarrying across relevant intervals of sliding speed, effective pressure, and bed slope. The parameter \(s_{b0}\) allows the power law to include negative (adverse) bed slopes down to \(-s_{b0}\). The form of this erosion law is motivated first of all by the documented influence of effective pressure, sliding speed, and bed slope (step heights) in the quarrying model [Iverson, 2012], second by the direct availability of these three parameters in large-scale glacial landscape evolution models, and third by the simplicity of a power law.

### 3.1. Quarrying of Subcell Topography

The model grid of the landscape evolution model used in this study is \(20 \times 40\) km and the cell size is \(160\) m. Regional elevation, \(b(x, y)\), and bed slope, \(s_b(x, y)\), represent the only topographic information from each grid cell, and individual bedrock steps prone to subglacial quarrying are hence not resolved by the model (Figure 2). In order to compute the average quarrying rate across a grid cell, we thus need to make assumptions about the distribution of topography at spatial scales below the cell resolution. For this, we use a power law model based on the theory of Von Karman to generate a detailed fractal topography based on a given topographic wavelength, \(\lambda\), and a Hurst number, \(v\), which describes the roughness of the subcell topography (Table 1). The resulting relief of the subcell topography is approximately \(8\) m. The subcell topography is tilted to account for the regional slope, \(s_b\) (Figure 2c). Here adverse slopes, with respect to sliding direction, are negative whereas slopes in direction of sliding are positive.

The three-dimensional fractal subgrid topography is then sampled along \(1000\) two-dimensional profiles in the direction of ice flow (\(0.16\) m between profiles). Each profile is \(160\) m long and includes \(80\) steps (Figures 2c and 2d). We use a fractal distribution of step lengths, \(L\), to define the positions of the topographic steps. The fractal step distributions of the \(1000\) profiles have the same fractal dimension (1.8) but with different random arrangements of the steps. A fractal step distribution is an assumption but is supported by measurements of small-scale, glacier bed surface roughness [Hubbard et al., 2000]. However, we are aware that tectonic fractures with characteristic spacings might not result in a self-similar distribution of step sizes. For each profile, the difference in elevation between steps defines the step height, \(h\) (Figure 2d). We compute an average rate of quarrying for all \(1000\) profiles, hereby obtaining an average quarrying rate for the subgrid topography.

An important element of computing the average erosion rate across the steps is that large cavities may engulf downglacier steps [Iverson, 2012]. Quarrying does not occur where steps are drowned by cavities (meaning that the cavity length, \(S\), associated with a given step exceeds the length of the step, \(L\)). Drowning of a step causes the cavity to extend farther to partly or completely cover the next step in the sequence. To compute the total length of the cavity drowning the step, the length, \(L\), and height, \(h\), of the drowned step are added to the length and the height of the next step in the sequence. Drowning of steps in a profile (resulting in zero erosion) is exaggerated by the two-dimensional setup of the quarrying model. However, sizes of large cavities depend on the relative positions of steps, so that the \(1000\) different profiles used experience slightly different degrees of step drowning for a given combination of effective pressure, sliding speed, and slope. The use of different arrangements of the steps thus makes the average erosion rate across the many profiles much less sensitive to step drowning than the erosion rate for any individual profile. Upglacier-facing topographic steps are assumed to not erode by quarrying. Yet positive step heights occur also for adverse regional slopes \((s_b < 0)\), although their frequency declines as the adverse slope steepens.

In order to constrain and test the multivariable power law (equation (6)), we calculate the average erosion rate across all \(1000\) profiles (i.e., \(8 \times 10^4\) steps) for \(100\) different values of effective pressure, \(N\), basal sliding rate, \(u\), and bed slope, \(s_b\) (Figure 3), resulting in consideration of \(10^6\) parameter combinations. We focus on conditions in alpine type glaciers and use the following parameter intervals: \(N = [0.1 – 5]\) MPa, \(u = [1 – 500]\) m yr\(^{-1}\), and \(s_b = [-0.5 – 1.0]\).
Figure 2. Modeling of topography at various scales. (a) Upper part of the initial topography in the landscape evolution model. (b) The spatial resolution of the landscape evolution model used in this study is 160 m. To calculate the average erosion rate across a grid cell, we thus need to make assumptions on the topography below cell resolution. (c) Subcell topography is generated based on the theory of Von Karman and tilted according to the regional slope in direction of ice flow, $s_b$, in the grid cell. Dashed lines from Figures 2b to 2c indicate that the subcell topography in Figure 2c does not exist in the DEM in Figures 2a and 2b but is a representation of what small-scale topography might look like. (d) Sampling the subcell topography in profiles aligned with the direction of ice flow enables us to calculate the average quarrying rate across a grid cell.

3.2. The Best Fit Power Law

We examine initially how effective pressure, basal sliding, and bed slope affect quarrying for a particular set of parameters contributing to equations (2)–(4) (i.e. $m$, $\sigma_0$, $V_0$, and $c$ listed in Table 1). We thereafter vary these parameters to test the robustness of the initial result. The four coefficients ($c_1$–$c_4$) of equation (6) are obtained by performing a least squares fit to the $10^6$ average erosion rates. For the initial parameter set the resulting best fit power law is

$$E_q = 7.5 \times 10^{-6} N^2 u^{1.0} (s_b + s_b0)^{1.9}$$

which implies a strong and nonlinear influence of effective pressure and slope (exponents 2.9 and 1.9, respectively). In contrast, the power law fit predicts a linear dependence on sliding speed (exponent 1.0).

Figure 3. Loop structure for computing quarrying rates. Average erosion rate is calculated across 1000 step profiles for 100 different values of bed slope $s_b$, effective pressure, $N$ and basal sliding speed, $u$. 
Figure 4. (a) Erosion rates calculated using the quarrying model. (b) The best fit power law. Effective pressure, \( N \), basal sliding speed, \( u \), and bed slope, \( s_b \), are along the three axes, and erosion rate is represented by the color scale. Highest erosion rates occur where values for \( s_b \), \( u \), and \( N \) are high.

The R-squared value (the fraction of the response variable variation that is explained by the linear model) for the power law fit (equation (7)) is 0.938.

The best fit power law can overall mimic variations within the set of calculated erosion rates (Figure 4). The fastest erosion occurs where values for basal sliding, bed slope, and effective pressure are high (Figure 4). The largest calculated erosion rate is very high (1 m yr\(^{-1} \)), but it occurs only for an extreme combination of high effective pressure, fast basal sliding, and steep positive bed gradient, which is probably very rare in nature. At the lower end of the slope interval (vertical axes in Figure 4), the power law overestimates erosion rates at steep reverse bed slopes where effective pressure and basal sliding are simultaneously high. The steep reverse slopes are dominated by steps with negative step height and zero erosion rate. The erosion rate is thus abruptly truncated by the geometry of the bed in this setting, and the smoothly varying power law cannot fully mimic this. Similarly, the calculated erosion rates decrease abruptly for very low levels of effective pressure when large cavities start to drown individual steps. Fitting these very low erosion rates of less than \( 10^{-6} \text{ mm yr}^{-1} \) is not possible with the power law, and only erosion rates above \( 10^{-6} \text{ mm yr}^{-1} \) are therefore used to construct the power law fit. This minimum threshold is justified by the fact that erosion rates below \( 10^{-6} \text{ mm yr}^{-1} \) are too small to play any significant role for landscape evolution, even when integrated over millions of years.

Keeping any two of the parameters, \( N \), \( u \), and \( s_b \), constant allows us to study (Figures 5–7) the isolated effect of each parameter on the erosion rate (along lines in the three-dimensional plot in Figure 4). The power law fit is still a best fit to all three parameters, but studying the fit along two-dimensional projections parallel to the \( N \), \( u \), and \( s_b \) axes facilitates a more detailed evaluation of how well the power law captures the variability within the set of calculated erosion rates. Along the \( N \) axis (Figure 5), erosion rate generally increases with effective pressure and the power law fit is reasonably well constrained for \( N \geq 0.5 \text{ MPa} \) at slow sliding (Figure 5, left column) and \( N \geq 1 \text{ MPa} \) at faster sliding (Figure 5, right column). At pressures below 0.2 MPa, the calculated rates of erosion drop rapidly and start to fluctuate. The low rates of erosion (\(< 10^{-6} \text{ mm yr}^{-1} \)) as well
Figure 5. Erosion rates, $E_q$, as a function of effective pressure, $N$, at constant values of slope, $s_b$, and basal sliding speed, $u$. Black lines are calculated average erosion rates (equation (8)). Yellow lines are the power law fit (equation (19)). Erosion rates below $10^{-6}$ mm yr$^{-1}$ (gray) are not considered in the power law fit.

as their fluctuation in the low-pressure domain are caused by the above mentioned growth of cavities that are large enough to drown individual steps. The power law cannot simultaneously fit the trends in both the low-pressure and the moderate- to high-pressure domains. Yet the moderate- to high-pressure trend is by far the most important to fit, as it associates with the highest rates of erosion and hence with the greatest impact on long-term landscape evolution. Furthermore, we note that fitting the highest erosion rates preferentially maximizes the strength of the overall fit. Alternatively, fitting the low erosion rates at the expense of the high rates would exaggerate the pressure exponent and overestimate the erosion rates by orders of magnitude (notice the logarithmic axes in Figures 5–7).

The small peak in erosion rate that persistently occurs at the transition between the two pressure domains is caused by rapid decrease of cavity length, $S$, as effective pressure increases from small to moderate levels. At the peak, $S$ has decreased enough to prevent step drowning, but it is still large enough to cause significant stress concentrations at the relatively small contact areas between ice and bedrock. For even higher pressures, the cavity lengths decrease more slowly, and the more gradual increase in erosion rate with pressure is mainly due to increasing load on the now relatively stable zones of ice bed contact. Superimposed on these effects of step drowning and ice bed contact pressure is the inverse relationship between rock strength and the area of ice bed contact, which promotes increased erosion rates if effective pressure is high and cavity sizes are small.
Figure 6. Erosion rates, $\dot{E}_q$, as a function of basal sliding speed, $u$, at constant values of slope, $s_b$, and effective pressure, $N$. Black lines are calculated average erosion rates (equation (8)). Yellow lines are the power law fit (equation (19)).

Studying the rates of erosion along the axes of sliding speed, $u$ (Figure 6), and slope, $s_b$ (Figure 7), also indicates how the power law cannot fully capture the small rates of erosion due to step drowning at low effective pressure and fast sliding (Figure 6, top row). Similarly, the power law clearly overestimates erosion on adverse slopes for fast sliding and low effective pressure (Figure 7, top row). On the other hand, the power law provides an acceptable fit over large parts of the pressure, sliding, and slope intervals, supporting the conclusion that the multivariable power law is our best fit approximation to the first-order trends predicted by the quarrying model.

3.3. Sensitivity Analysis

Before using the power law model to study long-term erosion in a landscape evolution model, we explore the robustness of the best fit exponents to the various model parameters, such as the Weibull modulus, $m$, the Weibull scale parameter, $\sigma_0$, the characteristic rock volume, $V_0$, and the stress scaling factor, $c$ (Table 1). These parameters are not equally well constrained, and a sensitivity analysis can expose the degree to which the power law coefficients represent a wide range of parameter settings. In order to perform the sensitivity analysis, we repeat the $10^6$ calculations of average erosion rate in cases where each of the four parameters mentioned above is respectively increased and decreased relative to their default setting (Table 1). The least squares fit of a power law is repeated in each case, and the resulting spread in coefficients is illustrated in Figure 8.
Figure 7. Erosion rates, $\dot{E}_q$, as a function of slope in direction of sliding, $s_b$, at constant values of effective pressure, $N$, and basal sliding speed, $u$. Black lines are calculated average erosion rates (equation (8)). Yellow lines are the power law fit (equation (19)).

The sensitivity test highlights that the power law exponents for effective pressure, $c_2 \approx 3$, basal sliding, $c_3 \approx 1$, and bed slope, $c_4 \approx 2$, are fairly robust to variations in the parameters of the quarrying law (Figure 8). However, the value of the pre-exponential factor, $c_1$, is difficult to constrain, as it spans several orders of magnitude when any of the model parameters are varied (Figure 8). The efficiency of quarrying is known to depend strongly on the availability of preexisting fractures, and it is hence not surprising that $c_1$ depends strongly on the homogeneity of bed strength $m$. However, $c_1$ also varies over reasonable ranges of the parameters $\sigma_0$, $V_0$, and $c$, which unfortunately means that the exact erosion rate is difficult to predict unless these parameters are well constrained.

Importantly, however, the less variable exponents $c_2 - c_4$ inform us on how variations in sliding rate, effective pressure, and bed slope affect the rate of quarrying. It is also worth noting that the exponents for effective pressure, $c_2$, and bed slope, $c_4$, are consistently higher than the exponent for basal sliding, $c_3$. This result clearly highlights that effective pressure and bed slope are just as, if not more, important to consider than sliding speed when predicting rates of subglacial quarrying, despite the absence of these two additional variables from many glacial landscape evolution models [MacGregor et al., 2000; Tomkin, 2003; Kessler et al., 2008; Egholm et al., 2009; MacGregor et al., 2009; Herman et al., 2011]. The R-squared values for the power law fits tested span 0.867–0.94. Importantly, the proposed power law is conditioned by a fractal step distribution, and a highly regular step distribution, for example, could yield a different result.
Figure 8. Sensitivity of parameters in the quarrying law (table 1). The exponents $c_2$, $c_3$, and $c_4$ of the quarrying law (equation (6)) are plotted on the right-hand axis, whereas $c_1$ is plotted on the left-hand logarithmic axis. The plot highlights that the power law exponents for effective pressure, $c_2$, basal sliding speed, $c_3$ and bed slope, $c_4$, vary mildly in response to variations in the parameters of the quarrying law parameters. However, the value of the pre-exponential factor, $c_1$, is difficult to constrain, as it spans several orders of magnitude.

Based on the sensitivity analysis, we suggest the following power law relation for scaling rates of quarrying:

$$\dot{E}_q = K_e N^{1.0} u^{1.3} (s_b - s_{b0})^{2.0}$$

where $K_e$ is a free scaling parameter that depends critically on rock strength variability due to fractures and therefore must be expected to vary by orders of magnitude between rocks of different lithology and tectonic history. We explore the consequences of this erosion law in the following sections.

4. Modeling Glacial Landscape Evolution

Computational landscape evolution models are useful tools for exploring long-term patterns of erosion [MacGregor et al., 2000, 2009; Hildes et al., 2004; Anderson et al., 2006; Herman and Braun, 2008; Tomkin, 2009], as well as feedbacks among the topographical properties that scale erosion over time [Whipple et al., 1999; Tomkin, 2007; Kessler et al., 2008; Egholm et al., 2009; Pedersen and Egholm, 2013]. However, many model components in a glacial setting, such as rules for ice flow and erosion, are based on simplified assumptions that are not fully supported by measurements or experiments. This is also true for the ice flow model and quarrying law used in this study. Landscape evolution models cannot, therefore, be used to make accurate predictions of past or future development of a landscape. Rather, landscape evolution models should be viewed as computer-based experiments that address “what if” questions [Codilean et al., 2006]. We conducted such experiments, aimed at exposing how the multivariable power law model for quarrying affects the development of topography over time (equation (8)). To make these experiments as transparent as possible, we ignored other erosion mechanisms, such as subglacial abrasion, stream incision, and frost weathering, which are known to also operate in a glacial setting.

In order to integrate quarrying over time, we need to model the temporal evolution of effective pressure and sliding along the bed of a glacier. For this we used the higher-order integrated second-order shallow ice approximation (iSOSIA) ice model [Egholm et al., 2011] coupled to cavity-based formulations for subglacial sliding and hydrology. Details of the model components are presented next.
4.1. Ice Flow Model

Glacial landscape evolution models require an ice flow model that to some extent solves the Stokes equations for the motion of the ice. The full set of the Stokes equations are computationally expensive to solve, and approximations that reduce the complexity of the equations, such as the shallow ice approximation [Hutter, 1983], have proved useful. We use in this study the integrated second-order shallow ice approximation (iSOSIA) [Egholm et al., 2011], which is a depth-integrated, higher-order ice model. iSOSIA is computationally efficient enough to perform million-year landscape simulations, but its higher-order terms increase the accuracy and the sensitivity to topography when compared to standard shallow ice models, and this allows us to use the model in fairly steep landscapes with a grid resolution of only few hundred meters [Egholm et al., 2011]. iSOSIA includes effects from longitudinal and lateral stress gradients in the ice. These stress components are of particular importance for capturing the effects of glacial erosion on valley morphology, and they provide important feedbacks when erosion modifies glacier beds [Egholm et al., 2012a]. The model uses the dynamic stress normal to the bed to represent basal ice pressure, which means that ice flow can enhance pressure on the stoss sides of bed bumps and decrease pressure on the lee sides. Egholm et al. [2011, 2012a, 2012b] and Brædstrup et al. [2016] provide validation of the iSOSIA ice model as well as a more in-depth description and discussion of the higher-order ice dynamical effects incorporated by the model.

4.2. Sliding Law

To be consistent with the quarrying model that is based on the formation of cavities, we use a cavity-based sliding law for linking the bed shear stress, $\tau_b$, to the sliding speed, $u$, and effective pressure, $N$ [Schoof, 2005]:

$$\frac{\tau_b}{N} = C \left( \frac{u}{u + N^3 \Lambda_0} \right)^2$$  \hspace{1cm} (9)

where $C = 0.25$ is related to the maximum adverse slope of bed bumps and $\Lambda_0 = 2 \times 10^{-17}$ m Pa$^{-3}$ yr$^{-1}$ depends on aspects of the bed roughness and ice rheology parameters. Importantly, this sliding law includes an upper limit to the shear stress that can be supported at the bed, which is expected if the real contact area between it and a glacier is limited by the distribution of cavities, as the quarrying theory assumes [Iken, 1981].

4.3. Hydrological Model

We assume that meltwater transport in each grid cell occurs either through a system of linked cavities [Kamb, 1987] or through a system of subglacial channels [Röthlisberger, 1972; Nye, 1976], depending on which system operates at the lowest water pressure. As for the quarrying model, the hydrological model thus also requires that we make assumptions about topography at spatial scales below the cell resolution. We do not model the hydrology of each individual cavity and channel but instead define a mean channel spacing, $L_m$, a mean cavity spacing, $L_{cm}$, and a mean cavity step height, $h_m$, in each cell. The mean cavity spacing and the mean channel spacing are treated as constants (Table 1). The mean step height is, however, modeled to depend linearly on regional slope in the direction of ice flow. Cavities also occur on adverse slopes, and we therefore assume that the step height decreases exponentially to a small but positive asymptotic value for negative (adverse) slopes (Figure 9):

$$h_m = \begin{cases} (L_m s_0 - h_0) \exp \left( \frac{h_0 - s_0}{d_s} \right) + h_0 & \text{for } s_0 \leq s_0 \\ L_m s_0 & \text{for } s_0 > s_0 \end{cases}$$  \hspace{1cm} (10)

Parameters $L_{cm}$, $s_0$, $h_0$, and $d_s$ are listed in Table 1.

In every time step of a simulation we route the water generated by ice melting (at the surface and at the ice bed) out through the glacier using the direction of the depth-averaged ice flow. We thus assume that ice and water follow the same path from anywhere in the glacier to the ice margin and hence that the hydrological gradient parallels the ice surface. This assumption is justified by the coarse grid of the landscape evolution model, which after all does not allow for resolving any differences between hydrological gradients and ice flow directions that may exist, for example, close to individual melt water channels. This strategy furthermore allows for efficient computation of water flux, without the need to time integrate the transient evolution of water storage in and under the ice. On the other hand, the strategy also prevents us from studying the effects of such transients in water storage on subglacial erosion. Such transients may be important for rapid variations in erosion rate caused by, for example, diurnal variations in meltwater input [Cohen et al., 2006; Anderson, 2014; Beaud et al., 2014] or when meltwater supercooling prevents water flow out of overdeepenings [Alley et al., 2003]. These effects are, however, outside the scope of the present study.
Step height, $h_m$, as a function of bed slope in direction of sliding, $s_b$. $h_m$ is modeled to depend linearly on $s_b$ for $s_b > s_0$. For $s_b < s_0$ the step height decreases exponentially to a small positive asymptotic value, $h_0$.

From the water flux in a grid cell we compute the cross-sectional area of the conduits using Darcy-Weisbach relations for the two drainage systems:

\[
A_s = \left[ \frac{L_s q_w}{k_s \sqrt{\nabla \Psi}} \right]^\frac{4}{3} \quad \text{and} \quad A_c = \left[ \frac{L_c q_w}{k_c \sqrt{\nabla \Psi}} \right]^\frac{4}{3}
\]

Here $A_s$ is the cross-sectional area of cavities and $A_c$ is the cross-sectional area of channels, $q_w$ is the water flux, $k_s$ is a parameter that relates to the Darcy-Weisbach friction factor [Schoof, 2010], and $\nabla \Psi$ is the hydrological gradient, which we assume parallels the ice surface (i.e., $\nabla \Psi = \rho_w g \nabla h$ where $h$ is the elevation of the ice surface). We note that $L_s q_w$ and $L_c q_w$ are water discharge through a cavity and a channel, respectively.

Effective pressure for both the cavity, $N_s$, and the channel, $N_c$, systems is calculated from the two balance equations governing the opening and closing rate of cavities and channels. The balance equation for cavities is

\[
\frac{\partial A_s}{\partial t} = uh_m - \frac{\pi}{8} \left( \frac{N_s}{B} \right)^n S^2
\]

where the first term on the right-hand side (the product of basal sliding rate, $u$, and bed step height, $h_m$) describes opening of cavities. The second term on the right-hand side represents closure of cavities by ice creep ($B$ and $n$ are ice creep parameters and $S$ is cavity length). We relate the cavity length to the cavity cross-sectional area by

\[
A_s = \alpha h_m S
\]

where $\alpha$ is a dimensionless parameter between 0 and 1. $\alpha = 0.5$ for a linear cavity roof and thus a triangular cross section, while $0.5 < \alpha < 1$ implies a convex cavity roof. We follow Anderson [2014] and use $\alpha = 0.7$. Equation (12) agrees with Kamb's cavitation model [Kamb, 1987], and its steady state form is fully consistent with equation (3), which has empirical support and is used to predict cavity length in the quarrying model [Iverson and Petersen, 2011; Iverson, 2012].

The balance equation for channels is [Nye, 1976]

\[
\frac{\partial A_c}{\partial t} = \frac{L_c q_w}{\rho_i L_i} \nabla \Psi - 2 \left( \frac{N_c}{nB} \right)^n A_c
\]

where the channel opening term is due to melting caused by heat from the water flow ($\rho_i$ is ice density and $L_i$ is latent heat of ice) and the closing term is again due to ice creep. Note that melting could also contribute to opening of cavities, but the water discharge through cavities is much smaller than through channels since cavities are considered more densely distributed ($L_s \ll L_c$). We can, therefore, safely ignore this effect in equation (12).
Rearranging equations (12) and (14) gives the effective pressure for the two drainage systems:

$$N_s = B \left( \frac{8}{\pi} \right)^{\frac{1}{2}} \left[ \frac{u h_m - a h_m \alpha}{S^2} \right]^{\frac{1}{2}}$$

(15)

and

$$N_c = nB \left[ \frac{\lambda \Psi - \frac{d h_m}{m}}{2A_c} \right]$$

(16)

The larger of these two pressures is assumed to represent the effective pressure in a grid cell. Note that effective pressure is thus determined by the cavity or channel size needed to drain the meltwater, and this depends mostly on water flux and bed roughness through $h_m$. Hence, effective pressure does not explicitly depend on ice thickness although ice thickness provides an upper limit for the effective pressure. In contrast, the direct influence of cavity and channel size on effective pressure opens for a strong effect of total water flux and its seasonal variability on rates of erosion [Beaud et al., 2014].

5. Computational Experiments

To investigate the effect of the quarrying law (equation (8)) on long-term landscape evolution, we perform three experiments:

1. Experiment 1 explores the spatial patterns of erosion that follow from implementing the quarrying law (equation (8)) in a three-dimensional landscape evolution model.
2. Experiment 2 investigates the general influence of hydrology in modulating effective pressure. The models in this experiment use a steady hydrological system operating with different degrees of drainage efficiency.
3. Experiment 3 addresses quarrying forced by seasonal variations in meltwater production.

All three experiments use the same initial bed topography; a synthetic preglacial landscape (Figure 10a) generated with a fluvial landscape evolution model based on an detachment-limited stream power erosion law [Whipple and Meade, 2006]. The experiments use a simple climate forcing where the mean annual atmospheric temperature decreases linearly with elevation:

$$T = T_{sl} - dT \times b$$

(17)

Here $T_{sl} = 6^\circ$C is the temperature at sea level, $dT = 6^\circ$C km$^{-1}$ is the lapse rate, and $b$ is the bedrock surface elevation.

Mass balance, $M$, is modeled as a simple linear function of the atmospheric temperature:

$$M = \begin{cases} -m_{acc} \alpha T, & \text{if } T \leq 0 \text{ (accumulation)} \\ -m_{abl} \alpha T, & \text{if } T > 0 \text{ (ablation)} \end{cases}$$

(18)

$m_{acc}$ is the accumulation gradient and $m_{abl}$ is the ablation gradient (Table 1). The same combination of initial topography and climate forcing was recently used in a comparative study of iSOSIA and the full-Stokes Elmer/Ice model [Brædstrup et al., 2016]. From this study, we know that iSOSIA accurately resolves variations in basal shear stress in spite of computation-efficient depth integration of the ice flow velocity.

The two first experiments have steady hydrology, meaning that the surface meltwater input to the subglacial hydrological system is constantly at the mean annual level. However, in experiment 3, we introduce hydrological transients by including seasonal variations in the surface melt production.

5.1. Experiment 1 — Spatial Patterns of Erosion

In this first experiment we study the patterns of erosion and the landforms produced by the erosion law for quarrying (equation (8)). Before glacial erosion, the initial model topography has valleys with V-shaped cross sections and concave longitudinal profiles as in a steady state fluvial landscape (Figures 11a, 11c, and 11d). The topography was then glacierized (Figure 10b), and climate kept constant for 100 kyr. Likewise, the hydrological forcing was constant with melt production equal to the mean annual rate of ablation (equation (18)). Water transport was relatively efficient ($k_0 = 0.1$ kg$^{-1}$ m$^{-2}$ in equation (11)), resembling a situation where the
subglacial hydrological system is open and well connected. The glacier was hence well drained and water pressure was relatively low.

The landscape was over time gradually transformed by subglacial quarrying (Figures 11c and 11d). The final landscape after 100 kyr has a deeper and broader trunk valley with a relatively smooth longitudinal profile including a few minor overdeepenings (Figures 11b – 11d). Several of the tributaries are now hanging valleys separated by truncated spurs (Figure 11b). Valley sides are generally steeper, and steep sections have developed in the upper part of the landscape near the heads of the tributary valleys. The change in topography is due to the patterns of erosion (Figure 12a), which in turn can be explained by first-order variations in the three parameters affecting it: effective pressure, sliding speed, and bed slope (Figures 12b – 12d).

The efficiency of the drainage system results in relatively high effective pressure around 2.5 – 3.0 MPa throughout most of the main valley and in the tributary valleys (Figure 12b). Because we aim for testing an end-member scenario in which the glacier is very well drained, these pressures are intentionally high when compared to measured effective pressures beneath valley glaciers [Lefevre et al., 2015]. In contrast, effective pressure is limited by thin ice in the uppermost parts of the valleys where bed topography is steep and the upglacier accumulation space is small.

The low effective pressure and the steep bed topography promote rapid sliding in the highest part of the landscape (Figure 12c). The rate of sliding then decreases in the tributary valleys before it again increases downglacier. Sliding is fastest in the ablation zone of the main valley, with maximum speed at the valley spurs.

Figure 10. (a) Initial fluvial eroded topography with dimensions 20 km × 40 km. (b) Initial ice configuration used in all experiments. Contour spacing is 30 m.
Figure 11. (a) Initial fluvial topography with concave valley profiles and V-shaped valley cross sections. (b) Glacially eroded topography at the end of experiment 1. (c) Cross section (A–B) showing how the main valley is widened and flattened. (d) Longitudinal profile following the center line of the initial fluvial valley (C–D). The longitudinal profile is rugged because the profile follows drainage in the initial fluvial valley. In Figures 11c and 11d blue lines represent the glacier surface and red lines represent the sliding speed. Colors get darker as time progresses. Black lines in Figures 11a and 11b indicate the positions of profiles in Figures 11c and 11d.

where the glacier cross-sectional area is reduced and hence forces ice acceleration accompanied by higher basal shear stresses (Figures 12c and 11c). Initially, in the starting fluvial landscape, the maximum sliding speed and the fastest erosion are on the spurs. Over time, the maximum sliding shifts toward the center of the valley as the cross section approaches a more U-shaped form, consistent with existing modeling studies by Harbor [1992]. Later, sliding speed drops in the ablation zone because of decreasing bed slopes in the trunk valley.
Figure 12. (a) Total depth of quarrying in experiment 1. (b) Quarrying is especially focused where effective pressure is high. (c) Basal sliding on the other hand has less influence on the erosion pattern. (d) Bed slope in direction of sliding is high at the end of spurs, which contributes to enhanced erosion. Figures 12b–12d are shown at the beginning of the experiment.
Figure 11d). However, in the accumulation zone the sliding speed increases over time, due to steepening of bed slopes.

Bed slope is generally positive in the upper part of the landscape where ice flow converges (Figure 12d). However, in the main valley negative (adverse) slopes occur when the ice must flow over the obstacles represented for example by the long spurs (ridges) of the initial fluvial landscape that reach all the way down to the center of the main valley. Some of the variation in effective pressure is due to the different orientations of bed slope. Positive slopes (where elevation decreases in the direction of sliding) are associated with higher steps \( (h_m) \) and therefore larger cavities, which can drain water at low water pressure. Negative (adverse) slopes have smaller cavities and therefore higher water pressure and lower effective pressure.

The most intense erosion is concentrated in areas where effective pressure is high and where the slope is positive (Figure 12). The dependence of quarrying on bed slope (equation (8)) gives rise to enhanced erosion on the lee sides of valley bends. This effect is reinforced as further erosion leads to steeper slopes. As a consequence, the erosion law for quarrying (equation (8)) broadens and flattens the main trunk valley, transforming it from a V-shaped valley to a U-shaped trough (Figure 11c). Spurs along the main valley are especially prone to intense erosion, as they represent areas where all three parameters (effective pressure, basal sliding, and slope in direction of sliding) are relatively high. Hence, the spurs become truncated and like the hanging valleys, the overdeepenings, and the U-shaped trough (Figures 11b–11d), they are characteristic glacial landforms [Sugden and John, 1976]. Quarrying is modest toward the terminus of the glacier, in spite of faster sliding (Figures 12a and 12c). This is because both effective pressure and bed slope decrease in that direction in this experiment.

5.2. Experiment 2—Influence of Hydrology

The exponent of 3 for effective pressure in the quarrying law highlights the major effect of this variable on quarrying. Effective pressure depends on the configuration and efficiency of the subglacial hydrological system. In this experiment we test the overall influence of hydrologic drainage efficiency in regulating the effective pressure and hence the rate of quarrying. The drainage efficiency is adjusted through the parameter \( k_0 \) in equation (11). Compared to experiment 1, \( k_0 \) is decreased an order of magnitude \( (k_0 = 0.01 \text{ kg}^{-1} \text{ m}^2) \), resembling a hydrological system where cavities are only weakly linked by small orifices [Kamb, 1987]. The glacier is thus poorly drained and water pressure is therefore relatively high.

The landscape in experiment 2 after 100 kyr of erosion is very different from the that in experiment 1 (Figures 11b and 13b). Landforms indicative of subglacial erosion are much less pronounced. The trunk valley is only slightly broadened (Figure 13c), and erosion in several of the tributaries is restricted to the upper parts of the catchment (Figure 13b). The total amount of erosion is markedly reduced, especially in the trunk valley and the lower parts of the tributaries (Figures 13c and 13d). The small amount of quarrying is reflected in the relatively stationary velocity field of both the longitudinal profile and the cross section (Figures 13c and 13d). Quarrying is focused at the spurs (Figures 13c and 13d), where effective pressures are sustained at levels around 2–3 MPa (Figure 14c).

The distribution of slopes at the beginning of the experiment is equal to the one in experiment 1 (Figure 12d), and except for faster sliding near the terminus, so is the pattern of sliding speed (Figures 14d and 13d). The overall decrease in erosion compared to experiment 1 is thus mainly due to changes in the magnitude and distribution of the effective pressure (Figure 14c). In large parts of the trunk valley, the effective pressure is reduced to 1–2 MPa. However, in the higher parts of the tributaries effective pressures can be maintained relatively high (around 3 MPa), due to the low discharge of meltwater (Figure 14c).

5.3. Experiment 3—Hydrological Transients

Experiments 1 and 2 use steady hydrology with a surface melt input specified as an annual average. In the following we address the influence of seasonal variations in water pressure on quarrying rates. The efficiency of the hydrological system is the same as in experiment 2 with \( k_0 = 0.01 \text{ kg}^{-1} \text{ m}^2 \). Surface melt rate is represented by a truncated cosine function of time, which idealizes the seasonal variation in water production. However, the integrated surface melt during a year is the same as in the steady state scenario in experiments 1 and 2. The total meltwater contribution to the hydrological system during a year is the sum of surface melt and basal melt (Figure 15a). The seasonal transients in surface melt clearly dominate the total water input, reaching maximum melt rates in the summertime around 2.7 m yr\(^{-1}\). In the wintertime on the other hand, the total melt rate is close to zero (Figure 15a).
Figure 13. (a) Initial fluvial topography with concave valley profiles and V-shaped valley cross sections. (b) Glacially eroded topography at the end of experiment 2. (c) Cross section (A–B) shows limited widening of the trunk valley. (d) Longitudinal profile along the center line of the initial fluvial valley (C–D). In Figures 13c and 13d blue lines represent the glacier surface and red lines represent the sliding speed. Colors get darker as time progresses. Black lines in Figures 13a and 13b indicate the positions of profiles in Figures 13c and 13d.
Figure 14. (a) Total depth of quarrying in experiment 2. Compared to experiment 1 erosion in the main valley is limited, while quarrying is maintained in the upper parts of the catchment. (b) Effective pressure is low in the main valley where water pressure is high in the inefficient drainage system. (c) Basal sliding is faster than for experiment 1, but the lower effective pressure dominates the resulting erosion rates. (d) Bed slope in direction of sliding. Figures 14b–14d are shown at the beginning of the experiment.
Figure 15. (a) Total meltwater rate for steady state hydrology and the seasonal hydrological transients. (b) Effective pressure and sliding speed for the steady case and the transient case. Black lines refer to the effective pressure axis on the left, whereas gray lines refer to sliding speed on the axis to the right. (c) Quarrying rates for the steady case and the transient case. (d–e) Total depth of quarrying in one year for the steady case and the transient case. (f) Difference between the total depth of quarrying for the steady case and the transient case. (g) Final topography after 100 kyr. (h) Total depth of quarrying after 100 kyr.
The hydrological transients directly influence two of the parameters affecting the rate of quarrying: effective pressure and basal sliding speed. In winter time, where melt rates are close to zero, the spatial average of effective pressure is elevated compared to the simulations with steady hydrology, reaching average levels just below 1.3 MPa (Figure 15b). During spring the relatively closed hydrological system needs high water pressures to transport the increasing volumes of meltwater. As a result, effective pressure decreases and drops below the steady state average to a minimum around 1.1 MPa.

As expected, effective pressure is anticorrelated with the total melt production rate (Figures 15a and 15b). The spatial average of sliding speed shows the opposite pattern: in winter, sliding rates are smaller than the steady hydrology average. In summer, when surface melt accesses the bed and facilitates sliding, the maximum average sliding speed is close to 15.4 m yr\(^{-1}\) (Figure 15b). The average sliding speed is computed across the entire model landscape where ice thickness is above 10 m. Large areas of the glacier have very low sliding rates that reduce the level of average sliding far below the maximum sliding rates.

At about day 140 the effective pressure starts to increase, while sliding speed decreases (15b). This response depicts the onset of meltwater channelization, initiated by the increase in water flux. Rates of quarrying more or less follow the trend of the effective pressure. The average erosion rate is highest during winter (0.55 mm yr\(^{-1}\)) and then decreases to levels of 0.35 mm yr\(^{-1}\) in summer.

Hydrological transients affect not only the averaged quarrying rate during the year but also the spatial distribution of quarrying within the catchment. In the simulation with steady hydrology, the total quarrying during 1 year is concentrated in the upper parts of the catchment (Figure 15d). Only very little erosion occurs in the main valley where water pressure builds up due to the inefficient drainage of the subglacial system keeping the effective pressure low. However, the spatial pattern of quarrying is different for the simulation with seasonal variation in meltwater input (Figure 15e). Due to low discharges of meltwater during winter, effective pressure is high in the ablation zone, and quarrying becomes active not only in the upper parts of the catchment but also in the main valley (Figure 15f). After 100 kyr the trunk valley has been broadened and deepened significantly in the ablation zone (Figure 15g) when compared to the results from experiment 2 (Figure 13b).

### 6. Discussion

Based on the model of Iverson [2012], we propose an erosion law for quarrying, where effective pressure, basal sliding speed, and bed slope affect the quarrying rate (equation (8)). Except for the prescaling factor, \(K_e\), the power law exponents \(c_2, c_3,\) and \(c_4\) are relatively insensitive to variations of the parameters in the quarrying model (Figure 8). The quarrying law produces realistic subglacial landscapes with recognizable large-scale glacial landforms (Figure 11). However, the high sensitivity to effective pressure \(c_2 = 3\) makes the quarrying law very sensitive to the configuration and efficiency of the subglacial hydrological system (Figure 13). Seasonal variations in meltwater production affect both the total amount of quarrying and its spatial distribution in the catchment (Figure 15).

#### 6.1. Importance of Hydrology

Our results identify the distribution and magnitude of subglacial water pressure as a key factor for scaling rates of subglacial quarrying, owing to the high sensitivity of quarrying rate to effective pressure (equation (8)). As demonstrated by the computational experiments of Beaud et al. [2014] and those presented here, variable meltwater production, driven, for example, by a combination of strong seasonality in climate and high levels of annual precipitation, can cause transiently high effective pressure (Figure 15).

Several empirical and theoretical studies have demonstrated the effect of shorter-term (diurnal) hydrological transients on quarrying rate [Iverson, 1991; Hallet, 1996; Cohen et al., 2006; Anderson, 2014]. A rapid drop in water pressure when cavities are largest increases the load on bedrock because the weight of the ice is transferred to the relatively small area of the bed still in contact with the ice. This amplifies the deviatoric stress in the bed and promotes crack growth. Imbalance between cavity size and water pressure hence drives episodes of elevated stress on the upglacier edge of steps, referred to as the "hammer effect" by Anderson [2014]. While this highly transient effect is potentially of importance also for the quarrying rates integrated over longer periods, it is not included in the model presented here, because it is based on a steady state relationship between cavity size and effective pressure (equation (3)). Future studies can perhaps expose whether...
short-term transients influence regional erosion averaged over many bedrock steps and if the power law erosion law proposed here is also valid on such conditions. This work is, however, outside the scope of the present study.

6.2. Fracture Distribution in the Bedrock
Observations demonstrate that rates of quarrying depend on fracture density and orientation \cite{Dühnforth2010, Krabbendam2011, Hooyer2012, Becker2014}. Hooyer et al. \cite{2012} compared the orientation of quarried lee surfaces to the direction of basal sliding. If sliding speed is the main factor affecting quarrying, orientations of quarried surfaces should be related to the direction of sliding. Instead, however, the orientations of the quarried lee surfaces correlate with the orientation of preexisting fractures in the bedrock \cite{Hooyer2012}. As noted, an important concept of the quarrying model of Iverson \cite{2012} is that larger rock volumes have lower strengths, because they have a higher probability of containing a large or optimally oriented flaw \cite{Jaeger1979}. In models where the bedrock is treated as homogeneous with only isolated cracks, the erosion rate depends only on the growth rate of these cracks, which in turn depends on the deviatoric stress that drives crack growth \cite{Iverson1991, Hallet1996}. Favorable conditions for quarrying in these models are thus rapid sliding and low effective pressure to maintain large cavities and thus minimize ice bed contact areas. Although this effect on increasing deviatoric stresses in the bed is included in the model of Iverson \cite{2012}, it is more than compensated for by the effective strengthening of bedrock that occurs as the ice bed contact area and volume of rock stressed by ice decrease. Quarrying is thus most rapid at high effective pressure, which both increases ice bed contact areas, so the glacier can exploit larger more favorably oriented cracks and increases deviatoric stresses by shifting more of the glacier weight from cavity roofs to bedrock.

However, we stress that although the "volume factor" in this model is more dominant than the "deviatoric stress factor" in determining the likelihood of a quarrying event (equation (4)), this may not necessarily always be the case. In a situation where equally spaced joints result in a periodic step sequence, the enhanced deviatoric stresses might become more important because stresses on the step edges are simultaneously increased when cavities cover a large fraction of the steps. Another situation that could affect the competing effects of volume and deviatoric stress is when subglacial water pressure varies rapidly, as discussed in section 6.1.

The parameter \( m \) in the quarrying model is inversely related to the heterogeneity of bedrock strength. Small values (approaching 1.0) mean that some elements of bedrock will be particularly weak, consistent with the presence of large, optimally oriented fractures. Although \( m \) was not varied in the experiments presented here, in future studies different values of \( m \) could represent zones of variable strength distribution. A difference in \( m \) can be simulated with the erosion power law (equation (8)) by varying \( K_0 \) over several orders of magnitude (Figure 8).

The regional slope, \( s_b \), in the quarrying law (equation (8)) is the slope in the direction of basal sliding and thus affects erosion only along a flow line. This is obviously a simplification as demonstrated by Krabbendam and Bradwell \cite{2011}, who suggest "lateral quarrying" with the direction of erosion perpendicular to ice flow. For example, the component of slope along a valley wall and perpendicular to flow could promote quarrying in that direction if joint or bedding-plane orientations are favorable \cite{2011}. This is a possible mechanism for widening valleys not accounted for in the quarrying model presented here.

6.3. Erosion Laws in Landscape Evolution Models
Implementing the physics of subglacial erosion in large-scale landscape evolution models is not a simple task but requires large-scale parametrization of the small-scale processes. Thus, although the quarrying law presented in this study (equation (8)) builds on theory from fracture mechanics and glaciology, it should not be considered a unique solution, but as an important step in advancing physics-based erosion laws in large-scale landscape models.

The ability to reproduce recognizable landforms is a weak test of physically based erosion laws. Ideally, the quarrying law (equation (8)) should be tested against a data set of quarrying rates, sliding speeds, effective pressure, and bed slope. Sliding speed can be estimated from surface velocity measurements \cite{Anderson2004, Shin2004, Bartholomau2008} and bed slope from seismic or radar surveys \cite{Moran2000, King2008}. Erosion rates can be inferred from sediment yield \cite{Herman2015, Koppes2015} and dating techniques such as cosmogenic nuclide analysis \cite{Small1997, Briner1998} and OSL-thermochronology \cite{Herman2010}. However, these methods do not
differentiate between erosion processes. Records of effective pressure measurements are extremely sparse, and those that exist are often limited to point measurements from boreholes [Jansson, 1995; Iken and Truffer, 1997] or load cells from a few locations at the base of the ice [Lefeuvre et al., 2015]. These limitations make it difficult to test the model, but further advances in techniques for determining erosion rates and effective pressure might make a test of the quarrying law possible in the future.

To study the sensitivity of landscape evolution to erosion laws, we have compared results using the new quarrying law (Figures 16a and 16b) with those using a simple erosion law in which erosion rate scales both linearly and nonlinearly with sliding speed (Figures 16c–16f). We compare the results of the three erosion laws for cases in which the maximum erosion in the main valley is approximately the same (500 m). All erosion laws produce characteristic large-scale glacial features: a U-shaped trunk valley, hanging valleys, and overdeepenings (Figures 16a, 16c, and 16e). However, although the linear erosion law results in a fairly distributed pattern of erosion, the quarrying law and the nonlinear sliding-based law focus erosion more in the trunk valley and the larger tributaries (Figures 16b, 16d, and 16f). In the highest parts of the catchment the linear erosion law

Figure 16. Results from experiment 1, showing (a) the final topography and (b) the total amount of quarrying. (c and d) Similar simulation but using an erosion law where erosion rate scales linearly with basal sliding speed. (e and f) Similar simulation but using an erosion law where erosion rate scales to the power of two with basal sliding speed. The linear erosion law produces fairly widespread erosion, but use of the quarrying law and the nonlinear erosion law focuses erosion in the trunk and tributary valleys. However, all three erosion laws produce large-scale recognizable glacial landforms: U-shaped trunk valley, hanging valleys, and overdeepenings.
and the nonlinear law predict relatively high erosion rates, as a result of steep bed gradients, which accelerate basal sliding (Figures 16d and 16f). The opposite is true for the quarrying law, because of low effective pressures in these areas (Figure 16c). Although all three erosion laws produce quite similar erosion patterns, they respond very differently to the hydrological forcing. Using the two sliding-based erosion laws, high water pressure (low effective pressure) results in fast erosion, whereas high water pressure decreases erosion rates in the quarrying law.

The erosion law presented here reflects only the process of subglacial quarrying. To fully model the topographic change in the landscape due to glacial activity, subglacial abrasion and freeze-thaw processes should also be considered. In a physics-based erosion rule for abrasion (Hallet, 1979) the rate of abrasion depends nonlinearly on sliding with an exponent of two. As demonstrated by MacGregor et al. (2009), coupling abrasion and quarrying laws and considering the clast concentration at the bed by tracking plucked material downglacier would increase erosion on adverse bed slopes and downglacier of places where quarrying rates are high. Abrasion would also increase erosion at the terminus where meltwater facilitates rapid sliding, as shown by for example Herman et al. (2011). Freeze-thaw processes would cause headwall retreat and provide tools for abrasion farther downglacier [MacGregor et al., 2009].

Including the above mentioned processes would most likely result in smoother landscapes with more distributed erosion patterns. Future studies that couple these erosion mechanisms could provide useful insight into the rate-limiting factors for glacial erosion.

7. Conclusions

Sensitivity analysis of quarrying rates to three first-order variables (basal sliding speed, effective pressure, and bed slope) indicates an erosion rule in the form of a multivariable power law

\[ \dot{E}_q = c_4 N^2 u^c (S_b + S_{b0})^{c_4} \]  \hspace{1cm} (19)

Importantly, the results documented here point to strong possible influences of effective pressure and bed slope, in addition to the well-recognized influence of basal sliding. The coefficients of this relationship vary over reasonable ranges of parameters in Table 1, but the variability of \( c_2, c_3, \) and \( c_4 \) is limited (Figure 8). The exponent for the effective pressure, \( c_2 \), is quite robust with a value around 3. The exponent for basal sliding speed, \( c_3 \), is close to 1, whereas the exponent for bed slope, \( c_4 \), is around 2. The coefficient \( c_4 \) cannot be determined as it depends on several factors such as lithology and degree of bedrock fracturing. The quarrying law is, owing to the strong dependence on effective pressure, very sensitive to the configuration of the subglacial hydrological system. Seasonal variations in water pressure affect the pattern, timing, and rate of quarrying. Implementing the erosion law for quarrying in a landscape evolution model results in large-scale landscapes characteristic of former glaciated landscapes: U-shaped valleys, hanging valleys, and overdeepenings. Erosion is focused in valleys where effective pressure is high, whereas erosion is limited in higher parts of the landscape where effective pressure is relatively low.

We stress that the results presented in this paper only reflect the process of quarrying. Including and coupling erosion processes such as abrasion and headwall retreat is a subject for further studies.

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