Electrical and Computer Engineering Publications   Electrical and Computer Engineering

2016

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Abstract

We consider a single hop wireless X network with K transmitters and J receivers, all with single antenna. Each transmitter conveys for each receiver an independent message. The channel is assumed to have constant coefficients. We develop interference alignment scheme for this setup and derived several achievable degrees of freedom regions. We show that in some cases, the derived region meets a previous outer bound and hence is the DoF region. For our achievability schemes, we divide each message into streams and use real interference alignment on the streams. Several previous results on the DoF region and total DoF for various special cases can be recovered from our result. We also presented several novel extensions that enable us to achieve more points of the DoF region. Next, we consider a network model such that each transmitter emits an arbitrary number of messages and each receiver can request an arbitrary subset of the all the emitted messages. We term this network the X network with multicast. We derived an achievability result for the DoF region for such networks, as well as an outer bound result. Finally we discuss some points on the outer bounds that are not achievable with any of the alignment schemes that we presented.

Index Terms

real interference alignment, degrees of freedom region, wireless X network, stream alignment

I. INTRODUCTION

A wireless X network models a single-hop wireless network such that every transmitter conveys an independent message for every receiver. The X network model includes the broadcast channels, multiple access channels, and the interference channels as special cases. It is therefore useful to quantify the capacity limits of X networks. However, this is a difficult problem because even the capacity region for the broadcast channel, which is a special case of the X network, has not been characterized in full generality.

Generally speaking, single-letter characterizations of capacity regions for many multi-user information-theoretic problems have eluded us. A recent line of attack focuses on Gaussian networks in the asymptotic regime where the signal to noise ratio (SNR) goes to infinity. The communication rates are normalized by $\log(\text{SNR})$ to yield a quantity defined as the *degrees of freedom* (DoF). The shape of the capacity region normalized by $\log(\text{SNR})$ as SNR goes to infinity is defined as the *DoF region*. The total DoF and in some cases the DoF region for several channels have been characterized recently. One important technique for proving the achievability results is *interference alignment*, which seeks to align the dimensions of interference signals so that more dimensions are available in the subspace unaffected by interference. Two commonly used interference alignment methods are vector interference alignment, e.g., [1], [2], and the real interference alignment [3]–[6].

For the DoF problem of wireless X network, several results are available. An outer bound for multiple-input multiple-output (MIMO) X network has been derived in [7], which also developed schemes for achieving the maximum total DoF for single antenna X network. For constant single-antenna channels, a real interference alignment scheme has been used in [6] to establish the maximum total DoF. For MIMO X networks, outer bounds and achievability schemes have been developed in [8] for the 2×2 MIMO X network. The DoF region for an $M \times 2$ X network with N_1 and N_2 antennas at the two receivers is available as a special case of the result in [9]. Antenna splitting argument has been used in [7] to establish a lower bound on the total DoF of MIMO X network. For the time-varying X network with multicast when each transmitter emits only one message, the DoF region has been obtained in [10].

The X network model can be made more general by allowing each transmitted message to be requested by more than one receiver. In fact, one may consider a model where each transmitter conveys an arbitrary

number of messages, and each receiver requests an arbitrary subset of all the messages sent by all transmitters. We termed such networks *wireless X network with multicast*.

In this paper, we first consider the single-antenna wireless X networks without multicast, and derive several achievability schemes based on real interference alignment. The achieved DoF regions are shown to be tight when the number of receivers is two. Certain points in the achievable region are shown to be optimal (they are on the boundary of the DoF region). Several previous results (or their constant channel counter parts) can be recovered as special cases. Several extensions that enable us to achieve more points in the DoF region are presented. Next, we investigate the single-antenna wireless X network with multicast and derive achievability results based on real interference alignment. We finally discuss some challenging examples.

II. SYSTEM MODEL

Notation: Throughout the paper, J and K will be integers and $\mathcal{J} = \{1, \ldots, J\}$, $\mathcal{K} = \{1, \ldots, K\}$. We use k, $\tilde{k}$, $\hat{k}$, k' as transmitter indices, and j, $\tilde{j}$, $\hat{j}$ as receiver indices. The set of integers and real numbers are denoted as $\mathbb{Z}$ and $\mathbb{R}$, respectively. We use $[d_{j,k}]$ to denote a matrix with element $d_{j,k}$ in the (j, k)th position, and use $[d_{j,k}]_{j=1,k=1}^{J,K}$ to make the size of the matrix explicit. Letter l will be reserved for the index of streams (parts of a message). Throughout the paper, a.e. means almost everywhere in the Lebesgue sense for the channel matrix. □

Consider a single-antenna wireless X network with K transmitters and J receivers. For each pair $(j, k) \in \mathcal{J} \times \mathcal{K}$, transmitter k conveys an independent message $m_{j,k}$ for receiver j. The channel from transmitter k to receiver j is denoted as $h_{j,k}$. The whole set of channel coefficients is denoted as a matrix

$$\mathbf{H} := [h_{j,k}]_{j=1,k=1}^{J,K}. \tag{1}$$

All the quantities are real in this paper. So $\mathbf{H} \in \mathbb{R}^{J \times K}$. The channel is assumed constant (non-fading) throughout the whole transmission. Each transmitter k transmits a symbol $x_{k,t}$ in one discrete time slot. Each transmitter has an average power constraint P so that for any transmission that spans $N \in \mathbb{Z}$ symbols, the transmitted symbols satisfy

$$\sum_{t=1}^{N} \frac{1}{N} |x_{k,t}|^2 \leq P, \quad \forall 1 \leq k \leq K. \tag{2}$$

The received signal at receiver j at time t can be written as

$$y_{j,t} = \sum_{k \in \mathcal{K}} h_{j,k} x_{k,t} + \nu_{j,t}, \quad \forall j \in \mathcal{J} \tag{3}$$

where $\{\nu_{j,t} | j \in \mathcal{J}\}$ is the set of additive noises, assumed to be independent and identically distributed according to zero mean Gaussian distribution with unit variance.

A code of length N and message sizes $[M_{j,k}]$ consists of

1) the encoders $\{f_k | k \in \mathcal{K}\}$, where f_k is a mapping from the set of messages to be conveyed by transmitter k, $[1, M_{1,k}] \times, \dots, \times [1, M_{J,k}]$, to the set of transmitted symbols (codewords) in $\mathbb{R}^N$. All codewords satisfy the power constraint.

2) the decoders $\{g_{j,k} | j \in \mathcal{J}, k \in \mathcal{K}\}$, where $g_{j,k}$ is a mapping from the set $\mathbb{R}^N$ of received symbols at receiver j to the set of messages $[1, M_{j,k}]$ intended for receiver j from transmitter k.

The rate of message $m_{j,k}$ is defined as

$$R_{j,k} = \frac{1}{N} \log_2(M_{j,k}), \tag{4}$$

and the DoF of message $m_{j,k}$ is defined as

$$d_{j,k} = \frac{R_{j,k}}{0.5 \log(1 + P)}. \tag{5}$$

We use $[W_{j,k}]$ to denote a set of messages such that $W_{j,k}$ is independently and uniformly chosen from $[1, M_{j,k}]$. The system decoding is viewed as correct if all of the messages are decoded correctly by their intended receivers. Otherwise, the decoding is viewed as erroneous. The probability of error P_e of the code is therefore

$$\Pr \left[g_{j,k} \left(\sum_{k \in \mathcal{K}} h_{j,k} f_k([W_{j,k}]_{j=1}^J) + [v_{j,t}]_{t=1}^N \right) \neq W_{j,k} \text{ for some } (j, k) \in \mathcal{J} \times \mathcal{K} \right].$$

The code we have thus defined will be denoted as a

$$(P, N, [M_{j,k}], [f_k], [g_{j,k}]) \tag{6}$$

code. A point $[d_{j,k}] \in \mathbb{R}^{J \times K}$ is said to be *achievable* if for any $\epsilon > 0$, and for all large enough P, there is a $(P, N, [M_{j,k}], [f_k], [g_{j,k}])$ code whose DoF for message $m_{j,k}$ according to (4) and (5) is at least $d_{j,k} - \epsilon$, and whose probability of error P_e is at most ϵ. The set of achievable DoF points is called the *DoF region*. It should be clear that the DoF region is a closed set by definition.

III. DoF Region of X Network without Multicast

In this section, we will present our result on DoF region for single-antenna X network without multicast. The case with multicast will be discussed in Sec. VI.

A. Statement of result

Theorem 1 (An achievable DoF region). *For a K-transmitter J-receiver constant-coefficient single-antenna wireless X network $\mathbf{H} \in \mathbb{R}^{J \times K}$, the DoF region $\mathcal{D}$ satisfies $\mathcal{D} \supset \mathcal{D}^{(in)}$ a.e., where $\mathcal{D}^{(in)}$ is a set of matrices $[d_{j,k}]_{j=1,k=1}^{J,K}$ such that*

1) *all entries are non-negative;*
2) *$\forall 1 \leq j \leq J$, the following inequality holds:*

$$\sum_{\tilde{k}=1}^{K} d_{j,\tilde{k}} + \sum_{\tilde{j} \in \mathcal{J}, \tilde{j} \neq j} \max_{k} d_{\tilde{j},k} \leq 1. \tag{7}$$

B. Main ideas

Our achievability proof uses the following ideas:

1) We use *real interference alignment*, a technique that has been initiated in [3], and further developed for interference problems in [4]–[6]. This technique models the single-antenna systems to pseudo multiple-antenna systems with an arbitrary number of pseudo antennas, where each antenna is capable of transmitting at a rate of a fractional degrees of freedom. These fractional "dimensions" are designed so that interference signals at a receiver are aligned in a way that is quite analogues to how the interference signals are aligned in a multi-antenna interference channel with vector interference alignment.

2) We split each message into *streams*, where all streams have the same DoF. This allows us to design achievability schemes for unequal DoFs. This idea has been used in e.g., [10].

3) The interference alignment at the receivers is stream-based. Several streams from different transmitters are aligned. Streams from the same transmitter are never aligned. Otherwise decodability of the aligned messages at other receivers will be compromised.

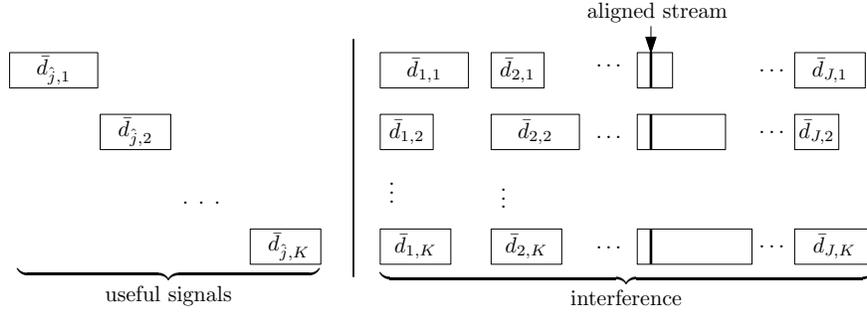4) We use a construction that involves "dimension padding" to guarantee that all streams have the same DoF.

Fig. 1. Interference alignment at receiver $\hat{j}$

C. The proof

We prove that for any $[d_{j,k}] \in \mathcal{D}^{(\text{in})}$, $[d_{j,k}]$ is achievable. We assume that all the elements of $[d_{j,k}]$ are rational numbers. Otherwise, if some elements are irrational, the proof here can be used to establish achievability of a point that is arbitrarily close to $[d_{j,k}]$, which means that $[d_{j,k}]$ also belongs to the DoF region by the definition of the DoF region. Under the rational assumption, we can find an integer κ such that for all $j \in \mathcal{J}$ and all $k \in \mathcal{K}$, $\bar{d}_{j,k} := \kappa d_{j,k}$ is a non-negative integer.

ENCODING: For each $(j,k) \in \mathcal{J} \times \mathcal{K}$, the message $m_{j,k}$ is divided into $\bar{d}_{j,k}$ parts as $\{m_{j,k,l}, l = 1, \ldots, \bar{d}_{j,k}\}$. Each part is called a *stream*. The signal emitted by transmitter k is in the following form

$$x_k = \sum_{j \in \mathcal{J}} x_{j,k} = \sum_{j \in \mathcal{J}} \sum_{l=1}^{\bar{d}_{j,k}} \alpha_{j,k,l} x_{j,k,l} \tag{8}$$

where $x_{j,k,l}$ carries the symbols of stream l of the message from transmitter k to receiver j, and $\{\alpha_{j,k,l}\}$ are design parameters that can be chosen randomly and independently according to certain continuous distribution, e.g., uniformly from the interval $[\frac{1}{2}, 1]$. The symbol $x_{j,k,l}$ is generated using elements (called *directions* [6]) specified in a set $\mathcal{T}_{j,k,l}$ (to be specified later) as follows:

$$x_{j,k,l} = \sum_{\delta_b \in \mathcal{T}_{j,k,l}} \delta_b u_{j,k,l,b} \tag{9}$$

where $u_{j,k,l,b} \in \{\lambda q | q \in \mathbb{Z}, -Q \le q \le Q\}$, and Q and λ are parameters to be specified appropriately later to satisfy the rate and power requirements. In the summation in (9), we have assumed that the directions in $\mathcal{T}_{j,k,l}$ have been indexed from 1, and b is the index of the direction of δ_b. The exact indexing scheme is of no importance.

STREAM ALIGNMENT: Consider an arbitrary receiver $\hat{j}$. The signal dimensions situation is shown in Fig. 1. The useful signals have DoF $\sum_{k \in \mathcal{K}} \bar{d}_{\hat{j},k}$. The interferences coming from different transmitters are shown on the right. The streams intended for the same receiver $j \neq \hat{j}$ are aligned together at receiver $\hat{j}$.

DIMENSION PADDING: To facilitate the construction of the transmission directions, we introduce an idea that we term *dimension padding*. Specifically, we notice that in the interference part in Fig. 1, the messages intended for the same receiver $j \neq \hat{j}$ in general do not have the same number of streams. To make sure that such disparity does not lead to difference in the achieved DoF for these messages, we introduce some fictitious streams so that with these additional streams the constructed transmission symbols for all *actual* streams use the same number of directions. These fictitious streams only aid in the construction of the transmission directions. No symbols are transmitted for these streams, otherwise the useful signal space dimension will become smaller (the interference space dimension remains unchanged though).

More specifically, we assume all messages $\{m_{j,k} | k \in \mathcal{K}\}$ intended for receiver j has the same number s_j of streams, where

$$s_j = \max_k \bar{d}_{j,k}. \tag{10}$$

For transmitter k, the first $\bar{d}_{j,k}$ of these s_j streams are actual transmitted streams. The remaining ones (if any) are virtual streams, whose transmitted symbols are all set to zero [c.f. (9)]:

$$u_{j,k,l,b} = 0, \quad \forall l \in [\bar{d}_{j,k} + 1, s_j]. \tag{11}$$

We assume that $\alpha_{j,k,l}$ is assigned for a virtual stream in the same way as for an actual stream.

TRANSMIT DIRECTIONS: Let n denote an integer. We design the directions $\mathcal{T}_{j,k,l}$ used by stream $m_{j,k,l}$ as follows

$$\left\{ \prod_{\hat{j} \in \mathcal{J}, \hat{j} \neq j} \prod_{\hat{k} \in \mathcal{K}} \left(h_{\hat{j},\hat{k}} \alpha_{j,\hat{k},l} \right)^{\beta_{\hat{j},\hat{k}}} \, \middle| \, \beta_{\hat{j},\hat{k}} \in [0, n-1], \hat{j} \in \mathcal{J}, \hat{j} \neq j, \hat{k} \in \mathcal{K} \right\} \tag{12}$$

It can be seen that there are totally $n^{K(J-1)}$ directions in $\mathcal{T}_{j,k,l}$ for all (j,k,l). The reason for doing dimension padding can be seen more clearly now as it leads to the same number of directions to be used by all streams. This will guarantee that each stream corresponds to the same DoF in the final result. We also remark that the set of directions as defined in (12) is actually independent of the transmitter index k, which means that for the same stream index l and receiver index j, all transmitters use same set of

directions to convey the l-th stream of the message intended for receiver j. This facilities the interference alignment at all receivers on the l-th stream.

ALIGNMENT VERIFICATION: The proposed design above guarantees that the interferences created by messages intended for the same receiver are aligned as desired at all receivers. To see this, for $j \in \mathcal{J}$, and $1 \leq l \leq s_j$, define $\hat{\mathcal{T}}_{j,l}$ as follows

$$\left\{ \prod_{\hat{j} \in \mathcal{J}, \hat{j} \neq j} \prod_{\hat{k} \in \mathcal{K}} \left(h_{\hat{j}, \hat{k}} \alpha_{j, \hat{k}, l} \right)^{\beta_{\hat{j}, \hat{k}}} \,\bigg|\, \beta_{\hat{j}, \hat{k}} \in [0, n], \hat{j} \in \mathcal{J}, \hat{j} \neq j, \hat{k} \in \mathcal{K} \right\} \tag{13}$$

The definition of the set $\hat{\mathcal{T}}_{j,l}$ is the same as that of $\mathcal{T}_{j,k,l}$ except that the range for $\beta_{\hat{j}, \hat{k}}$ is changed from $[0, n-1]$ to $[0, n]$. For a given j, for any receiver $\hat{j} \neq j$, the set $\hat{\mathcal{T}}_{j,l}$ contains all the interference directions due to the l-th message streams coming from all transmitters. At any particular $\hat{j} \neq j$ and stream l, not all directions in $\hat{\mathcal{T}}_{j,l}$ are present in the interference signal. A more refined analysis is possible to list more specifically which directions are actual interference directions and which are not. But our definition of $\hat{\mathcal{T}}_{j,l}$ is good enough to establish the achievability results.

According to (9), a symbol from stream (j, k, l) is transmitted in a direction of the form $\alpha_{j,k,l} T$ where $T \in \mathcal{T}_{j,k,l}$. This symbol will arrive at receiver $\hat{j}$, $\hat{j} \neq j$, in the direction of $\left(h_{\hat{j}, k} \alpha_{j,k,l} \right) T$, which is in $\hat{\mathcal{T}}_{j,l}$ because the power for $\left(h_{\hat{j}, k} \alpha_{j,k,l} \right)$ will be simply increased by one after the symbol goes through the channel.

DECODABILITY: The useful signals at receiver $\hat{j}$ will be generated by directions in $\mathcal{T}'_{\hat{j}}$, where

$$\mathcal{T}'_{\hat{j}} = \bigcup_{k \in \mathcal{K}} \left\{ \left(h_{\hat{j}, k} \alpha_{\hat{j}, k, l} \right) T \,\bigg|\, T \in \mathcal{T}_{\hat{j}, k, l} \right\} . \tag{14}$$

Since none of the $\mathcal{T}_{\hat{j}, k, l}$ contains a generator $\left(h_{\hat{j}, k} \alpha_{\hat{j}, k, l} \right)$ [recall the condition $\hat{j} \neq j$ in (12)], and for different k, $\left(h_{\hat{j}, k} \alpha_{\hat{j}, k, l} \right)$ are different, we conclude that for any $\hat{j} \in \mathcal{J}$, all directions in $\mathcal{T}'_{\hat{j}}$ are rationally independent of those in $\bigcup_{j,l} \mathcal{T}_{j,l}$, almost surely for all channel realizations, and with probability one for the choices of the $\alpha_{k,j,l}$. Such rational independence is sufficient to guarantee the desired DoF for the useful messages.

The total rational dimensions $D_{\hat{j}}$ of both the useful signals and the interference at any receiver $\hat{j}$ satisfies

$$D_{\hat{j}} \leq \sum_{\tilde{k}=1}^{K} \bar{d}_{\hat{j}, \tilde{k}} n^{K(J-1)} + \sum_{j \in \mathcal{J}, j \neq \hat{j}} \max_k \bar{d}_{j,k} (n+1)^{K(J-1)}.$$

We define

$$S = \max_{\hat{j} \in \mathcal{J}} \left(\sum_{\tilde{k}=1}^{K} \bar{d}_{\hat{j},\tilde{k}} + \sum_{j \in \mathcal{J}, j \neq \hat{j}} \max_{k} \bar{d}_{j,k} \right), \tag{15}$$

which is an upper bound on the total number of useful signal streams and interference streams (multiple aligned streams are counted as one), maximized over all receivers. For any DoF point in $\mathcal{D}^{(\text{in})}$ that satisfies (7), we have $S \leq \kappa$. As a result, we have

$$D_{\hat{j}} \leq S(n+1)^{K(J-1)} \leq \kappa(n+1)^{K(J-1)} \tag{16}$$

With reference to the constellation symbols in (9), if we choose

$$\lambda = P^{\frac{1}{2}}/Q \tag{17}$$

then we can guarantee that the power constraint is satisfied. In addition, if for any $\epsilon \in (0, 1)$ we choose as in e.g., [6],

$$Q = P^{\frac{1-\epsilon}{2(m+\epsilon)}}, \tag{18}$$

where m is an integer, then we can guarantee that the DoF per stream is $\frac{1-\epsilon}{m+\epsilon}$. Choosing $m = \kappa(n+1)^{K(J-1)}$, the hard decoding error probability for the constellation symbols decreases to zero as $P \to \infty$ due to the Khintchine-Groshev type Theorems, see the discussion in e.g., [5], [6], and the DoF of the message $m_{j,k}$ can be arbitrarily close to

$$\lim_{n \to \infty} \frac{\bar{d}_{j,k} n^{K(J-1)}}{\kappa(n+1)^{K(J-1)}} = \frac{\bar{d}_{j,k}}{\kappa} = d_{j,k}, \tag{19}$$

for all $j \in \mathcal{J}$ and $k \in \mathcal{K}$ by increasing n. This completes the proof. □

IV. SOME SPECIAL CASES

An outer bound for the wireless X channel has been derived in [8]. It states that $\forall j \in \mathcal{J}, \forall k \in \mathcal{K}$:

$$\sum_{\tilde{k}=1}^{K} d_{j,\tilde{k}} + \sum_{\tilde{j}=1}^{J} d_{\tilde{j},k} - d_{j,k} \leq 1. \tag{20}$$

This result can be written in an alternative form as

$$\sum_{\tilde{k}=1}^{K} d_{j,\tilde{k}} + \max_{k} \sum_{\tilde{j} \in \mathcal{J}, \tilde{j} \neq j}^{J} d_{\tilde{j},k} \leq 1, \quad \forall j \in \mathcal{J}. \tag{21}$$

1) $K \times 2$ X channel

Comparing (7) and (21), it can be seen that the inner bound does not meet the outer bound in general. However, there are some special cases where they do meet. One such case is when $J = 2$. In this case, both bounds are given by

$$\sum_{\tilde{k}=1}^{K} d_{1,\tilde{k}} + \max_{k} d_{2,k} \leq 1, \tag{22}$$

$$\sum_{\tilde{k}=1}^{K} d_{2,\tilde{k}} + \max_{k} d_{1,k} \leq 1. \tag{23}$$

We summarize the result in the following.

Theorem 2 (DoF Region of $K \times 2$ X Network)**.** *The DoF region of the $K \times J$ wireless X network when $J = 2$ is the set of $[d_{j,k}]_{j=1,k=1}^{2,K}$ that have non-negative entries and satisfy both (22) and (23).*

2) Some boundary points on the general DoF region

Another case where the two bounds (7) and (21) meet is when $d_{j,k} = d_{j,\hat{k}}$, for all $j \in \mathcal{J}$ and for all $k, \hat{k} \in \mathcal{K}$. We have:

Theorem 3 (Some Boundary Points)**.** *The DoF region of the $K \times J$ wireless X network has the following points on the boundary: $[d_{j,k}]_{j=1,k=1}^{J,K}$ such that*

 i) *all entries are non-negative;*

 ii) *$d_{j,k} = d_{j,\hat{k}}$, for all $j \in \mathcal{J}$ and for all $k, \hat{k} \in \mathcal{K}$;*

 iii) *$(K-1) \max_{j \in \mathcal{J}} d_{j,1} + \sum_{j \in \mathcal{J}} d_{j,1} = 1$.*

This is true for Lebesgue almost everywhere $\mathbf{H} \in \mathbb{R}^{J \times K}$.

If we set all $d_{j,k} = 1/(J + K - 1)$, then we recover the total DoF of $d^{(\text{total})} = JK/(J + K - 1)$ of [6], [7].

V. Extensions

The alignment scheme presented so far Sec. III is only one possible alignment schemes within the class of real alignment. We have aligned the messages intended for the same receiver. However this is not always optimal and not necessary either. We propose some extensions of the alignment scheme that can yield potentially larger achievable DoF regions.

A. Permuted alignment

To see the insufficiency of the alignment scheme in Sec. III, consider a 3×3 X network. If we set all messages $m_{j,k}$ to have rate zero whenever $j \neq k$, then it becomes a 3-user interference channel. It is known [2] that per user DoF $1/2$ is achievable. Therefore, the following DoF point is within the DoF region of the 3×3 X network:

$$[d_{j,k}]^T = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}. \tag{24}$$

However, it can be seen that this point cannot be achieved using the scheme in Sec. III. To achieve this point, we can arrange the individual DoFs in each row so that it looks as follows (c.f. Fig. 1):

$$\begin{bmatrix} d_{1,1} & d_{2,1} & d_{3,1} \\ d_{2,2} & d_{1,2} & d_{3,2} \\ d_{3,3} & d_{1,3} & d_{3,3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}. \tag{25}$$

Note the matrix has been shown in its transposed form to agree with the arrangement in Fig. 1. The permutations applied to different rows can be different. To see that this point is achievable, we can check e.g., the situation at receiver 1 as depicted (for illustration only) in the following

$$\begin{bmatrix} \frac{1}{2} & - & - \\ - & 0 & - \\ - & - & 0 \end{bmatrix} \Big\| \begin{bmatrix} - & 0 & 0 \\ \frac{1}{2} & - & 0 \\ \frac{1}{2} & - & 0 \end{bmatrix} \tag{26}$$

where the left part represents the signal dimensions, and the right part represents the interference dimensions. The minus signs are a place holder that means "no signal". The dimensions on the left $(\frac{1}{2}, 0, 0)$ are $(d_{1,1}, d_{1,2}, d_{1,3})$, the DoF's that receiver 1 needs. These entries have been removed from the right part (replaced with minus signs). Counting the total dimensions by taking the maximum of all the DoF on each column, treating minus as 0, we have

$$\frac{1}{2} + 0 + 0 + \frac{1}{2} + 0 + 0 = 1, \tag{27}$$

which is acceptable. Similar verification can be performed for receiver 2 and 3 as well. As a result, the point as in (24) is achievable. The same argument can be made in a more general setting by considering all possible alignment arrangements of the interference signals. Formally, we have

Theorem 4 (Permuted Alignment). *For a K-transmitter J-receiver constant-coefficient single-antenna wireless X network $\mathbf{H} \in \mathbb{R}^{J \times K}$, the DoF region $\mathcal{D}$ satisfies $\mathcal{D} \supset \mathcal{D}_0^{(in)}$ a.e., where $\mathcal{D}_0^{(in)}$ is a set of matrices $[d_{j,k}]_{j=1,k=1}^{J,K}$ such that*

1) *All entries of it are non-negative;*

2) *There exist K permutations of J objects $\{\sigma_k(\cdot)|k \in \mathcal{K}\}$ such that $\forall 1 \leq \hat{j} \leq J$, the following inequality holds:*

$$\sum_{\tilde{k}=1}^{K} d_{\hat{j},\tilde{k}} + \sum_{\tilde{j} \in \mathcal{J}} \max \mathcal{I}_{\tilde{j}} \leq 1, \tag{28}$$

where $\mathcal{I}_{\tilde{j}} := \{d_{j,k}|k \in \mathcal{K}, j \in \mathcal{J}, j \neq \hat{j}, \sigma_k(j) = \tilde{j}\}$.

It should be obvious that if we choose the permutations to be the identity mapping (no permutation), then the result in Sec. 1 is recovered. For the purpose of comparison, we will term the alignment scheme in Sec. III the *natural alignment*.

B. Staggered alignment

In both the natural alignment and the permuted alignment, any message from any single transmitter is aligned with one and only one message from another transmitter. However, this can also be generalized. It is possible to align two users' messages so that one message from the first user is aligned with multiple messages from the other user.

Staggered alignment can achieve DoF point that are not achievable using the natural or permuted alignments. Consider a 3×4 X network. The point $[d_{j,k}]$ as follows

$$[d_{j,k}]^T = \frac{1}{10} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{29}$$

is in the DoF region. This can be established using a staggered alignment scheme as shown in Fig. 2. Using permuted alignment without message staggering, a DoF point that is proportional to the matrix in (29) will have a coefficient $1/11$ instead of $1/10$ in front.

VI. X Network with multicast

In this section, we will consider the X network with multicast, such that each transmitter emits an arbitrary number of messages and each receiver can request an arbitrary combination of the transmitted

$$\overbrace{\qquad\qquad\qquad\qquad\qquad}^{4/10}$$

$$\boxed{\qquad\qquad d_{1,1} \qquad\qquad}$$

$$\boxed{\quad d_{1,2} \quad} \boxed{\quad d_{2,2} \quad}$$

$$\boxed{d_{1,3}} \boxed{d_{2,3}} \boxed{d_{3,3}} \boxed{d_{4,3}}$$
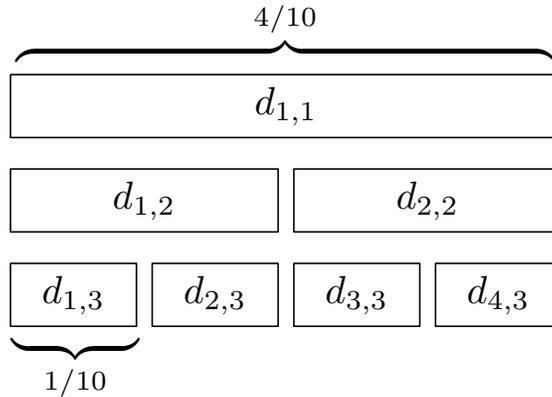
$$\underbrace{\qquad}_{1/10}$$

Fig. 2. Staggered alignment

messages from all transmitters. Since the number of messages from each transmitter is not necessarily equal to the number of receivers, we will use i, $\hat{i}$, i' as the indices of transmitted messages. In addition to l, we will also use $\hat{l}$ as the index of streams. G and I_k will be integers and $\mathcal{G} = \{1, \ldots, G\}$, $\mathcal{I}_k = \{1, \ldots, I_k\}$. The rest of notation will be as before.

Consider a single-antenna interference network with multicast such that there are K transmitters and J receivers. Assume transmitter $k \in \mathcal{K}$ has I_k messages with indices in $\mathcal{I}_k = \{1, \ldots, I_k\}$. Define

$$\mathcal{M} = \{(k,i) | 1 \le k \le K, 1 \le i \le I_k\}. \tag{30}$$

For each pair $(k,i) \in \mathcal{M}$, transmitter k sends a message $x_{k,i}$. As mentioned before, each receiver requests an arbitrary set of messages from multiple transmitters. Let $\mathcal{R}_j$ be the set of indices of the messages requested by receiver j;

$$\mathcal{R}_j = \{(k,i) \in \mathcal{M} | \text{receiver } j \text{ requests message } (k,i)\} \tag{31}$$

The channel model and power constraints are the same as the X network without multicast. The DoF region is also defined similar to that of the X network without multicast. Our goal is to quantify the DoF region of wireless X network with multicast.

We remark that the DoF region for the wireless X network with multicast when each transmitter only sends one messages has been derived in [10] for time-varying channels using vector interference alignment. Here, we present a more general case so that all transmitters can emit an arbitrary number of messages.
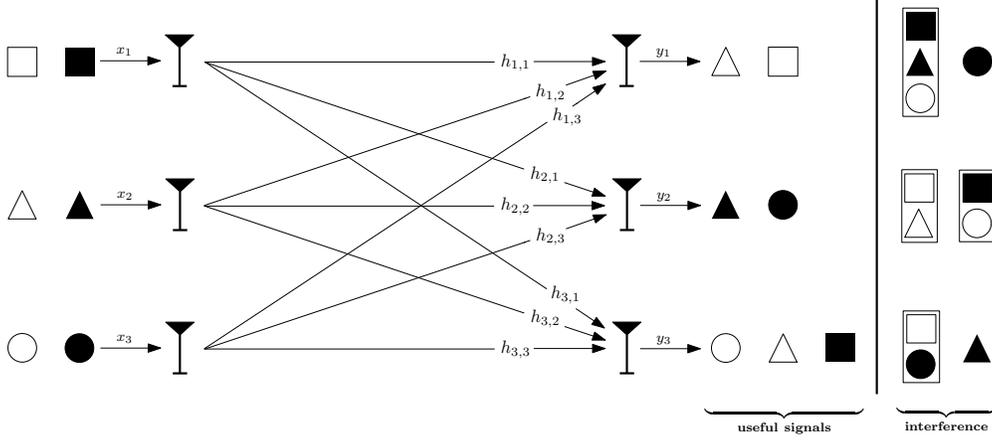
Fig. 3. 3×3 wireless X network with multicast

We will need the concept of partition. We say the sets $B_1, \ldots, B_N$ form a partition of a set B if

1) $\bigcup_{n=1}^{N} B_n = B$

2) $B_m \cap B_n = \Phi \; \forall m, n \in \{1, \ldots, N\}, \; m \neq n$

where Φ denotes the empty set.

A. Statement of result

Theorem 5. *Consider a K-transmitter J-receiver constant coefficient single antenna wireless X network with multicast, with the set of indices of messages as defined in (30). Let $\mathcal{X}_1, \ldots, \mathcal{X}_G$ be a partition of $\mathcal{M}$ such that $\mathcal{X}_g$, $g \in \mathcal{G}$ does not include more than one message emitted by any transmitter. The DoF region $\mathcal{D}$ satisfies $\mathcal{D} \supset \mathcal{D}^{(in)}(\{\mathcal{X}_g\})$ a.e., where $\mathcal{D}^{(in)}(\{\mathcal{X}_g\})$ contains all $\{d_{k,i} | (k, i) \in \mathcal{M}\}$ such that*

1) $d_{k,i} \geq 0, \quad \forall (k, i) \in \mathcal{M}$,

2) $\forall 1 \leq j \leq J$, *the following inequality holds:*

$$\sum_{\{(\tilde{k},\tilde{i}) \in \mathcal{R}_j\}} d_{\tilde{k},\tilde{i}} + \sum_{g \in \mathcal{G}} \max_{(\hat{k},\hat{i}) \in \mathcal{R}_j^c, (\hat{k},\hat{i}) \in \mathcal{X}_g} d_{\hat{k},\hat{i}} \leq 1 \tag{32}$$

where the set $\mathcal{R}_j^c$ contains all messages in $\mathcal{M}$ except those in $\mathcal{R}_j$.

Note that $\mathcal{X}_g$'s are arbitrary partition of the set $\mathcal{M}$. Different partitions may result in different achievable regions. The optimization with respect to the choice of partition will not be carried out in this paper.

B. The proof

We assume that all coefficients of $\{d_{k,i}|(k,i) \in \mathcal{M}\}$ are rational and prove that for any $\{d_{k,i}|(k,i) \in \mathcal{M}\} \in \mathcal{D}^{(in)}(\{\mathcal{X}_g\})$, $\{d_{k,i}|(k,i) \in \mathcal{M}\}$ is achievable. Based on rational assumption, it is possible to find κ such that for all $(k,i) \in \mathcal{M}$, $\bar{d}_{k,i} := \kappa d_{k,i}$ is a non-negative integer.

ENCODING: Analogous to the approach in Sec. III, for each $(k,i) \in \mathcal{M}$, the message $x_{k,i}$ is divided into $\bar{d}_{k,i}$ streams as $\{m_{k,i,l}|l = 1,\ldots,\bar{d}_{k,i}\}$. The message sent by transmitter k is shown as follow

$$x_k = \sum_{i=1}^{I_k} x_{k,i} = \sum_{i=1}^{I_k} \sum_{l=1}^{\bar{d}_{k,i}} \alpha_{k,i,l} x_{k,i,l} \tag{33}$$

where $x_{k,i,l}$ is the symbols of stream l of message $x_{k,i}$, and $\alpha_{k,i,l}$ is randomly, independently, and uniformly chosen from interval $[\frac{1}{2}, 1]$. Using directions in $\mathcal{T}_{k,i,l}$ which will be specified later, $x_{k,i,l}$ is generated in the following form

$$x_{k,i,l} = \sum_{\delta_b \in \mathcal{T}_{k,i,l}} \delta_b u_{k,i,l,b} \tag{34}$$

where $u_{k,i,l,b} \in \{\lambda q | q \in \mathbb{Z}, -Q \leq q \leq Q\}$, and Q and λ will be determined according to the rate and power constraints. Note that the directions of $\mathcal{T}_{k,i,l}$ are indexed and b is just the index of δ_b.

TRANSMIT DIRECTIONS: For the message $x_{k,i}$, we define $\mathcal{C}_{k,i}$ as the set of messages that are in the same group as does (k,i):

$$\mathcal{C}_{k,i} := \{(\hat{k},\hat{i})|(\hat{k},\hat{i}) \in \mathcal{X}_g, (k,i) \in \mathcal{X}_g, g \in \mathcal{G}\} \tag{35}$$

The stream $m_{k,i,l}$ only uses directions $\mathcal{T}_{k,i,l}$ that is designed in the following form

$$\prod_{\mathcal{C}_{k,i}} \prod_{\{\hat{j} \in \mathcal{J}|(\hat{k},\hat{i}) \in \mathcal{R}_{\hat{j}}^c\}} \left(h_{\hat{j},\hat{k}} \alpha_{\hat{k},\hat{i},l}\right)^{\beta_{\hat{k},\hat{i},\hat{j},k,i,l}} \tag{36}$$

where

$$0 \leq \beta_{\hat{k},\hat{i},\hat{j},k,i,l} \leq n-1, \tag{37}$$

$\forall (\hat{k},\hat{i}) \in \mathcal{M}, \forall \{\hat{j} \in \mathcal{J}|x_{\hat{k},\hat{i}} \in \mathcal{R}_{\hat{j}}^c\}$. Note that if for some $(\hat{k},\hat{i}) \in \mathcal{C}_{k,i}$, there is no $\hat{j}$ to be found in set $\{\hat{j} \in \mathcal{J}|(\hat{k},\hat{i}) \in \mathcal{R}_{\hat{j}}^c\}$, we simply do not include them in the set of directions. Clearly, the maximum number of total directions of $\mathcal{T}_{k,i,l}$ cannot be more than n^{KJ}. Considering dimension padding, we assume the total number of directions used by all streams are the same and equal to n^{KJ}.

ALIGNMENT VERIFICATION: To show our design guarantees that the interferences are aligned at all receivers, define $\hat{\mathcal{T}}_{k,i,l}$ similar to directions of (36) with (37) modified as follow

$$0 \leq p_{\hat{k},\hat{i},\hat{j},k,i,l} \leq n, \tag{38}$$

Based on our design, at receiver $\hat{j}$, we need to show messages that belong to each $\mathcal{X}_g$ are aligned as long as they are in set $\mathcal{R}_{\hat{j}}^c$. To verify, consider a symbol from stream $m_{k,i,l}$ transmitted by $\alpha_{k,i,l}T$ such that T is generated as (36) and the symbol also belongs to $\mathcal{X}_g$. We assume that the mentioned symbol is interference at receiver $\hat{j}$. This symbol will obviously arrive at receiver $\hat{j}$ in form of $\left(h_{\hat{j},k}\alpha_{k,i,l}\right)T$ which includes in $\hat{\mathcal{T}}_{k,i,l}$ since i) our design guarantees that $\left(h_{\hat{j},k}\alpha_{k,i,l}\right)$ is one of the elements used in (36) with maximum power $n-1$, and ii) the power of $\left(h_{\hat{j},k}\alpha_{k,i,l}\right)$ is increased by one after the symbol goes through the channel. Hence, the symbol from stream $m_{k,i,l}$ is aligned with common messages in sets $\mathcal{X}_g$ and $\mathcal{R}_{\hat{j}}^c$. Note that at receiver $\hat{j}$, the symbols of different $\mathcal{X}_g$ have no directions in common, because all $\alpha_{\hat{k},\hat{i},l}$ used to construct the directions of messages in each $\mathcal{X}_g$ are completely different when $\mathcal{X}_g$ changes.

DECODABILITY: We assume that a useful symbol from stream $m_{k',i',l}$ arrives at receiver $\hat{j}$ in form of $\left(h_{\hat{j},k'}\alpha_{k',i',l}\right)T$ where T is obtained as (36). It is obvious that $\hat{\mathcal{T}}_{k,i,l}$ and all interference directions at receiver $\hat{j}$ that are designed like $\hat{\mathcal{T}}_{i,k,l}$ do not contain a generator $\left(h_{\hat{j},k'}\alpha_{k',i',l}\right)$. In addition, this generator changes for different useful signals of $\mathcal{X}_g$s. Therefore, we conclude that $\left(h_{\hat{j},k'}\alpha_{k',i',l}\right)T$ is independent of interference and other useful signals directions. This proves all useful messages are decodable in the noiseless situation.

The total rational directions of both the useful signals and the interference messages at any receiver $\hat{j}$ denoted as $D_{\hat{j}}$ satisfies

$$D_{\hat{j}} \leq \sum_{(\tilde{k},\tilde{i}) \in \mathcal{R}_{\hat{j}}} \bar{d}_{\tilde{k},\tilde{i}} n^{KJ} + \sum_{g \in \mathcal{G}} \max_{(\hat{k},\hat{i}) \in \mathcal{R}_{\hat{j}}^c, (\hat{k},\hat{i}) \in \mathcal{X}_g} \bar{d}_{\hat{k},\hat{i}} (n+1)^{JK}.$$

Define S as follows

$$S := \max_{\hat{j} \in \mathcal{J}} \left(\sum_{(\tilde{k},\tilde{i}) \in \mathcal{R}_{\hat{j}}} \bar{d}_{k,i} + \sum_{g \in \mathcal{G}} \max_{(\hat{k},\hat{i}) \in \mathcal{R}_{\hat{j}}^c, (\hat{k},\hat{i}) \in \mathcal{X}_g} \bar{d}_{\hat{k},\hat{i}} \right)$$

which is an upper bound for total number of interference and useful messages streams. Points in $\mathcal{D}^{(\text{in})}$

satisfying (32) have $S \leq \kappa$ which results in

$$D_{\hat{j}} \leq S(n+1)^{KJ} \leq \kappa(n+1)^{KJ} \qquad (39)$$

Similar to the approach in Sec. III, we have

$$\lambda = P^{\frac{1}{2}}/Q \qquad (40)$$

and for any $\epsilon \in (0,1)$

$$Q = P^{\frac{1-\epsilon}{2(z+\epsilon)}} \qquad (41)$$

where z is an integer and the DoF per stream is $\frac{1-\epsilon}{z+\epsilon}$. Choosing $z = \kappa(n+1)^{KJ}$, as P goes to infinity, the hard decoding error probability tends to zero and the DoF of the message $x_{k,i}$ gets arbitrarily close to

$$\lim_{n \to \infty} \frac{\bar{d}_{k,i} n^{KJ}}{\kappa(n+1)^{KJ}} = \frac{\bar{d}_{k,i}}{\kappa} = d_{k,i}, \qquad (42)$$

for all $(k,i) \in \mathcal{M}$. $\qquad\square$

Example: Consider a 3×3 wireless X network with multicast as in Fig. 3. The transmitters send their messages as follows

$$x_1 = \alpha_{1,1}x_{1,1} + \alpha_{1,2}x_{1,2}$$

$$x_2 = \alpha_{2,1}x_{2,1} + \alpha_{2,2}x_{2,2}$$

$$x_3 = \alpha_{3,1}x_{3,1} + \alpha_{3,2}x_{3,2}$$

For simplicity, we have assumed l is always one in (33). $x_{1,1}$, $x_{2,1}$, and $x_{3,1}$ are shown by white square, triangle, and circle in Fig. 3. $x_{1,2}$, $x_{2,2}$, and $x_{3,2}$ are also indicated with black square, triangle, and circle. Additionally, we assume $\mathcal{X}_1 = \{x_{1,1}, x_{2,1}, x_{3,2}\}$ and $\mathcal{X}_2 = \{x_{1,2}, x_{2,2}, x_{3,1}\}$ as stated by theorem 5. Using (36), the transmit directions used for all signals in $\mathcal{X}_1$ are generated [6] by

$$V_1 = \{\alpha_{1,1}h_{2,1}, \alpha_{1,1}h_{3,1}, \alpha_{2,1}h_{2,2}, \alpha_{3,2}h_{1,3}, \alpha_{3,2}h_{3,3}\} \qquad (43)$$

In a similar fashion, transmit directions of messages in $\mathcal{X}_2$ are generated by the following generators

$$V_2 = \{\alpha_{1,2}h_{1,1}, \alpha_{1,2}h_{2,1}, \alpha_{2,2}h_{1,2}, \alpha_{2,2}h_{3,2}, \alpha_{3,1}h_{1,3}, \alpha_{3,1}h_{2,3}\} \qquad (44)$$

Our scheme requires interference messages in $\mathcal{X}_g$, $g = \{1,2\}$ to be aligned at each receiver. To see this, let us consider $x_{3,2}$ and the whole signals of $\mathcal{X}_2$ which are interference at receiver 1. $x_{3,2}$ arrives at

receiver 1 multiplied by factor $\alpha_{3,2}h_{1,3}$ belonging to V_1. The other interference signals, $x_{1,2}$, $x_{2,2}$, and $x_{3,1}$, get to the receiver 1 multiplied by $\alpha_{1,2}h_{1,1}$, $\alpha_{2,2}h_{1,2}$, and $\alpha_{3,1}h_{1,3}$, respectively. These factors are components of V_2. Therefore, interference messages can be aligned based on our scheme.

Moreover, the desired messages, $x_{2,1}$ and $x_{1,1}$, are multiplied by $\alpha_{2,1}h_{1,2}$ and $\alpha_{1,1}h_{1,1}$, respectively when they arrive at receiver 1. Our design guarantees that useful signals are decodable since V_1 contains neither $\alpha_{2,1}h_{1,2}$ nor $\alpha_{1,1}h_{1,1}$.

C. outer bound

We present an outer bound for the wireless X network channel with multicast.

Theorem 6. *For a K-transmitter J-receiver constant coefficient single antenna wireless X network with multicast, let $\mathcal{D}^{out}$ be $\{d_{k,i}|(k,i) \in \mathcal{M}\}$ such that $\forall j \in \mathcal{J}$, $\forall k \in \mathcal{K}$:*

$$\sum_{(\tilde{k},\tilde{i}) \in R_j} d_{\tilde{k},\tilde{i}} + \sum_{\hat{i} \in \mathcal{I}_k} d_{k,\hat{i}} - \sum_{(k,i') \in R_j} d_{k,i'} \leq 1. \tag{45}$$

Then $\mathcal{D} \in \mathcal{D}^{out}$ where $\mathcal{D}$ indicates the DoF region.

Remark 1: The above outer bound is established when all messages are requested by some receivers. In other words, we assume there is no message that is useless for all receivers.

Proof: Using the similar argument in [7], we determine an outer bound when R_j and $\{(k,\hat{i})|\hat{i} \in \mathcal{I}_k\}$ are the only messages that are communicated.

Consider a reliable coding scheme for the wireless X network channel with multicast when all messages except those in R_j and $\{(k,\hat{i})|\hat{i} \in \mathcal{I}_k\}$ are eliminated. Suppose a genie provides the set R_j to receivers $1, \ldots, j-1, j+1, \ldots, J$. Hence, receivers $1, \ldots, j-1, j+1, \ldots, J$ are able to discard the effect of transmitters $1, \ldots, k-1, k+1, \ldots, K$. Furthermore, using the coding scheme, receiver j is capable of decoding its desired messages. Therefore, receiver j can cancel the interference caused by transmitters $1, \ldots, k-1, k+1, \ldots, K$. Note that receivers $\hat{j} \neq j$ can decode their useful messages emitted by transmitter k. Now, we reduce the noise at receiver j so that we ensure the channel between transmitter k and receiver j is better than all channels among transmitter k and receivers $\hat{j} \neq j$. Such assumption guarantees that receiver j can decode all messages of transmitter k. Consequently, we have shown that receiver j is able to decode messages in the sets R_j and $\{(k,\hat{i})|\hat{i} \in \mathcal{I}_k\}$. The rest of proof will be analogous to [7].
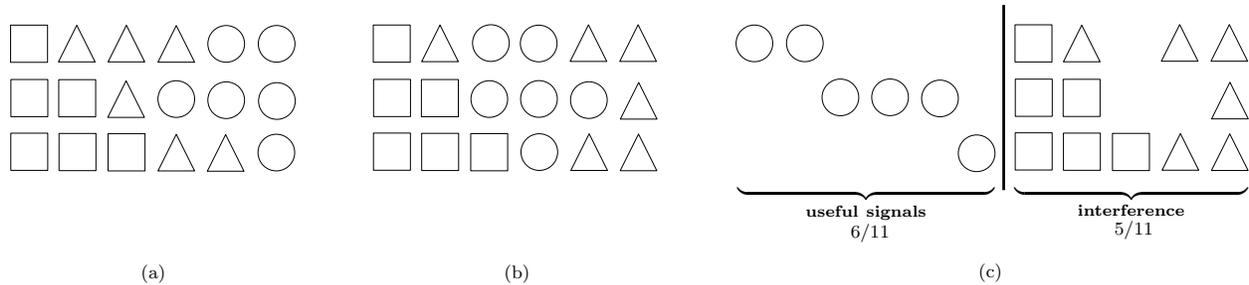
Fig. 4. Splitting alignment

$\square$

D. Splitting Alignment

The alignment scheme explained in this section can be extended to the case where messages are split. In fact, since messages have been divided into streams, we are capable of aligning streams. Thus, applying a small change to our scheme, we assume that streams are aligned instead of signals. This assumption needs $\mathcal{M}$ and $\mathcal{R}_j$ to be modified in the following manners

$$\mathcal{M} = \{(k,i,l)|1 \le k \le K, 1 \le i \le I_k, 1 \le l \le \bar{d}_{k,i}\} \tag{46}$$

$$\mathcal{R}_j = \{(k,i,l) \in \mathcal{M}|\text{receiver } j \text{ requests streams } (k,i,l)\} \tag{47}$$

Let $|A|$ denote the cardinality of the set A. We alter (32) as follows:

$$|\mathcal{R}_j| + |\{g \in \mathcal{G}|\mathcal{X}_g \cap \mathcal{R}_j^c \ne \phi\}| \le 1 \tag{48}$$

such that the set $\mathcal{R}_j^c$ contains all streams that are not in $\mathcal{R}_j$. The optimal G also changes as follow

$$G = \max_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}_k} d_{i,k} \tag{49}$$

The rest of the scheme is similar to part A and B, considering the fact that all (k,i)'s should be replaced by (k,i,l) because of aligning streams instead of messages.

VII. DISCUSSION

The alignment schemes presented in this paper are some of possible methods within the class of real alignment. They provide us with more possible ways to align signals. However, it is not shown whether

these schemes are sufficient to achieve all points in the DoF region. In this section, we investigate some examples to indicate even though proposed schemes offer more ways to have signals aligned, there are still points that are not achievable.
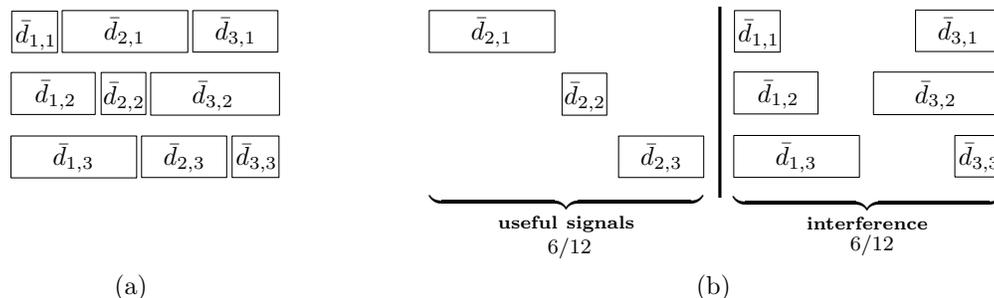

Fig. 5. Splitting alignment

Consider a 3×3 X network. The point $[d_{j,k}]$ as follow

$$[d_{j,k}]^T = \frac{1}{12} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \tag{50}$$

is in the DoF region. To investigate this point more, see Fig. 5 part (a) showing that one message from a user can be aligned with multiple messages from another user similar to the staggered alignment scheme. Part (b) of this figure clarifies how the coefficient $1/12$ appears in (50). Assume another point located on the outer bound that is proportional to the matrix mentioned in (50) with a coefficient $1/11$ instead of $1/12$. Neither permuted nor staggered alignment can achieve this point. However, using splitting alignment, we are able to split individual DoFs and meet the outer bound for this particular example. To make it more clear, consider Fig. 4 part (a) presenting that not only are messages aligned like Fig. 5, but also they are divided into the equal parts. Note that the individual DoFs at receiver 1, 2, and 3 are shown by square, triangle, and circle in Fig. 4. The splitting alignment here offers more ways to align signals such as part (b) of Fig. 4 which is the right signals alignment in order to meet the outer bound. Part c of this figure shows that the point on the outer bound is achievable. If we also looked at receiver 1 or 3 instead of 2 in part c, we would end up with the same result.

Although splitting alignment creates more possible ways to get messages aligned so that we can meet an outer bound, there are still cases where splitting alignment does not work. For instance, consider the

following point

$$[d_{j,k}]^T = \frac{1}{16} \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \\ 0 & 5 & 6 \end{bmatrix} \tag{51}$$

which is located on its outer bound. Unfortunately, none of the methods we have introduced in this paper can prove the above point belongs to DoF region.

The problem here is that to meet the outer bound, the maximum dimensions that can be devoted to the interference at receiver 3 is $5/15$ which needs $d_{1,1}$, $d_{1,2}$, and $d_{2,1}$ to be aligned with $d_{2,3}$. On the other hand, to have the outer bound achievable, receiver 2 needs to allot at most $8/15$ to interference dimensions, which necessitate aligning part of $d_{2,1}$ and $d_{1,2}$ (or $d_{1,1}$ and part of $d_{1,2}$) with $d_{3,3}$. This is impossible since $d_{1,1}$, $d_{1,2}$, and $d_{2,1}$ are already aligned with $d_{2,3}$.

VIII. CONCLUSIONS

We have derived some achievability results for the wireless X network with single antennas. Each message is split into multiple streams, and achievability is established using real interference alignment of the streams. The streams emitted by a single transmitter can be "shuffled" to determine the alignment position with respect to streams from other transmitters. Such rearrangement allow for higher DoF in some cases. We also showed that when the number of receivers is equal to two, then the achieved region is actually the DoF region. Moreover, we presented that certain boundary points in the general DoF region can be achieved using the proposed schemes. Next, we investigated wireless X network with multicast and some achievable points were established for such networks. An outer bound was also presented. We finally introduced the splitting alignment that gives us more possible ways to have signals aligned. Even though splitting technique was helpful to meet an outer bound in some cases, we presented an example showing that an point on the outer bound is not achievable using any of the the proposed schemes.

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