Option Pricing Methods for Estimating Capacity Shortages

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Capacity shortage, Option pricing, Optimal expansion policy

Disciplines
Operations Research, Systems Engineering and Industrial Engineering

Comments
Option Pricing Methods for Estimating Capacity Shortages

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Abstract

Uncertain demand combined with a positive lead time for adding capacity creates the risk of capacity shortage during the lead time. Accurate shortage estimation is very important to determine a capacity expansion policy. In this paper, we investigate four kinds of option pricing methods to estimate the shortage during a fixed lead time under the assumption of an exponential trend for demand growth. We compare the effect of the different shortage measures on the optimal parameters for an expansion policy over the long term.

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1. Introduction

Predicting demand is very important in capacity expansion problems but it is difficult because of random factors. If there is no lead time for adding capacity, despite the uncertainty of demand there would be no risk of capacity shortage, since the manager could simply wait until demand equals current capacity and then install new capacity. However, if a lead time for adding capacity exists, uncertain demand creates the risk of capacity shortage during the lead time. Postponing any capacity addition increases this risk of capacity shortage during the installation lead time.

In many new technology industries, demand for capacity grows according to an exponential trend. For example, the National Academy of Sciences [1] reported that lasers and optical fibers together have dramatically increased the capacity of the international telephone system, and fueled the exponential growth of the Internet. Both Internet hosts [2] and connections [3] are also predicted to increase exponentially.

In this paper, we investigate four kinds of option pricing methods to estimate the shortage during a fixed lead time under the assumption of an exponential trend for demand growth. With each method of estimating shortages, we use a dynamic programming model of the capacity expansion problem for uncertain demand growth and determine optimal expansion policies in two dimensions. The first dimension is timing, that is, setting a level of excess capacity that will trigger the initiation of a capacity addition. The second dimension concerns size; i.e., determining how much capacity to add at a given time in view of cost discounting and economies of scale.

In each case, these analytical estimates of the shortage depend only on the ratio of demand to existing capacity when an expansion begins. This suggests that expansions should be timed to maintain a constant proportional reserve margin of capacity. Ryan [4] has previously shown that under this timing policy it is optimal to always install the same multiple of existing capacity. Assuming this form for the policy, we can solve for the optimal policy parameters to minimize a weighted combination of expected shortage and total discounted expected expansion cost.

Chaouch and Buzacott [5] suggested how to find the optimal manufacturing plant size under the assumptions of fixed lead time and alternating periods of demand growth and stagnation. They showed that longer lead times increase the optimal trigger levels and sizes of capacity additions. Angelus, Porteus and Wood [6] solved a capacity expansion model with economies of scale in capacity expansion costs, dynamic sizing of expansions, and correlated stochastic demand for the case of contingent timing of expansions over a finite horizon in the semiconductor industry.

The focus of this research is to compare the use of European, Asian and Lookback option values to estimate the lead time capacity shortage and to suggest a new method of summing European option to predict the total shortage during the lead time. The notation used in this paper is as follows:
\( \mu \): constant drift of logarithmic demand growth
\( \sigma \): constant volatility of logarithmic demand growth
\( g = \mu + \sigma^2 / 2 \): mean exponential growth rate of demand
\( r \): annual risk-adjusted interest rate
\( D(t) \): demand for capacity at time \( t \)
\( H \): fixed installation lead time
\( K_n \): installed capacity after \( n \) additions are completed (\( K_0 \): initial capacity)
\( U_n = K_n - K_{n-1} \): size of \( n \)th capacity addition
\( S_n \): expected shortage at the end of \( n \)th lead time
\( \Phi(\cdot) \): the standard normal cumulative distribution function.

\[ (\cdot)^+ = \text{Max}[0,\cdot] \]

### 2. Option Pricing Approaches

Let \( t_n \) be the time point when \( n \)th capacity addition is initiated. Then assuming that at most one expansion is in progress at any one time, \( t_n + H < t_{n+1} \). The installed capacity at time \( t \) equals \( K_n \) for \( t_n + H \leq t < t_{n+1} + H \). Then the shortage at time \( t \) is \((D(t) - K_{n-1})^+\) for \( t_n + H \leq t < t_{n+1} + H \). The total shortage during the \( n \)th lead time is given by:

\[
\int_{t_n}^{t_{n+H}} (D(t) - K_{n-1})^+ \, dt
\]

However, the expectation of this integral cannot be evaluated in closed form. In this section, four kinds of shortage measures are introduced. Figure 1 illustrates the general idea of each option pricing method for estimating shortage.

#### 2.1 European Option

Ryan [4] showed that the value of a European call option on a stock is mathematically identical to the expected quantity of shortage at the end of lead time. Therefore, the expected shortage at the end of lead time using the Black-Scholes option pricing formula is given by [7],

\[
S_n^{(E)} = \mathbb{E}[(D(t_n + H) - K_{n-1})^+] = e^{\sigma \sqrt{H}} D(t_n) \Phi(h) - K_{n-1} \Phi(h - \sigma \sqrt{H}) ,
\]

where \( h = \log(D(t_n) / K_{n-1}) + (g^2 / 2H) \sigma \sqrt{H} \). The European option is not completely satisfactory, however, since it only focuses on the potential shortages at the end of the lead time.
2.2 Asian Option
An Asian option is a path-dependent option whose payoff depends on the path of asset prices over a pre-specified time horizon. The payoff of this option is a function of the average of prices taken at various points in time and hence “average option” is also a term frequently used to describe it. In contrast to the snapshot provided by the European option, the Asian option includes more information about demand during the lead time.

The value of a geometric Asian option is mathematically identical to the expected value of the difference between the geometric average of demand during the specified time interval and current capacity. By using the Asian option value we consider potential shortages during the lead time as well as at its end. The estimated shortage using Asian option is given by

\[ S^{(t)}_n = E[(G(t_n) - K_{n-1})^+] \]

\[ G(t_n) = D(t_n + H - N + 1)D(t_n + H - N + 2) \cdots D(t_n + H - 1)D(t_n + H) \]

: geometric average of demand for the time interval \([t_n + H - N + 1, t_n + H]\)

\[ S^{(t,N)}_n = e^\mu e^{\sigma^2} \Phi(x_i) - K_{n-1} \Phi(x_i - \sigma_G) = D(t_n)e^{\frac{\sigma^2}{2}} \Phi(x_i) - K_{n-1} \Phi(x_i - \sigma_G), \]

where

\[ x_i = \frac{\log D(t_n) + \left( \mu - \frac{1}{2} \sigma^2 \right) \left( \frac{H - t_i}{2} + t_i \right) + \sigma_G}{\sigma_G} \]

\[ \sigma_G = \sigma \left( \frac{(H - t_i)(2N - 1)}{6N} + t_i \right) \]

Though an arithmetic average might be preferred, no exact closed form expression for its option value is available.

2.3 Lookback Option
A lookback option is another example of a path-dependent option like the Asian option. In this section we use a fixed-strike lookback call option whose payoff is based on highest value of the index (demand) during the life of the option. Therefore a fixed-strike lookback call option can be seen as a European call option on the highest value of the demand during the lead time. The value of fixed-strike lookback call option is mathematically identical to the expected highest capacity shortage during the lead time [8]. According to the price of a full fixed-strike lookback call, the estimated highest shortage is given by,

\[ S^{(t)}_n = E[(M^{(t)}_n - K_{n-1})^+] \]

where \( M^{(t)}_n = \max(D_t : t_n \leq t \leq t_n + H) \). This leads to the following pricing formula

\[ S^{(t)}_n = e^{\mu t} \Phi(d_1) - K_{n-1} \Phi(d_2) + \frac{\sigma^2}{2r} D(t) \left( \frac{D(t)}{K_{n-1}} \right)^{2r} \Phi \left( d_1 - \frac{2rH}{\sigma \sqrt{H}} \right) + e^{\mu t} \Phi(d_1), \]

where

\[ d_1 = \frac{\log \left( \frac{D(t)}{K_{n-1}} \right) + \left( r + \frac{1}{2} \sigma^2 \right) H}{\sigma \sqrt{H}}, \quad d_2 = d_1 - \sigma \sqrt{H}. \]

2.4 Total Shortage by Summing European Options
The previous three sections give us three kinds of estimation for expected capacity shortage, but what we get from these methods are not total expected shortage but some aspect of shortages such as shortage at the end of lead time, shortages based on average demand for a fixed time interval and the highest shortage during the lead time. In this section, we suggest a method for summing of European option, with which we can approximate the total expected shortage during the entire lead time.

The total expected shortage at \( t^{th} \) lead time is

\[ \int_{t_n}^{t_{n+1}} E[D(t) - K_{n-1}]^+ dt. \]

This integral can be approximated by subdividing into short intervals.
where 

$$S_n^{(T)} = \sum_{j=1}^{H/\Delta t} e^{-\gamma \Delta t} E \left[ (D(t_n + i\Delta t) - K_{n-1})^+ \right] \Delta t = \sum_{j=1}^{H/\Delta t} \left( e^{-(\gamma - \sigma^2/2)i\Delta t} D(t_n) \Phi(h_j) - K_{n-1} \Phi(h_j - \sigma \sqrt{i\Delta t}) \right) \Delta t ,$$

and 

$$h_j = \frac{\log(D(t_n)/K_{n-1}) + (g + \sigma^2/2)i\Delta t}{\sigma \sqrt{i\Delta t}} .$$

We obtain a more accurate approximation of this integral by subdividing the lead time into smaller time units. As $H$ is subdivided more finely, Figure 2 shows that $S_n^{(T)}$ converges to the actual total expected shortage as the subintervals are shortened.

In numerical tests, the error percentage of daily-based with respect to the hourly based method does not exceed 10%. Using the optimal demand-to-capacity ratio as derived below, it does not exceed 5%. But the execution time to calculate the shortage and optimal parameters of hourly-based sum of European options is approximately 30 times longer than that for the daily based model. It is a reasonable compromise to use the daily-based sum of European options to estimate the total expected discounted shortage.

### 3. Capacity Expansion Policies and Costs

#### 3.1 Expansion Policies

Since we assume exponential growth of demand, the capacity installation trend should also be exponential in order that service not deteriorate. For this reason, shortage should be measured in terms of the ratio of potential shortage to existing capacity rather than in absolute terms. When expressed as a portion of installed capacity, our four kinds of shortage measure divided by existing capacity, $S_n/K_{n-1}$, all depend only on the ratio of demand to installed capacity, $D(t)/K_{n-1}$, independent of $n$. The observation suggests the following:

**Timing policy:** The $n^{th}$ expansion installment time ($t_n$) is the minimum value of time when $D(t) = \gamma K_{n-1}$. Here, $\gamma$ is a decision variable.

Since expansion lead times do not overlap each other under timing policy, for some small probability $\alpha$, the $n^{th}$ expansion size, $U_n$, must satisfy

$$\mathbb{P}(D(t_n + H) > \gamma (K_{n-1} + U_n)) < \alpha .$$

This leads to the following requirement for $U_n$:

$$U_n > \frac{D(t_n)}{\gamma} \left( e^{\gamma H} - 1 + z e^{\gamma H} \sqrt{e^{2\gamma H} - 1} \right).$$

Therefore we obtain:

**Size policy:** The $n^{th}$ expansion size is given by

$$U_n = K_{n-1} \left( e^{\gamma H} - 1 + z e^{\gamma H} \sqrt{e^{2\gamma H} - 1} \right),$$

where $z > z_0$.

#### 3.2 Infinite Horizon Discounted Shortage and Capacity Expansion Cost

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![Figure 2. Convergence of shortage estimates by summing European options.](image)
According to Ryan [4] with the previous policies, the infinite horizon expected discounted shortage is given by
\[ v(y,z) = \sum_{n=1}^{\infty} e^{-\rho H} \left( \frac{D(0)}{\gamma K_{n-1}} \right)^\rho \frac{S_n}{K_0} \left( \frac{D(0)}{K_0} \right)^\rho \frac{e^{-\rho H} y^\rho}{1 - V(z)^{1-\rho}} \frac{S_n}{K_0} = \left( \frac{D(0)}{K_0} \right)^\rho K_0 y(y,z), \]
where, \( V(z) = e^{\rho H} \left( 1 + z \sqrt{e^{2\rho H} - 1} \right) \) and \( \left( \frac{D(0)}{\gamma K_{n-1}} \right)^\rho \) is a discount factor for the \( n^{th} \) expansion. The infinite horizon expected capacity expansion cost is given by:
\[ u(y,z) = \sum_{n=1}^{\infty} \left( \frac{D(0)}{\gamma K_{n-1}} \right)^\rho U_n = \left( \frac{D(0)}{K_0} \right)^\rho K_0 \gamma^\rho (V(z)-1)^\rho \frac{S_n}{K_0} = \left( \frac{D(0)}{K_0} \right)^\rho K_0 f(y,z). \]

4. Optimal Policy Parameters
The objective is to find policy parameters for each method that can minimize the weighted sum of cost and shortage,
\[ w(y,z) = u(y,z) + p'v(y,z), \]
where \( p' \) represents the cost of unit of capacity shortage at the end of a lead time. The objective function can be restated as,
\[ w(y,z) = \left( \frac{D(0)}{K_0} \right)^\rho K_0 f(y,z) + p' \left( \frac{D(0)}{K_0} \right)^\rho K_0 y(y,z). \]

Thus, minimizing \( w \) is equivalent to minimizing
\[ w(y,z) = f(y,z) + p' y(y,z) \]
where \( p \) is a dimensionless penalty factor. For example, \( p=5 \) if a shortage in the amount of \( K_0 \) is five times as costly as installing \( K_0 \) units of capacity.

Figure 3 shows the optimal parameters found by summing European options with different subintervals of the lead time, \( H \). The plots converge to that of the smallest time unit (hourly). The plot of optimal parameters by daily-based sum of European options is very close to that of the hourly-based model. Considering the difference in execution time, the daily-based model appears reasonable for estimating the total shortages.

To compare the four option pricing methods, we used annual based parameter values \( \mu=0.05, \sigma=0.2, H=0.5 \) year, and \( \alpha=0.7 \). Figures 3 and 4 show that as \( p \) increases, the optimal \( \gamma \) decreases so that the expansion occurs earlier. The lookback option method yields smaller optimal \( \gamma \) values than either the European or the Asian option method, which is reasonable as its estimate of expected shortage is greater than European or Asian (Figure 1). The method of summing European options estimates the total shortage during the lead time and its optimal \( \gamma \) is smaller
than three other options. In fact, for representative policy parameters of $\gamma = 0.76$ and $z = 4.62$, the shortage estimate obtained by summing European options (daily) is nearly 1000 times that of the European option, while the Asian and lookback estimates are 0.3 and 2.5 times as large as the European estimate, respectively.

In an industry in which capacity shortage could strike a fatal blow, decision makers are quite risk-averse. Summing European options may provide the most appropriate estimate of lead time shortage.

![Figure 4. Comparison of optimal policy parameters found for different shortage measures with $p = 2, 3, 4, 5$ and 10.](image)

**Acknowledgements**

The National Science Foundation under grant number DMI-9996373 supported this work.

**References**


