ON BOUNDARY INTEGRAL EQUATION METHOD FOR FIELD DISTRIBUTION UNDER CRACKED METAL SURFACE

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INTRODUCTION

Two approaches to the problem of the AC magnetic field distribution inside conducting media are currently used. One consists of solving the quasi-stationary Maxwell differential equations subject to certain boundary conditions. Another one is based upon certain boundary integral equations that involve only the field components at the boundary surface [1,2]. Aside from some numerical advantage in treating unknowns of a lower dimension, the second method fits better the common formulation of the eddy current NDE, which aims at determination of the total impedance of a metal part (or its change due to a defect). The impedance is related to the integrated energy flux through the metal surface and as such it can be expressed in terms of the field components at the surface exclusively.

We consider in this report only the two-dimensional (2-D) case. Having little to do with real situations, this case is nevertheless instructive. Analytical development can be carried in some 2-D sitatuions "almost to the end" as opposed to the general 3-D configurations. Solvable 2-D problems may serve as test cases for more involved and less transparent general approaches.

INTEGRAL EQUATION

Let us consider the external magnetic field uniform in space and harmonic in time: $B_z = B_o \exp(-i\omega t)$. The metal part is a long (infinite) "rod" in the z direction with the surface parallel to the external field. The cross-section of such a part is shown in Fig. 1. The external field in this configuration remains uniform for any shape of the cross-section, thus making the external problem "solved". The internal problem, therefore, consists of solving the 2-D Maxwell equation

$$(\nabla^2+k^2)B(x,y) = 0, \quad k = (1+i)/\delta, \tag{1}$$

with δ being the skin depth. At the boundary contour $B(x,y)=B_o$.

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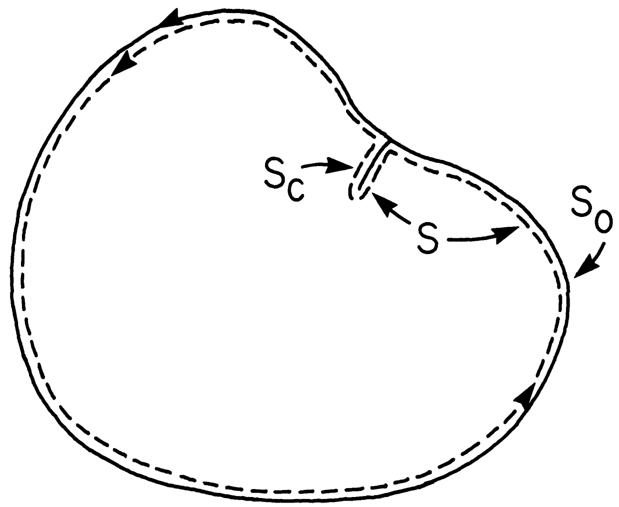

Fig. 1. Cross-section of a long metal part. S_o is the "uncracked" surface. Surface S consists of S_o and of the crack's faces, S_c.

Consider now a Green's function, $G(\vec{r},\vec{r}_o)$, which satisfies

$$(\nabla^2+k^2)G(\vec{r},\vec{r}_o) = -4\pi\delta(\vec{r}-\vec{r}_o) \tag{2}$$

under certain boundary conditions, and integrate the combination [Eq. (1)xG - Eq. (2)xB] over the volume $d^3\vec{r}$). After applying the Green's theorem one obtains

$$\int d\vec{s}\cdot(G\nabla B-B\nabla G) = 4\pi B(\vec{r}). \tag{3}$$

Here $d\vec{s}$ is the area element directed as the external normal to the metal surface. This equation gives the field $B(\vec{r})$ everywhere inside the metal in terms of the magnetic field at the surface (which is constant, B_o, in our geometry) and of its normal derivative (which is proportional to the tangential component of the electric field: $\partial B/\partial n=\sigma E_t$ with σ being the metal conductivity). We note that Eq. (3) holds irrespective of what specific Green's function is chosen (e.g., what boundary conditions are imposed upon G). The problem would have been solved, had the Green's function for a particular metal shape under the condition G=0 at the surface been known. In this case, Eq. (3) reduces to

$$B(\vec{r}_o) = (B_o/4\pi)\int d\vec{s}\cdot\nabla G. \tag{4}$$

The trivial example of such a case is the half-space y>0 filled with metal. The known solution, $B(y)=B_o\exp(iky)$, can be also obtained from Eq. (4) using the Green's function

$$G/i\pi=H_o(kR)-H_o(kR_1),\ R^2=(x-x_o)^2+(y-y_o),\ R_1{}^2=(x-x_o)^2+(y+y_o)^2 \tag{5}$$

where H_o's are Hankel functions of the first kind and of the zero order, and R_1 is the distance between $\vec{r}_o=(x_o,y_o)$ and the image of $\vec{r}$: $(x,-y)$.

Another example of a known Green's function is that of the 90° corner:

$$G/i\pi = H_o(kR) - H_o(kR_1) - H_o(kR_2) + H_o(kR_3),$$
$$R_2^{\,2} = (x+x_o)^2 + (y-y_o)^2, \quad R_3^{\,2} = (x+x_o)^2 + (y+y_o)^2, \tag{6}$$

and R, R_1 are defined in Eq. (5). One easily verifies that G=0 at both corner faces which coincide with positive x and y axes. Equation (4) now yields after some algebra [1]:

$$\frac{B(x,y)}{ikB_o} = y \int_o^{x/y} \frac{dv}{\beta} H_1(ky\beta) + x \int_o^{y/x} \frac{dv}{\beta} H_1(kx\beta), \quad \beta^2 = v^2 + 1. \tag{7}$$

Differentiate this with respect to y and take the limit $y \to 0$ to obtain:

$$\sigma E_x(x,0) = ik^2 B_o \int_o^x H_o(ku)du. \tag{8}$$

We now define the total surface impedance as the time average over the period of the integrated Pointing vector

$$Z = \int \vec{E} x \vec{B}^* \cdot d\vec{s}/2 \tag{9}$$

where the integral is taken over the metal surface with the normal directed into the metal. One can easily verify with the help of Maxwell equations that

$$\text{Re}Z = \int \vec{j} \cdot \vec{E}^* dv/2, \qquad \text{Im}Z = -\omega\mu \int dv |B|^2/2, \tag{10}$$

with the current density $\vec{j} = \sigma\vec{E}$. Thus, ReZ is the total dissipation power in the metal averaged over the period, while ImZ is proportional to the magnetic energy stored. It is worth noting that ReZ>0, while ImZ<0 [3].

For the corner, the total Z diverges. However, the difference, $Z-Z_o$, can be evaluated with Z_o being the total impedance of the plane surface ("unfolded corner").

$$Z - Z_o = B_o \int_o^\infty [E_x(x) - E_o]dx. \tag{11}$$

Here the electric field on the plane surface (or far from the corner's edge), $E_o = ikB_o/\sigma$. The integrand of (11) is

$$E_x(x) - E_o = -kE_o \int_o^\infty H_o[k(x+t)]dt \tag{12}$$

(this is verified with the help of Eq. (8) and using the identity $\int_0^\infty H_0(t)dt=1$). Substitute (12) in (11) and introduce polar coordinates $x=r\cos\phi$, $t=r\sin\phi$ to do the double integration. The result turns out to be a real number: $Z-Z_0 = -B_0^2/\pi\sigma$. One can normalize this on the dissipation power of 1 cm^2 of plane surface, $\mathrm{Re}z_0=\mathrm{Re}(E_0B_0/2)=B_0^2/2\sigma\delta$, to obtain

$$(Z-Z_0)/\mathrm{Re}z_0 = -(2/\pi)\delta = -0.637\delta. \tag{13}$$

One can say that the dissipative part of the corner impedance is depleted with respect to the plane surface as if the corner is "$2\delta/\pi$ shorter". This result has been obtained numerically by Kahn [1].

In most situations, the Green's function that vanishes at the metal part's surface is difficult to construct. One takes then $G=i\pi H_0(kR)$ (the simplest possible G), places the point $\vec{r}$ at the surface (where $B(\vec{r})=B_0$) to obtain from (3) an integral equation for the tangential component of the electric field:

$$\sigma\int ds G(\vec{s},\vec{s}_0)E_t(\vec{s}) = B_0(4\pi + \int d\vec{s}\cdot\nabla G) \tag{14}$$

($d\vec{s}$ is directed along the external normal, while ds is a scalar surface element). This can be solved numerically for E_t, thus making it possible to evaluate the total impedance (9) as well as the distribution $B(\vec{r}_0)$ (if the latter is needed) by substituting E_t in Eq. (3).

Let us turn now to the situation where the exact Green's function vanishing at the whole surface of the metal part, is unknown, while it is known for a surface "close" to the actual one. An example is the case of a tight crack normal to the plane metal surface (we take this example for simplicity, although the same approach can be used for other crack shapes). One can utilize then the Green's function (5) of the plane surface (y=0). Integral (3) splits in two: one over the plane surface (over x from $-\infty$ to ∞), and another one over the crack's face. The term $G\nabla B$ does not contribute to the integral over the plane surface, while $B\nabla G$ in this integral yields the unperturbed by the crack field $B_0\exp(iky)$. In the integral over the crack faces, the contribution of $B\nabla G$ cancels out ($B=B_0$ along the crack, $\nabla G\cdot d\vec{s}$ has opposite signs on two crack faces). One obtains

$$B(\vec{r}_0) = B_0e^{iky} - \frac{i\sigma}{2}\int_0^d E_y(y)\ [H_0(kR)-H_0(kR_1)]dy. \tag{15}$$

Here d is the crack depth, R and R_1 are defined in (5) (set crack's location as x=0), and $E_y(y)$ is the electric field at the right crack face x = +0. An integral equation for this quantity is obtained using the boundary condition $B = B_0$ at the crack face:

$$B_0(1-e^{iky}) = -\frac{i\sigma}{2}\int_0^d E_y(y)\{H_0(k|y-y_0|)-H_0[k(y+y_0)]\}. \tag{16}$$

This can be solved numerically for $E_y(y)$. The solution has been given by Kahn [4] (though he used $H_0(kR)$ as Green's function and, therefore, had to integrate over the metal surface in addition to the crack faces).

Given E(0,y) at the crack face, we can proceed in evaluation of the impedance change, ΔZ, due to the crack:

$$\Delta Z/B_o = \int_o^d E_y(0,y)dy + \frac{1}{2}\int_{-\infty}^{\infty} [E_x(x,0) - E_o]dx. \tag{17}$$

One could show by a direct evaluation of $E_x(x,0)$ using Eq. (15) for the magnetic field, that the second contribution to ΔZ in (17) is equal to $-B_o\int_o^d E_y(0,y)\exp(iky)dy$, i.e., it is expressed in terms of the quantity $E_y(0,y)$ we already know. It is instructive, however, to see that this result is a direct consequence of the general reciprocity theorem.

The theorem states that at a given frequency, any two solutions of the Maxwell equations (which could correspond to different boundary conditions) satisfy the identity

$$\int_S (\vec{E}_1 x\vec{B}_2 - \vec{E}_2 x\vec{B}_1)\cdot d\vec{s} = 0, \tag{18}$$

if the whole integration surface S is situated in a region where the material parameters (σ in our case) corresponding to these two situations are the same. Let us call $\vec{E}_1,\vec{B}_1$ the inside fields for the uncracked metal piece shown in Fig. 1. Let $\vec{E}_2,\vec{B}_2$ be the fields in the same piece with a crack. Then choosing the surface S as shown in the figure, we apply theorem (18). We further separate the surface S into the "uncracked" piece S_o and the crack face, S_c, to obtain

$$\int_{S_o} (\vec{E}_2-\vec{E}_1)x\vec{B}_o\cdot d\vec{s} = \int_{S_c} \vec{E}_2 x\vec{B}_1\cdot d\vec{s}. \tag{19}$$

We have used here that both $\vec{B}_2$ and $\vec{B}_1$ are equal to the constant $\vec{B}_o$ at the surface S_o.

Turning to the case of a closed crack in the plane surface, we obtain from (19) the above mentioned result:

$$-\int_{-\infty}^{\infty} [E_x(x,0)-E_o]dx = 2\int_o^d E_y(+0,y)e^{iky}dy. \tag{20}$$

(Note: $\vec{B}_1=B_o\hat{z}e^{iky}$, $\vec{B}_o=B_o\hat{z}$). Equation (17) now yields the impedance change due to the crack:

$$\Delta Z = B_o\int_o^d E_y(y)(1-e^{iky})dy. \tag{21}$$

Once again, we see that the total impedance change due to the crack is expressed in terms of the field $\vec{E}_t$ at the crack face. Thus, the whole problem reduces to that of finding $E_y(y)$ at the crack.

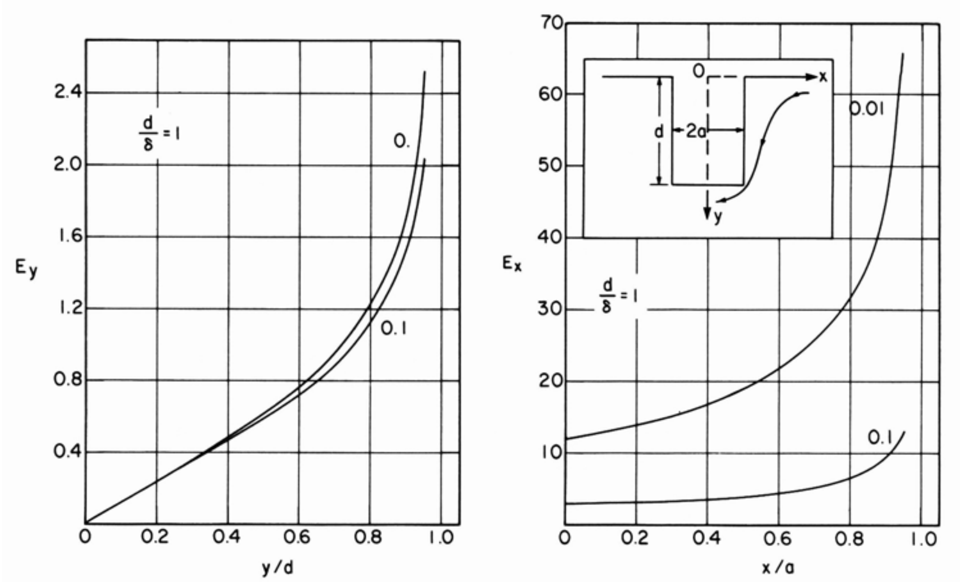

Fig. 2. Electric fields $ReE_y(y)$ and $ReE_x(x)$ at the surfaces of an open crack for $d/\delta=1$. The configuration and the coordinates are given in the insert. The numbers by the curves indicate the ratio of the width to the skin depth, $2a/\delta$. For simplicity, the field B_o and the conductivity σ are set equal to unity.

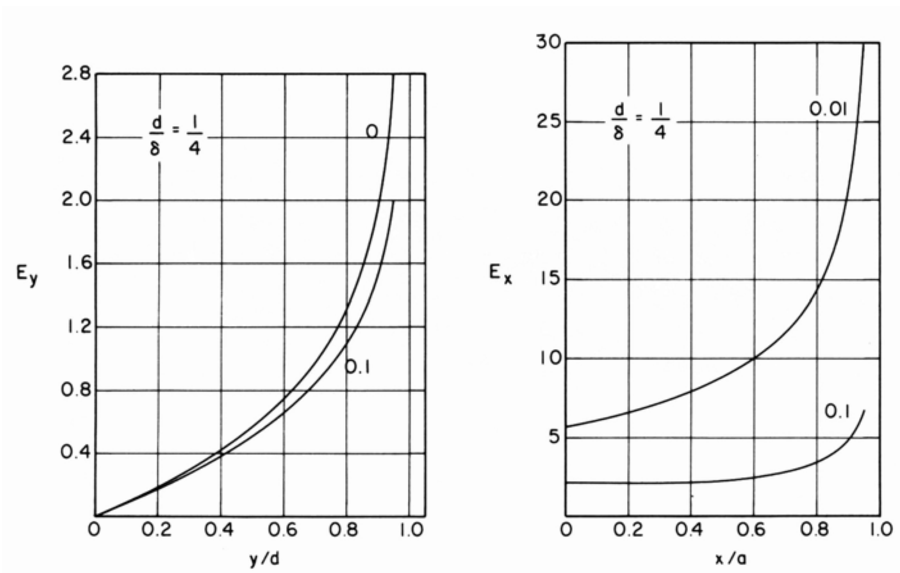

Fig. 3. Electric fields $ReE_y(y)$ and $ReE_x(x)$ at the surfaces of an open crack for $d/\delta=1/4$. The numbers by the curves indicate the ratio of the width to the skin depth, $2a/\delta$.

With essentially the same argument, one can deduce from the reciprocity relation (18) for any closed two dimensional crack

$$\Delta Z = \sum_{n=1}^{2} \int ds E_t(\vec{s},n)[B_o - B_o(\vec{s})]/2 \tag{22}$$

where n=1,2 corresponds to two opposite crack faces, $\vec{s}$ is the point at the crack, $B_o(\vec{s})$ is the "unperturbed" by the crack magnetic field (at the crack's location), and $E_t(\vec{s},n)$ is the tangential to the crack electric field at the side n. (Note that in general the field E_t, at two sides of the crack, are not necessarily equal in magnitude as in the case of a plane crack normal to the plane metal surface).

Equation (22) can be applied to evaluate the impedance change due to a closed crack in the 90° corner. In this situation, the field $B_o(\vec{s})$ is given in Eq. (7), where x and y take their values at the crack's location. The tangential component of the electric field $E_t(\vec{s},n)$ at the crack faces is found by solving an appropriate integral equation (similar to Eq. (16)). Numerical work on this problem is in progress.

The same approach can be exploited for open 2D cracks in the plane surface. Using the Green's function (5), one obtains the magnetic field distribution:

$$B(\vec{r}) = B_o e^{iky} - \frac{\sigma}{4\pi} \int_{S_c} ds\, E_t(\vec{s}) G(\vec{s},\vec{r}) - \frac{k^2 B_o}{4\pi} \int dA_o G(\vec{r},\vec{r}_o). \tag{23}$$

To derive this expression from Eqs. (3) and (5), we added and subtracted $B_o \int \Delta G \cdot d\vec{s}$ over the crack's opening ("crack's mouth"). This allows us to extract the contribution of the unperturbed field $B_o \exp(iky) = B_o \int_{-\infty}^{\infty} dx (\partial G/\partial y)_{y=0}$. The part $B_o \int \nabla G \cdot d\vec{s}$ over the crack faces complemented with the same integral over the "crack mouth", results in the last term in Eq. (23). For points $\vec{r}$ at the crack faces ($\vec{r}=\vec{s}_o$), $B=B_o$ and Eq. (23) yields an integral equation for the tangential electric field $\vec{E}_t$ at the crack faces and at the crack bottom. The preliminary numerical solutions for $ReE_y(y)$ (at the faces) and $ReE_x(x)$ (at the bottom) are given in Figs. 2, 3, and 4 for three different ratios of the crack depth d to the skin depth δ. It is worth noting that the real part of the electric field $E_y(y)$ at the crack faces depends rather weakly upon the crack width. In fact, for $d/\delta=4$ within the accuracy of about 10%, $ReE_y(y)$ are the same for $2a/\delta=0$, 0.01, and 0.1. The derivation of the impedance change due to open cracks along with details of the numerical procedure and accuracy estimates will be published elsewhere.

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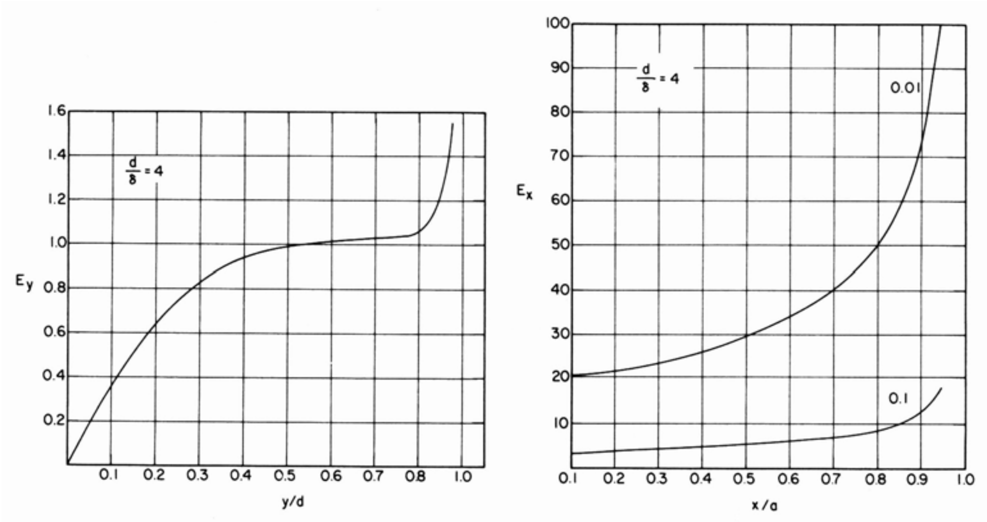

Fig. 4. Electric fields ReE_y(y) and ReE_x(x) at the surfaces of an open crack for $d/\delta=4$. The numbers by the curves indicate the ratio of the width to the skin depth, $2a/\delta$. Within about 10% accuracy, ReE_y(y) for values of $2a/\delta=0$, 0.01, and 0.1 are the same.

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