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Abstract
In this paper we study plenary train algebras of arbitrary rank. We show that for most parameter choices of the train identity, the additional identity (x^2 -w(x)x)^2 =0 is satisfied. We also find sufficient conditions for *A* to have idempotents.

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Idempotents in Plenary Train Algebras

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Abstract

In this paper we study plenary train algebras. We show that for most parameter choices of the train identity, the additional identity $(x^2 - \omega(x)x)^2 = 0$ is satisfied. In this case we prove that it has idempotents.

1 Introduction

Plenary powers are defined inductively by $x^{(1)} = x$ and $x^{(n+1)} = (x^{(n)})^2$. The pair (A, ω) is called a baric algebra if $\omega : A \to K$ is a nontrivial homomorphism. If a baric algebra (A, ω) satisfies an identity of the form

$$x^{(n)} = \alpha_1 \omega(x)^{(2^n-1)} x + \alpha_2 \omega(x)^{(2^n-2)} x^2 + \cdots + \alpha_{n-1} \omega(x)^{(2^{n-1})} x^{(n-1)} \qquad (1)$$

then we call it a plenary train algebra. We will further assume that A is commutative.

An important question in nonassociative algebras in general and in train algebras in particular is the existence of idempotents.

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2 Main Section

Lemma 1. *Let A be any baric algebra with weight function ω. If A satisfies the identity*

$$(x^2 - \omega(x)x)^2 = 0 \tag{2}$$

then, for any integers $i, j > 0$ and for any element x of weight 1:

$$(x^{(i)} - x^{(j)})^2 = 0 \tag{3}$$

Proof. We proceed by induction on $n = |i - j|$. The case $n = 0$ is obvious. The case $n = 1$ is a direct consequence of (2). We start by expanding and linearizing (2):

$$\begin{aligned} 4(xx)(xy) \quad - \quad & 2\omega(y)x(xx) - 2\omega(x)y(xx) - 4\omega(x)x(xy) \\ + \quad & 2\omega(x)\omega(y)(xx) + 2\omega(x)\omega(x)(xy) = 0 \end{aligned}$$

When $\omega(x) = \omega(y) = 1$ this shortens to

$$(4x^2 - 4x)(xy) + 2xy - 2x^2y - 2xx^2 + 2x^2 = 0 \tag{4}$$

Our inductive hypothesis is that (3) holds for all x of weight 1 and for all i, j such that $|i - j| < n$:

$$2x^{(i)}x^{(j)} = x^{(i+1)} + x^{(j+1)} \tag{5}$$

Replacing $y = x^{(n)}$ in (4) we get:

$$(4(x^2) - 4x)(xx^{(n)}) + 2xx^{(n)} - 2x^2x^{(n)} - 2xx^2 + 2x^2 = 0$$

using (5) on the first occurrence of $xx^{(n)}$

$$(2(x^2) - 2x)(x^2 + x^{(n+1)}) + 2xx^{(n)} - 2x^2x^{(n)} - 2xx^2 + 2x^2 = 0$$

again using (5) where appropriate

$$\begin{aligned} 2x^{(3)} + (x^{(3)} + x^{(n+2)}) - (x^2 + x^{(3)}) - 2xx^{(n+1)} + x^2 + x^{(n+1)} & \\ -(x^{(3)} + x^{(n+1)}) - (x^2 + x^{(3)}) + 2x^2 & = 0 \end{aligned}$$

collecting similar terms

$$x^{(n+2)} - 2xx^{(n+1)} + x^2 = (x^{(n+1)} - x)^2 = 0$$

finally substituting $x^{(i)}$ for x we get (3) for $|i - j| = n$. □

Theorem 2. *Let A be a plenary train algebra of rank n with defining identity:*

$$x^{(n)} = \alpha_1 \omega(x)^{(2^n-1)} x + \alpha_2 \omega(x)^{(2^n-2)} x^2 + \cdots + \alpha_{n-1} \omega(x)^{(2^{n-1})} x^{(n-1)} \qquad (6)$$

Let

$$\lambda = \sum_{i=1}^{n-1} (n-i)\alpha_i$$

Assume A satisfies $(x^2 - \omega(x)x)^2 = 0$. If $\lambda \neq 0$ then A has idempotents.

Proof. Let x be any weight one element of A and let

$$b_k = \sum_{i=1}^{k} \alpha_i \qquad b = \sum_{k=1}^{n-1} b_k x^{(k)}$$

Notice that $\sum b_k = \lambda$ and that $b_{n-1} = 1$. Next we calculate b^2:

$$\begin{aligned} b^2 &= \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} b_k b_j x^{(k)} x^{(j)} \\ &= \frac{1}{2} \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} b_k b_j (x^{(k+1)} + x^{(j+1)} - (x^{(k)} - x^{(j)})^2) \end{aligned}$$

using Lemma 1, $(x^{(k)} - x^{(j)})^2 = 0$,

$$b^2 = \frac{1}{2} \left(\sum_{k=1}^{n-1} \sum_{j=1}^{n-1} b_k b_j x^{(k+1)} + \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} b_k b_j x^{(j+1)} \right)$$

relabeling the indices of the second sum and using that $\sum b_k = \lambda$,

$$b^2 = \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} b_k b_j x^{(j+1)} = \lambda \sum_{j=1}^{n-1} b_j x^{(j+1)}$$

from the plenary identity and noticing that $b_{n-1} = 1$,

$$b^2 = \lambda \left(\sum_{j=1}^{n-2} b_j x^{(j+1)} + \sum_{k=1}^{n-1} \alpha_k x^{(k)} \right)$$

collecting terms and using the definition of the b_k,

$$b^2 = \lambda \left(\alpha_1 x + \sum_{k=2}^{n-1} (b_{k-1} + \alpha_k) x^{(k)} \right) = \lambda \left(\sum_{k=1}^{n-1} b_k x^{(k)} \right) = \lambda b$$

We conclude that $e = \dfrac{b}{\lambda}$ is an idempotent in A. $\square$

We may notice that in the previous proof the hypothesis $(x^2 - \omega(x)x)^2 = 0$ is not fully used. A sufficient condition would be $\sum_{k<j<n} b_k b_j (x^{(k)} - x^{(j)})^2 = 0$ where the b_k are defined as in the proof of the theorem.

Lemma 3. *Let A be a baric algebra. If all weight one elements $x \in A$ satisfy the equation:*

$$x^{(k)} = \sum_{i=1}^{n-1} \beta_i x^{(i)} \tag{7}$$

where $k \geq n$ and $\sum \beta_i = 1$, then they also satisfy

$$\sum_{1 \leq i < j < n} \beta_i \beta_j (x^{(i)} - x^{(j)})^2 = 0 \tag{8}$$

Proof. Let

$$S = 2 \sum_{1 \leq i < j < n} \beta_i \beta_j (x^{(i)} - x^{(j)})^2$$

We can turn (8) into a full double sum by adding some trivially zero terms where $i = j$:

$$S = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \beta_i \beta_j (x^{(i)} - x^{(j)})^2$$

expanding the squared terms

$$S = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \beta_i \beta_j (x^{(i)})^2 + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \beta_i \beta_j (x^{(j)})^2 - 2 \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \beta_i \beta_j x^{(i)} x^{(j)}$$

changing the summation order and factoring the sums

$$S = \sum_{j=1}^{n-1} \beta_j \sum_{i=1}^{n-1} \beta_i x^{(i+1)} + \sum_{i=1}^{n-1} \beta_i \sum_{j=1}^{n-1} \beta_j x^{(j+1)} - 2 \sum_{i=1}^{n-1} \beta_i x^{(i)} \sum_{j=1}^{n-1} \beta_j x^{(j)}$$

using (7) and that $\sum \beta_i = 1$

$$S = 2\sum_{j=1}^{n-1} \beta_j x^{(j+1)} - 2(x^{(k)})^2$$

using (7) again for x^2 in place of x

$$S = 2x^{(k+1)} - 2x^{(k+1)} = 0$$

□

Lemma 4. *Let A be a plenary train algebra of rank n with defining identity:*

$$x^{(n)} = \sum_{i=1}^{n-1} \alpha_i \omega(x)^{(2^n - 2^{i-1})} x^{(i)}$$

Consider an element $x \in A$ of weight one and let its plenary powers up to $x^{(n-1)}$ be the basis of a vectorspace where $x^{(i)} = (0 \dots 1 \dots 0)$ has a one in the ith position. Then we can express $x^{(k+1)}$ in this basis by $(1, 0, 0, 0 \dots 0)A^k$ where

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & & & \dots & & \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_{n-2} & \alpha_{n-1} \end{pmatrix}$$

Proof. The proof goes by induction on k. For $k = 0$ there is nothing to prove. So we assume

$$x^{(k)} = \sum_{i=1}^{n-1} \beta_i x^{(i)} = (\beta_1, \beta_2, \beta_3, \dots, \beta_{n-2}, \beta_{n-1}) = (1, 0, 0, 0 \dots 0)A^{k-1}$$

Replacing x by x^2 we have

$$\begin{aligned} x^{(k+1)} &= \sum_{i=1}^{n-1} \beta_i x^{(i+1)} = \sum_{i=2}^{n-1} \beta_{i-1} x^{(i)} + \beta_{n-1} \sum_{i=1}^{n-1} \alpha_i x^{(i)} \\ &= (0, \beta_1, \beta_2, \dots, \beta_{n-3}, \beta_{n-2}) + \beta_{n-1}(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{n-2}, \alpha_{n-1}) \\ &= (\beta_1, \beta_2, \beta_3, \dots, \beta_{n-2}, \beta_{n-1})A \\ &= (1, 0, 0, 0 \dots 0)A^k \end{aligned}$$

□

Theorem 5. *Let A be a plenary train algebra of rank n with defining identity:*

$$x^{(n)} = \sum_{i=1}^{n-1} \alpha_i \omega(x)^{(2^n - 2^{i-1})} x^{(i)}$$

Let $\lambda_1, \ldots \lambda_{n-1}$ be the eigenvalues of the matrix A defined in lemma 4 (the λ_k are the nonzero roots of the associative polynomial $x^n - \sum \alpha_i x^i$). If all the products $\lambda_i \lambda_j$ are distinct then A satisfies $(x^2 - \omega(x)x)^2 = 0$ and A has idempotents.

Proof. Using lemma 3 and lemma 4 we get identities

$$\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \beta_i^k \beta_j^k (x^{(i)} - x^{(j)})^2 = 0$$

where

$$(\beta_1^k, \beta_2^k, \beta_3^k, \ldots, \beta_{n-2}^k, \beta_{n-1}^k) = e_1 A^{k-1}$$

and k is any positive integer. So we have a homogeneous system of identities satisfied by the squares $(x^{(i)} - x^{(j)})^2$. In matrix form this can be written as:

$$\left\langle \left(e_1 A^{k-1}\right)^T e_1 A^{k-1}, U \right\rangle = 0$$

Where U is the symmetric matrix such that $U_{ij} = (x^{(i)} - x^{(j)})^2$, and the angled brackets stand for the Hadamard product of the matrices. Now consider $v_1 \ldots, v_{n-1}$ eigenvectors corresponding to the distinct eigenvalues $\lambda_1, \ldots \lambda_{n-1}$ of A. and write $e_1 = \sum c_i v_i$ as a linear combination of them. Since $e_k = e_1 A^{k-1} = \sum \lambda_i^{k-1} c_i v_i$ we notice that the $c_i v_i$ also form a basis of eigenvectors for A, so we may assume that $c_i = 1$ for every i. Then

$$\begin{aligned} 0 = \left\langle \left(e_1 A^k\right)^T e_1 A^k, U \right\rangle &= \left\langle \left(\sum_{i=1}^{n-1} \lambda_i^k v_i \right)^T \sum_{i=1}^{n-1} \lambda_i^k v_i, U \right\rangle \\ &= \left\langle \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (\lambda_i \lambda_j)^k v_i^T v_j, U \right\rangle \\ &= \sum_{1 \le i \le j < n} (\lambda_i \lambda_j)^k \left\langle \left(v_i^T v_j + v_j^T v_i \right), U \right\rangle \end{aligned}$$

since this holds for all k, the Vandermonde determinant says that for each $1 \leq i \leq j < n$ we have

$$\left\langle \left(v_i^T v_j + v_j^T v_i\right), U\right\rangle = 0$$

Using the symmetry of U,

$$2\left\langle \left(v_i^T v_j\right), U\right\rangle = 0$$

Since the v_i form a basis for the (n-1) dimensional rowspace, the matrices $v_i^T v_j$ form a basis for the space of all $(n-1)\times(n-1)$ matrices. To verify this, it suffices to show that they are linearly independent. In fact, if $\sum r_{ij} v_i^T v_j = 0$ then multiplying by any v_k on the left we get $\sum_j (\sum_i r_{ij} v_k v_i^T) v_j$. Since the v_j are linearly independent, $\sum_i r_{ij} v_k v_i^T = 0$ for every k, j. Now since the v_k form a basis $\sum_i r_{ij} v_i^T = 0$, and finally since the v_i^T are linearly independent, $r_{ij} = 0$ for every i, j.

Finally, this shows that U is orthogonal to a basis for the space of all matrices, so $U = 0$ and in particular $(x^{(i)} - x^{(j)})^2 = 0$ for every i, j. Finally, to use theorem 2 we need to chek that $\lambda = \sum (n-i)\alpha_i \neq 0$. We will show that this just means that 1 is not a repeated eigenvalue of A and so it is part of the hypothesis. We want to factor the plenary polynomial:

$$x^n - \sum_{i=1}^{n-1} \alpha_i x^i = \sum_{i=1}^{n-1} \alpha_i (x^n - x^i) = \sum_{i=1}^{n-1} \sum_{k=i}^{n-1} \alpha_i (x^{k+1} - x^k) = (x-1) \sum_{i=1}^{n-1} \sum_{k=i}^{n-1} \alpha_i x^k$$

Evaluating the right factor at $x = 1$ we get

$$\sum_{i=1}^{n-1} \sum_{k=i}^{n-1} \alpha_i = \sum_{i=1}^{n-1} (n-i)\alpha_i$$

So A has idempotents. □

As an illustration we consider some small cases:

Example 1 (n=3)**.** Let A be a plenary train algebra satisfying:

$$x^{(3)} = \alpha x + (1-\alpha)x^2$$

The nonzero roots of the polynomial $x^3 - (1-\alpha)x^2 - \alpha x$ are 1 and α so by theorem 5 we can guarantee that A has an idempotent as long as $1, \alpha, \alpha^2$ are all different, that is $\alpha \notin \{0, 1, -1\}$. Furthermore, for every x of weight 1, we now an idempotent to be

$$\frac{1}{\alpha+1}\left(\alpha x + x^2\right)$$

Notice that when $\alpha = 0$, x^2 is an idempotent and that when $\alpha = 1$ we do not find an idempotent in this way but it is known (Etherington...) that there are idempotents.

Example 2 (n=4)**.** Let A be a plenary train algebra satisfying:

$$x^{(4)} = \alpha x + \beta x^2 + \gamma x^{(3)}$$

where $\alpha + \beta + \gamma = 1$. Lets assume that $1, \lambda, \mu$ are the nonzero roots of $x^4 - \gamma x^3 - \beta x^2 - \alpha x = 0$ so that $\alpha = \lambda\mu$, $\beta = -(\lambda\mu + \lambda + \mu)$, $\gamma = \lambda + \mu + 1$. Theorem 5 says that A has an idempotent as long as $1, \lambda, \mu, \lambda\mu, \lambda^2, \mu^2$ are all distinct, that is $\lambda\mu(\lambda^2 - 1)(\mu^2 - 1)(\lambda^2 - \mu^2)(\lambda - \mu^2)(\lambda^2 - \mu) \neq 0$.

Furthermore, in this case, we now an idempotent to be

$$\frac{1}{3\alpha + 2\beta + \gamma}\left(\alpha x + (\alpha+\beta)x^2 + (\alpha+\beta+\gamma)x^{(3)}\right)$$

One may notice again that the given condition is not really necessary since to really answer the question, we need to solve a linear algebra problem. We need to know whether the vector

$$\begin{pmatrix} \alpha(\alpha+\beta) & \alpha(\alpha+\beta+\gamma) & (\alpha+\beta)(\alpha+\beta+\gamma) \end{pmatrix}$$

is in the rowspace of the following matrix:

$$\begin{pmatrix} \alpha\beta & \alpha\gamma & \beta\gamma \\ \alpha\gamma(\alpha+\beta\gamma) & \alpha\gamma(\beta+\gamma^2) & (\alpha+\beta\gamma)(\beta+\gamma^2) \\ \alpha(\beta+\gamma^2)(\alpha\gamma+\beta(\beta+\gamma^2)) & \alpha(\beta+\gamma^2)(\alpha+\beta\gamma+\gamma(\beta+\gamma^2)) & (\alpha\gamma+\beta(\beta+\gamma^2))(\alpha+\beta\gamma+\gamma(\beta+\gamma^2)) \end{pmatrix}$$

It turns out that this is the case as long as $(\beta - 1)(\alpha - 1) \neq 0$. Which in terms of the eigenvalues leaves the final condition as $(\lambda^2 - 1)(\mu^2 - 1)(\lambda\mu - 1) \neq 0$

This result was obtained recently by Labra and Suazo (see...).