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3-coloring triangle-free planar graphs with a precolored 9-cycle

Ilkyoo Choi[1∗] Jan Ekstein[2†] Přemysl Holub[2‡] Bernard Lidický[3§]

July 25, 2017

Abstract

Given a triangle-free planar graph G and a 9-cycle C in G, we characterize situations where a 3-coloring of C does not extend to a proper 3-coloring of G. This extends previous results when C is a cycle of length at most 8.

1 Introduction

Given a graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. We will also use $|G|$ for the size of $E(G)$. A *proper k-coloring* of a graph G is a function $\varphi : V(G) \to \{1, 2, \ldots, k\}$ such that $\varphi(u) \neq \varphi(v)$ for each edge $uv \in E(G)$. A graph G is *k-colorable* if there exists a proper k-coloring of G, and the minimum k where G is k-colorable is the *chromatic number* of G.

Garey and Johnson [18] proved that deciding if a graph is k-colorable is NP-complete even when $k = 3$. Moreover, deciding if a graph is 3-colorable is still NP-complete when restricted to planar graphs [12]. Therefore, even though planar graphs are 4-colorable by the celebrated Four Color Theorem [5, 6, 22], finding sufficient conditions for a planar graph to be 3-colorable has been an active area of research. A landmark result in this area is Grötzsch's Theorem [20], which is the following:

Theorem 1 ([20]). *Every triangle-free planar graph is 3-colorable.*

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We direct the readers to a nice survey by Borodin [8] for more results and conjectures regarding 3-colorings of planar graphs.

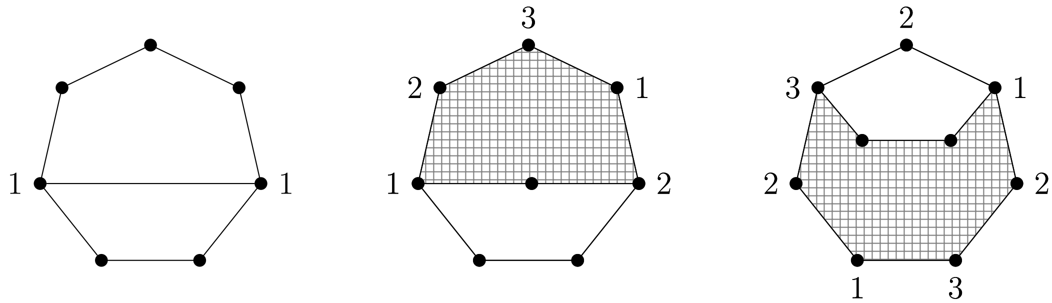
A graph G is *k-critical* if it is not $(k-1)$-colorable but every proper subgraph of G is $(k-1)$-colorable. Critical graphs are important since they are (in a certain sense) the minimal obstacles in reducing the chromatic number of a graph. Numerous coloring algorithms are based on detecting critical subgraphs. Despite its importance, there is no known characterization of k-critical graphs when $k \geq 4$. On the other hand, there has been some success regarding 4-critical planar graphs. Extending Theorem 1, the Grünbaum–Aksenov Theorem [1, 7, 21] states that a planar graph with at most three triangles is 3-colorable, and we know that there are infinitely many 4-critical planar graphs with four triangles. Borodin, Dvořák, Kostochka, Lidický, and Yancey [9] were able to characterize all 4-critical planar graphs with four triangles.

Given a graph G and a proper subgraph C of G, we say G is *C-critical for k-coloring* if for every proper subgraph H of G where $C \subseteq H$, there exists a proper k-coloring of C that extends to a proper k-coloring of H, but does not extend to a proper k-coloring of G. Roughly speaking, a C-critical graph for k-coloring is a minimal obstacle when trying to extend a proper k-coloring of C to a proper k-coloring of the entire graph. Note that $(k+1)$-critical graphs are exactly the C-critical graphs for k-coloring with C being the empty graph.

In the proof of Theorem 1, Grötzsch actually proved that any proper coloring of a 4-cycle or a 5-cycle extends to a proper 3-coloring of a triangle-free planar graph. This implies that there are no triangle-free planar graphs that are C-critical for 3-coloring when C is a face of length 4 or 5. This sparked the interest of characterizing triangle-free planar graphs that are C-critical for 3-coloring when C is a face of longer length. Since we deal with 3-coloring triangle-free planar graphs in this paper, from now on, we will write "C-critical" instead of "C-critical for 3-coloring" for the sake of simplicity.

The investigation was first done on planar graphs with girth 5. Walls [25] and Thomassen [23] independently characterized C-critical planar graphs with girth 5 when C is a face of length at most 11. The case when C is a 12-face was initiated in [23], but a complete characterization was given by Dvořák and Kawarabayashi in [15]. Moreover, a recursive approach to identify all C-critical planar graphs with girth 5 when C is a face of any given length is given in [15]. Dvořák and Lidický [14] implemented the algorithm from [15] and used a computer to generate all C-critical graphs with girth 5 when C is a face of length at most 16. The generated graphs were used to reveal some structure of 4-critical graphs on surfaces without short contractible cycles. It would be computationally feasible to generate graphs with girth 5 even when C has length greater than 16.

The situation for planar graphs with girth 4, which are triangle-free planar graphs, is more complicated since the list of C-critical graphs is not finite when C has size at least 6. We already mentioned that there are no C-critical triangle-free planar graphs when C is a face of length 4 or 5. An alternative proof of the case when C is a 5-face was given by Aksenov [1]. Gimbel and Thomassen [19] not only showed that there exists a C-critical triangle-free planar graph when C is a 6-face, but also characterized all of them. A *k^--cycle,*



(a) (b) (c)

Figure 1: Critical graphs with a precolored 7-face.

k^+*-cycle* is a cycle of length at most k, at least k, respectively. A cycle C in a graph G is *separating* if $G - C$ has more connected components than G.

Theorem 2 (Gimbel and Thomassen [19])**.** *Let G be a connected triangle-free plane graph with outer face bounded by a 6^--cycle $C = c_1c_2\cdots$. The graph G is C-critical if and only if C is a 6-cycle, all internal faces of G have length exactly four and G contains no separating 4-cycles. Furthermore, if φ is a 3-coloring of C that does not extend to a 3-coloring of G, then $\varphi(c_1) = \varphi(c_4)$, $\varphi(c_2) = \varphi(c_5)$, and $\varphi(c_3) = \varphi(c_6)$.*

Aksenov, Borodin, and Glebov [3] independently proved the case when C is a 6-face using the discharging method, and also characterized all C-critical triangle-free planar graphs when C is a 7-face in [4]. The case where C is a 7-face was used in [9].

Theorem 3 (Aksenov, Borodin, and Glebov [4])**.** *Let G be a connected triangle-free plane graph with outer face bounded by a 7-cycle $C = c_1\cdots c_7$. The graph G is C-critical and ψ is a 3-coloring of C that does not extend to a 3-coloring of G if and only if G contains no separating 5^--cycles and one of the following propositions is satisfied up to relabelling of vertices (see Figure 1 for an illustration).*

(a) The graph G consists of C and the edge c_1c_5, and $\psi(c_1) = \psi(c_5)$.

(b) The graph G contains a vertex v adjacent to c_1 and c_4, the cycle $c_1c_2c_3c_4v$ bounds a 5-face and every face drawn inside the 6-cycle $vc_4c_5c_6c_7c_1$ has length four; furthermore, $\psi(c_4) = \psi(c_7)$ and $\psi(c_5) = \psi(c_1)$.

(c) The graph G contains a path c_1uvc_3 with $u, v \notin V(C)$, the cycle $c_1c_2c_3vu$ bounds a 5-face and every face drawn inside the 8-cycle $uvc_3c_4c_5c_6c_7c_1$ has length four; furthermore, $\psi(c_3) = \psi(c_6)$, $\psi(c_2) = \psi(c_4) = \psi(c_7)$, and $\psi(c_1) = \psi(c_5)$.

Dvořák and Lidický [13] used a correspondence of nowhere-zero flows and colorings to give simpler proofs of the case when C is either a 6-face or a 7-face, and also characterized C-critical triangle-free planar graphs when C is an 8-face. For a plane graph G, let $S(G)$ denote the set of multisets of lengths of internal faces of G with length at least 5.

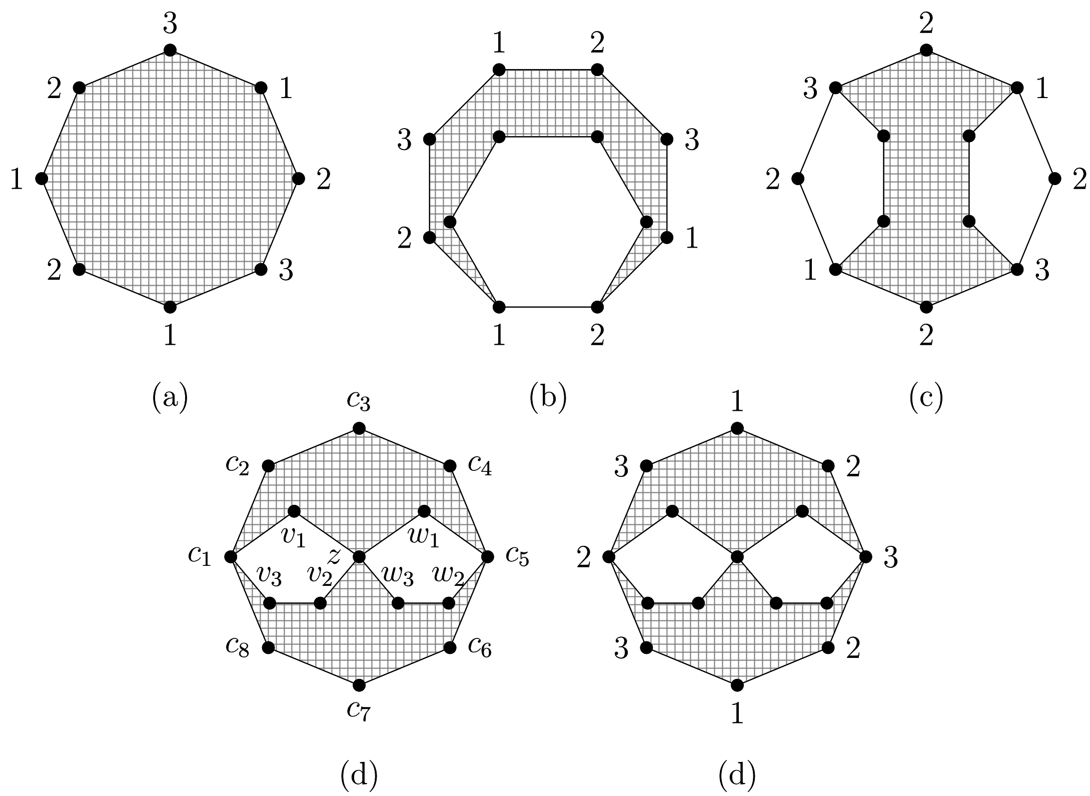


Figure 2: Graphs described by Theorem 4 and examples of 3-colorings of C that do not extend.

Theorem 4 (Dvořák and Lidický [13])**.** *Let G be a connected triangle-free plane graph with outer face bounded by an 8-cycle C. The graph G is C-critical if and only if G contains no separating 5^{-}-cycles, the interior of every non-facial 6-cycle contains only 4-faces, and one of the following propositions is satisfied (see Figure 2 for an illustration).*

(a) $S(G) = \emptyset$.

(b) $S(G) = \{6\}$ *and the* 6*-face of* G *intersects* C *in a path of length at least one.*

(c) $S(G) = \{5, 5\}$ *and each of the* 5*-faces of* G *intersects* C *in a path of length at least two.*

(d) $S(G) = \{5, 5\}$ *and the vertices of* C *and the* 5*-faces* f_1 *and* f_2 *of* G *can be labelled in clockwise order along their boundaries so that* $C = c_1c_2 \cdots c_8$*,* $f_1 = c_1v_1zv_2v_3$*, and* $f_2 = zw_1c_5w_2w_3$ *(where* w_1 *can be equal to* v_1*,* v_1 *can be equal to* c_2*, etc).*

Theorem 4 has the following corollary that was not explicitly stated in [13].

Corollary 5 ([13])**.** *Let G be a triangle-free plane graph and let v be a vertex of degree 4 in G. Then there exists a proper 3-coloring of G where all neighbors of v are colored the same.*

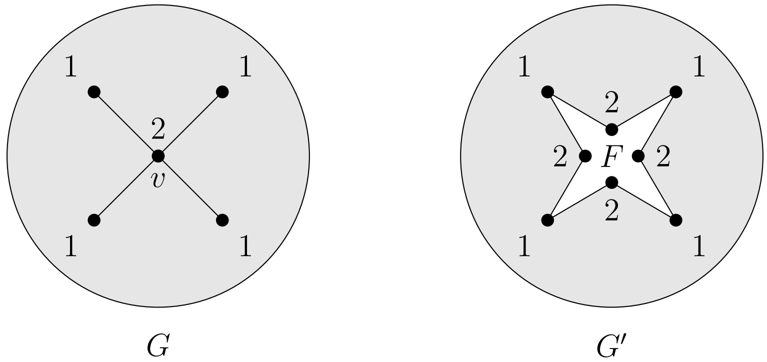


Figure 3: The coloring of a graph G, where all neighbors of a 4-vertex v have the same color, can be obtained by extending a precoloring of an 8-face F in G', where G' is obtained from G by splitting v into four vertices of degree 2.

The corollary can be proven by splitting v into four vertices of degree 2 that are in one 8-face F and precoloring F by two colors, see Figure 3.

In this paper, we push the project further and characterize all C-critical triangle-free planar graphs when C is a 9-face.

Theorem 6. *Let G be a connected triangle-free plane graph with outer face bounded by a 9-cycle C. The graph G is C-critical for 3-coloring if and only if for every non-facial 8^--cycle of K the subgraph of G drawn in the closed disk bounded by K is K-critical and one of the following propositions is satisfied (see Figure 4 for an illustration).*

(a) $S(G) = \{5\}$ *and the 5-face of G intersects C in a path of length at least two.*

(b) $S(G) = \{7\}$.

(c) $S(G) = \{5, 6\}$ *and the 5-face and 6-face of G intersects C in a path of length at least two and one, respectively.*

(d) $S(G) = \{5, 6\}$ *and G is depicted as (d1) or (d2) in Figure 4.*

(e) $S(G) = \{5, 5, 5\}$ *and G is depicted as (Bij) in Figure 4 for all i, j.*

(f) *G contains a chord.*

The proof of Theorem 6 involves enumerating all integer solutions to several small sets of linear constraints. It would be possible to solve them by hand but we have decided to use computer programs to enumerate the solutions. Both computer programs and enumerations of the solutions are available online at `http://orion.math.iastate.edu/lidicky/pub/9cyc/`.

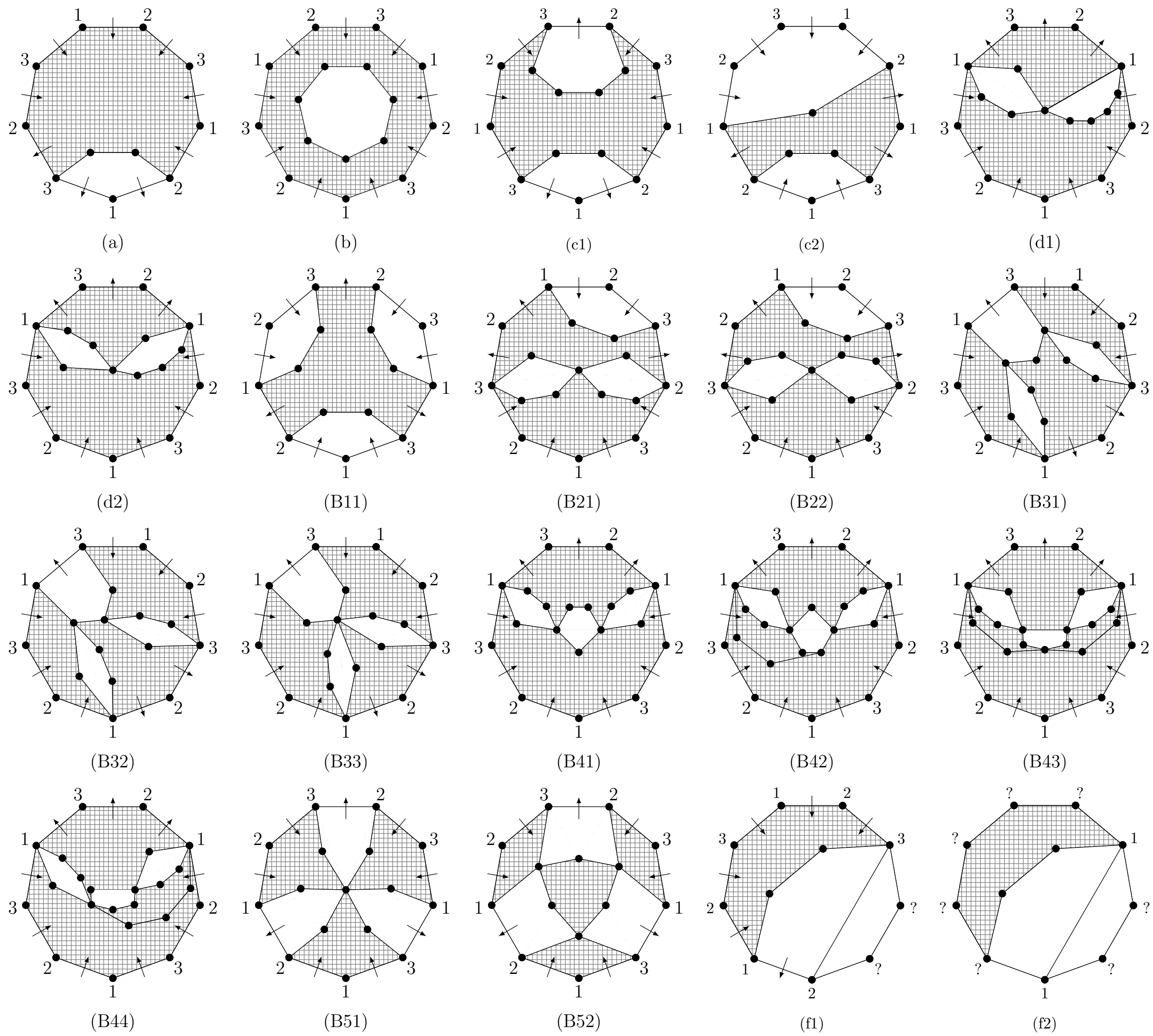


Figure 4: All C-critical triangle-free plane graphs where C is a 9-cycle bounding the outer face. Note that each figure actually represents infinitely many graphs, including ones that can be obtained by identifying some of the depicted vertices. The arrows correspond to source edges and sink edges that are defined in the Preliminaries.

2 Preliminaries

Our proof of Theorem 6 uses the same method as Dvořák and Lidický [13]. The main idea is to use the correspondence between colorings of a plane graph G and flows in the dual of G. In this paper, we give only a brief description of the correspondence and state Lemma 7 from [13], which is used throughout this paper. A more detailed and general description can be found in [13].

Let $G^\star$ denote the dual of a 3-colorable plane graph G. Let φ be a proper 3-coloring of the vertices of G by colors $\{1, 2, 3\}$. For every edge uv of G, we orient the corresponding edge e in $G^\star$ in the following way. Let e have endpoints f, h in $G^\star$, where f,v,h is in the clockwise order from vertex u in the drawing of G. The edge e will be oriented from f to h if $(\varphi(u), \varphi(v)) \in \{(1, 2),(2, 3),(3, 1)\}$, and from h to f otherwise. See Figure 5 for an example of the orientation.

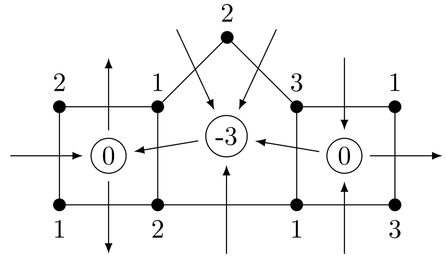


Figure 5: A 3-coloring of a graph G and the corresponding orientation of the edges in $G^\star$.

Since φ is a proper coloring, every edge of $G^\star$ has an orientation. Tutte [24] showed that this orientation of $G^\star$ defines a nowhere-zero $\mathbb{Z}_3$-flow, which means that the in-degree and the out-degree of every vertex in $G^\star$ differ by a multiple of three. Conversely, every nowhere-zero $\mathbb{Z}_3$-flow in $G^\star$ defines a proper 3-coloring of G up to the rotation of colors.

Let h be the vertex in $G^\star$ corresponding to the outer face of G. Edges oriented away from h are called *source edges* and the edges oriented towards h are called *sink edges*. The orientations of edges incident to h depend only on the coloring of C, where C is the cycle bounding the outer face of G. Denote by n^s the number of source edges and by n^t the number of sink edges. For a subgraph Z of G or a subset Z of $E(G)$, we will use n^s_Z and n^t_Z to denote the number of source edges and sink edges in $G^\star$ whose dual is in Z, respectively. Recall that only edges in C have source edges or sink edges in the dual.

For a vertex f of $G^\star$, let $\delta(f)$ denote the difference of the out-degree and in-degree of f. Possible values of $\delta(f)$ depend on the size of the face corresponding to f, denoted by $|f|$. Clearly $|\delta(f)| \leq |f|$ and $\delta(f)$ has the same parity as $|f|$. Hence if $|f| = 4$, then $\delta(f) = 0$. Similarly, if $|f| \in \{5, 7\}$, then $\delta(f) \in \{-3, 3\}$ and if $|f| = 6$ then $\delta(f) \in \{-6, 0, 6\}$.

We call a function q assigning an integer to every internal face f of G a *layout* if $q(f) \leq |f|$, $q(f)$ is divisible by 3, and $q(f)$ has the same parity as $|f|$. Notice that $q(f)$ satisfies the same conditions as $\delta(f)$. Therefore it is sufficient to specify the q-values for faces of size at least 5, since $q(f) = 0$ if f is a 4-face. A layout q is *ψ-balanced* if $n^s + m = n^t$, where m is the sum of the q-values over all internal faces of G.

Our main tool is the following lemma from [13].

Lemma 7 ([13]). *Let G be a connected triangle-free plane graph with outer face C bounded by a cycle and let ψ be a 3-coloring of C that does not extend to a 3-coloring of G. If q is a ψ-balanced layout in G, then there exists a subgraph $K_0 \subseteq G$ such that either*

i) K_0 is a path with both ends in C and no internal vertex in C, and if P is a path in C joining the end vertices of K_0, n^s is the number of source edges of P, n^t is the number of the sink edges of P, and m is the sum of the q-values over all faces of G drawn in the open disk bounded by the cycle $P + K_0$, then $|n^s + m - n^t| > |K_0|$. In particular, $|P| + |m| > |K_0|$.
Or,

ii) K_0 is a cycle with at most one vertex in C, and if m is the sum of the q-values over all faces of G drawn in the open disk bounded by K_0, then $|m| > |K_0|$.

For a multiset of numbers F, let $\ell(F)$ denote the smallest integer ℓ such that there exists a triangle-free plane graph G with outer face bounded by an ℓ-cycle C, such that G is C-critical and $S(G) = F$. It is known from [17] that $\ell(\{i\}) = i + 2$ and $\ell(\{5, 6\}) = 9$.

The next lemma from [16] describes interiors of cycles in critical graphs and will be used frequently in this paper.

Lemma 8 ([16]). *Let G be a plane graph with outer face C. Let K be a non-facial cycle in G, and let H be the subgraph of G drawn in the closed disk bounded by K. If G is C-critical for k-coloring, then H is K-critical for k-coloring.*

Next we include several definitions used throughout the rest of the paper. For the definitions, we assume that G is a graph with outer face bounded by a cycle C.

An x, y-path is a path with endpoints x and y. Given $a, b, c, d \in V(C)$, let $C(a, b; c, d)$ denote the a, b-subpath of C that does not contain vertices c and d as internal vertices. An x, y-path K is an *$(x, y; f)$-cut* if x, y are on C, no internal vertices of K are on C, and the face f is in the region bounded by K and the clockwise x, y-subpath of C.

Let K_1 and K_2 be two distinct paths with endpoints on C that are internally disjoint from C. For $i \in \{1, 2\}$ let P_i be a subpath of C with the same endpoints as K_i and label the endpoints of K_i by u_i and v_i, where u_i is the first vertex of P_i when traversing the cycle formed by K_i and P_i clockwise. The *order* of u_1, v_1, u_2, v_2 is an ordering of these vertices when traversing along C in the clockwise order. If $x_1 \in \{u_1, v_1\}$ and $x_2 \in \{u_2, v_2\}$ are the same vertex, we define the order of x_1 and x_2 in the following way. Let $x_1 = y_0, \ldots, y_m$ be the longest common subpath of K_1 and K_2. We consider the neighbors N of y_m in the counterclockwise order ending with y_{m-1} or a vertex of C if $m = 0$. If a vertex of K_1 appears in N before every vertex of K_2, then x_1 is before x_2 in the ordering, otherwise x_2 is before x_1.

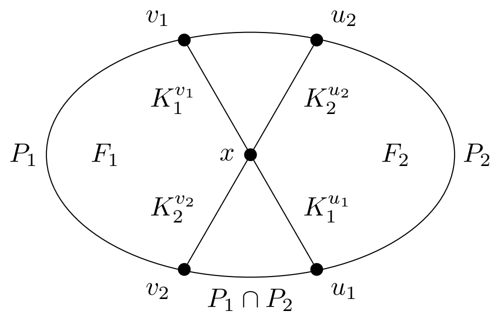


Figure 6: Kind (11) and a common point x.

Every order, or the pair K_1, K_2, is assigned a *kind* (t_1t_2), where t_i is the number of vertices from $\{u_{3-i}, v_{3-i}\}$ that are in P_i. Hence there are only five possible kinds; namely (00), (02), (20), (22), and (11). If K_1 is the same path as K_2, we can pick any order that will give kind (00), (02), or (20).

Suppose that the order is u_1, v_2, v_1, u_2, which gives kind (11). By planarity, there exists a vertex x such that x is an internal vertex of both K_1 and K_2. Denote by K_i^y a subpath of K_i with endpoints x and y for $i \in \{1,2\}$ and $y \in \{u_i, v_i\}$; see Figure 6. Let F_i be the set of 5^+-faces that are in the interior of the cycle bounded by P_i and K_i and in the exterior of the cycle bounded by P_{3-i} and K_{3-i} for $i \in \{1,2\}$. The vertex x is a *common point* of K_1 and K_2 if every face in F_1 is in an interior face of the subgraph of G induced by $P_1, K_1^{v_1}, K_2^{v_2}$ and every face in F_2 is in an interior face of the subgraph of G induced by $P_2, K_1^{u_1}, K_2^{u_2}$.

It is possible to show that there always exists a common point for the kind (11) if K_1 and K_2 are not too long.

Lemma 9. *Let G be a triangle-free plane graph with outer face C where every 4-cycle bounds a face and let K_1 and K_2 be paths in G with endpoints u_1, v_1, u_2, v_2 in C. Let K_1 and K_2 be internally disjoint with C and the order of u_1, v_1, u_2, and v_2 form (11). Then there exists a common point for K_1 and K_2 if either* $\max\{|K_1|, |K_2|\} \leq 7$ *and* $\min\{|K_1|, |K_2|\} \leq 6$ *or* $|K_1| = |K_2| = 7$ *and the endpoints of K_1 and K_2 are the same.*

Proof. We describe operations that eliminate candidates for common points. Eventually, we show that the situation is equivalent to the case where K_1 and K_2 share exactly one vertex and then it is easy to see it is the common point.

By planarity and the kind (11), there must be at least one vertex of G that is internal vertex of both K_1 and K_2. When traversing K_2 from v_2 to u_2 we label the internal vertices of K_2 that are also vertices of K_1 by $c_1, c_2, c_3, \ldots$. These vertices are candidates to be common points. We order them by their distance from u_1 on the path K_1. Let P_1 be the clockwise path in C from u_1 to v_1.

An edge of K_2 is *inside* if it is drawn inside of the open disk bounded by the cycle formed by P_1 and K_1 or if it is incident with v_2. An edge of K_2 is *outside* if it is drawn outside of the closed disk bounded by the cycle formed by P_1 and K_1 or if it is incident with u_2. Notice that if an edge of K_2 is neither inside nor outside, it is also an edge of K_1 and we call it *shared.*

Now we do several modifications to G, K_1 and K_2 such that the 5^+-faces are not affected but some candidates for common vertices are eliminated.

For some i, if c_i is adjacent to one shared edge and one not shared edge h, then we split c_i into two vertices c_i^1 and c_i^2, creating a new 4-face containing c_i^1, c_i^2 and the two other neighbors of c_i in K_1. We can replace c_i by c_i^1 and c_i^2 in K_1 and K_2 such that c_i^2 is inside or outside of the new cycle formed by P_1K_1 if h is inside or outside, respectively. By performing this operation, we decrease the number of vertices in the intersection of K_1 and K_2 and we can assume K_2 has no shared edges.

If both edges of K_2 incident with c_i for some i are inside (or outside) we split c_i into two vertices c_i^1 and c_i^2, creating a new 4-face containing c_i^1, c_i^2 and the other neighbors of c_i in K_1. We can replace c_i by c_i^1 and c_i^2 in K_1 and K_2, respectively. We label the vertices such that c_i^2 is in the interior (or exterior, respectively) of the cycle bounded by K_1 and P_1. By performing this operation, we decrease the number of vertices in the intersection of K_1 and K_2 and assume that c_i is incident to one inside edge and one outside edge for all i.

If c_i and c_{i+1} are consecutive in the order given by the distance from u_1 and the subpaths of K_1 and K_2 with endpoints c_i and c_{i+1} form a 4-cycle K (hence a 4-face), then we can reroute the paths such that the length of one of the paths is decreased or we create two vertices that are both incident with only inside or only outside edges. If one of the two paths forming K has length one, the other one has length three and replacing the longer one by an edge decreases the length of K_1 or K_2 (it also creates a new shared that we can eliminate). If both paths have length two, we swap them and now both c_i and c_{i+1} are incident to two edges that are both inside or both outside and they can be eliminated.

Notice that these operations do not increases the length of K_1 or K_2, do not create new vertices in the intersection of K_1 or K_2, do not affects locations or number of 5^+-faces of G with respect to regions formed by K_1C and K_2C. Hence a common point in the result would be a common point in the original configuration.

With use of computer, we generate all possible patterns where none of the above operations can be applied. In all of the patters with $\max\{|K_1|, |K_2|\} \leq 7$ and $\min\{|K_1|, |K_2|\} \leq 6$, there is only one vertex shared by K_1 and K_2, which is the common point.

If $|K_1| = |K_2| = 7$, there are eight patterns with more than one internal vertex in the intersection of K_1 and K_2. Four of them do not actually form (11) and none of the three operations was used on them which is a contradiction. The other four contain 4-cycles that do not bound a face which is also a contradiction. The program including the eight patterns is available with all the other programs used in this paper. □

3 Proof of Theorem 6

Let $\mathcal{S}_k$ be the set of possible multisets of lengths of 5^+-faces in a connected plane graph of girth at least 4 where the length of the precolored face is k. The result of Dvořák, Kráľ, and Thomas [17] implies among others that $\mathcal{S}_6 = \{\emptyset\}$, $\mathcal{S}_7 = \{\{5\}\}$, $\mathcal{S}_8 = \{\emptyset, \{6\}, \{5, 5\}\}$, and

$\mathcal{S}_9 = \{\{7\}, \{5\}, \{6,5\}, \{5,5,5\}\}$.

By the previous paragraph, we have four cases to consider when C has length 9. The case of one 7-face was already resolved by Dvořák and Lidický [13], and it is described in Theorem 6(b). We restate the result from [13] in the next subsection as Theorem 11. We resolve the remaining three cases in Lemmas 12, 14, 21, 22, 23, and 24 in the following three subsections. In order to simplify the cases, we first solve the case when C has a chord.

If G is C-critical and C has a chord, then Lemma 8 implies that G can be obtained by identifying two edges of the outer faces of two different smaller critical graphs or cycles. Lemma 10 shows that the converse is also true.

Lemma 10. *Let G_i be either a cycle C_i or a triangle-free plane C_i-critical graph, where $|C_i| \geq 4$ for $i \in \{1,2\}$. Let G be the graph obtained by identifying $e_1 \in E(C_1)$ and $e_2 \in E(C_2)$ and let C be the longest cycle formed by $E(C_1) \cup E(C_2)$ after the identification. Then G is C-critical, where $|C| = |C_1| + |C_2| - 2$.*

Proof. Let $e \in E(G) \setminus E(C)$.

Suppose first that $e \in E(G_i) - e_i$ for some $i \in \{1,2\}$. Since G_i is either a cycle or a C_i-critical triangle-free plane graph and it contains e that is not on the boundary, G_i is C_i-critical. Hence there exists a 3-coloring φ of C_i that extends to a proper 3-coloring of $G_i - e$ but does not extend to a proper 3-coloring of G_i. Since G_{3-i} is triangle-free, there exists a proper 3-coloring ϱ of G_{3-i} by Grötzsch's Theorem [20]. By permuting colors we may assume that φ and ϱ agree on e_i and e_{3-i}. This gives a proper 3-coloring of C showing that G is C-critical with respect to e.

The other case is when e is the result of the identification of e_1 and e_2. Let u, v be the vertices of e. Since $G - e$ is a triangle-free planar graph, there exists a proper 3-coloring φ of $G - e$ such that $\varphi(u) = \varphi(v)$; this is a result of Aksenov et al. [2] that was simplified by Borodin et al. [10]. Let ϱ be the restriction of φ to C. Clearly, ϱ can be extended to a proper 3-coloring of $G - e$ but not to a proper 3-coloring of G. □

Therefore, we can enumerate C-critical triangle-free plane graphs G where C has a chord and has length 9 by identifying edges from two smaller graphs with outer faces of lengths either 4 and 7 or 5 and 6. Since there are no C-critical graphs when $|C| \in \{4,5\}$, we just use a 4-cycle and a 5-cycle. The resulting graphs are depicted in Figure 4 (a), (b), (c1), (c2), (f1), and (f2), where some of the vertices may be identified.

3.1 One 7-face

The case of one 7-face is solved by a more general result from [13]. The result works for graphs with an outer face of length k and one internal face of length $k-2$. Let $r(k) = 0$ if $k \equiv 0 \pmod 3$, $r(k) = 2$ if $k \equiv 1 \pmod 3$, and $r(k) = 1$ if $k \equiv 2 \pmod 3$.

Theorem 11 ([13]). *Let G be a connected triangle-free plane graph with outer face bounded by a 7^+-cycle C of length k. Suppose that f is an internal face of G of length $k-2$ and that all other internal faces of G are 4-faces. The graph G is C-critical if and only if* *(a) $f \cap C$ is a path of length at least $r(k)$ (possibly empty if $r(k) = 0$),*

(b) G contains no separating 4-cycles, and

(c) for every $(k-1)^-$-cycle $K \neq f$ in G, the interior of K does not contain f.

Furthermore, in a graph satisfying these conditions, a precoloring ψ of C extends to a 3-coloring of G if and only if $E(C) \setminus E(f)$ contains both a source edge and a sink edge with respect to ψ.

In our case, we apply Theorem 11 with $k = 9$. Since $r(9) = 0$, the 7-face does not have to share any edges with the outer face. The description is in Theorem 6(b) and it is depicted in Figure 4(b).

3.2 One 5-face and one 6-face

Lemma 12. *Let G be a connected triangle-free plane graph with outer face bounded by a chordless 9-cycle C. Moreover, let G contain one 5-face f_5 and one 6-face f_6, all other internal faces are 4-faces, and all non-facial 8^--cycles K in G bound K-critical subgraphs. If ψ is a 3-coloring of C that does not extend to a 3-coloring of G, then ψ extends to $G - e$ for every $e \in E(G) \setminus E(C)$.*

Proof. Let $e \in E(G) \setminus E(C)$. We want to show that ψ extends to a proper 3-coloring of $G-e$. Suppose that ψ does not extend to a 3-coloring of $G-e$. Then there exists a C-critical subgraph H of $G - e$, such that the 3-colorings of C that extend to $G - e$ are exactly the 3-colorings of C that extend to H. Since H is C-critical, its multiset of 5^+-faces is one of $\{5\}, \{7\}, \{5, 6\}, \{5, 5, 5\}$. Since all non-facial 8^--cycles K in G bound K-critical subgraphs, Lemma 8 implies that every 5-face of H is a 5-face of G, every 7-face of H contains exactly one 5-face of G, and a 6-face of H contains no 5-faces in the interior. Hence, H contains one odd 5^+-face and one even 6^+-face or one odd 9^+-face, and the only option for the multiset of 5^+-faces of H is $\{5, 6\}$. That would mean that G is the same graph as H, and this is a contradiction. □

Notice that Lemma 12 implies that in order to prove C-criticality, it is enough to find one coloring that does not extend. In Figure 4 we depict colorings that do not extend.

Now we prove the other direction of the Theorem 6. We start by the following lemma that we prove separately for future reference and then continue with the main part Lemma 14.

Lemma 13. *Let G be a connected triangle-free plane graph with outer face bounded by a chordless 9-cycle C. Moreover, let G contain one 5-face f_5 and one 6-face f_6 and all other internal faces are 4-faces. If G is C-critical ψ is a 3-coloring of G with 9 source edges then ψ extends to a 3-coloring of G.*

Proof. Suppose for a contradiction that ψ does not extend to a 3-coloring of G. Hence there is just one ψ-balanced layout q with $q(f_5) = -3$ and $q(f_6) = -6$. Let K_0 be obtained from Lemma 7.

If K_0 is a cycle, then Lemma 7 implies $9 > |K_0|$. Let m denote the sum of the q-values of the faces in the interior of K_0. By Lemma 7, $|m| > k_0$. If both f_5, f_6 are in the interior of K_0, then $|m| = |q(f_5) + q(f_6)| = 9$, contradicting the fact that $|m| > k_0$ since $k_0 \geq \ell(\{5, 6\}) = 9$. If f_5 is in the interior of K_0, but f_6 is not, then $|m| = 3$, while $\ell(\{5\}) = 5$, a contradiction again. Similarly, we obtain a contradiction when f_6 is in the interior of K_0 but f_5 is not, since $\ell(\{6\}) = 6$ and $|m| \leq 6$.

Therefore K_0 is always a path joining two distinct vertices of C. These endpoints of K_0 partition the edges of C into two paths X and Y intersecting at the endpoints of K_0. For $Z \in \{X, Y\}$, recall that n^s_Z and n^t_Z denotes the number of source edges and sink edges, respectively, among the edges of Z in coloring ψ. The described structure is shown in Figure 7. Let R_X and R_Y be the subgraph of G induced by vertices in the closed interior of the cycle formed by K_0, X and K_0, Y respectively.

Note that $n^s_X + n^s_Y = 9$ and $n^t_X + n^t_Y = 0$. If both f_5, f_6 belong to R_X, then Lemma 7 implies $9 - n^s_X > k_0$ and Lemma 8 implies $n^s_X + k_0 \geq 9$ since $\ell(\{5, 6\}) = 9$, which is a contradiction. By symmetry, R_Y does not contain both f_5 and f_6.

Without loss of generality, suppose f_6 belongs to R_X and f_5 belongs to R_Y. Lemma 8 implies that $n^s_X + k_0 \geq 6$, which gives $k_0 \geq 6 - n^s_X$. Lemma 7 implies $|n^s_X - 6| > k_0$. Combining the inequalities give $|n^s_X - 6| > 6 - n^s_X$, which implies $n^s_X > 6$. Hence $n^s_Y < 3$. Analogously, we obtain $n^s_Y + k_0 \geq 5$ and $|n^s_Y - 3| > k_0$, whose combination gives $3 - n^s_Y > 5 - n^s_Y$, which is a contradiction. □

Lemma 14. *Let G be a connected triangle-free plane graph with outer face bounded by a chordless 9-cycle C. Moreover, let G contain one 5-face f_5 and one 6-face f_6 and all other internal faces are 4-faces. If G is C-critical, then G is described by Theorem 6(c),(d), and is depicted in Figure 4(c1),(c2),(d1), and (d2).*

Proof. Since G is C-critical, from Lemma 8 follows that every non-facial 8^--cycle K in G bounds a K-critical subgraph. Since G is C-critical, there exists a 3-coloring ψ of C that does not extend to a proper 3-coloring of G.

By symmetry, we assume that C has more source edges than sink edges. Hence C has either 9 or 6 source edges. Lemma 13 eliminates the case of 9 source edges hence C has 6 source edges. Let q be a ψ-balanced layout of G. Let $K_0 \subset G$ be obtained by Lemma 7 and let $k_0 = |K_0|$.

First suppose that K_0 is a cycle. Let m denote the sum of the q-values of the faces in the interior of K_0. By Lemma 7, $|m| > k_0$. If both f_5, f_6 are in the interior of K_0, then $|m| = |q(f_5) + q(f_6)| = 6$, contradicting the fact that $|m| > k_0$ since $k_0 \geq \ell(\{5, 6\}) = 9$. If f_5 is in the interior of K_0, but f_6 is not, then $|m| = 3$, while $\ell(\{5\}) = 5$, a contradiction again. Similarly, we obtain a contradiction when f_6 is in the interior of K_0 but f_5 is not, since $\ell(\{6\}) = 6$ and $|m| \leq 6$.

Therefore K_0 is always a path joining two distinct vertices of C. These endpoints of K_0 partition the edges of C into two paths X and Y intersecting at the endpoints of K_0. For $Z \in \{X, Y\}$, recall that n^s_Z and n^t_Z denotes the number of source edges and sink edges, respectively, among the edges of Z in coloring ψ. The described structure is shown in

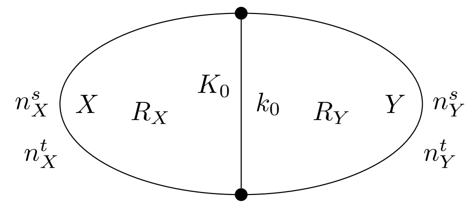


Figure 7: The structure of a cut in G.

Figure 7. Let R_X and R_Y be the subgraph of G induced by vertices in the closed interior of the cycle formed by K_0, X and K_0, Y respectively.

Claim 15. *If q is a ψ-balanced layout either with $q(f_5) = -3$ and $q(f_6) = 0$ or with $q(f_5) = 3$ and $q(f_6) = -6$, then both R_X and R_Y contain exactly one of f_5 and f_6.*

Proof. By Lemma 13, C contains 6 source edges, hence $n_X^s + n_Y^s = 6$ and $n_X^t + n_Y^t = 3$. By symmetry, suppose for a contradiction that both f_5, f_6 belong to R_X. Notice that $q(f_5) + q(f_6) = -3$ in both layouts. By Lemma 8, $n_X^s + n_X^t + k_0 \geq \ell(\{5, 6\}) = 9$, and by Lemma 7, $|n_X^s - 3 - n_X^t| > k_0$.

If $n_X^s - 3 - n_X^t > k_0$, then we obtain $n_X^s - 3 - n_X^t > k_0 \geq 9 - n_X^s - n_X^t$. This gives $n_X^s > 6$, which is a contradiction.

If $-n_X^s + 3 + n_X^t > k_0$, then we obtain $-n_X^s + 3 + n_X^t > k_0 \geq 9 - n_X^s - n_X^t$. This gives $n_X^t > 3$, which is a contradiction. □

Since C has 6 source edges, we have two different ψ-balanced layouts. Let q_1 and q_2 be the layout where $q_1(f_5) = -3$, $q_1(f_6) = 0$, and $q_2(f_5) = 3$, $q_2(f_6) = -6$, respectively. Let K and L be the subgraph of G obtained by Lemma 7 applied to q_1 and q_2, respectively, and let $k = |K|$ and $l = |L|$. Note that we already showed that each of K and L is a path joining pairs of distinct vertices of C; let K and L be a $(v_1, v_2; f_5)$-cut and $(w_1, w_2; f_6)$-cut, respectively. The paths K and L form a structure of kind (00), (11), (22), (20), or (02), see Figure 8, Figure 9, and Figure 10 for illustration. We discuss these cases in separate claims.

Claim 16. *If K and L are of kind (00), then G is depicted in Figure 4(c1).*

Proof. Note that K, L are not necessarily disjoint. By symmetry, let X be $C(w_1, w_2; v_1, v_2)$ such that the disk bounded by L and X contains f_6. Similarly, let Z be $C(v_1, v_2; w_1, w_2)$ such that the disk bounded by K and Z contains f_5. Denote by Y the edges of C that are neither in X nor in Z. See Figure 8 (00).

By the assumption that C has no chord, $k \geq 2$ and $l \geq 2$. By Claim 13, we know $n_X^s + n_Y^s + n_Z^s = 6$ and $n_X^t + n_Y^t + n_Z^t = 3$.

Lemma 8 implies $l + n_X^s + n_X^t \geq \ell(\{6\}) = 6$. Moreover, by parity, $l + n_X^s + n_X^t$ must be even. Similarly, Lemma 8 implies that $k + n_Z^s + n_Z^t \geq \ell(\{5\}) = 5$ and it is odd. Lemma 7 applied to q_1 and q_2 implies $|n_X^s + n_Y^s - n_X^t - n_Y^t| > k$ and $|3 + n_Z^s + n_Y^s - n_Z^t - n_Y^t| > l$, respectively.

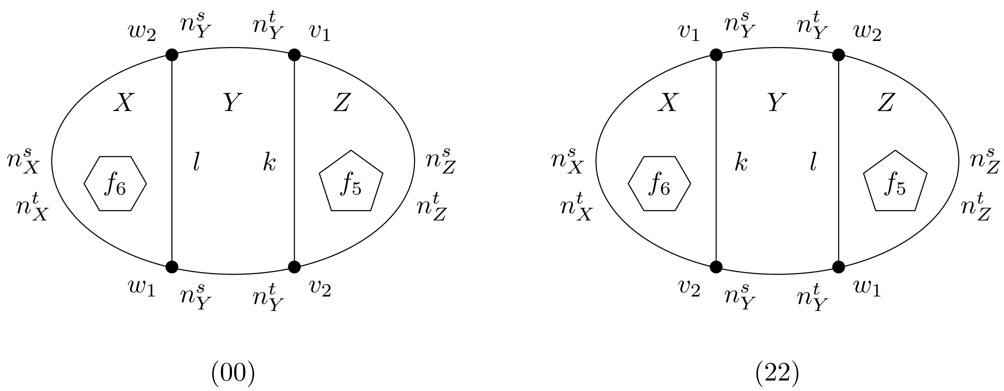


Figure 8: The cases where K and L are of kinds (00) and (22).

Here is the summary of the constraints:

$$\begin{aligned}
|n^s_X + n^s_Y - n^t_X - n^t_Y| &> k \\
|3 + n^s_Z + n^s_Y - n^t_Z - n^t_Y| &> l \\
l + n^s_X + n^t_X &\geq 6 \text{ and even} \\
k + n^s_Z + n^t_Z &\geq 5 \text{ and odd} \\
n^s_X + n^s_Y + n^s_Z &= 6 \\
n^t_X + n^t_Y + n^t_Z &= 3 \\
\min\{k, l\} &\geq 2
\end{aligned}$$

All integer solutions to these constraints are in the following table:

n^s_X	n^t_X	n^s_Y	n^t_Y	n^s_Z	n^t_Z	k	l
0	1	5	0	1	2	2	5
0	1	6	0	0	2	3	5
1	1	4	0	1	2	2	4
1	1	5	0	0	2	3	4
2	1	3	0	1	2	2	3
2	1	4	0	0	2	3	3
3	1	2	0	1	2	2	2
3	1	3	0	0	2	3	2

From these eight solutions we obtain the graph depicted in Figure 4(c1), up to identification of vertices. □

Claim 17. *If K and L are of kind (22), then G is depicted in Figure 4(c2).*

Proof. By symmetry, let X be $C(v_1, v_2; w_1, w_2)$ such that the disk bounded by K and X contains f_6. Similarly, let Z be $C(w_1, w_2; v_1, v_2)$ such that the disk bounded by L and Z contains f_5. Denote by Y the edges of C that are in neither X nor Z. See Figure 8 (22).

By the assumption that C has no chord, $k \geq 2$ and $l \geq 2$. By Claim 13, we know $n^s_X + n^s_Y + n^s_Z = 6$ and $n^t_X + n^t_Y + n^t_Z = 3$.

Lemma 8 implies $k + n^s_X + n^t_X \geq \ell(\{6\}) = 6$. Moreover, by parity, $k + n^s_X + n^t_X$ must be even. Similarly, Lemma 8 implies $l + n^s_Z + n^t_Z \geq \ell(\{5\}) = 5$ and it is odd. Lemma 7 applied to q_1 and q_2 implies $|n^s_X - n^t_X| > k$ and $|n^s_Z + 3 - n^t_Z| > l$, respectively.

Here are the constraints:

$$\begin{aligned}
|n^s_X - n^t_X| &> k \\
|n^s_Z + 3 - n^t_Z| &> l \\
k + n^s_X + n^t_X &\geq 6 \text{ and even} \\
l + n^s_Z + n^t_Z &\geq 5 \text{ and odd} \\
n^s_X + n^s_Y + n^s_Z &= 6 \\
n^t_X + n^t_Y + n^t_Z &= 3 \\
\min\{k, l\} &\geq 2
\end{aligned}$$

All integer solutions to these constraints are in the following table:

n^s_X	n^t_X	n^s_Y	n^t_Y	n^s_Z	n^t_Z	k	l
4	0	0	2	2	1	2	2
4	0	0	3	2	0	2	3

From these two solutions we obtain the graph depicted in Figure 4(c2). □

Claim 18. *If K and L are of kind (11), then G is depicted in Figure 4(d1) or (d2).*

Proof. Assume that K and L are of kind (11), so that the clockwise order of their endpoints on C is v_1, w_2, v_2, w_1. Let v_1, w_2, v_2, w_1 partition C into four paths X, Y, Z, W in the clockwise order such that X is an w_1, v_1-path. Moreover, the disk bounded by X, Y, L contains f_6 and the disk bounded by K, Y, Z contains f_5. See Figure 9 for an illustration.

First we show that $|K| \leq 6$ and $|L| \leq 7$. We obtain the following set of constraints by applying Lemma 7 and Lemma 8.

$$\begin{aligned}
|n^s_X + n^s_W - n^t_X - n^t_W| &> |K| \\
|n^s_Z + n^s_W + 3 - n^t_Z - n^t_W| &> |L| \\
n^s_X + n^t_X + n^s_Y + n^t_Y + |L| &\geq \ell(\{6\}) = 6 \text{ and even}
\end{aligned}$$

In all solutions, t $|K| \leq 6$ and $|L| \leq 7$. Hence Lemma 9 applies and K and L have a common point v.

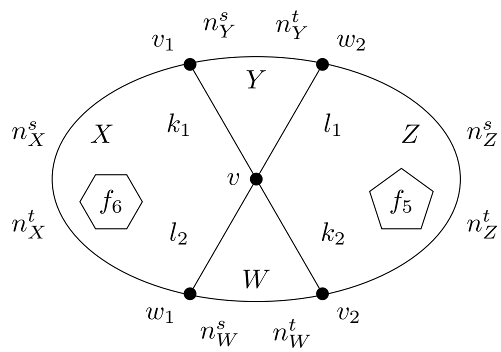


Figure 9: Case where K and L are of kind (11).

Partition L into paths L_1 and L_2 such that L_1 and L_2 is a v, w_2-path and a v, w_1-path, respectively. Do a similar partition of K into K_1 and K_2. Since v is a common point, f_6 and f_5 is contained in interior faces of subgraphs of G induced by X, K_1, L_2 and Z, L_1, K_2, respectively. Let $k_i = |K_i|$ and $l_i = |L_i|$ for $i \in \{1, 2\}$.

Note that $\min\{k_1, k_2, l_1, l_2\} \geq 1$ since v is an internal vertex.

We obtain the following set of constraints by applying Lemma 7 and Lemma 8.

$$|n_X^s + n_W^s - n_X^t - n_W^t| > k_1 + k_2 \tag{1}$$
$$|n_Z^s + n_W^s + 3 - n_Z^t - n_W^t| > l_1 + l_2 \tag{2}$$
$$k_1 + l_2 + n_X^s + n_X^t \geq \ell(\{6\}) = 6 \text{ and even} \tag{3}$$
$$l_1 + k_2 + n_Z^s + n_Z^t \geq \ell(\{5\}) = 5 \text{ and odd} \tag{4}$$
$$l_2 + k_2 + n_X^s + n_X^t + n_Y^s + n_Y^t + n_Z^s + n_Z^t \geq \ell(\{5, 6\}) = 9 \text{ and odd} \tag{5}$$
$$n_X^s + n_Y^s + n_Z^s + n_W^s = 6 \tag{6}$$
$$n_X^t + n_Y^t + n_Z^t + n_W^t = 3 \tag{7}$$

Inequalities (1) and (2) come from Lemma 7. Inequalities (3)–(5) come from the fact that interiors of cycles are also critical graphs.

This system of equations has 68 solutions. In all of them, $n_X^s + n_X^t + k_1 + l_2 = 6$ and $n_Z^s + n_Z^t + k_2 + l_1 = 5$. Hence the region bounded by X, K_1, L_2 is a 6-face and the region bounded by Z, L_1, K_2 is a 5-face. In order to generate only general solutions, where faces share as little with C as possible, we add constraints $n_X^s + n_X^t = 0$ and $n_Z^s + n_Z^t = 0$. Then the system has only two solutions.

n_X^s	n_X^t	n_Y^s	n_Y^t	n_Z^s	n_Z^t	n_W^s	n_W^t	k_1	k_2	l_1	l_2
0	0	0	3	0	0	6	0	1	3	2	5
0	0	0	3	0	0	6	0	2	2	3	4

From these solutions we obtain graphs depicted in Figure 4(d1) and (d2). We also checked that the 68 solutions can indeed be obtained from these two by identifying some

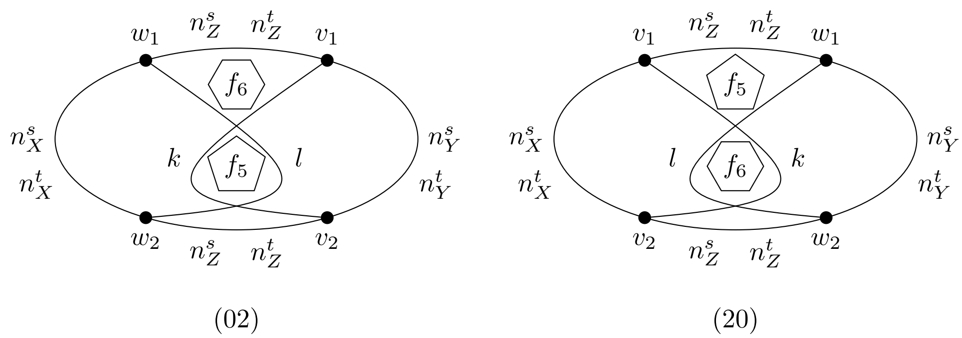


Figure 10: Case where K and L are of kinds (02) and (20).

vertices. The solutions were obtained by a computer program that is available at `http://orion.math.iastate.edu/lidicky/pub/9cyc/`.

□

Claim 19. *The case where K and L are of kind (02) does not occur.*

Proof. Assume that K and L are of kind (02), so that the clockwise order of their endpoints on C is v_1, v_2, w_2, w_1. Let Y and X be the clockwise v_1, v_2-subpath and w_2, w_1-subpath, respectively, of C. Let Z be the edges of C that are in neither X nor Y. See Figure 10 (02) for an illustration.

Observe that (by the structure of K and L) the subgraph of G formed by Z, K, and L contains in the internal faces both f_5 and f_6 and at least one additional 4-face. Hence $k + l + |Z| \geq 15$. We obtain the following set of constraints by applying Lemma 7 and Lemma 8.

$$\begin{aligned} |n^s_X - n^t_X| &> k \\ |n^s_Y - 6 - n^t_Y| &> l \\ k + l + n^s_Z + n^t_Z &\geq 15 \\ n^s_X + n^s_Y + n^s_Z &= 6 \\ n^t_X + n^t_Y + n^t_Z &= 3 \end{aligned}$$

This set of equations has no solution. □

Claim 20. *The case where K and L are of kind (20) does not occur.*

Proof. Assume that K and L are of kind (20), so that the clockwise order of their endpoints on C is w_1, w_2, v_2, v_1. Let Y and X be the clockwise w_1, w_2-subpath and v_2, v_1-subpath, respectively, of C. Let Z be the edges of C that are in neither X nor Y. See Figure 10 (20) for an illustration.

Observe that (by the structure of K and L) the subgraph of G formed by Z, K, and L contains in the internal faces both f_5 and f_6 and at least one additional 4-face. Hence

$k + l + |Z| \geq 15$. We obtain the following set of constraints by applying Lemma 7 and Lemma 8.

$$\begin{aligned}
|n^s_Y - 3 - n^t_Y| &> k \\
|n^s_X + 3 - n^t_X| &> l \\
k + l + n^s_Z + n^t_Z &\geq 15 \\
n^s_X + n^s_Y + n^s_Z &= 6 \\
n^t_X + n^t_Y + n^t_Z &= 3
\end{aligned}$$

This set of equations has no solution. □

This finishes the proof of Lemma 14. □

3.3 One 5-face

Lemma 21. *Let G be a connected triangle-free plane graph with outer face bounded by a chordless 9-cycle C. Moreover, let G contain one 5-face f_5 that shares a path of length at least two with C, all other internal faces of G are 4-faces, and all non-facial 8^--cycles K in G bound a K-critical graph. Then G is C-critical.*

Proof. Let $e \in E(G) \setminus E(C)$. We want to find a 3-coloring ψ of C that does not extend to a proper 3-coloring of G but does extends to a proper 3-coloring of $G - e$. Note that if $e \notin E(f_5)$, then $G - e$ has a 5-face and a 6-face, and if $e \in E(f_5)$, then $G - e$ has a 7-face.

If every coloring of C extends to $G - e$, then we can let ψ be a coloring with 9 source edges since ψ does not extend to G as there is no ψ-balanced layout for G. If not all colorings of C extend to $G - e$, then there is a C-critical subgraph H of $G - e$ where the same set of precolorings of C extends to $G - e$ as well as to H. The property that every 8^--cycle K either bounds a face or a K-critical subgraph gives that H contains either a 5-face and a 6-face or a 7-face.

Case 1: H contains a 5-face and a 6-face.
Let ψ be a 3-coloring of C containing 9 source edges; in other words, the colors of the vertices around C are $1, 2, 3, 1, 2, 3, 1, 2, 3$. Then ψ extends to a 3-coloring of H by Claim 13. However, ψ does not extend to a 3-coloring of G since it is not possible to create a ψ-balanced layout for G.

Case 2: H contains a 7-face f_7.
By Theorem 11, if ψ is a 3-coloring of C containing 9 source edges, then ψ does not extend to a proper 3-coloring of H, and if ψ is a 3-coloring of C containing 6 source edges and 3 sink edges, then ψ extends to a proper 3-coloring of H if $E(C) \setminus E(f_7)$ contains both a sink edge and a source edge with respect to ψ. Now it remains to observe that there exists a coloring ψ of C such that $E(f_5) \cap E(C)$ contains two sink edges and the third sink edge is in $E(C) \setminus E(f_7)$. The other edges of C are source edges. Such a coloring does not extend to G but it does extend to $G - e$.

□

Lemma 22. *Let G be a connected triangle-free plane graph with outer face bounded by a chordless 9-cycle. Moreover, let G contain one 5-face f_5 and all other internal faces of G are 4-faces. If G is C-critical, then G is described by Theorem 6(a) and is depicted in Figure 4(a).*

Proof. Let G be C-critical. By Lemma 8, every 8^--cycle K bounds a face or a K-critical subgraph in G. Let $e \in E(G) \setminus E(C)$ such that $G - e$ contains a 7-face f_7. Let ψ be a 3-coloring of C that extends to $G - e$ but does not extend to G.

By Theorem 11, if ψ is a 3-coloring of C containing 9 source edges, then ψ does not extend to a proper 3-coloring of $G - e$. Hence ψ is a 3-coloring of C containing 6 source edges and 3 sink edges. Let q be a ψ-balanced layout of G. The only possibility is $q(f_5) = -3$.

Since ψ does not extend to G and q is ψ-balanced layout of G, Lemma 7 can be applied. Notice that Lemma 7 cannot give that K_0 is a cycle since $|m| \leq 3$ and there is no cycle of length at most 2. Hence K_0 is a path, and let $k_0 = |K_0|$.

Let the endpoints of K_0 partition C into two paths X and Y that are internally disjoint and have the same endpoints as K_0. Since ψ has six source edges, we obtain $n^s_X + n^s_Y = 6$ and $n^t_X + n^t_Y = 3$. By symmetry assume that f_5 is in the region bounded by Y and K_0. Lemma 7 implies $|n^s_Y - 3 - n^t_Y| > k_0$. Since Y contains f_5, $k_0 + n^s_Y + n^t_Y \geq \ell(\{5\}) = 5$ and odd. Because C has no chords, $k_0 \geq 2$. We solve this system of constraints by a computer program. The solutions are in Table 1. Sketches of the solutions are in Figure 11.

#	n^s_X	n^t_X	n^s_Y	n^t_Y	k_0
(a)	6	1	0	2	3
(b)	6	0	0	3	2
(c)	5	1	1	2	2
(d)	6	0	0	3	4
(e)	5	0	1	3	3
(f)	4	0	2	3	2

Table 1: Solutions in Lemma 22

From the first three solutions we obtain that Y is part of a 5-face f_5 sharing at least two sink edges with C. This is the desired conclusion.

The other three solutions give that Y, K_0 form a 7-cycle sharing at least three sink edges with C. We need to rule out this case. Since the cycle formed by Y, K_0 does not bound a face in G, it must contain an edge e' in its interior. Since G is C-critical, there exists a proper 3-coloring ϱ of C that does not extend to G but does extend to $G - e'$. Notice that the solutions (d), (e), and (f) also describe all patterns of a 3-coloring of C that do not extend to G, in particular for ϱ. In all three cases, X contains only source edges and $|X| > |K_0|$. Since the cycle formed by X, K_0 contains only 4-faces in its interior, it is not

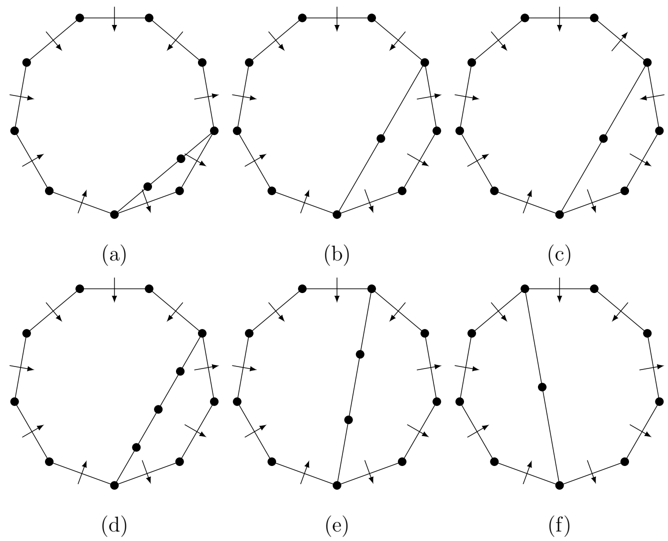


Figure 11: Solutions in Lemma 22.

possible to create a ρ-balanced layout in its interior. Hence ϱ does not extend to subgraph of $G - e'$ bounded X, K_0 . Therefore ϱ does not extend G. This contradicts the C-criticality of G. Hence the cases (d), (e), and (f) do not correspond to C-critical graphs.

This finishes the proof of Lemma 22. □

3.4 Three 5-faces

Lemma 23. *Let G be a connected triangle-free plane graph with outer face bounded by a chordless 9-cycle C. Moreover, let G contain three 5-faces, all other internal faces are 4-faces, and all non-facial 8^--cycles K in G bound a K-critical graph. If there is a proper 3-coloring ψ of C that does not extend to a 3-coloring of G, then G is C-critical.*

Proof. Let $e \in E(G) \setminus E(C)$. We want to show that ψ extends to a proper 3-coloring of $G - e$. Suppose that ψ does not extend to a 3-coloring of $G - e$. Then there exists a C-critical subgraph H of $G - e$, such that the 3-colorings of C that extend to $G - e$ are exactly the 3-colorings of C that extend to H. Since H is C-critical, its multiset of 5^+-faces is one of $\{5\}, \{7\}, \{5, 6\}, \{5, 5, 5\}$. Since all non-facial 8^--cycles K in G bound K-critical subgraphs, Lemma 8 implies that every 5-face of H is a 5-face of G, every 7-face of H contains exactly one 5-face of G, and a 6-face of H contains no 5-faces in the interior. Hence, H contains three odd faces, and the only option for the multiset of 5^+-faces of H is $\{5, 5, 5\}$. That would mean that G is the same graph as H, and this is a contradiction. □

Lemma 24. *Let G be a connected triangle-free plane graph with outer face bounded by a chordless 9-cycle C. Moreover, let G contain three 5-faces and let all other internal faces of G be 4-faces. If G is C-critical, then G is described by Theorem 6(e) and is depicted in Figure 4(Bij) for some i and j.*

Proof. Let G be a C-critical graph containing three 5-faces. Hence there is a proper 3-coloring ψ of C that does not extend to a proper 3-coloring of G. Without loss of generality, assume C has more source edges than sink edges in the coloring ψ. Either C contains 9 source edges and no sink edges or C contains 6 source edges and 3 sink edges.

Given $i \in \{0, 1, 2, 3\}$, let $\ell_5(i) = \ell(S)$ where S is a multiset of cardinality i containing only elements 5. Observe that $\ell_5(0) = 4, \ell_5(1) = 5, \ell_5(2) = 8$, and $\ell_5(3) = 9$.

Claim 25. *There are 6 source edges in C.*

Proof. Suppose for a contradiction that there are 9 source edges. Hence there is just one ψ-balanced layout q assigning -3 to every 5-face. Let K_0 and m be obtained from Lemma 7, which says $|m| > |K_0|$. Let $k = |K_0|$.

Suppose K_0 is a cycle. When i of the 5-faces are in the interior of K_0, then $3i = |m| > k \geq \ell_5(i)$, which is a contradiction for all $i \in \{0, 1, 2, 3\}$.

Therefore, K_0 is a path. Let C be partitioned into paths X and Y that both have the same endpoints as K_0. Note that $n^s_X + n^s_Y = 9$ and $n^t_X + n^t_Y = 0$, which implies $n^t_X = n^t_Y = 0$. Since C is chordless, $k \geq 2$. By symmetry assume that X, K_0 form a cycle that has $i \in \{0, 1\}$ of the three 5-faces in its interior. Lemma 7 implies that $|n^s_X - 3i| > k$ and $n^s_Y + k \geq \ell_5(3-i)$. This set of equations gives a contradiction for all $i \in \{0, 1\}$. □

Hence C contains 6 source edges and 3 sink edges. Let q be a ψ-balanced layout, and we know that the three 5-faces of G are assigned q-values $3, -3, -3$. Notice there are three different ψ-balanced layouts. Let K_0 be obtained from Lemma 7.

Claim 26. *K_0 is a path with both endpoints in C.*

Proof. Suppose for a contradiction that K_0 is a cycle. Denote by m the sum of the q-values of the faces in the interior of K_0. Lemma 7 implies that $|m| > |K_0|$. When i of the 5-faces are in the interior of K_0, then $3i \geq |m| > |K_0| \geq \ell_5(i)$, which is a contradiction for all $i \in \{0, 1, 2, 3\}$. □

Claim 26 says that K_0 is a path. Let C be partitioned into paths X and Y that both have the same endpoints as K_0. Denote by R_X and R_Y the induced subgraph of G whose outer face is bounded by K_0, X and K_0, Y, respectively.

Claim 27. *Each R_X and R_Y contains at least one 5-face.*

Proof. Note that $n^s_X + n^s_Y = 6$ and $n^t_X + n^t_Y = 3$. Without loss of generality, R_X contains three 5-faces. Hence $n^s_X + n^t_X + k \geq 9$, and Lemma 7 gives $|n^s_X - 3 - n^t_X| > k$. This set of constraints has no solution which is a contradiction. □

By Claim 27 and by symmetry, we may assume that R_X contains exactly one 5-face f; we call f *lonely* with respect to K_0. If $q(f) = -3$, then we call this configuration *type A* and if $q(f) = 3$, then we call it *type B*; see Figure 12.

Denote the three different ψ-balanced layouts by q_1, q_2, and q_3. For $i \in \{1, 2, 3\}$, let K_i be K_0 obtained from Lemma 7 when applied to q_i. By Claim 27, we can define f_i to be the lonely face for K_i. Notice that f_1, f_2, and f_3 are not necessarily pairwise distinct faces.

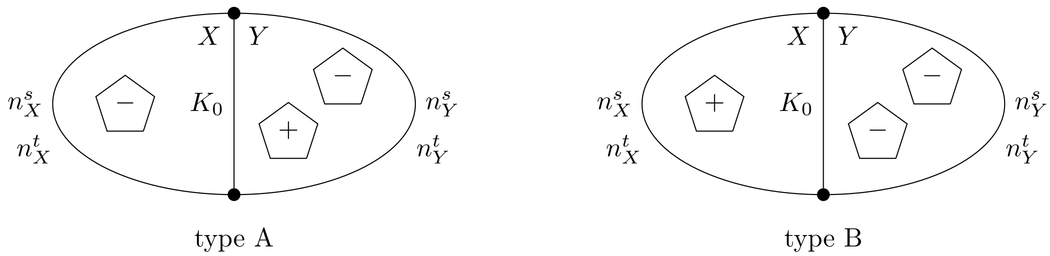


Figure 12: When C has 6 source edges and has three 5-faces. The possible configurations are of type A (left) and type B (right).

Label the endpoints of K_i by u_i and v_i such that K_i is a (u_i, v_i, f_i)-cut. Define k, l, m to be the length of K_1, K_2, K_3, respectively.

First we show that configurations of type A do not exist.

Claim 28. *Let q_1 be a configuration of type A and let q_2 be a layout where $q_2(f_1) = 3$. Then q_2 is not a configuration of type A.*

Proof. Suppose for a contradiction that both q_1 and q_2 give a configuration of type A, so $q_2(f_1) = 3$ and $q_2(f_2) = -3$. Since $q_2(f_1) = 3$, and $q_2(f_2) = -3$, we have that f_1 and f_2 are distinct. Let f_0 be the third 5-face.

By symmetry, paths K_1 and K_2 give one of four possible kinds (11), (00), (22), and (20). The kind (02) is symmetric with (20). For an illustration, see Figure 13.

Suppose K_1 and K_2 are of kind (11). The situation is depicted in Figure 13 (AA11). Let X, A, Y, Z be $C(u_2, u_1; v_2, v_1), C(u_1, v_2; v_1, u_2), C(v_2, v_1; u_2, u_1), C(v_1, u_2; u_1, v_2)$ respectively. We obtain the following constrains that must be satisfied by using Lemma 7 and Lemma 8.

$$|n^s_X + n^s_Z - n^t_X - n^t_Z| > k_1 + k_2 \tag{8}$$
$$|n^s_Y + n^s_Z - n^t_X - n^t_Y| > l_1 + l_2 \tag{9}$$
$$n^s_X + n^t_X + n^s_A + n^t_A + l \geq 7 \text{ and odd} \tag{10}$$
$$n^s_Y + n^t_Y + n^s_A + n^t_A + k \geq 7 \text{ and odd} \tag{11}$$
$$n^s_X + n^t_X + n^s_Y + n^t_Y + k + l \geq 10 \tag{12}$$

Inequalities (8) and (9) follow from Lemma 7. Inequalities (10), (11), and (12) follow from Lemma 8 and the structure of the (11) kind.

Suppose K_1 and K_2 are of kind (00). The situation is depicted in Figure 13 (AA00). Let X and Y be $C(u_2, v_2; u_1, v_1)$ and $C(u_1, v_1; u_2, v_2)$ respectively. Let Z be edges of C that are in neither X nor Y.

As in the previous case we obtain the following set of constraints that must be satisfied.

$$|n^s_X + n^s_Z - n^t_X - n^t_Z| > k \tag{13}$$
$$|n^s_Y + n^s_Z - n^t_X - n^t_Y| > l \tag{14}$$
$$n^s_Y + n^t_Y + k \geq 5 \text{ and odd} \tag{15}$$
$$n^s_X + n^t_X + l \geq 5 \text{ and odd} \tag{16}$$

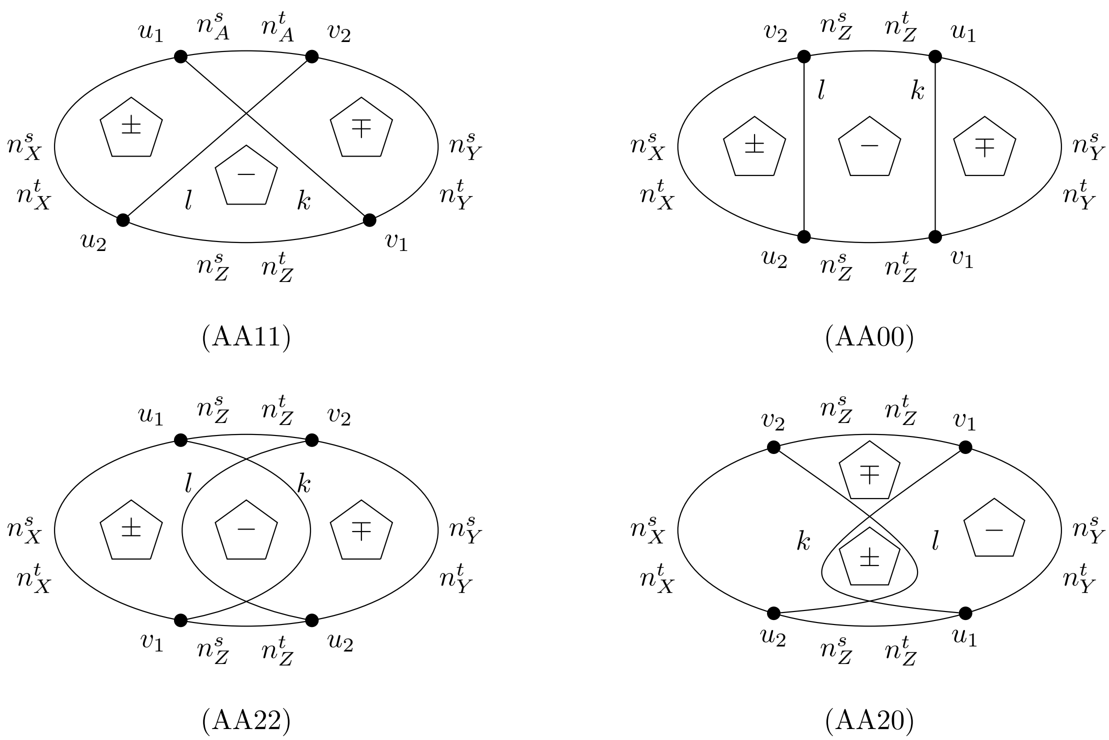


Figure 13: Four different cases of two types A. Face f_1, f_2, and f_3 has symbol $\mp$, $\pm$, and $-$, respectively.

Inequalities (13) and (14) are obtained from Lemma 7. The other inequalities come from Lemma 8. Recall that we assumed that C has no chords, so we also include that $\min\{k,l\} \geq 2$. The above set of constraints has no solution. Hence K_1 and K_2 cannot be of kind (00).

The next case (22) is depicted in Figure 13 (AA22). Let X and Y be $C(v_1,u_1;v_1,u_2)$ and $C(v_1,u_2;v_1,u_1)$, respectively. Let Z be edges of C that are in neither X nor Y.

Using Lemmas 7 and 8 we obtain the following set of constraints that must be satisfied:

$$\begin{aligned} |n_X^s + n_Z^s - n_X^t - n_Z^t| &> k \\ |n_Y^s + n_Z^s - n_X^t - n_Y^t| &> l \\ n_Y^s + n_Y^t + l &\geq 8 \text{ and even} \\ n_X^s + n_X^t + k &\geq 8 \text{ and even} \end{aligned}$$

This system has no solution. This finishes the case (22) of Claim 28.

The last case (20) is depicted in Figure 13 (AA20). Let X and Y be $C(u_2,v_2;v_1,u_1)$ and $C(v_1,u_1;u_2,v_2)$, respectively. Let Z be edges of C that are in neither X nor Y. Using

Lemmas 7 and 8 we obtain the following set of constraints that must be satisfied:

$$\begin{aligned}
|n^s_Y - n^t_Y| &> k \\
|n^s_X - 3 - n^t_X| &> l \\
n^s_Y + n^t_Y + k &\geq 8 \text{ and even} \\
n^s_X + n^t_X + l &\geq 5 \text{ and odd} \\
k + l + n^s_Z + n^t_Z &\geq 10
\end{aligned}$$

The last equation was obtained from the fact that in the kind (20), the subgraph of G bounded by K_1, K_2, and Z contains at least two 5^+-faces. This system has no solutions. This finishes the proof of Claim 28. □

Claim 29. *Let q_1 be a configuration of type A and let q_2 be a layout where $q_2(f_1) = 3$. Then q_2 is not a configuration of type B.*

Proof. Suppose for a contradiction that q_1 gives a configuration of type A and q_2 gives a configuration of type B, where $q_2(f_1) = 3$, hence $f_1 = f_2$. We have four kinds depending on the order of the endpoints of K_1 and K_2. The cases are depicted in Figure 14. The kind (22) is not possible if $f_1 = f_2$.

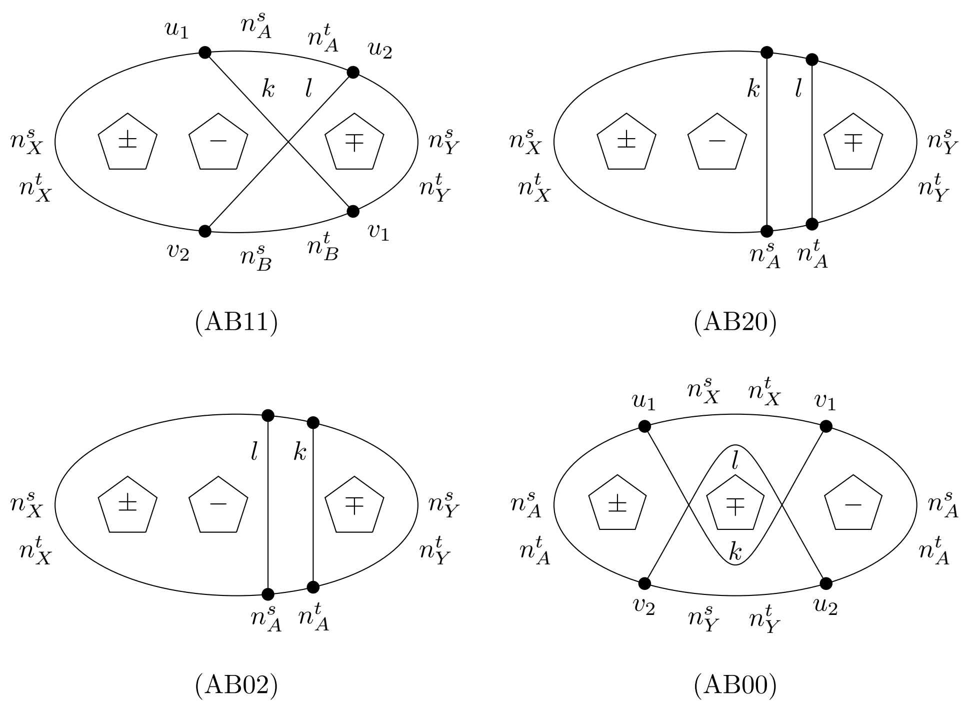


Figure 14: Sketches of kinds (11), (20), (02), and (00) for one configuration of type A and one configuration of type B.

Depending on the case, from by Lemma 7 and Lemma 8. we obtain a set of constraints that must be satisfied.

(AB11):

$$
\begin{aligned}
|n_X^s + n_B^s - n_X^t - n_B^t| &> k \\
|n_X^s + n_A^s - 6 - n_X^t - n_A^t| &> l \\
n_Y^s + n_Y^t + n_A^s + n_A^t + k &\geq 7 \text{ and odd} \\
n_X^s + n_X^t + n_Y^s + n_Y^t + k + l &\geq 13 \text{ and odd}
\end{aligned}
$$

(AB20):

$$
\begin{aligned}
|n_X^s - n_X^t| &> k \\
|n_Y^s + 3 - n_Y^t| &> l \\
n_Y^s + n_Y^t + l &\geq 5 \text{ and odd} \\
n_X^s + n_X^t + k &\geq 8 \text{ and even}
\end{aligned}
$$

(AB02):

$$
\begin{aligned}
|n_Y^s - 3 - n_Y^t| &> k \\
|n_X^s - 6 - n_X^t| &> l \\
n_Y^s + n_Y^t + k \text{ is } &\geq 5 \text{ and odd} \\
n_X^s + n_X^t + l \text{ is } &\geq 8 \text{ and even}
\end{aligned}
$$

(AB00):

$$
\begin{aligned}
|n_X^s - 3 - n_X^t| &> k \\
|n_Y^s + 3 - n_Y^t| &> l \\
n_X^s + n_X^t + k &\geq 5 \text{ and odd} \\
n_Y^s + n_Y^t + l &\geq 5 \text{ and odd} \\
n_A^s + n_A^t + k + l &\geq 13 \qquad (17)
\end{aligned}
$$

Inequality (17) comes from the fact that the subgraph bounded by K_1, K_2 and A must contain all three 5-faces of G in its interior faces. In addition, we include that $\min\{k_1, k_2, l_1, l_2\} \geq 1$ since v is not a vertex of C and $\min\{k, l\} \geq 2$ since C has no chords.

None of the four sets of constraints has any solution, which is a contradiction. □

By Claim 28 and Claim 29, every layout gives a configuration of type B. Thus, we know that for each $i \in \{1, 2, 3\}$, $q_i(f_i) = 3$, and f_1, f_2, f_3 are pairwise distinct. Let P_i be the subpath of C such that K_i and P_i bound a cycle that contains f_i.

Let $i, j \in \{1, 2, 3\}$ and $i \neq j$. Based on the order of u_i, v_i, u_j, v_j on C, and K_i and K_j we get four possible kinds (BB00), (BB11), (BB22), (BB20); see Figure 15. Note that (BB02) is symmetric to what would be (BB20).

Claim 30. *For all $i, j \in \{1, 2, 3\}$ and $i \neq j$ we get that K_i and K_j do not form (BB22).*

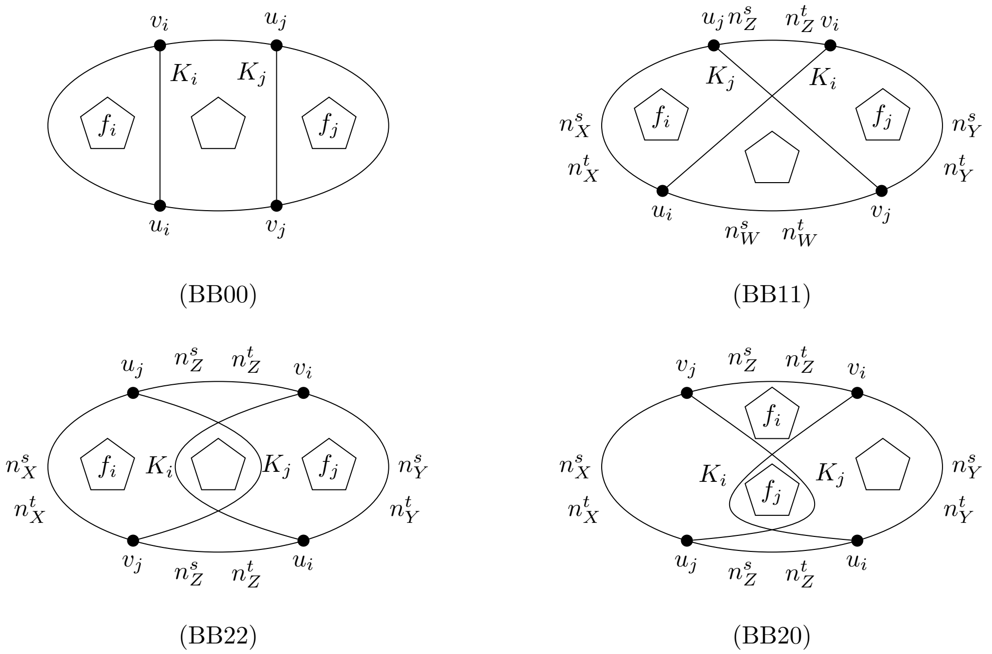


Figure 15: Possible configurations of two cuts of type B.

Proof. Suppose for a contradiction that K_i and K_j do form (BB22). See Figure 15 (BB22) for a sketch of the situation. Let X and Y be $C(v_j, u_j; v_i, u_i)$ and $C(v_i, u_i; v_j, u_j)$, respectively. Let Z be the edges of C that are in neither X nor Y.

Let $t = 6 - i - j$. Since K_i and K_j form (BB22), the subgraph of G bounded by $K_i \cup Y$ contains faces that contain 5-faces f_j and f_t, and the subgraph of G bounded by $K_j \cup X$ contains faces that contain 5-faces f_i and f_t. This, Lemma 7, and Lemma 8 give the following set of constraints.

$$
\begin{aligned}
|n_Y^s - 6 - n_Y^t| &> k_i \\
|n_X^s - 6 - n_X^t| &> k_j \\
k_i + n_Y^s + n_Y^t &\geq 8 \text{ and even} \\
k_j + n_X^s + n_X^t &\geq 8 \text{ and even}
\end{aligned}
$$

This set of constraints has no solution. □

Claim 31. *For all $i, j \in \{1, 2, 3\}$ and $i \neq j$ we get that K_i and K_j do not form (BB20) or there exist alternative paths that form (BB11) and no new (BB20) is created.*

Proof. Suppose for a contradiction that K_i and K_j form (BB20). See Figure 15 (BB20) for a sketch of the situation. Let X and Y be $C(u_j, v_j; v_i, u_i)$ and $C(v_i, u_i; u_j, v_j)$, respectively. Let Z be the edges of C that are in neither X nor Y.

First we will obtain a few potential solutions. The first four inequalities follow from Lemmas 7 and 8. The inequality (18) comes from the description of (BB20) where the subgraph of G bounded by K_i, K_j, and Z contains at least three interior faces where at least two are 5-faces. The inequality (19) comes from (BB20) saying that X and K_j do not form the boundary of f_j.

$$|n_Y^s + 3 - n_Y^t| > k_i$$
$$|n_X^s + 3 - n_X^t| > k_j$$
$$k_j + n_X^s + n_X^t \geq 5 \text{ and odd}$$
$$k_i + n_Y^s + n_Y^t \geq 8 \text{ and even}$$
$$k_i + k_j + n_Z^s + n_Z^t \geq 14 \tag{18}$$
$$k_j + n_X^s + n_X^t \geq 7 \tag{19}$$

This set of constraints has the following four solutions.

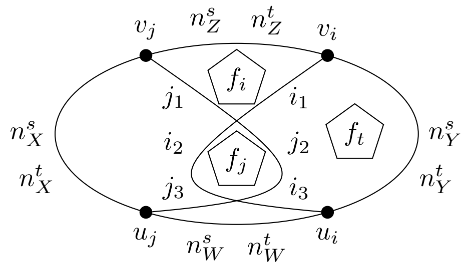
n_X^s	n_X^t	n_Y^s	n_Y^t	n_Z^s	n_Z^t	k_i	k_j
3	0	0	3	3	0	7	4
4	0	0	3	2	0	7	5
5	0	0	3	1	0	7	6
6	0	0	3	0	0	7	7

Notice that in all the solutions $k_i + k_j + n_Z^s + n_Z^t = 14$. Hence the subgraph of G bounded by K_i, K_j, and Z has two 5-faces and one 4-face. We create a more detailed instance where we split Z into two paths $C(v_j, v_i; u_j, u_j)$ that we keep calling Z and $C(u_i, u_j; v_j, v_i)$ that we call W. Moreover, we partition K_i and K_j into three subpaths of lengths i_1, i_2, i_3 and j_1, j_2, j_3 respectively. See Figure 16. This leads to the following constraints, where the first six are the same as before.

$$|n_Y^s + 3 - n_Y^t| > k_i$$
$$|n_X^s + 3 - n_X^t| > k_j$$
$$k_j + n_X^s + n_X^t \geq 5 \text{ and odd}$$
$$k_i + n_Y^s + n_Y^t \geq 8 \text{ and even}$$
$$k_i + k_j + n_Z^s + n_Z^t \geq 14$$
$$k_j + n_X^s + n_X^t \geq 7$$
$$i_1 + j_2 + i_3 + n_Y^s + n_Y^t \geq 5 \text{ and odd} \tag{20}$$
$$i_2 + j_2 \geq 5 \text{ and odd} \tag{21}$$
$$i_1 + j_1 + n_Z^s + n_Z^t \geq 5 \text{ and odd} \tag{22}$$

The system has 14 solutions. Create a path K_j' from K_j by dropping the piece corresponding to j_3 and replacing it by i_3 and potentially deleting repeated edges. The path K_j' is a path with endpoints in C and it makes f_j lonely. Moreover, all 14 solutions satisfy $|n_X^s + n_W^s +$

$3 - n_X^t - n_W^t| > j_1 + j_2 + i_3$. Hence K_j' can be used instead of K_j, and K_j' and K_i form configuration (BB11).



(BB20)

Figure 16: More details for configuration (BB20).

Finally, we need to show that no new (BB20) or (BB02) is created by replacing K_j by K_j'. All 14 solutions satisfy $i_3 + j_3 + n_W^s + n_W^t = 4$, $i_2 + j_2 = 5$, $i_1 + j_1 + n_Z^s + n_Z^t = 5$, $n_Y^s + n_Y^t = 3$, and $n_Z^s + n_Z^t \le 2$. Hence the subgraph of G induced by W, Z, K_i, and K_j contains a 4-face and two 5-faces as internal faces and one of the 5-faces is sharing with C vertices v_j and v_i.

Let K_a and K_b form (BB20) or (BB20) for some $a, b \in \{1, 2, 3\}$. The previous paragraph implies that for any $c \in \{a, b\}$ one of the edges of K_c incident to v_c and u_c is incident to a 4-face and the other edge is incident to a 5-face f_a or f_b. Moreover, one of f_a and f_b is disjoint from C and the other one is sharing at most two edges with C.

Let $t = 6 - i - j$ and K_t be in $\{K_1, K_2, K_3\}$ with endpoints u_t and v_t. Suppose for contradiction that a new (BB20) or (BB02) is created by replacing K_j by K_j'. Since K_i is not changed, the new (BB20) or (BB02) is formed by K_j' and K_t. Hence K_j and K_t is neither (BB20) nor (BB02). The new (BB20) or (BB02) must satisfy the constraints from the previous paragraph. Since the edge of K_j' incident to v_j is incident to a 4-face and f_i, the edge e of K_j' incident to u_i must be incident to f_t. Notice that e is also incident to a 4-face h that is incident to W. Hence f_t must be on the opposite side of e than h. Let $x \in \{u_t, v_t\}$ be incident to an edge of K_t that is incident to f_t. Since $n_Y^s + n_Y^t = 3$ and f_t is sharing at most two edges with C, we obtain that $x \in Y$ and the order around C is $u_i v_j x$. Since K_j' and K_t form (BB20) or (BB02) and we know the order for x, the order of the endpoints of K_j' and K_t is $u_i v_j v_t u_t$. Hence K_j' and K_t form (BB02). Observe that the order of endpoints of K_j and K_t is $u_j v_j v_t u_t$. Hence K_j and K_t form (BB02), a contradiction.

□

Claim 32. *For all $i, j \in \{1, 2, 3\}$ and $i \neq j$ if K_i and K_j form (BB11) then they have a common point.*

Proof. In order to apply Lemma 9 we need to verify that $\max\{|K_i|, |K_j|\} \le 7$ and if $|K_i| = |K_j| = 7$ then K_i and K_j have common endpoints. Let K_i and K_j form (BB11), see Figure 15 (BB11) for illustration. Let X, Z, Y, and W be $C(u_i, u_j; v_i, v_j)$, $C(u_j, v_i; v_j, u_i)$,

$C(v_i, v_j; u_i, u_j)$, and $C(v_j, u_i; u_j, v_i)$, respectively. Denote $|K_i|$ and $|K_j|$ by k_i and k_j, respectively. Lemma 7 and Lemma 8 imply that the following constraints are satisfied.

$$|n_X^s + n_Z^s + 3 - n_X^t - n_Z^t| > k_i$$
$$|n_Y^s + n_Z^s + 3 - n_Y^t - n_Z^t| > k_j$$
$$n_X^s + n_X^t + n_Z^s + n_Z^t + k_i \geq 7 \text{ and odd}$$
$$n_Y^s + n_Y^t + n_Z^s + n_Z^t + k_j \geq 7 \text{ and odd}$$

All solutions to these constraints satisfy that $\max\{k_i, k_j\} \leq 7$. Moreover, if $k_i = k_j = 7$ then $n_X^s + n_X^t + n_Y^s + n_Y^t = 0$. Hence Lemma 9 applies and there is a common point. □

Now we know that we have only configurations (BB00) and (BB11) with common points.

Denote the length of the path K_1, K_2, and K_3 by k, l, and m, respectively. We will use k_1, k_2, k_3 to denote the lengths of subpaths of k if some of the paths form (BB11); l_1, l_2, l_3, m_1, m_2, m_3 will be used similarly. See Figure 17.

If there is a pair of layouts giving configuration (BB00), we distinguish the following cases:

(B1) all pairs form (BB00).

(B2) one pair forms (BB11).

(B3) two pairs form (BB11)

If all three pairs of layouts give (BB11), then we define v_K and v_L to be the common point of K_3 with K_1 and K_2, respectively. The vertex v_K is *before* v_L if v_K appears before v_L when traversing the cycle formed by K_3 and P_3 in the clockwise order and the starting point is on C.

(B4) P_1, P_2, and P_3 have a common edge and v_K is before v_L or $v_K = v_L$.

(B5) There is no common edge of P_1, P_2, and P_3 and v_K is before v_L or $v_K = v_L$.

(B6) There is no common edge of P_1, P_2, and P_3, and v_L is before v_K and $v_K \neq v_L$

(B7) P_1, P_2, and P_3 have a common edge and v_L is before v_K and $v_K \neq v_L$.

See Figure 17 for an illustration of the cases (B1)–(B7). Since one layout may contain several different configurations of type B, pick K_1, K_2, K_3 such that the number of (B11) pairs is minimized.

Next we give constraints for each of the cases (B1)–(B7). Solutions to these constraints were obtained by simple computer programs. Critical graphs obtained from (Bi) are depicted in Figure 4 as (Bij) for all i, j.

Endpoints of K_1, K_2, and K_3 partition C into several internally disjoint paths. The paths have names in $\{X, Y, Z, W, A, D, E, F\}$ with exception of W in (B1) and (B2), where W refers to a union of up to three and two paths, respectively. To simplify the write-up we refer the reader to Figure 17 for the labelings of the paths.

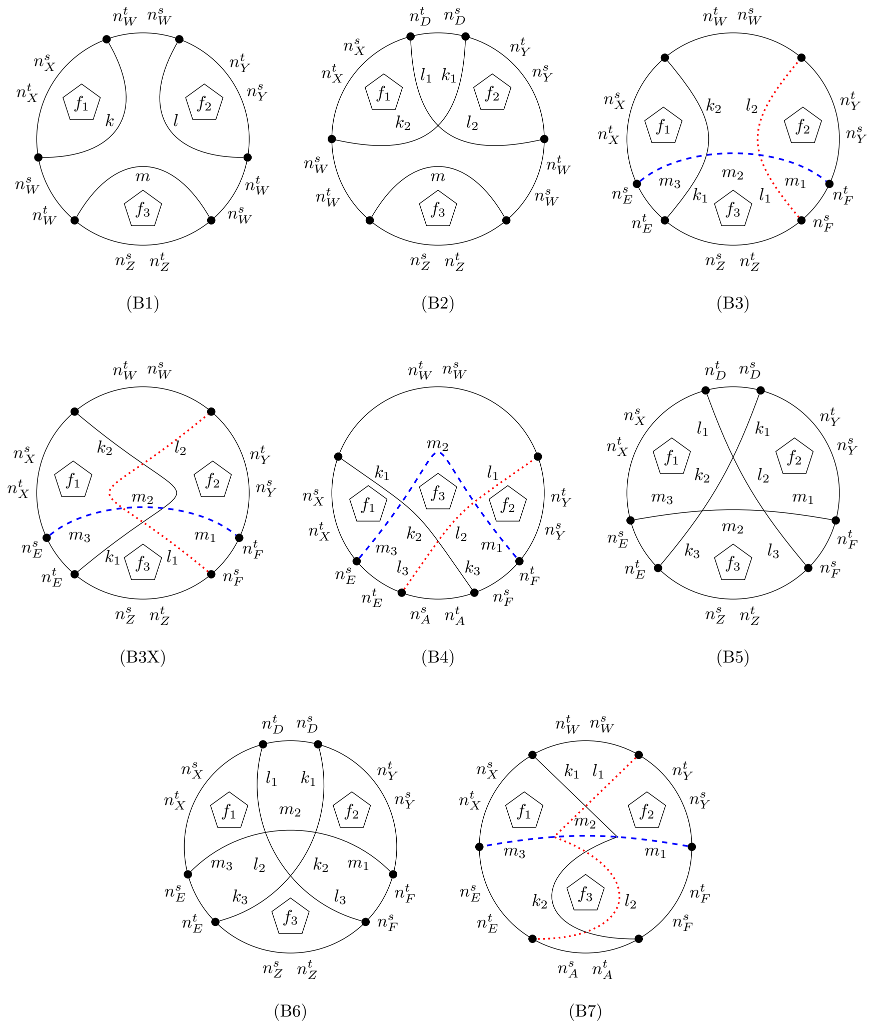


Figure 17: Possible configurations of cuts K_1, K_2, K_3.

Claim 33. *The configuration (B1) results in a critical graph where every 5-face shares at least two edges with the boundary. Moreover, in every non-extendable* 3*-coloring of the outer face, every 5-face contains two source edges.*

Proof. We refer the reader to Figure 17 (B1) for the labelings of the paths. By Lemma 7 we get the first three equations and by Lemma 8 we get the remaining equations.

$$\begin{aligned}
|n_X^s + 3 - n_X^t| &> k, \\
|n_Y^s + 3 - n_Y^t| &> l, \\
|n_Z^s + 3 - n_Z^t| &> m, \\
k + n_X^s + n_X^t &\geq 5 \text{ and odd}, \\
l + n_Y^s + n_Y^t &\geq 5 \text{ and odd}, \\
m + n_Z^s + n_Z^t &\geq 5 \text{ and odd},
\end{aligned}$$

In addition, we also include some constraints to break symmetry; for example $n_X^t + n_X^s \geq n_Y^t + n_Y^s \geq n_Z^t + n_Z^s$. All solutions to this system of equations are in Table 2. By inspecting

n_X^s	n_X^t	n_Y^s	n_Y^t	n_Z^s	n_Z^t	n_w^s	n_W^t	k	l	m
2	0	2	0	2	0	0	3	3	3	3
2	1	2	0	2	0	0	2	2	3	3
2	1	2	1	2	0	0	1	2	2	3
2	1	2	1	2	1	0	0	2	2	2

Table 2: Solutions from Claim 33.

the solutions from Table 2, we conclude that they satisfy the statement of the claim. □

For the remaining cases, we give the sets of constraints but we skip detailed justification since they all come from the description of the configurations, Lemma 7, Lemma 8, and the fact that C has no chords. We provide computer programs online for solving the sets of equations and to help with checking the solutions.

The description of the configuration is the following, see Figure 17. If there is exactly one (B11) pair, then we get configuration (B2), where we assume it is pair q_1 and q_2.

If there are two (B11) pairs, then assume that q_3 is in both pairs. There are two common points on K_3, where one is shared with K_1 and the other is shared with K_2. Depending on the order of these points we get either (B3) or (B3X). The last option is that all three pairs are (B11). By considering the order of the endpoints of K_1, K_2, K_3 and the order of the common points on K_3, we get (B4)–(B7).

Claim 34. *Configurations (B2)–(B7) result in critical graphs (B21)–(B52). Every graph in Figure 4 represents several graphs that can be obtained from the depicted graph by identifying edges and vertices and by filling every face of even size by a quadrangulation with no separating 4-cycles. Moreover, the 5-faces in (B21) and (B22) that share two edges with C can be moved along C as long as they stay neighboring with a region with three sink edges.*

Proof Outline: We slightly abuse notation and use k_i, l_i, m_i for subpaths of K_1, K_2, K_3 respectively as well as for lengths of these subpaths, where $i \in \{1, 2, 3\}$. For a path in $\{X, Y, Z, W, A, D, E, F\}$, we use its lower case letter to denote its length.

For each case we include constraints that all three layouts give configurations of type B using Lemma 8 and Lemma 7 analogously to Claim 33. In addition, we add the following set of constraints depending on the case:

(B2):

$$x + k_2 + l_1 \geq 5 \text{ and odd} \qquad y + k_1 + l_2 \geq 5 \text{ and odd}$$
$$z + w + k_2 + l_2 \geq 7 \text{ and odd if } w > 0 \qquad x + d + y + k_2 + l_2 \geq 8 \text{ and even}$$
$$x + y + z + w + l_1 + k_1 \geq 9 \text{ and odd}$$

(B3):

$$e + x + k_1 + k_2 \geq 7 \qquad f + y + l_1 + l_2 \geq 7$$
$$y + m_1 + l_2 \geq 5 \text{ and odd} \qquad z + k_1 + m_2 + l_1 \geq 5 \text{ and odd}$$
$$x + m_3 + k_2 \geq 5 \text{ and odd} \qquad y + z + f + k_1 + m_2 + l_2 \geq 8 \text{ and even}$$

(B3X):

$$\min\{m_1, m_2, m_3, k_1, k_2, l_1, l_2\} \geq 1 \qquad k_1 + l_1 + z \geq 6$$
$$\text{if } l_2 = 1 \text{ then } x + m_3 + k_2 \geq 6 \qquad y + m_1 + m_2 + l_2 \geq 5 \text{ and odd}$$
$$\text{if } k_2 = 1 \text{ then } y + m_1 + l_2 \geq 6 \qquad x + k_2 + m_2 + m_3 \geq 5 \text{ and odd}$$

(B4):

$$x + e + l_3 + k_1 + k_2 \geq 5 \text{ and odd} \qquad y + f + k_3 + k_2 + m_2 + l_1 \geq 8 \text{ and even}$$
$$k_2 + l_2 + m_2 \geq 5 \text{ and odd} \qquad f + y + w + x + m_3 + k_2 + k_3 \geq 9 \text{ and odd}$$
$$x + k_1 + m_3 \geq 5 \text{ and odd} \qquad e + x + k_1 + m_2 + l_2 + l_3 \geq 8 \text{ and even}$$
$$y + l_1 + m_1 \geq 5 \text{ and odd} \qquad y + w + x + e + l_3 + l_2 + m_1 \geq 9 \text{ and odd}$$
$$f + y + w + x + e + l_3 + k_3 \geq 9 \text{ and odd}$$

(B5):

$$\min\{k_2, l_2, m_2\} \geq 1 \text{ or } k_2 = l_2 = m_2 = 0$$

$$y + k_1 + l_2 + m_1 \geq 5 \text{ and odd} \qquad x + l_1 + k_2 + m_3 \geq 5 \text{ and odd}$$
$$z + k_3 + m_2 + l_3 \geq 5 \text{ and odd} \qquad y + f + l_3 + l_2 + k_1 \geq 7 \text{ and odd}$$
$$x + e + k_3 + k_2 + l_1 \geq 7 \text{ and odd} \qquad m_1 + m_2 + k_3 + f + z \geq 7 \text{ and odd}$$
$$m_3 + m_2 + l_3 + e + z \geq 7 \text{ and odd} \qquad l_1 + l_2 + m_1 + d + y \geq 7 \text{ and odd}$$

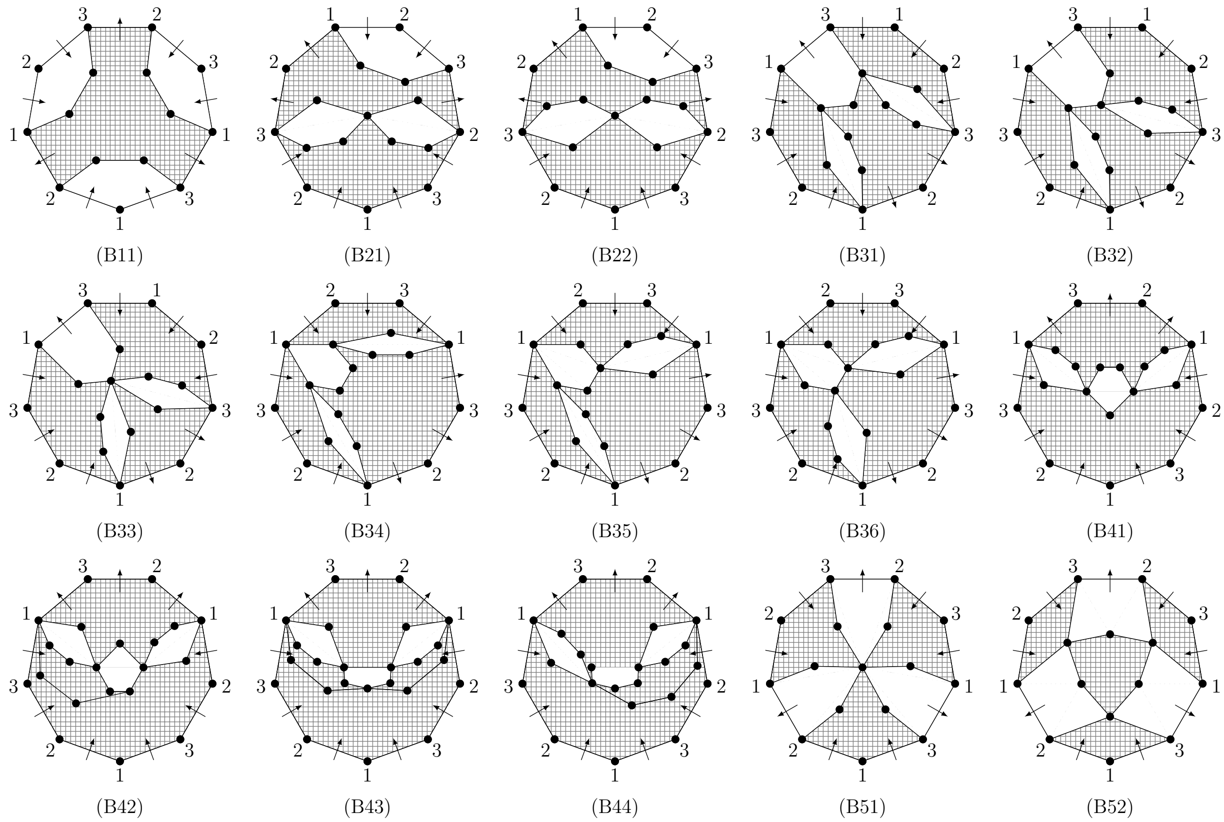


Figure 18: All solutions to cases (B1)–(B7).

(B6):

$$m_2 \geq 1 \qquad y + k_1 + m_1 \geq 5 \text{ and odd}$$
$$x + l_1 + m_3 \geq 5 \text{ and odd} \qquad z + k_3 + l_3 \geq 5 \text{ and odd}$$

(B7):

$$k_1 + l_1 + m_1 + m_3 + x + y \geq 10 \tag{23}$$
$$e + x + w + y + f + l_2 + k_2 - 5 \geq 9 \tag{24}$$

We enumerated all solutions to all seven sets of constraints, and we checked that the resulting graphs are depicted in Figure 4. In order to eliminate mistakes in computer programs, we have two implementations by different authors and we checked that they give identical results. Sources for programs for cases (B2)–(B7) together with their outputs can be found at `http://orion.math.iastate.edu/lidicky/pub/9cyc/`.

The most general solution for each of the sets of equations is depicted in Figure 18. Notice that (B3X), (B6), and (B7) have no solutions. In (B7), inequality (23) comes from a

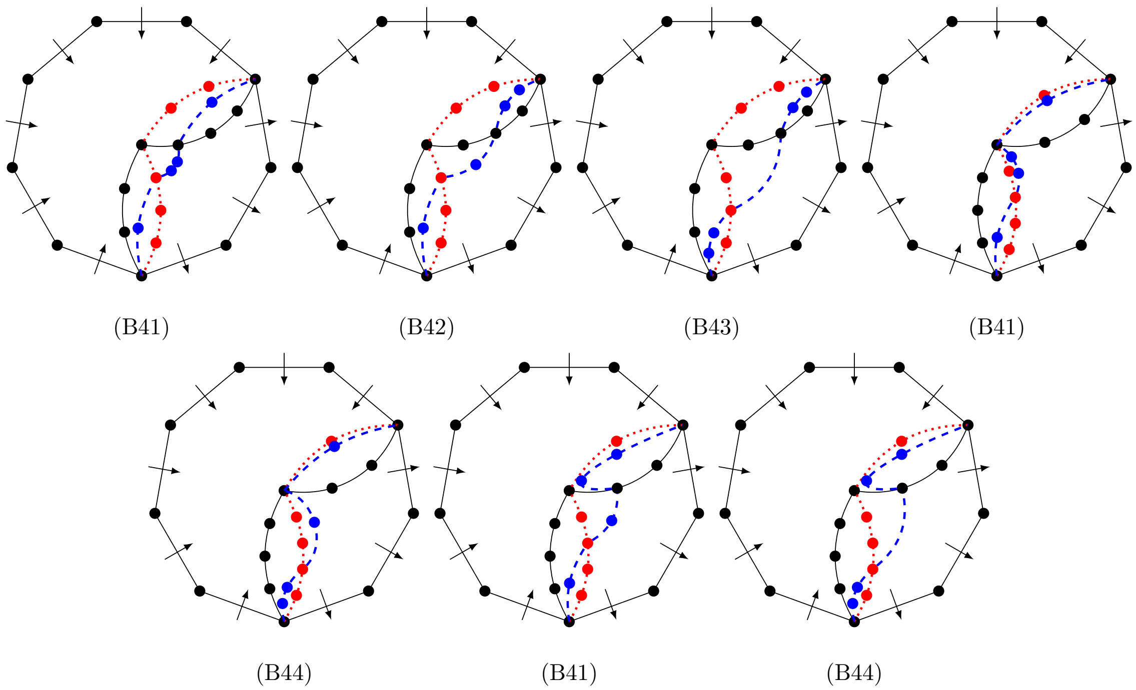


Figure 19: Sketches of solutions to case (B4) generated by our program. The style of paths K_1, K_2, and K_3 correspond to the style in Figure 17 (B4).

subgraph having two faces where each contains a 5-face in the interior and (24) comes from Lemma 8 and the -5 appears due to k_2 and l_2 enclosing f_3. Observe that (B34), (B35), and (B36) are special cases of (B41), (B42), and (B43), respectively. Hence we dropped (B34), (B35), and (B36) from Figure 4. One can think of (B41), (B42), and (B43) as being obtained from (B34), (B35), and (B36) by duplicating a subpath P of C where all dual edges of P are oriented inside. Notice that by using this operation, (B44) could be obtained from (B21), also (B22) from (B11) and (B41) from (B22). We suspect that it is part of a more general description of C-critical graphs, where C is larger.

We think the case (B4) is the most complicated case. We again used the trick to identify general solutions quickly by observing that regions bounding faces contain only the face and obtained seven solutions. We include sketches of the solutions generated by our program in Figure 19. Although there are seven solutions, they give only four distinct cases due to some vertex identifications.

□

This finishes the proof of Lemma 24. □

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