

9-1993

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## Abstract

An experiment with the nonassociative algebra program Albert led to the discovery of the following surprising theorem. *Let  $G$  be a groupoid satisfying the identity  $(xy)z = y(zx)$ . Then for products in  $G$  involving at least five elements, all factors commute and associate.* A corollary is that any semiprime ring satisfying this identity must be commutative and associative, generalizing a known result of Chen.

## Keywords

identity, groupoid, ring

## Disciplines

Algebra | Mathematics

## Comments

This article is published as Hentzel, Irvin Roy, David P. Jacobs, and Sekhar V. Muddana. "Experimenting with the identity  $(xy)z = y(zx)$ ." *Journal of symbolic computation* 16, no. 3 (1993): 289-293. [10.1006/jsc.1993.1047](https://doi.org/10.1006/jsc.1993.1047). Posted with permission.

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## Experimenting with the Identity $(xy)z = y(zx)$

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*(Received 28 October 1993)*

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An experiment with the nonassociative algebra program **Albert** led to the discovery of the following surprising theorem. *Let  $G$  be a groupoid satisfying the identity  $(xy)z = y(zx)$ . Then for products in  $G$  involving at least five elements, all factors commute and associate.* A corollary is that any semiprime ring satisfying this identity must be commutative and associative, generalizing a known result of Chen.

**Key words:** identity, groupoid, ring.

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### 1. Introduction

The nonassociative algebra program **Albert**, (Jacobs, Muddana and Offutt, 1993) has recently provided insight that has led to several interesting theorems on commutativity and associativity (Hentzel and Jacobs, 1992; Hentzel, Jacobs and Kleinfeld, 1993). These papers, along with the present paper, provide a growing body of evidence that computational experiments can suggest interesting conjectures in abstract algebra that can then be proven by conventional methods. The results of this paper were motivated by experiments with the identity

$$(xy)z = y(zx). \tag{1.1}$$

Clearly any binary operation that is both commutative and associative will satisfy this identity. While the converse is not true, as we will see, it is almost true. Using **Albert**, we noticed that in the presence of (1.1), a product involving a sufficient number of elements seemed to be independent of the way the elements were ordered or associated.

Recall that a *groupoid* is a nonempty set with a single binary operation. For a positive integer  $k$ , let us say that a groupoid is *k-nice* if the product of any  $k$  elements is the same, regardless of their association or order. With this, commutativity is then equivalent to being 2-nice. A groupoid is commutative and associative if and only if it is both 2-nice and 3-nice. It is clear that a groupoid is  $k$ -nice for every positive  $k$ . In this note we study

groupoids satisfying the identity Identity (1.1) defines a generalization of commutative-associative groupoids. Our main result, proven in Section 2, is that groupoids satisfying identity (1.1) are  $k$ -nice for each  $k \geq 5$ . In Section 3, we see that this yields the corollary that any semiprime ring satisfying (1.1) must be commutative-associative. Chen (1970) showed that rings satisfying identity (1.1), the right alternative law, and containing no zero divisors must be commutative-associative. Thus, our corollary generalizes Chen's result.

## 2. Niceness of groupoids

Henceforth, let  $G$  denote a groupoid satisfying (1.1). If  $a, b, c, d, e \in G$ , there are fourteen ways in which a product on these letters can be associated. We number these as follows.

- |                   |                   |
|-------------------|-------------------|
| 1. $a(b(c(de)))$  | 2. $a(b((cd)e))$  |
| 3. $a((bc)(de))$  | 4. $a((b(cd))e)$  |
| 5. $a(((bc)d)e)$  | 6. $(ab)(c(de))$  |
| 7. $(ab)((cd)e)$  | 8. $(a(bc))(de)$  |
| 9. $((ab)c)(de)$  | 10. $(a(b(cd)))e$ |
| 11. $(a((bc)d))e$ | 12. $((ab)(cd))e$ |
| 13. $((ab)c)d$    | 14. $((a(bc))d)e$ |

By permuting  $a, b, c, d, e$  we can form  $1680 = 14 * 5!$  words. Each word is defined by an association  $t$ ,  $1 \leq t \leq 14$ , and a permutation  $\pi \in S_5$ , the symmetric group on five letters. The permutation  $\pi$  acts on the letters and may be regarded as a renaming. We denote that word by  $w(t, \pi)$ . For example, if  $\pi = (13)(24)$  and  $t = 6$ , then  $w(t, \pi) = (cd)(a(be))$ . Our goal is now to show that all words in  $\{w(t, \pi) \mid 1 \leq t \leq 14, \pi \in S_5\}$  are equal. Observe that for  $a, b, c, d \in G$ , by applying (1.1) in different ways we obtain  $((ab)c)d = (b(ca))d = (ca)(db) = a((db)c) = a(b(cd)) = ((cd)a)b = (d(ac))b = (ac)(bd) = c((bd)a) = c(d(ab))$ . In particular, we have

$$((ab)c)d = a(b(cd)), \quad (2.1)$$

$$(b(ca))d = (d(ac))b, \quad (2.2)$$

$$a((db)c) = c((bd)a). \quad (2.3)$$

The last two equations say that for certain associations, the two innermost letters can be interchanged while interchanging the two outermost letters.

Next, consider  $G'$  the groupoid on the same set of elements as  $G$ , but with product  $x \circ y$  redefined as  $yx$ . It is easy to see that  $G$  satisfies (1.1) if and only if  $G'$  satisfies (1.1). This implies that any identity such as  $((ab)c)d = (b(ca))d$  will derive its mirror image, i.e.  $d(c(ba)) = d((ac)b)$ . We make heavy use of this technique and will often merely say "by symmetry".

Note also that if  $w(t, \pi) = w(t', \pi')$  is an identity, then for any  $\sigma \in S_5$ ,  $w(t, \sigma\pi) = w(t', \sigma\pi')$  is also an identity.

**LEMMA 2.1.** *For each  $\pi \in S_5$ ,  $w(1, \pi) = w(2, \pi) = \dots = w(14, \pi)$ .*

PROOF.  $\emptyset$  By the previous observation, we may assume  $\pi$  is the identity permutation  $i$ . Multiplying both sides of equation (2.1) on the right by  $e$ , we obtain  $((ab)c)d)e = (a(b(cd)))e$ , and so

$$w(13, i) = w(10, i). \tag{2.4}$$

Next, we substitute  $ab$  for  $a$  in (2.1) and rename  $b, c, d$  as  $c, d, e$ . We obtain  $((ab)c)d)e = (ab)(c(de))$ , or

$$w(13, i) = w(6, i). \tag{2.5}$$

Similarly, by renaming  $c, d$  as  $d, e$  in equation (2.1) and putting  $bc$  in place of  $b$ , we get

$$w(14, i) = w(3, i). \tag{2.6}$$

By symmetry, equations (2.4)–(2.6) yield respectively,

$$w(1, i) = w(5, i), \tag{2.7}$$

$$w(1, i) = w(9, i), \tag{2.8}$$

$$w(2, i) = w(12, i). \tag{2.9}$$

Using (1.1), (2.3), (2.9), (1.1), (1.1) and (2.2), we derive  $((ab)c)d)e = d(e((ab)c)) = d(c((ba)e)) = ((dc)(ba))e = (ba)(e(dc)) = a((e(dc))b) = a((b(cd))e)$ , or

$$w(13, i) = w(4, i). \tag{2.10}$$

Next, using identity (1.1) repeatedly we obtain  $(a(bc))(de) = ((ca)b)(de) = (e((ca)b))d = ((be)(ca))d = ((a(be))c)d = c(d(a(be))) = c(d((ea)b)) = (((ea)b)c)d = (b(c(ea)))d = (c(ea))(db) = (ea)((db)c) = (ea)(b(cd)) = a((b(cd))e)$ , or

$$w(8, i) = w(4, i). \tag{2.11}$$

Next,  $(ab)((cd)e) = b(((cd)e)a) = b(e(a(cd))) = ((a(cd))b)e = (((da)c)b)e = (c(b(da)))e = (b(da))(ec) = (da)((ec)b) = a(((ec)b)d) = a(b(d(ec))) = a(b((cd)e))$ , gives

$$w(7, i) = w(2, i). \tag{2.12}$$

By symmetry, equations (2.10)–(2.12) yield respectively,

$$w(1, i) = w(11, i), \tag{2.13}$$

$$w(7, i) = w(11, i), \tag{2.14}$$

$$w(8, i) = w(14, i). \tag{2.15}$$

Finally,  $((ab)c)(de) = (e((ab)c))d = ((ce)(ab))d = ((b(ce))a)d = a(d(b(ce))) = a(d((eb)c)) = a((cd)(eb)) = a((b(cd))e)$ , and we have

$$w(9, i) = w(4, i). \tag{2.16}$$

Combining equations (2.4)–(2.16) completes the proof.  $\square$

LEMMA 2.2. A groupoid satisfying  $(xy)z = y(zx)$  is 5-nice.

PROOF. Consider the set

$$T = \{\sigma \in S_5 \mid w(1, i) = w(1, \sigma)\}.$$

Now using Lemma 2.1, (1.1), and Lemma 2.1 again we obtain  $a(b(c(de))) = (ab)((cd)e) = b(((cd)e)a) = b(c(d(ea)))$ , or  $(12345) \in T$ . Next, using identity (1.1) three times and then Lemma 2.1 we obtain  $a(b(c(de))) = a(((de)b)c) = a((e(bd))c) = a((bd)(ce)) = a(b(d(ce)))$ , or  $(34) \in T$ . It is well-known (Herstein, p. 69) that given a cycle of length  $n$  and given a transposition, any element in  $S_n$  can be expressed as a product involving these two permutations. It is easy to show that  $T$  is closed under multiplication. Hence we have  $T = S_5$ . Finally, let  $w(t, \pi)$  be an arbitrary word. By Lemma 2.1,  $w(t, \pi) = w(1, \pi)$ . Since  $\pi \in T$ , we have  $w(1, \pi) = w(1, i)$ .  $\square$

LEMMA 2.3. *If  $k \geq 3$ , a  $k$ -nice groupoid is  $(k + 1)$ -nice.*

PROOF. We show this holds when  $k = 5$ . Passing to the general case will be clear. Let  $\Pi$  represent an arbitrary product involving  $a_i$ ,  $1 \leq i \leq 6$ . It suffices to show  $\Pi = (((a_1 a_2) a_3) a_4) a_5) a_6$ . For some ordering  $\Pi$  can be regarded as a product of the five elements

$$(a_{i_1} a_{i_2}), a_{i_3}, a_{i_4}, a_{i_5}, a_{i_6}. \tag{2.17}$$

Case 1:  $i_1 \neq 6, i_2 \neq 6$ . We first apply 5-niceness to the elements in (2.17), obtaining  $\Pi = (((a_{i_1} a_{i_2}) a_{i_3}) a_{i_4}) a_{i_5} a_6$ . We then apply it to  $a_1$  through  $a_5$ , obtaining  $((((a_1 a_2) a_3) a_4) a_5) a_6$ .

Case 2:  $i_1 = 6$ . We apply 5-niceness first to the elements in (2.17), obtaining  $\Pi = (((a_6 a_{i_2}) a_{i_3}) a_{i_4}) a_{i_5} a_{i_6}$ . We then apply 5-niceness to the five leftmost letters, obtaining  $((((a_{i_3} a_{i_2}) a_6) a_{i_4}) a_{i_5}) a_{i_6}$ . (Note here that we made use of  $k \geq 3$ .) We now return to case 1.

Case 3:  $i_2 = 6$ . This is similar to case 2.  $\square$

Lemma 2.2 and Lemma 2.3 now yield

THEOREM 2.1. *A groupoid satisfying  $(xy)z = y(zx)$  is  $k$ -nice for each  $k \geq 5$ .*

### 3. Semiprime nonassociative rings

Recall a *nonassociative ring* has an additive structure that forms an abelian group, but multiplication is not necessarily commutative or associative. We say a ring  $R$  is *semiprime* if it has no nonzero ideal  $I$  for which  $I^2 = 0$ . Chen (1970, Theorem 3.2) showed that a right alternative ring satisfying  $(xy)z = y(zx)$ , and containing no zero divisors, must be commutative-associative. We now observe that “right alternative” is unnecessary, and “containing no zero divisors” can be replaced by the weaker condition of “semiprime”. From here on, let  $R$  denote a semiprime ring satisfying  $(xy)z = y(zx)$ . We first show that  $R$  is associative. Let  $J$  be the ideal in  $R$  generated by all *associators*, i.e. elements of the form  $(uv)w - u(vw)$ . It is known that  $J$  is the additive span of elements of the form  $(uv)w - u(vw)$  and  $((uv)w - u(vw))x$ . By our theorem, the product of any two such elements vanishes, and so we have  $J^2 = 0$ . Since  $R$  is semiprime, we must have  $J = 0$ . We may now assume  $R$  is associative. Next let  $K$  be the ideal generated by all *commutators*, i.e. elements of the form  $uv - vu$ . By associativity, it is easy to see that  $K$  is the additive span of elements of the form  $uv - vu$  and  $(uv - vu)w$ . However, by 5-niceness the product of any two elements vanishes with the possible exception of  $(uv - vu)(rs - sr)$ . But note that associativity and identity (1.1) imply that any three

elements can be cyclicly shifted. Hence  $(uv)(rs) = u(vrs) = u(svr) = (usv)r = (svu)r = s(vu)r = (vu)rs = (vu)(rs)$ . Similarly  $(uv)(sr) = (vu)(sr)$  and so  $(uv - vu)(rs - sr) = 0$ . It now follows that  $K^2 = 0$ , and by assumption  $K = 0$ , and  $R$  is commutative. We have shown

**COROLLARY 3.1.** *A semiprime ring satisfying  $(xy)z = y(zx)$  is commutative and associative.*

Experience shows that with most identities, the number of words increases as the degree increases. Thus, the results in this paper were surprising to us. Computer experiments show that other variants of our identity, such as  $(xy)z = y(xz)$ , studied by Thedy (1967), or  $x(yz) = z(yx)$ , studied by M. Kleinfeld (1978), do not have the property of  $k$ -niceness, at least at degree 5. Finally, we wish to thank E. Kleinfeld for providing a useful reference to us.

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