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The Dynamics of Ski Skating

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The Dynamics of Ski Skating

Abstract
We discuss the dynamics of ski skating, in particular, the optimization problem: Maximize average speed for a given power. We begin with a mathematical model of ski skating. To simplify the analysis, we limit our attention to ski skating with no poles on a level plane. We also limit our attention to the physics of ski skating; we ignore most biomechanical considerations. In the early days of skating, ski instructors often said: “Take a long glide on a flat ski.” Many skiers interpreted this to mean: “Take a passive glide, then give a short hard push to the side.” As an application of our theory, we show that this advice is wrong. We show that the skier should start pushing to the side as soon as possible.

Keywords
cross-country skiing, nordic skiing, freestyle, skating, dynamics, model, ordinary differential equation, control, optimization, bio mechanics, physics, friction

Disciplines
Applied Mathematics | Sports Studies

Comments

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The Dynamics of Ski Skating

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Abstract

We discuss the dynamics of ski skating, in particular, the optimization problem: Maximize average speed for a given power.

We begin with a mathematical model of ski skating. To simplify the analysis, we limit our attention to ski skating with no poles on a level plane. We also limit our attention to the physics of ski skating; we ignore most biomechanical considerations.

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Introduction

By limiting the scope of inquiry, researchers can attain greater certainty about questions they do answer. —William C. McGrew [1966, 76]

In the years following the 1980 Winter Olympics, cross-country skiing experienced a rapid revolution in skiing technique. The classic technique, wherein the skis stay in a track parallel to the direction of motion, was replaced by a skating technique, where the ski moves at an angle to the direction of motion. This change in technique brought about lower times in races (see Karvonen et al. [1989] and Bilodeau et al. [1992]), possibly lower metabolic costs [Smith 1992], and eventually a split of the sport into separate classic and freestyle races. Along with the split in races, a renewed effort to optimize ski performance has also occurred.

Traditional Approaches

Traditionally, biomechanists have used two methods to optimize performance:

- The first, and most common, is to assume that elite athletes are performing at or near optimal performance. This method has been used extensively to analyze cross-country skiing. Three variables of interest have been the stride frequency, stride length, and the angle between the ski and the direction of motion. Several authors [Bilodeau et al. 1992; Smith et al. 1989; Smith et al. 1989] have reported values for athletes in classic skiing as well as three different methods of freestyle, or skate skiing.

  - Stride frequencies ranged from 0.68 Hz to 0.81 Hz for the three skating techniques on flat surfaces as compared to 0.80 Hz for the classic technique. (Recall that “Hz” is the standard abbreviation for “hertz” or “cycles per second”.) Higher stride frequencies were found for skating up an incline (0.81 to 0.84 Hz).

  - Stride lengths were greater for the skating techniques (from 7.19 m to 8.69 m) as compared to the classic technique (5.95 m), with a decrease in length when skiing up an incline (2.99 m to 3.84 m when on a 7° and 11° slope, respectively).

  - Ski angle was likewise dependent on slope, with smaller ski angles found on a lower grade (24°) than on a steeper grade (28.9°).

Strong positive correlations were found by Bilodeau et al. [1992] between velocity and the distance traveled per cycle (which they call cycle length) for all strides, while the significant correlations between velocity and frequency were negative when significant. This analysis seems to suggest that skiers would gain an advantage by taking longer, slower strides. By contrast, Smith
et al. [1989] found significant positive correlations between frequency and velocity, although these values were obtained from skiing up a slope. Both correlations were found using elite athletes, which suggests a problem with this type of analysis: All elite athletes, especially when the techniques are relatively new, do not show identical movement patterns.

- The second method that can be used to optimize performance is to model performance and try to find optimal results that way. An example of the success of this strategy comes from swimming. For a long period of time, it was assumed that swimmers propelled themselves through the water using the drag forces acting on the hands and feet as a propulsive force. The result of this assumption was that swimmers were taught to move their hands in a linear path through the water. However, when the forces acting on the hand were modeled, a different result was found [Brown and Counsilman 1971]: A more effective means of propulsion is to use a combination of lift and drag forces acting on the hand, and this discovery had the effect of changing the instruction to an S-shaped path used by elite swimmers today. Thus far, theoretical work of this kind has not been done on ski skating. In this paper we propose to use this method to gain further understanding of how to optimize ski skating performance.

Our Goal and Outline

We discuss the dynamics of ski skating. In particular, we try to solve the following optimization problem:

\[ \text{Maximize average speed for a given power.} \]

We first need to develop a mathematical model of ski skating. To simplify the analysis, we limit our attention to ski skating with no poles. We also limit our attention to the physics of ski skating; we ignore most biomechanical considerations.

In **Preliminaries**, we briefly review Newton’s Second Law of Motion and its application to skiing. We also review some of this law’s elementary consequences that we use repeatedly.

In **Straight-Line Skiing**, we consider a simple model of classical (diagonal) skiing and review the model of friction that we use.

In the main section, **Ski Skating**, we introduce our model of ski skating with no poles on a level plane. Our analysis produces a number of relationships between the variables over which the skier has control. These variables include tempo, glide time, step length, push force, push time and the angle between the skis. We then solve the constrained optimization problem: Maximize average speed.

We include **Suggestions for Future Work** and two appendices. In **Appendix 1: Measuring the Dynamic Coefficient of Friction**, we consider sliding
friction. For a skier coasting down an incline, we describe an experiment that can be used to measure the dynamic coefficient of friction between the snow and the skis. In Appendix 2: Dimensional Analysis, we show how to form dimensionless parameters from the mass of the skier, the power of the skier, the glide time and the acceleration of gravity.

We are deliberately somewhat repetitious in this report, so that the sections are somewhat independent of one other; we want to try to accommodate the “grasshopper” style of reading.

We have tried to minimize the mathematical background needed to understand this paper, since we hope that skiers and ski coaches will read it. Elementary college physics and calculus are the only prerequisites for understanding most of it (see, for example, Finney and Thomas [1990]). We have also included most of the details in our calculations.

**Preliminaries**

We discuss the dynamics of skiing. We base our discussion on Newton’s Second Law of Motion—force equals mass times acceleration—and apply it to cross-country skiing. (For a discussion of Newton’s law applied to downhill skiing, see Lind and Sanders [1996].)

**Notation**

We use the following notation:

- \( r(t) \) is the position of the center of gravity of the skier; this position is a vector quantity that is a function of time;
- \( t \) denotes time;
- \( v(t) := dr/dt \) is the velocity of the center of gravity; \( a(t) := dv/dt \) is the acceleration of the center of gravity;
- \( m \) is the mass of the skier; and
- \( F(t) \) is the vector sum of all the external forces acting on the center of gravity.

We set forth all of our notation for easy reference in Table 1.

**Newton’s Law and Its Consequences**

Using our notation, we can write Newton’s law as \( F = ma \). However, we prefer to work with the following system of equations:

\[
\begin{align*}
    m \frac{dv}{dt} &= F, \\
    \frac{dr}{dt} &= v.
\end{align*}
\]
Table 1.
Notation, by section.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Dimension</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Preliminaries</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>r(t)</strong></td>
<td>Position of the center of gravity of the skier</td>
<td>L</td>
<td>m (meter)</td>
</tr>
<tr>
<td><strong>r(t)</strong></td>
<td>Position of the center of gravity of the skier</td>
<td>L</td>
<td>m (meter)</td>
</tr>
<tr>
<td><strong>t</strong></td>
<td>Time</td>
<td>T</td>
<td>s (second)</td>
</tr>
<tr>
<td><strong>v(t) := dr/dt</strong></td>
<td>Velocity of the center of gravity</td>
<td>L/T</td>
<td>m/s</td>
</tr>
<tr>
<td><strong>a(t) := dv/dt</strong></td>
<td>Acceleration of the center of gravity</td>
<td>L/T²</td>
<td>m/s²</td>
</tr>
<tr>
<td><strong>m</strong></td>
<td>Mass of the skier</td>
<td>M</td>
<td>kg (kilogram)</td>
</tr>
<tr>
<td><strong>F(t)</strong></td>
<td>Vector sum of all external forces acting on the center of gravity of the skier</td>
<td>ML/T²</td>
<td>N (newton)</td>
</tr>
<tr>
<td>✷</td>
<td>End-of-remark notation</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>R</strong></td>
<td>Force exerted by the ground as a reaction to the skier’s pushing</td>
<td>ML/T²</td>
<td>N</td>
</tr>
<tr>
<td><strong>W_a</strong></td>
<td>Work done by a force when a particle moves along a curve from time ( t = a ) to time ( t = b )</td>
<td>ML²/T²</td>
<td>J (joule)</td>
</tr>
<tr>
<td>( \mathbf{v} \cdot \mathbf{v} := \mathbf{v} \cdot \mathbf{v} )</td>
<td>The inner product operation between two vectors</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>K(t) := \frac{1}{2} m v(t)^2</strong></td>
<td>Kinetic energy of a particle at time ( t )</td>
<td>ML²/T²</td>
<td>J</td>
</tr>
<tr>
<td><strong>P(t) := dW_a/dt</strong></td>
<td>Power, the rate at which work is done, is the derivative of work with respect to time</td>
<td>ML²/T³</td>
<td>W (watt)</td>
</tr>
<tr>
<td><strong>T</strong></td>
<td>Half the period of a periodic function</td>
<td>T</td>
<td>s</td>
</tr>
</tbody>
</table>

**Straight-Line Skiing**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Dimension</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v := dy/dt )</td>
<td>Speed of the skier’s center of gravity</td>
<td>L/T</td>
<td>m/s</td>
</tr>
<tr>
<td>( y(t) )</td>
<td>y-coordinate of the skier’s center of gravity</td>
<td>L</td>
<td>m</td>
</tr>
<tr>
<td><strong>R(t)</strong></td>
<td>Force exerted by the ground as a reaction to pushing</td>
<td>ML/T²</td>
<td>N</td>
</tr>
<tr>
<td><strong>S(t)</strong></td>
<td>Snow friction force</td>
<td>ML/T²</td>
<td>N</td>
</tr>
<tr>
<td><strong>V</strong></td>
<td>Average speed of the skier</td>
<td>L/T</td>
<td>m/s</td>
</tr>
<tr>
<td><strong>μ</strong></td>
<td>Coefficient of friction</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td><strong>N</strong></td>
<td>Force normal to the travel plane</td>
<td>ML/T²</td>
<td>N</td>
</tr>
<tr>
<td><strong>g</strong></td>
<td>Acceleration of gravity</td>
<td>L/T²</td>
<td>m/s²</td>
</tr>
<tr>
<td>( p := P/(mg) )</td>
<td>Power-to-weight ratio of the skier</td>
<td>L/T</td>
<td>m/s</td>
</tr>
<tr>
<td><strong>z(t)</strong></td>
<td>Vertical height of the center of gravity of the skier</td>
<td>L</td>
<td>m</td>
</tr>
<tr>
<td>( z'(t) := dz/dt )</td>
<td>Vertical velocity of the center of gravity of the skier</td>
<td>L/T</td>
<td>m/s</td>
</tr>
<tr>
<td>( z''(t) := dz'/dt )</td>
<td>Vertical acceleration of the center of gravity of the skier</td>
<td>L/T²</td>
<td>m/s²</td>
</tr>
</tbody>
</table>

**Ski Skating**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Dimension</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>S(t)</strong></td>
<td>Snow friction force</td>
<td>ML/T²</td>
<td>N</td>
</tr>
<tr>
<td><strong>α</strong></td>
<td>Angle between the direction of travel and the glide direction of the right ski</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>( t_r )</td>
<td>Time of weight transfer to the right foot</td>
<td>T</td>
<td>s</td>
</tr>
<tr>
<td>( t_l )</td>
<td>Time of weight transfer to the left foot</td>
<td>T</td>
<td>s</td>
</tr>
<tr>
<td><strong>F_r</strong></td>
<td>Size of the component of the force perpendicular to the ski</td>
<td>ML/T²</td>
<td>N</td>
</tr>
<tr>
<td><strong>F_a</strong></td>
<td>Size of the component of the force along the ski</td>
<td>ML/T²</td>
<td>N</td>
</tr>
<tr>
<td>( \mathbf{w} )</td>
<td>Velocity vector that results from the push</td>
<td>L/T</td>
<td>m/s</td>
</tr>
<tr>
<td>( w_r )</td>
<td>Size of the velocity component perpendicular to the ski</td>
<td>L/T</td>
<td>m/s</td>
</tr>
<tr>
<td>( w_a )</td>
<td>Size of the velocity component along the ski</td>
<td>L/T</td>
<td>m/s</td>
</tr>
<tr>
<td><strong>δ</strong></td>
<td>The angle ( \pi/2 ) minus the angle between the ski tracks</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>( \Delta t )</td>
<td>Push time</td>
<td>T</td>
<td>s</td>
</tr>
<tr>
<td>( \mathbf{e}_r )</td>
<td>Unit vector in the direction of the right ski glide</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mathbf{e}_l )</td>
<td>Unit vector perpendicular to ( \mathbf{e}_r ) and to the skier’s left side</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>s(t)</strong></td>
<td>The “ski speed” or “foot speed”</td>
<td>L/T</td>
<td>m/s</td>
</tr>
<tr>
<td><strong>r(t)</strong></td>
<td>The “side speed” (effected by the skier’s push with the right foot)</td>
<td>L/T</td>
<td>m/s</td>
</tr>
<tr>
<td><strong>A</strong></td>
<td>Average speed in the direction of travel</td>
<td>L/T</td>
<td>m/s</td>
</tr>
<tr>
<td><strong>ρ</strong></td>
<td>Size of the push force relative to the skier’s weight</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>( τ := (T − t_p)/T )</td>
<td>Amount of the push time relative to the half period</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>( f(α, τ) )</td>
<td>“Normalized speed function”</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>
### Symbol Meaning Dimension Unit

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Dimension</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l$</td>
<td>Size of the step to the left that the skier takes as a result of the side push by the right leg</td>
<td>$L$</td>
<td>m</td>
</tr>
<tr>
<td>$L$</td>
<td>Maximum step size</td>
<td>$L$</td>
<td>m</td>
</tr>
<tr>
<td>$\lambda(t) := \mu^3 g l / 2 p^2$</td>
<td>Normalized step size</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$\Lambda := \lambda(L)$</td>
<td>Normalized maximum step size</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$U$</td>
<td>Feasible region where we wish to maximize average speed $A$</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

### Appendix: Measuring the Dynamic Coefficient of Friction

- $y = f(x)$: Function representing the incline down which the particle is sliding
- $e_2$: Unit vector $(0, 1)$
- $T$: Unit vector tangent to the graph
- $\theta$: Angle between the incline and vertical
- $s$: Distance down the incline from some reference position
- $s' := ds/dt$: Speed of the particle along the curve $y = f(x)$
- $s'' = F(x)$: Acceleration of the particle due to gravity and friction
- $\gamma$: Path defined by $(x, f(x))$
- $h(x, y) = \mu g x - g y$: "Kinetic energy" of the particle
- $b$: Average slope

These are the basic ordinary differential equations that determine the trajectory of the center of gravity of the skier. We concentrate on the first of these equations.

**Remark** Concentrating on the center of gravity of the athlete is a standard viewpoint in biomechanics. See, for example, Alexander [1992] and/or McMahon [1984]. In particular, this is the viewpoint adopted by Svensson [1994] for skiing.

We can decompose the forces affecting the motion of the skier into a number of main components:

- sliding friction between the snow and the ski,
- drag between the air and the skier,
- gravity,
- muscular action forces, and
- ground reaction forces.

The force vector $\mathbf{F}$ is the vector sum of these forces. In particular, a force $\mathbf{R}$ is exerted by the ground as a reaction to the skier’s pushing. (We ignore the biomechanical details that produce this force.)

In later sections, we use a relation between kinetic energy and work that follows from Newton’s law. If a particle moves along a curve $\mathbf{r}(t)$ between time $a$ and time $b$ and is acted on by a force $\mathbf{F}(t)$, then the **work** $W_a^b$ is given by

$$ W_a^b = \int_a^b \mathbf{F} \cdot \mathbf{v} \, dt, $$
where the dot denotes the inner (dot) product of the two vectors. (Note, for example, that if $\mathbf{F}$ is constant, then $W^b_a = \mathbf{F} \cdot (\mathbf{r}(b) - \mathbf{r}(a))$, so that in this case work equals the inner product of force and distance.) Using Newton’s law, we have

$$W^b_a = \int_a^b \mathbf{F} \cdot \mathbf{v} \, dt = \int_a^b m \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} \, dt$$

$$= \frac{1}{2} m \int_b^a \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v}) dt = \frac{1}{2} m \mathbf{v}^2 |_{t=b} - |_{t=a},$$

where $v^2 := \mathbf{v} \cdot \mathbf{v}$. In summary, we have

$$W^b_a = K(b) - K(a) \quad (*)$$

where $K(t) := \frac{1}{2} m \mathbf{v}(t)^2$ is the kinetic energy at time $t$. Thus the work equals the difference in the kinetic energies. We refer to the relation (*) as the relation between the work and the kinetic energy.

Power is the rate at which work is done, commonly measured in watts or horsepower. In particular, instantaneous power is the derivative of work with respect to time. We have

$$P(t) := \frac{d}{dt} W^t_{a}.$$

Using the formula for work, we get

$$P(t) = \frac{d}{dt} \int_a^t \mathbf{F} \cdot \mathbf{v} \, dt = \mathbf{F}(t) \cdot \mathbf{v}(t).$$

We are particularly interested in the average power of the skier. If $\mathbf{R}(t)$ and $\mathbf{v}(t)$ are periodic functions with period $2T$, then the average power of the skier is defined by

$$P := \frac{1}{2T} \int_0^{2T} \mathbf{R}(t) \cdot \mathbf{v}(t) dt.$$

(Defining the average of a function is more complicated if the function is not periodic.) Note that only the component of $\mathbf{R}$ parallel to $\mathbf{v}$ enters here.

**Straight-Line Skiing**

We consider straight-line skiing on a plane and fashion a mathematical model of classical (or diagonal-stride) skiing. Later, we apply to ski skating some of the results that we derive in this simpler setting. (The mathematics that we develop in this section can also be used to model simple double poling or travel by means of a Norwegian kick sled that is propelled like a scooter; these are simpler applications.)

We introduce Cartesian coordinates as follows:
• The $xy$-plane is the travel plane;
• the direction of travel is the positive $y$-axis;
• the $x$-axis is in the travel plane and directed to the right of the direction of travel; and
• the $z$-axis is perpendicular to the travel plane and pointing up.

We assume that each ski travels in a straight line in the direction of the $y$-axis, and that the skis are on opposite sides of the $y$-axis. It follows that the skier’s center of gravity remains (approximately) in the $yz$-plane. In particular, we consider the following scalar differential equation:

$$m \frac{dv}{dt} = R - S,$$

where

• $m$ is the mass of the skier,
• $v := dy/dt$ is the speed of the skier’s center of gravity,
• $y(t)$ is the $y$-coordinate of the skier’s center of gravity,
• $R(t)$ is the force that is exerted by the ground as a reaction to the skier’s pushing,
• and $S(t)$ is the snow friction force.

(We are ignoring air drag.) We are interested in solving the following optimization problem:

**Maximize average speed for a given power.**

We assume that the functions $R(t)$ and $v(t)$ are periodic with period $2T$, in particular, that $v(0) = v(2T)$. We expect these functions to be approximately periodic if the skier is traveling in a straight line on a plane and is kicking and poling at a steady rate. We use $P$ to denote the average power of the skier; in symbols,

$$P := \frac{1}{2T} \int_{0}^{2T} R(t)v(t)dt.$$

It is clear that for periodic velocity, the average power input by the skier (determined by the function $R(t)$) equals the average power loss (determined by the function $S(t)$). The following mathematical argument corresponds to this intuition. Using the relation between work and kinetic energy, we have

$$\int_{0}^{2T} R v \, dt = \int_{0}^{2T} S v \, dt = \int_{0}^{2T} (R - S)v \, dt.$$
The Dynamics of Ski Skating

\[ = \frac{1}{2} m \, v(2T)^2 - \frac{1}{2} m \, v(0)^2 = 0 \]

and hence
\[ P = \frac{1}{2T} \int_0^{2T} R \, v \, dt = \frac{1}{2T} \int_0^{2T} S \, v \, dt. \]

Model for Constant Snow Friction

We now consider the case when \( S \) is constant. We then obtain the relation \( P = SV \), where \( V \) is the average speed, given by
\[ V := \frac{1}{2T} \int_0^{2T} v \, dt = \frac{y(2T) - y(0)}{2T}. \]

From the equation \( V = P/S \), we see that the average speed of the skier is completely determined by the skier’s power \( P \) and the constant frictional force \( S \). Thus, average speed does not depend on the shape of the function \( R(t) \). Furthermore (on the basis of this model), there is no mathematical reason for the skier to adopt any particular tempo; the same average speed is achieved by using short fast strides or by using long slow (but more forceful) strides. (We will see that for ski skating, the situation is quite different.) However, there are undoubtedly biomechanical reasons for choosing a certain tempo. For example, bicycling experiments show that power depends on tempo (see Whitt and Wilson [1974]).

We consider several specific examples. The usual model for ski friction is \( S = \mu N \) where \( \mu \) is the coefficient of friction and \( N \) is the force normal to the travel plane. (This model appears in physics textbooks; see, for example, Sears [1958, 34ff]. It also appears in Svensson [1994]; see especially p. 254 Figure A-2, p. 257, and p. 259 Figure A-4. There are more-sophisticated models of friction; Krim [1996] discusses the work by physicists on friction.) For skiing on a level plane, \( N \) equals the weight of the skier, that is, \( N = mg \) where \( g \) is the acceleration of gravity. Then \( V = p/\mu \) where \( p := P/(mg) \) is the power-to-weight ratio of the skier.

Example We compute the power-to-weight ratio and the average speed for some typical values. A typical value for \( \mu \) is 0.05 [Svensson 1994, 257]. The coefficient of friction is dimensionless, so there are no associated units. A typical weight for a male skier is 750 Newtons (= 75 kg \( \times \) 10 m/s\(^2\) \( \approx \) 165 lbs since 1 Newton =1 kg \( \times \) 10 m/s\(^2\) \( \approx \) (1/4.46) lbs). A typical power for recreational athletes (bicycle riders) is 75 to 225 watts [Whitt and Wilson 1974, 22], which is equivalent to 0.1 to 0.3 hp (since 745 watts \( \approx \) 1 hp). Hence, the power-to-weight ratio \( p \) is typically between 75/750 = 0.1 m/s and 225/750 = 0.3 m/s. (Recall the following relations between units: Watt = Joule/s, Joule = Newton–meter.) It follows that \( V \) is typically between
\[ 0.1/0.05 = 2 \text{ m/s} \approx 7.2 \text{ km/h} \approx 4.5 \text{ mi/h} \]
and

\[ 0.3/0.05 = 6 \text{ m/s} \approx 21.6 \text{ km/h} \approx 13.5 \text{ mi/hr}. \]

Bilodeau et al. [1992] report that the speeds of elite skiers are about $4.75 \text{ m/s}$, in this range, though our top theoretical value seems a bit high.

We easily find the solution of the differential equation \( m \frac{dv}{dt} = -S \), where \( S = \mu mg \); we will use this solution later. Substituting for \( S \) and canceling \( m \) gives \( \frac{dv}{dt} = -\mu g \). Integrating both sides of this differential equation gives

\[ v(t) - v(t_0) = \int_{t_0}^{t} (-\mu g) \, dt = -\mu g (t - t_0). \]

From this result, we see that if the skier is freely gliding on a horizontal plane then the change in velocity of the skier is proportional to the glide time with constant of proportionality equal to \( \mu g \). We can also easily compute the distance traveled as function of time. Since \( v = \frac{dy}{dt} \), we have

\[ \frac{dy}{dt} = v(t_0) - \mu g (t - t_0). \]

Integrating, we get

\[ y(t) - y(t_0) = \int_{t_0}^{t} (v(t_0) - \mu g (t - t_0)) \, dt \]

\[ = v(t_0)(t - t_0) - \frac{1}{2} \mu g (t - t_0)^2. \]

**Effect of Vertical Motion**

We consider the effect of any vertical motion of the center of gravity of the skier while the skier is gliding on a horizontal plane. We want to neglect such vertical motion when we analyze skating. Our argument here shows its effect to be negligible. We just derived formulas for \( v(t) \) and \( y(t) \) when the normal force \( N \) is constant and equal to the weight \( mg \) of the skier. Suppose now that \( N = mg + mz''(t) \), where \( z(t) \) is the vertical height of the center of gravity of the skier, \( z' := dz/dt \) is the vertical velocity and \( z'' = dz'/dt \) is the vertical acceleration. We consider the differential equation

\[ m \frac{dv}{dt} = -\mu (mg + mz''), \]

or

\[ \frac{dv}{dt} = -\mu (g + z''). \]

Integrating, we get

\[ v(t) - v(t_0) = \int_{t_0}^{t} -\mu (g + z'') \, dt \]
\[-\mu g(t - t_0) - \mu (z'(t) - z'(t_0)).\]

Note that if \(z'(t) = z'(t_0)\), then \(v(t) - v(t_0) = -\mu g(t - t_0)\), which is the same formula that we got when \(N = mg\). Integrating again, we get

\[
y(t) - y(t_0) = \int_{t_0}^{t} [v(t_0) - \mu g(t - t_0) - \mu (z'(t) - z'(t_0))] dt
\]

\[
v(t_0)(t - t_0) - \frac{1}{2}\mu g(t - t_0)^2 - \mu (z(t) - z(t_0)) + \mu z'(t_0)(t - t_0).
\]

If \(z(t) = z(t_0)\) and \(z'(t_0) = 0\), then we get

\[
y(t) - y(t_0) = v(t_0)(t - t_0) - \frac{1}{2}\mu g(t - t_0)^2,
\]

which is the same formula that we got when \(N = mg\). In other words, if the skier starts the glide with no vertical velocity and returns to the same vertical height, then the skier travels the same distance as if the center of gravity had stayed at a constant height. This analysis shows that vertical motion generally has only a minor effect on forward motion. Consequently, we often ignore the vertical motion of the center of gravity of the skier when motion is on a horizontal plane.

However, this analysis also shows that there is a way for the skier to use vertical motion to increase average speed: The skier can lower the center of gravity during the glide, to reduce friction during this phase, and then quickly raise it when the ski stops. In other words, the skier can take advantage of the stopping of the ski to increase efficiency. This may be the source of the directive “Take long glides” that we hear from a number of ski coaches. Since in order to provide a propulsive force during straight-line skiing, one of the skis must stop, the skier can take advantage of this stop time to accelerate the body upwards. This procedure reduces \(N\) and hence the frictional force during the glide phase. Furthermore, increasing \(N\) during the kick phase increases the frictional force, which allows the skier to apply a greater forward propulsive force.

**Ski Skating**

We consider ski skating with no poles on a level plane. In particular, we consider the following vector differential equation (Newton’s law):

\[
m \frac{dv}{dt} = R - S
\]

where

- \(m\) is the mass of the skier,
- \(v(t)\) is the velocity of the center of gravity of the skier,
- \(R(t)\) is the ground reaction force that is the result of the skier’s muscular action, and
\( S(t) \) is the snow friction force.

(We ignore air drag.) We are interested in the following optimization problem:

\[
\text{Maximize average speed for a given power.}
\]

We assume that the skier is traveling with a steady rhythm. More precisely, we assume that the velocity vector \( \mathbf{v}(t) \) and the reaction force vector \( \mathbf{R}(t) \) are periodic functions of time. We use \( 2T \) to denote the cycle time—a time allotment \( T \) for the right foot and a time allotment \( T \) for the left foot. We assume right-left (or bilateral) symmetry of the skier’s motion. We also assume that the right and left skis travel in straight lines while the skis are gliding. We use \( \alpha \) to denote the angle between the direction of travel and the glide direction of the right ski. It follows from the bilateral symmetry assumption that \( \alpha \) is also the angle between the direction of travel and the glide direction of the left ski (Figure 1). In this figure, \( \alpha \) is the angle between the line segments \( Q_1Q_2 \) and \( Q_1B_2 \). (This figure is a variation of ones that appear in Svensson [1994], for example, on p. 101. See also the overhead pictures in Caldwell [1987], for example, on pp. 76–81.)

**Remark** We have only rarely encountered any objection to our assumption that the skis travel in a straight line during the glide. However, when one of us (Driessel) mentioned this assumption to Antonina Anikin (a well-known coach of Russian and American cross-country ski racers) during a visit to Duluth, Minnesota in January, 1997, she objected. She claimed that the skis change direction during the push at the end of the glide. We have observed some ski tracks which turn farther away from the line of travel in the last part of the glide. We have only rarely observed such tracks.

We introduce Cartesian coordinates as follows:

- The \( xy \)-plane is the travel plane,
- the centerline of travel as the \( y \)-axis directed in the direction of travel,
- the \( x \)-axis is in the travel plane and directed to the right of the direction of travel (see Figure 1), and
- the \( z \)-axis is perpendicular to the travel plane and pointing up.

We restrict our attention to the \( xy \)-components of the vectors that we consider. In other words, we ignore the vertical motion of the skier. We do this mainly to simplify the analysis. However, we also believe that such vertical motion is not very important. (See also our discussion of vertical motion in the section *Straight-line Skiing.* Throughout the rest of this section all vectors should be regarded as vectors in the \( xy \)-plane. In particular, we are concerned with the motion of the projection of the center of gravity of the skier onto the \( xy \)-plane.
Figure 1. Ski tracks.
The following suggestion by Svensson [1994, 229] seems to be relevant here: “To ski effectively and maintain speed, the ski skater should strive for minimizing the vertical fluctuation by keeping the level of the center of gravity relatively high during the skate cycle.”

The Skate Cycle

We analyze the portion of the skate cycle between the time of weight transfer to the right foot and the time of weight transfer to the left foot. We temporarily use \( t_r \) and \( t_l \) to denote these times of weight transfer to the right and to the left. In Figure 1, we represent the locations of the right and left skis at time \( t_r \) by the line segments \( A_1B_1 \) and \( C_1D_1 \); we represent the locations of the right and left skis at time \( t_l \) by the line segments \( A_2B_2 \) and \( C_2D_2 \).

We assume that the skier completes a full periodic cycle by stepping to the right, to the left, and then again to the right. Using the bilateral symmetry assumption, we then see that the half-cycle time is equal to the time between the weight transfer to the right foot and the weight transfer to the left foot; in symbols, we have \( T = t_l - t_r \). In other words, \( T \) equals the glide time on the right ski. We now assume that \( t_r = 0 \) and \( t_l = T \). We can shift the time scale so that these two equations are satisfied.

During the glide on the right ski, the skier pushes sideways with the right leg. This push is perpendicular to the right ski. If the ski stops at the end of the glide, then the skier can also push backwards along the ski. It appears that there are two styles of skate skiing—one for which the ski stops and one for which the ski does not stop. We limit our attention to the nonstopping style. In particular, we assume that the skier’s push is perpendicular to the ski.

Aside We believe that a ski-skater’s push is usually perpendicular to the ski. However when one of us (Driessel) mentioned this assumption to Scott Hauser during a visit to the Steamboat Ski Touring Center in December 1996, Hauser questioned this assumption. We then performed two simple informal experiments.

- For the first experiment the skier stood still on level ground, leaned forward and pushed off the right ski. (Of course, poles were not used.) We measured the angle between the left and right ski tracks that the skier produced. We found this angle to be about 70° (which is significantly different than 90°).
- For the second experiment the skier skied up an incline (with a slope of about 3%) with no poles. The skier was instructed to let each ski glide to a momentary stop (or near stop) before stepping off it. We again measured the angle between the ski tracks; again we found it to be about 70°.

We performed these experiments with only one intermediate skier; we expect that the angles probably vary between skiers.
We believe that the skier is using the static friction between the ski and the snow to get a velocity component in the direction of the ski. The following sample calculation seems to support this belief. We assume that the skier exerts a constant force $F$ during the push. Let $F_r$ denote the size of the component of this force perpendicular to the ski and let $F_s$ denote the size of the component along the ski. (We use “s” for “snow”.) Let $w$ denote the velocity vector that results from the push. We assume that the skier starts from rest, that is $w(0) = 0$. From Newton’s law (namely, $F = m\frac{dv}{dt}$), we get $w = (F/m)\Delta t$ where $\Delta t$ is the push time. In terms of components we have $w_r = (F_r/m)\Delta t$ and $w_s = (F_s/m)\Delta t$, where $w_r$ and $w_s$ are respectively the sizes of the velocity components perpendicular to the ski and along the ski. Let $\delta$ be $\pi/2$ minus the angle between the ski tracks. Then

$$\tan \delta = \frac{w_s}{w_r} = \frac{F_s}{F_r}.$$

We assume that $F_s = \mu_smg$, where $\mu_s$ is the coefficient of static friction. (In the rest of this report, we use $\mu$ to denote the coefficient of sliding friction.) Then we have

$$w_s = \mu_s g \Delta t, \quad w_r = \frac{w_s}{\tan \delta}.$$

For example, take $\mu_s := 0.1$, $g := 10$ m/s$^2$, $\Delta t := 0.1$ s, and $\delta := 0.1$ radian. Then

$$w_s = (0.1)(10 \text{ m/s}^2)(0.1 \text{ s}) = 0.1 \text{ m/s}$$

and (using $\tan \delta \simeq \delta$)

$$w_r \simeq (0.1 \text{ m/s})/0.1 = 1 \text{ m/s}.$$

(Recall that 1 m/s = 3.6 km/h = 2.24 mi/h.) These speeds are reasonable. In particular, we see that the skier can get some speed in the direction of the right ski.

The following comment by Svensson [1994, 259] is relevant here: “During the final skate push-off . . . the ski is stationary for a short time.” When one of us (Driessel) mentioned this comment to Antonina Anikin during a visit to Duluth, Minnesota in January 1997, she said: “No, the ski should not stop.”

\begin{itemize}
\item Constraints
\end{itemize}

Let $e_s$ denote a unit vector in the direction of the right ski glide. The snow friction force vector is in the opposite direction. Let $e_r$ denote a unit vector perpendicular to $e_s$ and to the skier’s left side. The ground reaction force is in the direction of the vector $e_r$. We resolve the velocity $v(t)$ of the skier’s center of gravity vector into components in these two directions as follows:

$$v(t) = r(t)e_r + s(t)e_s.$$
This equation defines the functions \( r(t) \) and \( s(t) \). We call \( s(t) \) the ski speed or foot speed; it is effected by the snow friction. We call \( r(t) \) the side speed or reaction speed; it is effected by the ground reaction force caused by the skier’s push with the right foot.

We assume that the skier is balanced over the right foot when the weight is transferred to the right ski. It follows that the velocity of the skier’s center of gravity equals the velocity of the right foot at this time; in symbols, we have \( v(0) = s(0)e_s \), or \( r(0) = 0 \).

Let \( 2W \) denote the work done by the skier during a cycle and let \( P \) be the average power of the skier. Then \( P = 2W/(2T) = W/T \). Now \( r(T)e_r \) is the velocity vector that results from the skier’s push with the right leg. Since this vector is the result of the work \( W \) done by this leg during the glide of the right ski, we also have (using the relation between work and kinetic energy)

\[
W = \frac{1}{2} mr(T)^2.
\]

From the two equations involving the work, we get

\[
PT = \frac{1}{2} m r(T)^2.
\]

We call this relation the power constraint.

Since the skier is balanced over the left ski when the skier’s weight is transferred to this ski, we have

\[
v(T) = r(T)e_r + s(T)e_s.
\]

Using the bilateral symmetry assumption, we have \( |v(T)| = |v(0)| = s(0) \).

In Figure 2, we represent these various velocities: The line segment \( AD \) represents the vector \( v(0) \). The line segment \( AC \) represents the vector \( v(T) \). The line segment \( BC \) represents the vector \( r(T)e_r \). Note that \( \alpha \) equals angle \( EAD \).

In the figure, \( |AC| = |AD| = s(0) \), \( |BC| = r(T) \) and \( \sin 2\alpha = |BC|/|AC| \). Consequently, we have

\[
\sin 2\alpha = \frac{r(T)}{s(0)}.
\]

We also have \( 0 < 2\alpha \leq \pi/2 \), or

\[
0 \leq \alpha \leq \pi/4,
\]

since \( s(T) \geq 0 \). We call these relations the geometric constraints.

Recall that \( s(t) \) denotes the speed of the right ski during the glide. We assume that the snow friction is constant and equal to \( \mu mg \). From the previous section, we then have

\[
s(t) = s(0) - \mu gt.
\]

(This formula for the slow-down holds only approximately, since the model of friction that we use is only an approximation. More important, perhaps, is
the fact that the coefficient of friction $\mu$ probably changes as the ski is edged during the glide. We don’t know of any experimental results that quantify this change. However, Svensson [1994, 259] does discuss this matter.)

From the slow-down formula for $s(t)$, we get $|AB| = s(T) = s(0) - \mu g T$ and hence $|BD| = \mu g T$. Note that $\angle BCD$ equals $\alpha$. Hence,

$$\tan \alpha = \frac{\mu g T}{r(T)}.$$  

We call this relation the slow-down constraint.

In summary, we have the following constraints:

$$PT = \frac{1}{2} m \ r(T)^2,$$  \hspace{1cm} \text{(power)}

$$\sin 2\alpha = \frac{r(T)}{s(0)} \quad \text{and} \quad 0 \leq \alpha \leq \frac{\pi}{4},$$  \hspace{1cm} \text{(geometric)}

$$\tan \alpha = \frac{\mu g T}{r(T)},$$  \hspace{1cm} \text{(slow-down)}
Reducing to a Single Parameter

We regard the quantities $m$, $P$, $\mu$ and $g$ as parameters, the values of which usually remain unchanged during a ski outing:

- The mass $m$ and power $P$ are determined by the skier’s body.
- The coefficient of friction $\mu$ is determined by the interface between the ski and snow.
- The acceleration of gravity depends slightly on the location at which the outing takes place.

We regard the quantities $T$, $r(T)$, $s(0)$, and $\alpha$ as variables. The skier may vary these quantities during the outing. However, they must satisfy the constraints.

In fact, we can use the constraints to reduce the number of variables. Since there are three constraints and four variables, we expect that we might be able to express all of them in terms of one of them; indeed, we find formulas for all of them in terms of $\alpha$.

We can solve for $r(T)$ in terms of $\alpha$ as follows. From the power and slowdown constraints, we get

$$\mu g PT = \frac{1}{2} \mu m g r(T)^2$$

and

$$\mu g PT = P r(T) \tan \alpha.$$ 

Hence, we have

$$\frac{1}{2} \mu m g r(T)^2 = P r(T) \tan \alpha$$

and

$$\frac{1}{2} \mu m g r(T) = P \tan \alpha$$

provided $r(T) \neq 0$. Thus,

$$r(T) = \frac{2p}{\mu} \tan \alpha$$

where $p := P/(mg)$ is the power-to-weight ratio (introduced in the previous section).

We can now express $s(0)$ in terms of $\alpha$. From the geometric constraint and the formula for $r(T)$ in terms of $\alpha$, we get

$$s(0) = \frac{r(T)}{\sin 2\alpha} = \frac{2p \tan \alpha}{\mu \sin 2\alpha},$$

or

$$s(0) = \frac{p}{\mu \cos^2 \alpha}$$

since

$$\frac{\tan \alpha}{\sin 2\alpha} = \frac{\sin \alpha / \cos \alpha}{2 \sin \alpha \cos \alpha} = \frac{1}{2 \cos^2 \alpha}.$$
We can also express the half-period time $T$ in terms of $\alpha$. From the formula for $r(T)$ and the slow-down constraint, we get

$$T = \frac{r(T)}{\mu g} \tan \alpha = \frac{(2p/\mu) \tan^2 \alpha}{\mu g},$$

or

$$T = \frac{2p}{\mu^2 g} \tan^2 \alpha.$$

**Remark** Svensson [1994, 232] makes the following statement: “Skiers therefore should strive to skate with optimal smallest angle and long glide which is more efficient.” We don’t understand how it is possible for a skier to have a long glide time $T$ and a small angle $2\alpha$ between the skis. The formula for $T$ in terms of $\alpha$ shows that if $\alpha$ is small, then $T$ must be small. 

**Average Speed**

We now consider the average speed of the skier in the direction of travel. Recall that $v(t)$ denotes the velocity of the skier. Note that $e_2 := (0, 1)$ is a unit vector in the direction of travel. Then the inner product $v(t) \cdot e_2$ is the speed of the skier in the direction of travel. The average speed in this direction is

$$A := \frac{1}{T} \int_0^T v(t) \cdot e_2 dt.$$

Recall that we represent the velocity of the skier during the glide on the right ski as

$$v(t) = r(t)e_r + s(t)e_s.$$ 

Note that $e_r \cdot e_2 = \sin \alpha$ and $e_s \cdot e_2 = \cos \alpha$. Hence,

$$A = \left( \frac{1}{T} \int_0^T r(t) dt \right) \sin \alpha + \left( \frac{1}{T} \int_0^T s(t) dt \right) \cos \alpha.$$

In particular, $A$ is a weighted sum of the average of $r(t)$ and the average of $s(t)$.

From above, we have $s(t) = s(0) - \mu gt$ and hence (using the formulas for $s(0)$ and $T$) we get

$$\frac{1}{T} \int_0^T s(t) dt = \frac{1}{T} \int_0^T (s(0) - \mu gt) dt$$

$$= \frac{1}{T} \left[ s(0)t - \frac{\mu gt^2}{2} \right]_0^T$$

$$= \frac{1}{T} \left( s(0)T - \frac{\mu gT^2}{2} \right).$$
\frac{p}{\mu \cos^2 \alpha} - \frac{1}{2} \mu g \left( \frac{2p}{\mu^2 g} \tan^2 \alpha \right) = \frac{p}{\mu} \left( \frac{1}{\cos^2 \alpha} - \tan^2 \alpha \right) = \frac{p}{\mu}.

We also want to compute the average of the function \( r(t) \). We assume that the skier pushes perpendicular against the right ski with a constant force starting at time \( t_p \). We use the expression \( \rho mg e_r \) to represent this constant force. In particular, we have normalized this force using the skier’s weight; the dimensionless number \( \rho \) indicates the size of this push force relative to the skier’s weight. In other words, we consider reaction force functions \( R(t)e_r \) where \( R(t) := 0 \) if \( t \) is between 0 and \( t_p \) and \( R(t) := \rho mg e_r \) if \( t \) is between \( t_p \) and \( T \).

**Remark** This model of the skier’s side push function is quite simple. More-general functions can be treated using the calculus of variations; see, for example, Alexander[1996] or Weinstock[1974]. We expect that the conclusions using more-general functions will be similar to the ones that we derive here.

From Newton’s law (that is, from \( mdr/dt = \rho mg \)), we get

\[ r(t) = \rho g (t - t_p) \]

for \( t_p \leq t \leq T \). Note that

\[ r(T) = \rho g \tau T, \]

where \( \tau := (T - t_p)/T \), and that \( 0 \leq \tau \leq 1 \). In particular, \( \tau \) is the percentage of the glide time that the skier is pushing.

The parameters \( \rho \) and \( \tau \) are related to the power of the skier. From the power constraint, we get

\[ PT = \frac{1}{2} m (\rho g \tau T)^2, \]

or

\[ p = \frac{P}{mg} = \frac{1}{2} g \rho^2 \tau^2 T. \]

There is a trade off between \( \rho \) and \( \tau \) for fixed \( p \) and \( T \). If \( \rho \) is large, then \( \tau \) must be small. Using the formula for \( T \) in terms of \( \alpha \), we get

\[ p = \frac{1}{2} g \rho^2 \tau^2 \left( \frac{2p}{\mu^2 g} \tan^2 \alpha \right) = \left( \frac{\rho^2 \tau^2}{\mu^2} \tan^2 \alpha \right) p. \]

Hence,

\[ \rho = \frac{\mu}{\tau \tan \alpha}. \]
We can now compute the average of \( r(t) \). We have (using the formulas for \( T \) and \( \rho \) in terms of \( \alpha \))

\[
\frac{1}{T} \int_0^T r(t) dt = \frac{1}{T} \int_{t_p}^T \rho g(t - t_p) dt \\
= \frac{1}{T} \rho g \frac{1}{2} (t - t_p)^2 |_{t=t_p} \\
= \frac{1}{T} \rho g \frac{1}{2} (T - t_p)^2 \\
= \frac{1}{T} \rho g \frac{1}{2} \tau^2 T^2 = \\
\frac{1}{2} \rho g \tau^2 T \\
= \frac{1}{2} \left( \frac{\mu}{\tau \tan \alpha} \right) g \tau^2 \left( \frac{2p}{\mu^2 g} \tan^2 \alpha \right) \\
= \frac{\tau p}{\mu} \tan \alpha.
\]

Thus,

\[
A = \left( \frac{1}{T} \int_0^T r(t) dt \right) \sin \alpha + \left( \frac{1}{T} \int_0^T s(t) dt \right) \cos \alpha \\
= \frac{\tau p}{\mu} \tan \alpha \sin \alpha + \frac{p}{\mu} \cos \alpha,
\]
or

\[
A = \frac{p}{\mu} (\tau \tan \alpha \sin \alpha + \cos \alpha).
\]

**Optimization**

We regard \( p \) and \( \mu \) as fixed parameters. We want to maximize the average speed \( A = A(\alpha, \tau) \) subject to the constraints \( 0 \leq \alpha \leq \pi/4 \) and \( 0 \leq \tau \leq 1 \). We consider the function

\[
f(\alpha, \tau) := \tau \tan \alpha \sin \alpha + \cos \alpha
\]
on the rectangle \([0, \pi/4] \times [0, 1]\). We call this function the *normalized speed function*. (Figure 3 shows a plot of this function. Figure 4 shows its level curves; the arrows in this figure point uphill.)

Note that if \( \alpha \neq 0 \), then

\[
\frac{\partial f}{\partial \tau} = \tan \alpha \sin \alpha > 0.
\]

It follows that the maximum of \( f \) occurs on the boundary of the rectangle. Now

\[
f(0, \tau) = 1,
\]
Figure 3. Normalized speed function.

Figure 4. Level curves for normalized speed function.
\[ f(\pi/4, \tau) = \tau, \]
\[ f(\alpha, 0) = \cos \alpha, \quad \text{and} \]
\[ f(\alpha, 1) = \tan \alpha \sin \alpha + \cos \alpha = \frac{1}{\cos \alpha}. \]

We see that the maximum of \( f \) occurs at \( \alpha = \pi/4, \tau = 1 \), with \( f(\pi/4, 1) = \sqrt{2} \), since \( \cos \pi/4 = \sqrt{2}/2 \). It follows that the maximum of \( A(\alpha, \tau) \) on the rectangle is \( \sqrt{2} p/\mu \). Note that \( \tau = 1 \) implies that \( t_p = 0 \); in other words, the skier should immediately start pushing to the side.

### Examples and Adjustment of Model

**Example: Intermediate Athlete**  
We consider an example to see if we have missed any constraints. Let \( p := 0.2 \text{ m/s}, \mu := 0.05, a := \pi/4, \) and \( \tau := 1 \). Then

\[
A = \frac{p}{\mu} (\tau \tan \alpha \sin \alpha + \cos \alpha) \\
= \frac{0.2}{0.05} \left( \tan \frac{\pi}{4} \sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right) \\
= 4\sqrt{2} \text{ m/s} \approx 20 \text{ km/h} \approx 12 \text{ mi/h}.
\]

This value seems rather high. We also compute the half-period time \( T \) (with \( g := 10 \text{ m/s}^2 \)); we get

\[
T = \frac{2p}{\mu^2 g} \tan^2 \alpha = \frac{2(0.2)}{(0.05)^2 10} = 16 \text{ sec}.
\]

This value is not reasonable.  

We can also compute the size of the step to the left that the skier takes as a result of the side push by the right leg. We use \( l \) to denote the length of this step. In the figure representing the ski tracks, this step is represented by the line segment \( R_2L_2 \). We have

\[
l = \int_{t_p}^{T} r(t) dt = \int_{t_p}^{T} \rho g(t - t_p) dt \\
= \frac{1}{2} \rho g \tau^2 T^2 \\
= \frac{1}{2} \left( \frac{\mu}{\tau \tan \alpha} \right) g \tau^2 \left( \frac{2p \tan^2 \alpha}{\mu^2 g} \right)^2
\]

or

\[
l = \frac{2p^2 \tau}{\mu^2 g} \tan^3 \alpha.
\]
Example: Intermediate Athlete (continued) With the values in the last example, we get
\[ l = \frac{2(0.2)^2}{(0.05)^310} = 64 \text{ m.} \]
This step size is clearly impossible. ♦

It is now obvious that we need to add another constraint to our optimization problem. We add the constraint \( l \leq L \), where \( L \) denotes the maximum step size. We also introduce a dimensionless version of these quantities; in particular, we let
\[ \lambda(l) := \frac{\mu^3 gl}{2p^2} \quad \text{and} \quad \Lambda := \lambda(L). \]
We can rewrite the equation for \( l \) as follows:
\[ \lambda(l) = \tau \tan^3 \alpha. \]

We want to maximize \( A(\alpha, \tau) \) on the region \( U \) defined by the constraints \( 0 \leq \alpha \leq \pi/4 \), \( 0 \leq \tau \leq 1 \), and \( \tau \tan^3 \alpha \leq \Lambda \). In Figure 5, we plot such a region.

As before, we see that the maximum must occur on the boundary—in fact, it must occur on the boundary curve
\[ \{ (\alpha, \tau) : \Lambda = \tau \tan^3 \alpha, 0 \leq \alpha \leq \pi/4, 0 \leq \tau \leq 1 \}. \]
Note, in particular, that \( \tau \) is a decreasing function of \( \alpha \) on this curve. We consider the function \( \alpha \mapsto A(\alpha, \tau(\alpha)) \), where \( \tau(\alpha) := \Lambda/ \tan^3 \alpha \). We claim that this function is decreasing on the interval \( (0, \pi/4] := \{ \alpha : 0 < \alpha \leq \pi/4 \} \). It will
follow that the maximum of \( A(\alpha, \tau) \) on \( U \) occurs when \( \tau = 1 \) and \( \tan^3 \alpha = \Lambda \). We have

\[
A(\alpha, \tau(\alpha)) = \frac{p}{\mu} (\tau(\alpha) \tan \alpha \sin \alpha + \cos \alpha)
= \frac{p}{\mu} \left( \frac{\Lambda}{\tan^3 \alpha} \tan \alpha \sin \alpha + \cos \alpha \right)
= \frac{p}{\mu} h(\alpha),
\]

where \( h(\alpha) := \Lambda(1/ \sin \alpha - \sin \alpha) + \cos \alpha \), since

\[
\frac{\sin \alpha}{\tan^2 \alpha} = \frac{\cos^2 \alpha}{\sin \alpha} = \frac{1 - \sin^2 \alpha}{\sin \alpha}.
\]

We have

\[
h'(\alpha) = \frac{dh}{d\alpha} = \frac{d}{d\alpha} \left[ \Lambda \left( \frac{1}{\sin \alpha} - \sin \alpha \right) + \cos \alpha \right]
= \Lambda \left( -\frac{\cos \alpha}{\sin^2 \alpha} - \cos \alpha \right) - \sin \alpha.
\]

Clearly, \( h'(\alpha) < 0 \) on \((0, \pi/4]\).

**Final Model: Intermediate Athlete vs. Elite Athlete**

**Example: Intermediate Athlete (continued)** We reconsider the previous example, with the values \( p := 0.2 \text{ m/s}, \mu := 0.05, g := 10 \text{ m/s}^2, \) and \( L := 1 \text{ m} \). Then

\[
\Lambda = \lambda(1) = \frac{\mu^3 g}{2p^2} = \frac{(0.05)^3 10}{2(0.2)^2} = \left( \frac{1}{4} \right)^3.
\]

From \( \Lambda = \tau \tan^3 \alpha \) and \( \tau = 1 \), we get \( \tan \alpha = \frac{1}{4} \) or \( \alpha \approx \frac{1}{4} \) radians \( \approx 14^\circ \). We can now easily compute the corresponding average speed \( A \); we get

\[
A = \frac{p}{\mu} (\tau \tan \alpha \sin \alpha + \cos \alpha)
\simeq 4((0.25)^2 + 0.97) = 4.12 \text{ m/s} \approx 14.8 \text{ km/h} \approx 9.2 \text{ mi/h}.
\]

We can also compute the corresponding half-period \( T \); we get

\[
T = \frac{2p}{\mu^2 g} \tan^2 \alpha = \frac{2(0.2)}{(0.05)^2 10} \left( \frac{1}{4} \right)^2 = 1 \text{ s}.
\]
Example: Elite Skier  The value $p := 0.2 \text{ m/s}$ is appropriate for an intermediate athlete. If we repeat the calculation in the previous paragraph with $p := 0.3 \text{ m/s}$, which is appropriate for an elite athlete (and using the same values for $\mu$, $g$, and $L$ as above), then we get

\[
\Lambda \approx 6.94 \times 10^{-3}, \\
\tan \alpha \approx \alpha \approx 0.191 \text{ radians} \approx 11^\circ, \\
A \approx 6.10 \text{ m/s}, \quad \text{and} \\
T \approx 0.866 \text{ s}.
\]

From Bilodeau et al. [1992], we have the following typical observed values for real elite skiers (using poles) traveling on a flat site:

- average velocity = $A = 5.8 \text{ m/s}$,
- cycle time = $2T = 1.4 \text{ s}$, and
- cycle length = $2TA = 8.1 \text{ m}$.

Our values for the theoretical elite skier (not using poles) are

- $A = 6.10 \text{ m/s}$,
- $2T = 1.7 \text{ s}$, and
- $2TA = 10.6 \text{ m}$.

We see that real elite skiers travel at about the same speed and use a somewhat faster tempo than our theoretical elite skiers. The angle of about 1$^\circ$ for the theoretical elite appears reasonable for flat plane skiing. Smith et al. [1988] report a mean angle of about 24$^\circ$ for skiers climbing a 7% slope; angles are generally larger for climbing steeper slopes.

Suggestions for Future Work

We have discussed a simple mathematical model of ski skating. This model explains some aspects of ski skating. But we would like to understand more, and there are many possibilities for future work that are related to our model. In this section, we discuss a few of these possibilities.

Testing the Theory

Our model agrees qualitatively with data gathered for actual skiers. Unfortunately, we have no data for ski skating without poles on a level plane. Is there reasonable quantitative agreement between our model and the performance of actual skiers who are not using poles? We need experimental data to answer this question. The experiments could be done on a snow-covered lake so that the level-plane assumption is satisfied.
**Hills**

We model a skier traveling on a level plane. This situation is rare; skiers are almost always climbing or descending. We believe that our mathematical model can be generalized to inclined-plane situations.

Here is a question related to such a situation: Is there a climbing steepness at which the skier should glide to a stop with each stride? We conjecture that there is such an angle.

There are several different hill situations that can be modeled using an inclined plane. In one situation the line of travel is straight uphill—in other words, the line in the travel plane perpendicular to the direction of travel is level. In this situation, bilateral symmetry is again a reasonable assumption.

Here is another situation involving an inclined plane. The line of travel is level but the line in the travel plane perpendicular to the line of travel is not level. This is a simple “side-hill” situation. In this situation, bilateral symmetry is not reasonable.

Of course, skiers sometimes climb in side-hill situations. Here the line of travel is not directly uphill and not level.

**More-General Push Functions**

We consider simple push (that is, reaction force) functions. In particular, we consider functions that equal zero until a push starting time \( t_p \) and equal a positive constant value after that time. We find that among these push functions, the one with \( t_p = 0 \) maximizes the average speed in the direction of travel. The question arises: Can an optimum function be found when more general functions are allowed? (We mentioned this generalization in a remark in the section *Ski Skating*.)

A related question is: What more-general class of push functions should be considered?

**Three Dimensions**

We consider a two-dimensional model of ski skating. In particular, we regard the center of gravity to be in a plane and the push (that is, ground reaction force) vector to be in this plane. We should consider an analogous three-dimensional model, with the center of gravity and the push vector in three-dimensional space.

Our two-dimensional analysis implies that the skier should start pushing to the side at the beginning of the glide. But in three dimensions, if the skier’s center of gravity is directly above the foot and the push vector is directed along the line from the foot to the center of gravity, then a push at the beginning of the glide only raises the center of gravity and does not cause a motion to the side.
Multi-linked Chains


We should consider simple multi-linked chain models of ski skating as one way of incorporating the three-dimensional aspect of ski skating.

Poling

We consider a model of ski skating with no poles. We should find a model that includes poling.

Friction of an Edged Ski

In the section Ski Skating, we allude to the following question: Is the coefficient of friction for a flat ski different than the coefficient of friction for an edged ski? We don’t know any experimental results concerning this question.

We indicate in an appendix how to measure the dynamic coefficient of friction of a pair of skis. Here is a way to answer the question concerning edged skis: Build a sled to which a pair of skis can be attached at various edging angles. Then use the theory in the appendix to measure the coefficient of friction for skis edged at various angles.

Appendix 1: Measuring the Dynamic Coefficient of Friction

We consider sliding friction. For a skier coasting down an incline, we describe an experiment to measure the dynamic coefficient of friction between the snow and the ski.

We begin by constructing a mathematical model. We represent the skier as a particle with mass $m$. We represent the incline by the graph $y = f(x)$ of a smooth function $f$. We choose our coordinate axes so that the gravitational force vector is in the direction of the negative $y$-axis; that is, the gravitational force vector is represented by $-ge_2$, where $g$ is the acceleration of gravity and $e_2 := (0, 1)$. We assume that the particle moves from left to right down the incline; that is, we assume that the particle moves in the direction of increasing $x$. If the particle is at a point $(x, y)$ on the incline, then the component of the particle’s weight in the direction tangent to the incline acts to accelerate (or
decelerate) the particle. Let

$$T := \frac{(1, f'(x))}{\sqrt{1 + f'(x)^2}};$$

note that $T$ is a unit vector tangent to the graph. The size of the weight component in the direction of $T$ is

$$mg(-e_2 \cdot T) = -mg \frac{f'(x)}{\sqrt{1 + f'(x)^2}} = mg \cos \theta,$$

where $\theta$ is the angle between the incline and vertical. (We call this force the *tangential gravitational force*.) In Figure 6, we picture the incline; the slope is represented by the curved line. At the point $P$ on the graph, the tangent is represented by the line segment $AB$ and the normal is represented by the line segment $CD$. The vertical direction $e_2$ at $P$ is represented by the line segment $EF$. The angle $\theta$ is represented by $APF$.

![Figure 6. Incline.](image)

We assume that the friction force between the particle and the incline is determined by the coefficient of friction $\mu$ and the component of the particle’s weight in the direction perpendicular to the incline. (We call this component of the weight the *normal gravitational force*.) In other words, the frictional force is given by $-\mu mg \sin \theta$. (This model of the frictional force is standard; for example,
it appears in Svensson [1994, 236].) We assume that gravity and friction are the only two significant forces acting on the particle (in particular, we ignore air drag).

In summary, we use the following differential equation to model the motion of the sliding particle:

\[ ms'' = mF(x), \quad \text{or} \]

\[ s'' = F(x), \]

where \( s \) is the distance down the incline from some reference position, \( F(x) := g \cos \theta - \mu g \sin \theta, \)

\[ \cos \theta = \frac{-f'(x)}{[1 + f'(x)^2]^{1/2}}, \quad \text{and} \]

\[ \sin \theta = \frac{1}{[1 + f'(x)^2]^{1/2}}. \]

We regard the quantities \( x \) and \( \theta \) as functions of \( s \).

Note that the change in speed of the particle is related to the work done. In particular, from the differential equation, we have

\[ \frac{d}{dt} \left( \frac{1}{2} (s')^2 \right) = s's'' = F(x)s' \]

and hence

\[ \frac{1}{2} (s')^2|_{t=t_0}^{t_1} = \int_{t_0}^{t_1} Fs' dt. \]

We can compute the work done and obtain the following simple formula.

**Proposition.** Let \( f \) be a continuously differentiable function. Assume that a particle moves along the graph of \( f \) according to the differential equation given above and the particle occupies points \( (x_0, y_0) \) and \( (x_1, y_1) \) at times \( t_0 \) and \( t_1 \) respectively where \( t_0 < t_1 \) and \( x_0 < x_1 \). Then the work done in moving between these points is given by

\[ -g(y_1 - y_0) - \mu g(x_1 - x_0). \]

**Remark.** Note that the first term in this formula is positive if the particle moved downhill (that is, \( y_1 < y_0 \)). Note that the second term is negative.

\[ \diamond \]

**Proof:** Recall the following relation between \( x \) and \( s \):

\[ \frac{ds}{dx} = [1 + f'(x)^2]^{1/2}. \]
Using the formula for change of variables, we then get

\[
\int_{t_0}^{t_1} F s' \, dt = \int_{s_0}^{s_1} F \, ds \\
= \int_{x_0}^{x_1} F(x) [1 + f'(x)^2]^{1/2} \, dx \\
= \int_{x_0}^{x_1} (-g f'(x) - \mu g) \, dx \\
= -g (f(x_1) - f(x_0)) - \mu g(x_1 - x_0) \\
= -g(y_1 - y_0) - \mu g(x_1 - x_0).
\]

We conclude that the change in speed of the particle in moving from \((x_0, y_0)\) to \((x_1, y_1)\) is determined by the difference in elevation and the horizontal distance traveled. In terms of formulas, we have

\[
\frac{1}{2} (s')^2|_{t_1} = -g(y_1 - y_0) - \mu g(x_1 - x_0).
\]

In particular, the detailed shape of the graph (that is, of the incline) is not relevant. Also note that the mass of the particle does not appear in this relation.

**Aside** The work is given by the following line integral:

\[
\int_{s_0}^{s_1} F \, ds = \int_{s_0}^{s_1} (-g, -\mu g) \cdot T \, ds \\
= \int_{\gamma} (-\mu g) \, dx - g \, dy
\]

where \(\gamma\) is the path defined by \(\gamma(s) := (x, f(x))\). This line integral also represents the work done in moving a particle from \((x_0, y_0)\) to \((x_1, y_1)\) in the force field \((-\mu g, -g)\). Let \(h(x, y) := -\mu gx - gy\). Then

\[
dh = -\mu g \, dx - g \, dy, \quad \text{or} \quad \nabla h = (-\mu g, -g).
\]

It follows that the line integral is path-independent.

Let \(V(x, y) := -mh(x, y) = \mu mgx + mgy\). Then we have

\[
\frac{1}{2} m (s')^2|_{t_0}^{t_1} = -V(x_0, y_0) + V(x_1, y_1)
\]

or

\[
K(t_0) + V(x_0, y_0) = K(t_1) + V(x_1, y_1)
\]

where \(K(t) := \frac{1}{2} ms'(t)^2\) is the “kinetic energy” of the particle. In other words the quantity \(K + V\) is conserved.

We want to consider some examples. We rewrite the relation between speed and work as follows:

\[
\frac{1}{2} (s')^2|_{t=t_0} = g(x_1 - x_0)(b - \mu)
\]
where \( b : = \frac{-(y_1 - y_0)}{(x_1 - x_0)} \) is the “average slope”. Note that if \( b = \mu \) then there is no change in the skier’s speed between \((x_0, y_0)\) and \((x_1, y_1)\).

In the following examples we assume that \( g = 10 \text{ m/s}^2 \) and \( x_1 - x_0 = 10 \text{ m} \). Then \( g(x_1 - x_0) = 10^2 \text{ m}^2/\text{s}^2 \). We also assume that \( t_0 = 0 \text{ s} \).

**Example** Assume that \( s'(0) = 0 \text{ m/s} \), that is, the skier starts at rest. Also assume that \( b = 0.1 \) (a 10% down slope) and \( \mu = 0.05 \). Then
\[
\frac{1}{2}s'(t_1)^2 = (10^2 \text{ m}^2/\text{s}^2)(0.1 - 0.05)
\]
and hence
\[
s'(t_1) = \sqrt{10} \text{ m/s} = 3.16 \text{ m/s} \approx 11.4 \text{ km/h} \approx 7.08 \text{ mi/h}.
\]

**Example** Assume that \( s'(0) = 1.0 \text{ m/s} \), \( b = 0.01 \), and \( \mu = 0.05 \). Then
\[
\frac{1}{2}s'(t_1)^2 - \frac{1}{2}(1 \text{ m}^2/\text{s}^2) = 5 \text{ m}^2/\text{s}^2
\]
and hence
\[
s'(t_1) = \sqrt{11} \text{ m/s} = 3.32 \text{ m/s}11.9 \text{ km/h} \approx 7.44 \text{ mi/h}.
\]

We can solve the relation between speed and work for the coefficient of friction \( \mu \):
\[
\mu = b - \frac{1}{2} \frac{v_1^2 - v_0^2}{g(x_1 - x_0)},
\]
where \( v_1 := s'(t_1) \) and \( v_0 := s'(t_0) \).

We can perform the following experiment to determine \( \mu \): We use surveying equipment to measure the difference in elevations \( y_1 - y_0 \) and the horizontal distance \( x_1 - x_0 \). We have a skier coast down the incline across the points \((x_0, y_0)\) and \((x_1, y_1)\). We measure the skier’s speeds \( v_0 \) and \( v_1 \) at these points. We can then easily compute \( \mu \) from the last displayed formula.

**Appendix 2: Dimensional Analysis**

We want to form a dimensionless parameter from the quantities \( m, g, P, \) and \( T \). Recall that \( m \) denotes the mass of the skier, \( g \) denotes the acceleration of gravity, \( P \) denotes the power of the skier and \( T \) denotes the half-period of the skier’s body motion. We use \( M, L, \) and \( T \) to denote the mass, length, and
time dimensions. (We hope the reader can easily distinguish by context our two different uses of the symbol $T$.) We have $\dim m = M$, $\dim g = L/T^2$, $\dim P = ML^2/T^3$, and $\dim T = T$. Hence

$$\dim(m^w g^x P^y T^z) = M^w (L/T^2)^x (ML^2/T^3)^y T^z = M^{w+y} L^{x+2y} T^{-2x-3y+z}.$$ Consequently, we want to solve the following linear equations:

$$w + y = 0,$$
$$x + 2y = 0,$$
$$2x + 3y - z = 0.$$

We get

$$w = -y,$$
$$x = -2y,$$
$$z = 2x + 3y = -4y + 3y = -y.$$ If we take $y = 1$, we get $w = -1, x = -2$ and $z = -1$. The corresponding dimensionless quantity (which we call the power parameter) is

$$m^{-1} g^{-2} P T^{-1} = \frac{p}{g T},$$

where $p := P/mg$. Recall that in the main part of this report we called $p$ the power-to-weight ratio. Note that

$$\dim p = \dim P/\dim(mg)$$
$$= (ML^2/T^3)/(ML/T^2)$$
$$= L/T,$$

which is a velocity dimension, and

$$\dim(gT) = (L/T^2)T = L/T,$$

which is also a velocity dimension. Recall the formula $T = 2p \tan^2 \alpha/\mu^2 g$ for the half-period from the section on ski skating. From this formula, we get $\mu^2 = 2(p/gT) \tan^2 \alpha$, in which we see the dimensionless parameter found above.

References


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