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THE NUCLEUS IN ALTERNATIVE RINGS WITH IDEMPOTENT

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It is difficult to construct examples of nonzero alternative rings R whose nucleus N is zero. Ževlakov, Slin'ko, Šestakov and Širšov[6] have given one such example. Since the nucleus plays such a central role in the structure theory of alternative rings, it seems reasonable to ask for some sufficient conditions on R that will guarantee $N \neq 0$. Somehow characteristic two seems to be different and it is necessary to impose characteristic not two on R, by which we understand that there should exist no elements whose additive order is two. Then any one of the following three conditions turns out to be sufficient:

(i) That R contain an idempotent $e \neq 0$,

(ii) that R be an algebra over a field, whose nil radical is finitely generated,

(iii) that R have descending chain conditions on two-sided ideals.

No doubt there must be many other sufficient conditions, but these are enough to establish our statement that indeed $N = 0$ happens only rarely for alternative rings. Perhaps the most striking of these conditions is the first, since it does not involve any structure theory, either in the statement or the proof.

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We now give some brief clues about the proof of condition (i). A detailed account of this and other assertions will appear elsewhere. We remind the reader that in every alternative ring with idempotent e one has the Peirce decomposition of R into a direct sum of submodules

$$R = R_{11} \oplus R_{10} \oplus R_{01} \oplus R_{00},$$

where e acts as a left identity or annihilator, depending on whether the left subscript of the submodule is 1 or 0, and as a right identity or annihilator, depending on whether the right subscript of the submodule is 1 or 0. Further details may be found in [1]. Basic to our proof is the identity that for all

a,b,c,d,f in R_{10}

(1) $(a,[cd]f+f[cd],b) = 0.$

Interestingly enough this identity fails for rings of characteristic two, but holds in rings of characteristic not two. This also has an implication for free alternative rings with idempotent and free generators a,b,c,d,f in the designated submodule, for then identity (1) results in a torsion element of order two. Next we use (1) to establish

(2) $n = [a_{10}b_{10}][x_{01}y_{01}] + [x_{01}y_{01}][a_{10}b_{10}]$ lies in N.

In general however n need not be in the center of R [3]. If N = 0, then all elements of the type given by (2) are zero. Then one can show in successive stages that all elements of the form $x_{10}(y_{10}z_{10}) + (y_{10}z_{10})x_{10}$ must be in N, hence zero until finally R_{10} and R_{01} are in N and hence zero, at which point e must be in N, hence zero. We have reached a contradiction, so that (i) is established. Conditions (ii) and (iii) may be proved by appealing to [2], [4], [5], [6] and (i). Two further identities are worth singling out:

(3) $p = a_{10}([b_{10}c_{10}]d_{10}) - b_{10}([c_{10}d_{10}]a_{10}) + c_{10}([d_{10}a_{10}]b_{10}) - d_{10}([a_{10}b_{10}]c_{10})$ lies in $N \cap R_{10}$,

(4) $(a_{10},b_{10},c_{10})(d_{10},f_{10},g_{10})$ lies in the commutative center of R.

By skillfully combining (3) and (4) it can be proved that in every free alternative ring with idempotent, assuming characteristic $\neq 2,3$ and at least four generators, any four elements of R_{10} satisfy a dependence relation over the center. These results also lead to a new proof of Albert's classification of simple alternative rings with idempotent [1], replacing simplicity with more general hypotheses.

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