Fall 2018

Happy numbers

Elizabeth Todd
Iowa State University

Elizabeth Todd
Iowa State University
Master of School Mathematics Graduate Program
Creative Component

Table of Contents

1 Introduction to Happy Numbers

Happy numbers were first introduced to me through the Holl Mini-Workshop for MSM students at Iowa State University. The presenter, Leslie Hogben, led our group in an exploration of happy numbers that invoked our curiosities and led to many interesting discoveries. While happy numbers do not seem to have a practical application (that I've discovered at least), they are an enjoyable recreational math concept.

Definition

Happy numbers are numbers greater than one such that when the digits of the number are squared and summed repeatedly, the number eventually goes to 1. For example, take the number 82. $8^2 + 2^2 = 68$; $6^2 + 8^2 = 100$; $1^2 + 0^2 + 0^2 = 1$. Since 82 eventually went to 1, 82 is a happy number. However, not all numbers exhibit this type of behavior. Consider the number 4. 4 yields a different, and most interesting result. $4^2 = 16$; $1^2 + 6^2 = 37$; $3^2 + 7^2 = 58$; $5^2 + 8^2 = 89$; $8^2 + 9^2 = 145$; $1^2 + 4^2 + 5^2 = 42$; $4^2 + 2^2 = 20$; $2^2 + 0^2 = 4$. From this we see that 4 is part of a cycle, because applying the function of squaring the digits and taking their sum will eventually lead back to the initial number. Once you reach a number in the cycle, you will stay in the cycle indefinitely. There are eight numbers (4, 15, 37, 58, 89, 145, 42, 20) in this cycle. A number may also "fall into" the cycle. A number like 90 is not part of the cycle, but by applying the function, it will eventually reach a number in the cycle. $9^2 + 0^2 = 81$; $8^2 + 1^2 = 65$; $6^2 + 5^2 = 61$; $6^2 + 1^2 = 37$. 37 is a part of the cycle, so 90 fell into the cycle.

Happy Tree

At the Holl workshop, our group started by investigating the numbers 1 - 100 and determined if they were happy, part of the cycle, or go to the cycle. The tree in Figure 1 shows all the numbers less than 100 that are happy, and how they get to 1. The remaining numbers less than 100 are in the cycle or fall into the cycle (4, 15, 37, 58, 89, 145, 42, 20). The tree uses numbers greater than 100 (such as 130) where necessary in order to complete the tree. This "happy tree" is based on the drawing of Matt Parker from Numberphile Youtube video series. [6]

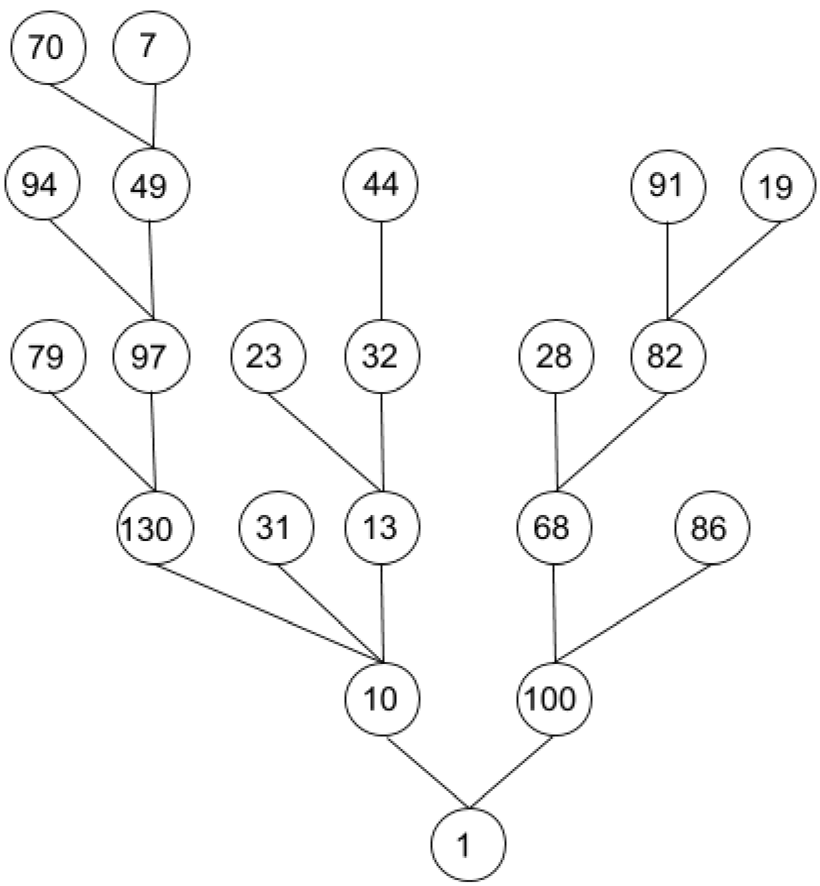

Figure 1: Happy Tree

Naturally this poses the question, will this tree "grow" forever? In other words, are there infinitely many happy numbers? Well to become happy, the sums of squares need to eventually lead to 1. Any power of 10 would fit that description, as all powers of 10 begin with a digit of one, followed by zeros. Therefore there are infinitely many happy numbers, as there are an infinitely many powers of 10.

Cycle

The happy tree contains only a small fraction of the numbers less than 100. Happy numbers has a lower density of 0.1138 and the upper density of 0.18577. [5] During the workshop we found that the rest of the numbers were either part of the cycle or go to the cycle. Matt Parker [6] dubbed this cycle, and the numbers that go into it, the "melancoil." Figure 2 shows all the numbers less than 100 that are either part of the cycle or go to the cycle. It also includes three digit numbers when necessary to complete the diagram.

This diagram shows some interesting discoveries. The trees that span from the cycle vary greatly in length, with no numbers leading to 42 (except those within the cycle) and forty-two numbers leading to 89. It begs the question- if we were to continue towards infinity, would there still be no numbers leading to 42? Would there still be a larger amount of numbers leading to 89? These would be interesting questions for further research.

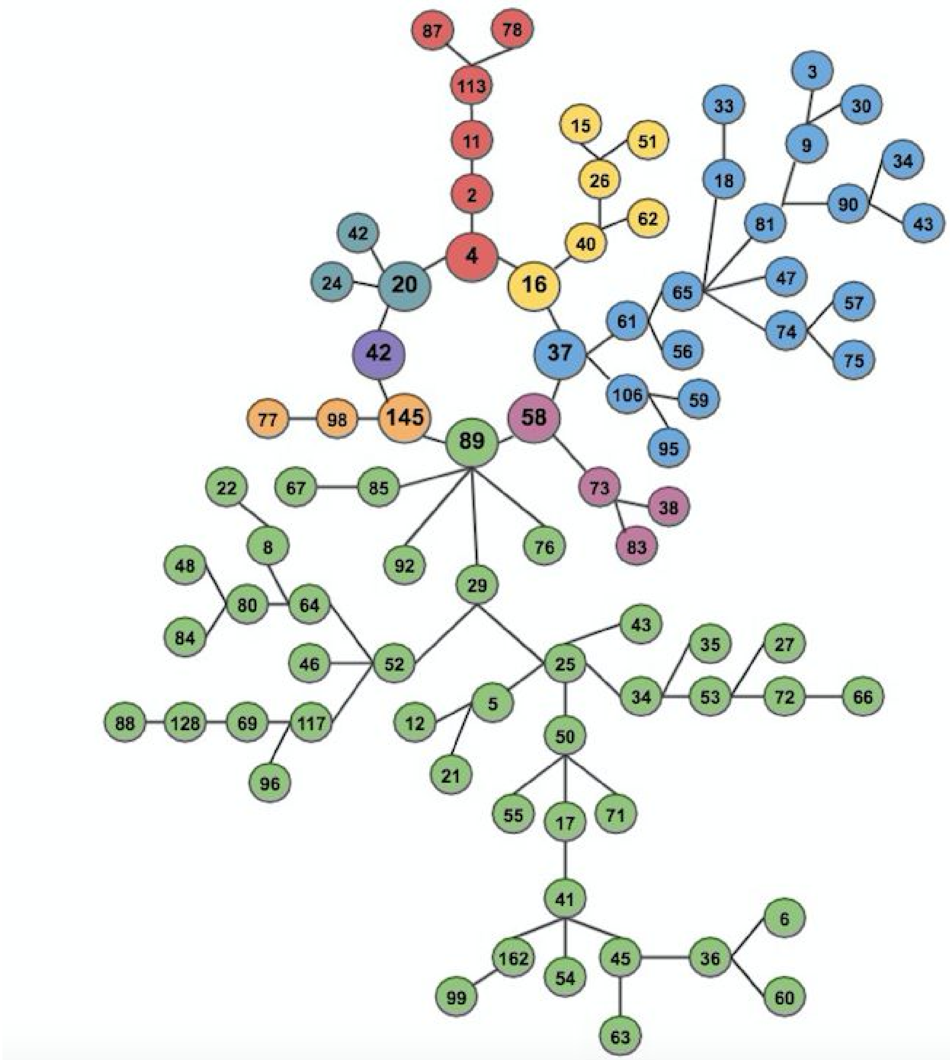

Figure 2: Melancoil

Behavior of Numbers

At the Holl workshop, we determined the classification of numbers less than 100 and then posed the question, can we prove that *all* numbers are either one, happy, part of the cycle, or go to the cycle? If we could show that applying the function of summing the squares of the digits to a number greater than 100 *always* produced a smaller value, we could conclude that all numbers eventually led to two digit numbers, and given that all two digit numbers are either one, happy, part of a cycle, or goes to the cycle, we would be able to conclude all numbers do so as well. By testing several numbers, we saw that it did, indeed, appear to decrease in value. The theorem and its proof follows below.

A number with $n+1$ digits can be written as $a_n a_{n-1} ... a_0$ (and juxtaposition does not imply multiplication). For $a = a_n a_{n-1} ... a_0$, define $S_2(a) = a_n^2 + a_{n-1}^2 + ... + a_0^2$

Lemma 1: For $a \geq 100$, $S_2(a) < a$

Proof: Assume $n \geq 2$. Consider a number a with $n+1$ digits and assume that $S_2(a) > a$. Or in another way, $a_n^2 + a_{n-1}^2 + ... + a_0^2 \geq 10^n a_n + 10^{n-1} a_{n-1} + ... + a_0$

Given that $a_n, a_{n-1}, ... a_0$ are single digit positive integers and the leading digit $\neq 0$, then $10a_n > a_n{}^2$. Therefore, $10a_n + 10a_{n-1} + ... + 10a_0 > a_n^2 + a_{n-1}^2 + ... + a_0^2$ and by substitution $10a_n + 10a_{n-1} + ... + 10a_0 \geq 10^n a_n + 10^{n-1} a_{n-1} + ... + a_0$. Simplifying the expression we have $9a_0 \geq (10^n - 10)a_n + (10^{n-1} - 10)a_{n-1} + ... + 90a_2$. This is a contradiction because $9a_0 \leq 81$ and $(10^n - 10)a_n \geq 90$.

Theorem 2: All numbers are either one, happy, part of the cycle, or goes to the cycle.

Proof: By Lemma 1, repeatedly applying the sum of squares of digits function to a base 10 number greater than 100 will eventually lead to 2 digit number, indicating all numbers are either one, happy, part of the cycle, or go to the cycle. $\square$

2 Other exponents

Consider what happens if the iterated function used an exponent other than 2; what changes would occur? This question led me to the discovery of fixed points. With an exponent of 2, 1 was the only number that returned itself. With an exponent of 3, that is not the case. While 1 still returns 1, there are other numbers that do the same thing. There are five fixed numbers to be exact- 1, 153, 370, 371, and 407. This will be established below and is known [2]. For all of these numbers, if their digits are cubed, the sums of the cubes returns the same number. The last four numbers are also known as "narcissistic numbers." Wolfram Math World defines narcissistic numbers as "An *n*-digit number that is the sum of the *n*th powers of its digits is called an *n*-narcissistic number. It is also sometimes known as an Armstrong number, perfect digital invariant, or plus perfect number." [7]

Thus we now have five options for all numbers; they are either fixed, happy, part of the cycle, go to the cycle, or go to a fixed point. To find the fixed points, we need to apply the function to numbers one by one and identify the numbers that return themselves. (This is best done through the aid of programming.) Fortunately, just like the function with an exponent of 2, applying the function with higher exponents will eventually decrease the number as well. Therefore to find the fixed points for any exponent, we do not need to search all numbers, just those below a certain bound. The higher the exponent, e, the higher the bound. A proof of the bound of 10^{e+1} is reproduced below.

For $a = a_n a_{n-1} ... a_0$, define $S_e(a) = a_n^e + a_{n-1}^e + ... + a_0^e$ given an exponent e.

Theorem 3: $S_e(a) < a$ when $a \geq 10^{e+1}$ [1]

Proof: Suppose that $a \geq 10^{e+1}$. Then $a = 10^n a_n + 10^{n-1} a_{n-1} + ... + a_0$, where $a_n,\ a_{n-1},\ ...\ a_0$ are single positive integers, $a_n \neq 0$, and $n \geq e+1$. Given that a is $n+1$ digits long, and each digit is at most 9, $S_e(a) \leq (n+1)9^e$

$$10^n = (9+1)^n$$
$$> 9^n + (n)9^{n-1}$$
$$> 9^e + (n)9^e$$
$$= (n+1)9^e$$

So then $10^n > (n+1)9^e$, and by substitution $S_e(a) \le 10^n$. Since $a \ge 10^{e+1}$, then $S_e(a) < a$. ▢

We now turn our attention to the cycle, or rather, the cycles. When the exponent was 2, there was one cycle of eight numbers. However, using the exponent of 3 yields four small cycles of numbers. They are: (55, 250, 130); (136, 244); (160, 217, 352); (919, 1459) [2] By Theorem 2, we know that these are the only cycles given that all numbers greater than 10,000 will decrease under the operation of summing cubes of digits, and these were the only cycles found for numbers less than 10,000.

The happy numbers of exponent 3 also have unique qualities. While there is no pattern to happy numbers (that is currently known at least), cubic happy numbers share that quality that they are all congruent to 1 modulo 3. [2] The proof is reproduced below.

Proof: Consider the cubic happy number $a_n a_{n-1} ... a_0$ (where juxtaposition does not imply multiplication) or written another way $10^n a_n + 10^{n-1} a_{n-1} + ... + a_0$. Given that fact that $a^3 \equiv a \pmod 3$ and $10^n \equiv 1 \pmod 3$, then $a_n{}^3 + a_{n-1}{}^3 + ... + a_0{}^3 \equiv a_n + a_{n-1} + \ ... \ + a_0 \pmod 3$ and $10^n a_n + 10^{n-1} a_{n-1} + ... + a_0 \equiv a_n + a_{n-1} + \ ... \ + a_0 \pmod 3$. Given that the sum of the digits cubed of happy numbers eventually equal 1, and $1 \equiv 1 \pmod 1$, then all happy numbers are congruent to 1 modulo 3. ▢

There are many differences between the behavior for numbers with exponents of two and exponents of three. This raises the question, what happens with other exponents? How many fixed points and cycles will they yield? Is there any way to predict these number of fixed points and cycles? These questions are best answered with the assistance of a program.

Given an exponent e, the program will need to search through 10^{e+1} for all fixed points and cycles. As the exponent increases, the program will have ten times more numbers to search through, and therefore at some point comprehensive list of fixed points will not be possible. However, the program was able to find all the fixed points for exponents 2-7, and cycles for 2-6 in a reasonable amount of time. The results are listed in the table below.

Exponent	Fixed Points	Cycles
2	1	(4, 15, 37, 58, 89, 145, 42, 2)
3	1 153 370 371 407	(55, 250, 130) (136, 244) (160, 217, 352) (919, 1459)
4	1 1634 8208 9474	(13139, 6725, 4338, 4514, 1138, 4179, 9219) (2178, 6514)
5	1 4150 4151 54748 92727 93084 194979	(244, 2080, 32800, 33043, 1753, 20176, 24616, 16609, 74602, 25639, 70225, 19996, 184924, 93898, 183877, 99394, 178414, 51625, 14059, 63199, 126118, 40579, 80005, 35893, 95428, 95998, 213040, 1300) (92873, 108899, 183635, 44156, 12950, 62207, 24647, 26663 23603) (8294, 9044, 61097, 83633, 41273, 18107, 49577, 96812, 99626, 133682, 41063) (8299, 150898, 127711, 33649, 68335, 44155) (9045, 548525, 313179, 650550, 63198, 99837, 167916, 91410, 60075, 27708, 66414, 17601, 24585, 40074, 18855, 71787, 83190, 92061, 66858, 84213, 34068, 41811, 33795, 79467, 101463) (10933, 59536, 73318, 50062)

		(24594, 37973, 93149, 119366, 74846, 59399, 180515, 39020, 59324, 63473, 26093, 67100) (58618, 76438) (89883, 157596)
6	1 548834	(17148, 383891, 1057188, 657564, 246307, 169194, 1113636, 94773, 771564, 301676, 211691, 578164, 446171, 172499, 1184692, 844403, 275161, 179996, 1758629, 973580, 927588, 1189067, 957892, 1458364, 333347, 124661, 97474, 774931, 771565, 313205) (23949, 1083396, 841700, 383890, 1057187, 513069, 594452, 570947, 786460, 477201) (63804, 313625) (93531, 548525, 313179, 650550)
7	1 1741725 4210818 98000817 9926315 14459929	

Table 1

3 Applications in the Classroom

I have the pleasure of teaching Algebra 1 to 8th grade students and wanted to give them an opportunity to develop number sense, look for patterns, and build an understanding of recursive functions through exploring happy numbers. In the lesson, we started off by building an understanding of what it means to "sum the squares of the digits of a number." Students were not accustomed to seeing functions rewritten descriptively, rather than with an equation.

I took the number 23 and summed the squares of the digits. I asked the students what they think would happen if I kept doing this. Most students, inaccurately, thought that the number would mostly increase and keep increasing. I think that came from an understanding that squaring a number makes it bigger, but they did not think about how the biggest square of a digit we would have is 81.

After students saw that 23 lead to 1 through the recursive function, I defined all such numbers as "happy." For middle schoolers, clear definitions are important, as well as being able to describe them in their own words. The first thing I had students do in their groups was write their own definition of happy numbers. This allowed me to check understanding of happy numbers before moving on.

I then challenged students to find as many happy numbers as they can. This showed students that there are many happy numbers, that there are patterns to happy numbers, and it helped students develop number sense and an understanding of happy numbers. This part of the lesson provoked curiosity and creativity from the students. Many students realized that the numbers that we found from 23 on our way to 1 (13, 10, and 1) are also happy numbers. They also quickly learned that the reverse of these numbers were also happy. In other words, the order of the digits did not matter. Other students realized that all the powers of 10 are happy, or started taking the numbers 23, 13, etc and adding zeros onto the end. One student realized if the sum of the squares was 10, the number was happy. The easiest route for him to do this was the number 1,111,111,111. Another student, having the same realization, stuck to the digits of 0, 1, 2, and 3 and found many unique numbers others did not consider, such as 2,211 or 301. She transposed these numbers into many other numbers.

After giving students sufficient time to find these patterns, we discussed and highlighted the key findings as a class. When posed with the question "Are there an infinite amount of happy numbers?" a few students seemed stumped or unsure, but many quickly surmised that because there are an infinite amount of powers of 10, there are naturally an infinite amount of happy numbers. Then I asked the class "Are there non-happy numbers?" Many realized that there are non-happy numbers, and one student predicted that non-happy sum to single digits other than 1. Another student disproved this with the number 7, and we also talked about how we do not stop summing the digits with a recursive function just because we reached a single digit.

To introduce the cycle, I used the number 4, and said we will investigate what happens when we sum the squares of the digits. We continued to do this until we reached the number 4 again. Some students were flummoxed why I stopped, while other students immediately let out a knowledgeable "ohhhhhhhhh." Using the age-old wise teaching words of "what do you notice?" students pointed out that we ended up back where we started, that the numbers got stuck in a loop. From here other questions arose: "Are there other cycles?" "What about the numbers that are not part of cycle?"

While we did not reach a conclusion to the first question (some students did hang out after class to ask more questions about that), we did figure out what happens to numbers that were not happy or part of the cycle. Using another example, students discovered that other numbers lead to the cycle. Students used the last few remaining minutes to fill out a table and label numbers a happy, part of the cycle, or goes to the cycle. Based off of the natural curiosity and questions that they posed, a good follow lesson would be to look at other exponents.

4 Lesson Plan

Overview of Lesson

Students will introduced to the concept of a recursive function through happy numbers. Students will explore the different outcomes that will occur through this function, and make predictions. Students will be guided to prove their predictions. Students can also extend their exploration of happy numbers by using different exponents or bases.

Common Core State Standards for Mathematical Practice

MP1 Make sense of problems and persevere in solving them.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.
MP8 Look for and express regularity in repeated reasoning.

Learning Objectives

The objective of this lesson is focused on reasoning and problem solving skills. Students will be making sense of the recursive function for happy numbers. Using that structure, students will see how the function results in two different outcomes. Students will construct an argument as to whether there it is possible to have more than those two outcomes. This relates to the Iowa Core Standard HSF.BF.A.1.a
"Determine an explicit expression, a recursive process, or steps for calculation from a context."

Prerequisites

Students should have background knowledge on using exponents. Knowledge of other bases can help extend the lesson, but is not required.

Time Required

45- 90 minutes. Basics of the lesson can be done in one class period, but further explorations and extensions could make this two class periods.

Materials and Preparation Required

Student handout, calculators

Lesson Plan

Introduction

Start with the number 23. Define what it means to "sum the squares of the digits." Ask the students to make a prediction about what will happen. (Will the value increase? Decrease? Stay the same?) Repeat summing the squares until you get to one. Ask the students what happens when we sum the squares of 1. Ask students if they think this will happen with any other numbers. Define these numbers as "happy numbers."

Cycle

Ask students to share their happy numbers, and highlight any patterns, such as 10, 100, 1000, etc. Then ask students to share numbers that were not happy. Highlight one of the numbers that is part of the cycle. Use the number to introduce the cycle. Ask students to explain why a cycle occurred, and predict if any other numbers would enter into this cycle. Ask students to predict if there are any other cycles.

Testing Numbers

Students should define what the possible results of recursive function are. Then have students work collectively to complete the table. Ask students to define any patterns they see, such as numbers like 23 and 32 both are happy.

Proofs, Predictions, and Extension

After completing the table, ask students if they all numbers will behave the same way. Guide students to prove their hypothesis. Or consider having students make and test predictions about changing the function to an exponent of 3 or using a different base.

Student Handout

Write your own definition of a happy number

With your group, find as many happy numbers as you can

What do you notice about the non-happy numbers?

What are the possible results of the recursive function?

Determine if the following numbers in the table are happy, part of the cycle, or go to the cycle

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Do you notice any patterns with the numbers that are happy?

5 Python Code for Fixed Points and Cycles

```
import math

def getNumberPlaceValue(number):
    placeValue = 1
    i = 0

    while placeValue <= number:
        placeValue *= 10
        i += 1

    return i

def numberToArray(number):
    placeValue = getNumberPlaceValue(number)
    digits = [1] * placeValue

    while placeValue > 0 :
        tens = math.pow(10, placeValue - 1)
        digits[placeValue - 1] = int (number / tens)
        number -=  digits[placeValue - 1] * tens
        placeValue -= 1
    return digits

def sumsOfPowers(digits, exponent):
    sum = 0
    for digit in digits:
        sum += math.pow(digit, exponent)

    return int(sum)

def happyFunction(number, exponent):
    digitArray = numberToArray(number)
    happyReturn = sumsOfPowers(digitArray, exponent)
    return happyReturn

def cycleToString(cycleList):
    cycleList.sort()
    return str(cycleList)

def printDict(dictonary):
    for key in dictonary.keys():
        print(key)
        numbers = dictonary[key]
        numbers.sort()
        print(numbers)
        print()

def printDictKeys(dictionary):
    for key in dictionary.keys():
        print(key)
        print()
```

```python
from HappyMath import happyFunction
from HappyMath import printDict
from HappyMath import cycleToString
from HappyMath import printDictKeys
import math
import time

def fixedPoints(exponent):
    upperBound = int(math.pow(10, exponent + 1)) + 1
    fixedPointsList = []

    for i in range(1, upperBound + 1):
        if i == happyFunction(i, exponent) :
            fixedPointsList.append(i)
    return fixedPointsList


def cycleSearch(exponent):
    upperBound = int(math.pow(10, exponent + 1)) + 1
    numberDict = dict()

    for i in range(1, upperBound):
        currentNumber = i
        currentBranch = [currentNumber]
        while currentBranch:
            nextNumber = happyFunction(currentNumber, exponent)

            if nextNumber < i:
                currentBranch = []
                break
            if nextNumber in currentBranch:
                index = currentBranch.index(nextNumber)
                cycle = cycleToString(currentBranch[index:])

                if cycle in numberDict.keys():
                    for number in currentBranch:
                        if number not in numberDict[cycle]:
                            numberDict[cycle].append(number)
                else:
                    numberDict.update({cycle: currentBranch})

                currentBranch = []
            else:
                currentBranch.append(nextNumber)
                currentNumber = nextNumber

    return numberDict
```

6 References

[1] Budden, C., Chalmers, J., Hogben, L., Reece, B., Wilson, U., & Yoder, L. Adventures with numbers and iteration. Preprint. (2014)

[2] Grundman, H. G., & Teeple, E. A. Generalized Happy Numbers. *The Fibonacci Quarterly,* 39(2001), 462-466.

[3] Grundman, H. G., & Teeple, E. A. Iterated Sums of Fifth Powers of Digits *Rocky Mountain Journal of Mathematics,* 38(2008), 1139- 1146

[4] Hogben, L. (2017, June 14). *Happy Numbers*. Lecture presented at Holl Mini-Workshop for MSM students at Iowa State University, Ames.

[5] OEIS, The Online Encyclopedia of Integer Sequences. Sequence A007770 (2018)

[6] Parker, Matt (2012, February 26). 145 and the Melancoil. Retrieved November 06, 2017, from https://www.youtube.com/watch?v=_DpzAvb3Vk4

[7] Weisstein, Eric W. "Narcissistic Number." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/NarcissisticNumber.html