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The Hurwicz decision rule's relationship to decision making with the triangle and beta distributions and exponential utility

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By Sarat Sivaprasad and Cameron A. MacKenzie

Abstract Non-probabilistic approaches to decision making have been proposed for situations in which an individual does not have enough information to assess probabilities over an uncertainty. One non-probabilistic method is to use intervals in which an uncertainty has a minimum and maximum but nothing is assumed about the relative likelihood of any value within the interval. The Hurwicz decision rule in which a parameter trades off between pessimism and optimism generalizes the current rules for making decisions with intervals. This article analyzes the relationship between intervals based on the Hurwicz rule and traditional decision analysis using a few probability distributions and an exponential utility functions. This article shows that the Hurwicz decision rule for an interval is logically equivalent to: (i) an expected value decision with a triangle distribution over the interval; (ii) an expected value decision with a beta distribution; and (iii) an expected utility decision with constant absolute risk aversion with a uniform distribution. These probability distributions are not exhaustive. There are likely other distributions and utility functions for which equivalence with the Hurwicz decision rule can also be established. Since a frequent reason for the use intervals is that intervals assume less information than a probability distribution, the results in this article call into question whether decision making based on intervals really assumes less information than subjective expected utility decision making.

1. Introduction

Decision analysis helps an individual make better decisions, and these decisions impact the individual's future prospects. Decision trees are a tool to help structure a decision and select an alternative. For each branch of the tree extending from an uncertain node, an associated probability quantifies the uncertainty. Using probabilities enables the decision tree to be solved by calculating the expected value or expected utility of each possible alternative. The probability for each decision tree is defined by a quantitative value.

Probability distributions can be assessed using data or expert opinion. Assessing probabilities from experts can be difficult and reflect biases (Fischhoff et al. 1982, Hubbard 2009, Fredrickson and Kahneman 1993, Tversky 1974, Johnson et al. 1993, Tversky and Kahneman 1971). Biases such as representativeness, availability, and anchoring and adjustment can lead to incorrect and misjudged probabilities (Tversky and Kahneman 1971, Bedford and Cooke 2001). Using probability distributions may also assume more information about uncertainty than is warranted. These errors in estimating probabilities can lead to overconfidence and a failure to protect against highly consequential risks (Taleb 2007).

Rather than assuming probabilities when little information is available, making decisions based on intervals has been suggested as an alternative (Barker and Wilson 2012, Cao 2014). The interval-based approach to decision making defines an uncertainty with a minimum and maximum value and makes no assumption about whether any real number within the interval is more or less likely than any other real number contained in the interval. An interval-valued decision tree requires assessing the probability of each possible interval for an outcome. Barker and Wilson (2012) propose single and multiple objective interval-valued decision trees where the latter type of decision tree has multiple objectives, each of which can be expressed with intervals. The authors provide rules to construct and solve a decision tree where only the bounds and not the distributions of uncertain outcomes are known. Cao (2014) extends their work to demonstrate that each of Barker and Wilson's decision rules can be described with the Hurwicz (1952) equation. The Hurwicz equation balances pessimism and optimism for a decision maker where a single parameter provides a trade-off weight between the minimum and maximum values in an interval.

This article builds on the previous work on interval-valued decision making by exploring the relationship between the Hurwicz decision rule for intervals and prescriptive decision analysis following the von Neumman and Morgenstern and Savage axioms. Specifically, this article proves that making a decision based on the Hurwicz equation is mathematically equivalent to making a decision based on some common probability distributions and/or an exponential utility function. The Hurwicz decision rule can always result in the same optimal alternative as an expected value decision maker who assumes a triangle distribution or a beta distribution. The Hurwicz decision rule can also always result in the same optimal alternative as a decision maker who uses an exponential utility function and assumes a uniform distribution.

The significant contribution of this article is that if an individual uses an interval to make a decision based on the Hurwicz rule, he or she will get the exact same decision as a risk neutral decision maker using a triangle or beta distribution or an expected utility decision maker with constant absolute risk aversion using a uniform distribution. As will be discussed in the conclusion, other probability distributions or utility functions may also be equivalent to the Hurwicz decision rule. This contribution is important because the result calls into question whether making decisions based on intervals really assumes less information than traditional prescriptive decision analysis. If an individual is using intervals to select an alternative—for which the interval decision rule can be described with the Hurwicz equation—then the individual's decision rule can be mapped to an expected-value decision with a triangle or beta distribution or an expected utility decision with constant absolute risk aversion and a uniform distribution.

This article is organized as follows. Section 2 reviews the debate about assigning probabilities to uncertainties with little information and non-probabilistic approaches to decision making. Section 3 describes interval analysis and the Hurwicz equation. Section 4 proves the relationship between the Hurwicz decision rule and the triangle and beta probability distributions and an exponential utility function. Section 5 provides a numerical example based on Barker and Wilson (2012) for how intervals can be translated to probability distributions. Concluding remarks appear in section 6.

2. Literature Review

Uncertainty has traditionally been categorized into two types (Aven 2010): aleatory (objective) and epistemic (subjective). Aleatory uncertainty results from natural variation in a population, and epistemic uncertainty results from a lack of knowledge about the system. Ferson and Ginzburg (1996) suggest using intervals to model epistemic uncertainty and using probabilities to model aleatory uncertainty. Probability theory and interval analysis could be combined by bounding probabilities and analyzing uncertainty using all possible probability distributions that satisfy the bounds of an interval.

Aleatory and epistemic uncertainties—and distinguishing between them—appear in various fields. Der Kiureghian and Ove Ditlevsen (2009) discuss the influence of these two types of uncertainties in reliability assessment, risk-based decision making, and codified design. Aleatory and epistemic uncertainties have been analyzed in hazardous waste management (Hora 1996), radioactive waste disposal (Helton et al. 2011), seismic risk assessments of buildings (Dolšek 2012), power system reliability (Billinton and Huang 2008), and nuclear power plants (Rao et al. 2007). Aleatory and epistemic uncertainties have also been used in design optimization (Huang and Zhang 2009, Zhang and Huang 2010). One critique to representing both aleatory and epistemic uncertainties with probability is that the resulting probability distribution will comingle these uncertainties, and a decision maker will not understand if the uncertainty is due to aleatory or epistemic uncertainty (Dubois 2010, Aven and Zio 2011).

Non-probabilistic methods have been proposed to model uncertainty due to ignorance and lack of knowledge (i.e., epistemic uncertainty) and to correspond to more naturalistic ways that people think about uncertainty. These methods include possibility theory, evidence theory, fuzzy decision making, and interval analysis (Helton et al. 2004, 2010). Possibility theory can be used for rare events because little information is required, and it offers a qualitative model for epistemic uncertainty (Dubois et al. 2004, Georgescu 2012). Possibility theory has been proposed for fuzzy reasoning in decision making and optimization (Carlsson and Fuller 2002), in fuzzy cooperative games (Mares 2013), in answer set programming (Bauters et al. 2015), in fingerprint verification systems (Guesmi et al. 2015), and to model operational risk in finance and banking (Chaudhari and Ghosh 2016).

Evidence theory, developed by Shafer (1976), is based on the idea of belief functions (Kohlas and Monney 1994). The Demspter-Shafer mathematical theory of evidence collates information from various sources and helps an individual reach a degree of belief that takes into consideration all of the available evidence (Maseleno et al. 2015). Evidence theory has been applied to human reliability assessment (Su et al. 2015), information fusion (Li et al. 2017), reliability analysis (Zhang et al. 2015, Tang et al. 2015a), complex engineering systems (Shah et al. 2015), and in aeronautics (Tang et al. 2015b).

Fuzzy decision making categorizes an uncertain outcome into different grades of membership (Zadeh 1965, Klir and Yuan 1996). Fuzzy set theory can use intervals by assigning levels of possibilities to intervals (Moore and Lodwick 2003, Dubois and Prade 2012). Fuzzy sets for epistemic uncertainty have been used in automotive crashes in structural dynamics and to simulate landslide failure in geotechnical science and engineering (Hanss and Turin 2010).

An interval-valued decision trees is a non-probabilistic decision making tool for epistemic uncertainties. In an interval-valued decision tree, each uncertain branch of the tree is assigned a probability, but the uncertain outcome at the end of the branch is described by an interval. As will be discussed in further detail, Barker and Wilson (2012) discuss five decision rules that can be used to solve an interval-valued decision tree, and Cao (2014) demonstrates how each of these decision rules can be reduced to the Hurwicz (1952) decision rule. Other papers have applied the Hurwicz decision rule to belief functions (Strat 1990) and expert systems and intelligent control (Nguyen et al. 1997). Danielson and Ekenberg (2007) present algorithms for calculating an optimal value in interval decision trees, and Rodríguez and Alonso (2004) propose interval decision trees within dynamic decision contexts. Intervals have been used in some multi-criteria decision-making methods like TOPSIS (Jahanshahloo et al. 2006), grey theory (Zhang et al. 2005), and VIKOR (Sayadi et al. 2009). Lertworaprachaya et al. (2010, 2014) extend interval-valued decision trees by creating interval-valued fuzzy decision trees.

Many decision and risk analysts are skeptical of non-probabilistic approaches to quantify uncertainty (Aven and Zio 2011). Probabilities are well understood and established as a proper tool for decision making with uncertainty. North (2010) rejects non-probabilistic decision rules such as interval analysis and argues that risk analysts should use standard probability theory rather than resorting to other methods that are difficult to understand. Howard (1988) argues that long-run frequency concepts are not required for probabilities and an individual should assign a probability for an uncertainty that represents what he or she believes the long-run frequency would be.

This article enters into the debate between probabilistic and non-probabilistic approaches to decision making by exploring the relationship between decisions based on intervals and traditional prescriptive decision analysis based on subjective probabilities and risk preference. To our knowledge, no research has explored to what extent making decisions based on intervals results in the same or different decisions based on the expected value of probability distributions or expected utility. If proponents of non-probabilistic methods suggest methods for making decisions that result in identical decisions to that of subjective expected utility, then those proponents should perhaps be willing to use subjective expected utility.

3. Background on Making Decisions with Intervals

In the interval-based approach for decision making, an uncertain outcome has a lower and upper bound, but the decision maker has no other knowledge about the outcome. An interval number can be defined as an ordered pair of real numbers $[a, b]$ with $a \leq b$. Several approaches have been made to compare two intervals, including the mean value of intervals (Dubois and Prade 1987, de Campos Ibáñez and Muñoz 1989, Ishibuchi and Tanaka 1990, Chanas and Kuchita 1996) and minimum distance measures (Heilpern 1997, Yao and Wu 2000, Asady and Zendehnam 2007).

Barker and Wilson (2012) propose five rules for making decisions with intervals. We compare two intervals $[a, b]$ and $[c, d]$ where $a \leq b$ and $c \leq d$ and assume the decision maker prefers to maximize the outcome. For each of the five rules, the decision maker should prefer interval $[a, b]$ to interval $[c, d]$:

$a > c$, Worst case rule

$b > d$, Best case rule

$a + b > c + d$, Laplace rule

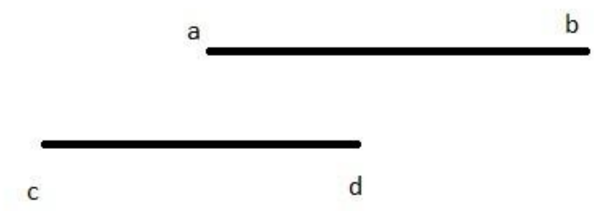
$d - a < b - c$, Minimum regret rule

$\alpha a + (1 - \alpha)b > \alpha c + (1 - \alpha)d$, Hurwicz's rule, where $\alpha \in [0,1]$

If $a > c$ and $b > d$, then the decision maker should always prefer $[a, b]$ to $[c, d]$ for each of the five decision rules and for any value of $\alpha \in [0,1]$ in the Hurwicz rule. Figure 1 depicts such a case for two intervals in which $[a, b]$ can be said to dominate $[c, d]$ if the objective is to maximize the outcome.

Similarly, if $c > a$ and $d > b$, the decision maker should always prefer $[c, d]$ to $[a, b]$ for each decision rule. (If the both of the greater-than inequalities are replaced with greater-than-or-equal inequalities, the preference relationship is a weak preference.) Thus, the mathematical justifications in this article will always assume that $c < a < b < d$ to avoid the trivial case in which one interval is always preferred over another interval.

Figure 1 Two Intervals in which $[\boldsymbol{a}, \boldsymbol{b}]$ Dominates $[\boldsymbol{c}, \boldsymbol{d}]$



Cao (2014) shows that each of these five decision rules for intervals can be reduced to the Hurwicz rule. The decision maker determines α based on whether he or she is more optimistic or pessimistic about the outcome. The parameter α can be defined as the parameter that trades off between the best case (optimism) and the worst case (pessimism). The worst case rule corresponds to $\alpha = 1$, the best case rule corresponds to $\alpha = 0$, and both the Laplace rule and the minimum regret correspond to $\alpha = 0.5$. Thus, this article will focus exclusively on the Hurwicz rule since it covers the other four decision-making rules for intervals.

4. Relationship between Hurwicz and Prescriptive Decision Analysis

This section shows how the Hurwicz rule for decision making results in the same optimal alternative as (i) an expected value decision rule with a triangle probability distribution, (ii) an expected value decision rule with a beta distribution, and (iii) an expected exponential utility decision rule with a uniform distribution. Each of these three instances is addressed separately. Each case compares the intervals $[a, b]$ and $[c, d]$ where $c < a < b < d$ and a decision maker wants to maximize the outcome. The article only demonstrates the relationship between the Hurwicz rule and these three cases. There are almost certainly other distributions and utility functions which have a similar relationship to the Hurwicz decision rue.

The parameter α in the Hurwicz equation and the minimum and maximum for an interval are used to define corresponding parameters for the triangle and beta distributions. With the selection of the correct parameters, making a decision based on maximizing the expected value of these distributions will correspond exactly to the decision in the Hurwicz rule. The triangle and beta distributions are chosen because both distributions have lower and upper bounds that can correspond to an interval. The skewness of these distributions can correspond to the parameter α.

Since the expected value of a uniform distribution is the midpoint, the expected value of a uniform distribution only results in the same alternative as the Hurwicz rule if $\alpha = 0.5$. The uniform distribution is used with an exponential utility function where the risk preference parameter is chosen to align with α.

4.1. Expected Value with a Triangle Distribution

In the following discussion, we show that if an uncertain outcome follows a triangle distribution, then the expected-value decision rule is equivalent to the Hurwicz decision rule. The triangle distribution is denoted $Tri(a, m, b)$ where a is the minimum value, m is the mode, and b is the maximum value of the distribution. It must be true that $a \leq m \leq b$. Three different cases exist to define the parameters for the triangle distribution based on the intervals $[a, b]$ and $[c, d]$ and value of α from the Hurwicz equation. If $\alpha \in \left[\frac{1}{3}, \frac{2}{3}\right]$, the modes of the triangle distributions for interval $[a, b]$ and for $[c, d]$ are:

$$m_x = 2b - a - 3\alpha(b - a) \quad \text{and} \quad m_y = 2d - c - 3\alpha(d - c) \tag{1}$$

such that $a \leq m_x \leq b$ and $c \leq m_y \leq d$.

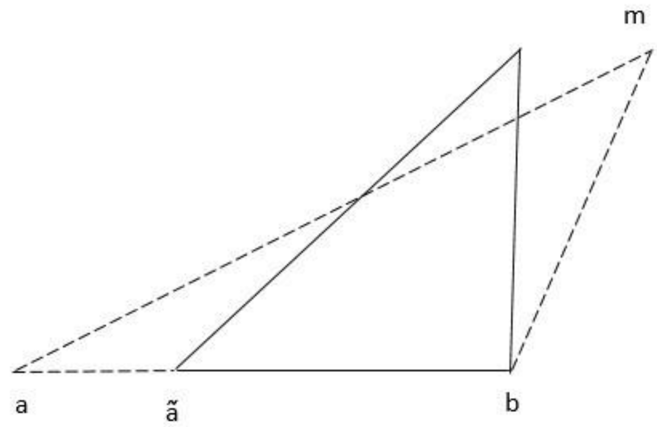
If $\alpha < \frac{1}{3}$, then using Equation (1) for m_x and m_y results in $m_x > b$ and $m_y > d$, which is invalid for the triangle distribution. To bracket the mode within the interval $[a, b]$, we calculate a new minimum value for the triangle distribution, $\tilde{a}$, as shown in Figure 2, and set the mode of this triangle equal to b. In order to calculate $\tilde{a}$, the mean of the new triangle distribution is equated to the Hurwicz equation:

$$\frac{\tilde{a} + 2b}{3} = \alpha a + (1 - \alpha)b \tag{2a}$$

$$\tilde{a} = 3\alpha(a - b) + b \tag{2b}$$

The minimum value $\tilde{c}$ of the triangle distribution for the interval $[c, d]$ is calculated in a similar fashion.

Figure 2 Bracketing the Mode between a and b when $\alpha < \frac{1}{3}$



If $\alpha > \frac{2}{3}$, then m_x and m_y will result in $m_x < a$ and $m_y < c$ based on Equation (1). Instead, we calculate a new maximum value $\tilde{b}$ for the interval $[a, b]$ and set the mode of the distribution equal to a. In order to calculate $\tilde{b}$, the mean of the new triangle distribution is equated to the Hurwicz equation:

$$\frac{2a+\tilde{b}}{3} = \alpha a + (1 - \alpha)b \tag{3a}$$

$$\tilde{b} = (3\alpha - 2)a + 3b(1 - \alpha) \tag{3b}$$

We similarly calculate the maximum value $\tilde{d}$ of the triangle distribution for $[c, d]$.

Using these rules for the triangle distribution, if a decision maker is risk neutral and selects the alternative that maximizes the expected value of the triangle distribution, the decision maker should always select the same alternative as given by the Hurwicz decision rule. The following theorem formalizes and proves this result.

Theorem 1: If X and Y are two random variables where each follows a triangle distribution according to the following cases:

Case 1: $\frac{1}{3} \leq \alpha \leq \frac{2}{3}$

$X \sim Tri\ (a, m_x, b)\ where\ m_x \ \ = \ 2b - a - 3\alpha(b - a)$

$Y \sim Tri\ (c, m_y, d)\ where\ m_y \ \ = \ 2d - c - 3\alpha(d - c)$

Case 2: $\alpha \ < \ \frac{1}{3}$

$X \sim Tri(\tilde{a}, b, b)\ where\ \ \tilde{a} = 3\alpha(a - b) + b$

$Y \sim Tri(\tilde{c}, d, d)\ where\ \tilde{c} = 3\alpha(c - d) + d$

Case 3: $\alpha > \ \frac{2}{3}$

$X \sim Tri(a, a, \tilde{b})\ where\ \ \breve{b} = (3\alpha - 2)a + 3b(1 - \alpha)$

$Y \sim Tri(c, c, \check{d})\ where\ \ \tilde{d} = (3\alpha - 2)c + 3d(1 - \alpha)$

then $E(X) > E(Y)$ *if and only if* $\alpha a + (1 - \alpha)b > \alpha c + (1 - \alpha)d$ *where* $\alpha \in [0,1]$.

PROOF. Case 1. First, if $\frac{1}{3} \leq \alpha \leq \frac{2}{3}$, then $a \leq m_x \leq b$ and $a \leq m_y \leq b$, which means that the triangle distributions for X and Y are well defined. Second, we show that $E(X) > E(Y)$ if and only if $\alpha a + (1 - \alpha)b > \ \alpha c + (1 - \alpha)d$:

$$E(X) > E(Y) \tag{4a}$$

$$\frac{a+b+m_x}{3} > \frac{c+d+m_y}{3} \tag{4b}$$

$$\frac{a+b+2b-a-3\alpha(b-a)}{3} > \ \frac{c+d+2d-c-3\alpha(d-c)}{3} \tag{4c}$$

$$3b - 3\alpha(b - a) > \ 3d - 3\alpha(d - c) \tag{4d}$$

$$\alpha a + (1 - \alpha)b > \ \alpha c + (1 - \alpha)d \tag{4e}$$

Case 2. With the defined values for $\tilde{a}$ and $\tilde{c}$, we show that the expected value of X is greater than the expected value of Y if and only if the $\alpha a + (1 - \alpha)b > \ \alpha c + (1 - \alpha)d$:

$$E(X) > E(Y) \tag{5a}$$

$$\frac{\tilde{a}+b+b}{3} > \frac{\tilde{c}+d+d}{3} \tag{5b}$$

$$\frac{3\alpha(a-b)+3b}{3} > \ \frac{3\alpha(c-d)+3d}{3} \tag{5c}$$

$$\alpha(a-b)+b \; > \; \alpha(c-d)+d \tag{5d}$$

$$\alpha a+(1-\alpha)b > \; \alpha c+(1-\alpha)d \tag{5e}$$

Case 3. With the defined values for $\tilde{b}$ and $\tilde{d}$, we show that the expected value of X is greater than the expected value of Y if and only if the $\alpha a+(1-\alpha)b > \; \alpha c+(1-\alpha)d$:

$$E(X) > E(Y) \tag{6a}$$

$$\frac{a+a+\tilde{b}}{3} > \frac{c+c+\tilde{d}}{3} \tag{6b}$$

$$\frac{2a+(3\alpha-2)a+3b(1-\alpha)}{3} > \frac{2c+(3\alpha-2)c+3d(1-\alpha)}{3} \tag{6c}$$

$$\frac{2a+3\alpha a-2a+3b(1-\alpha)}{3} > \frac{2c+3\alpha c-2c+3d(1-\alpha)}{3} \tag{6d}$$

$$\alpha a+(1-\alpha)b > \; \alpha c+(1-\alpha)d. \tag{6e}$$

Thus, $E(X) > E(Y)$ if and only if $\alpha a+(1-\alpha)b > \alpha c+(1-\alpha)d$. End proof.

This result shows that a decision maker who is selecting an alternative based on an interval decision-making rule will select the same alternative as a risk-neutral decision maker who assumes that each interval follows a triangle distribution according to the above guidelines. If the decision maker using an interval with the Hurwicz decision rule does not believe the corresponding triangle distribution is a possible representation of the uncertainty, then perhaps the decision maker should question whether he or she should use the interval to make a decision based on that value of α.

4.2. Expected Value with a Beta Distribution

The previous subsection showed that a triangle distribution with risk neutrality can be formed from an interval that will always result in the same optimal alternative as the Hurwicz decision rule. This subsection shows that the beta distribution can also be used with risk neutrality and that maximizing the expected value of the beta distribution is equivalent to the Hurwicz decision rule.

A four-parameter beta distribution is used such that the beta distribution for random variable X is denoted by $Beta(\alpha_x, \beta_x, a, b)$ where a is the minimum value, b is the maximum value, and α_x and β_x are the two traditional parameters for the beta distribution. Increasing α_x moves the mass of the distribution closer to b and increasing β_x moves the mass of the distribution closer to a.The mean of the beta distribution can be equated to the Hurwicz equation to derive the relationship between α_x and β_x:

$$\frac{\alpha_x b+\beta_x a}{\alpha_x+\beta_x} = \alpha a + (1-\alpha)b \tag{7a}$$

$$\beta_x = \alpha_x * \frac{\alpha}{1-\alpha} \tag{7b}$$

Theorem 2: If 2 random variables X and Y follow beta distributions, where $X \sim Beta(\alpha_x, \beta_x, a, b)$ *and* $\sim Beta(\alpha_y, \beta_y, c, d)$*, and where*

$$\beta_x = \alpha_x * \frac{\alpha}{1-\alpha} \tag{8}$$

$$\beta_y = \alpha_y * \frac{\alpha}{1-\alpha} \tag{9}$$

then $E(X) > E(Y)$ *if and only if* $\alpha a + (1-\alpha)b > \alpha c + (1-\alpha)d$ *where* $\alpha \in [0,1]$.

PROOF. To prove this, it suffices to show that expectation of X is greater than the expectation of Y if and only if $\alpha a + (1-\alpha)b > \alpha c + (1-\alpha)d$:

$$E(X) > E(Y) \tag{10a}$$

$$\frac{\alpha_x b+\beta_x a}{\alpha_x+\beta_x} > \frac{\alpha_y d+\beta_y c}{\alpha_y+\beta_y} \tag{10b}$$

$$\frac{\alpha_x b+\alpha_x * \frac{\alpha}{1-\alpha} a}{\alpha_x+\alpha_x * \frac{\alpha}{1-\alpha}} > \frac{\alpha_y d+\alpha_y * \frac{\alpha}{1-\alpha} c}{\alpha_y+\alpha_y * \frac{\alpha}{1-\alpha}} \tag{10c}$$

$$\alpha a + (1-\alpha)b > \alpha c + (1-\alpha)d \tag{10d}$$

End proof.

The result for this beta distribution is similar to that of the triangle distribution. A decision maker who selects an alternative based on an interval decision-making rule will always select the same alternative as risk-neutral decision maker who assumes that each interval follows a beta distribution. If the decision maker does not believe the corresponding beta distribution is a possible representation of the uncertainty, then the decision maker should be leery of using the interval to make a decision based on that value of α.

4.3. Expected Utility with Constant Absolute Risk Aversion with a Uniform Distribution

The decision rules for the triangle and beta distributions assume a risk neutral decision maker, and the skewness of each distribution provides a means to mirror the influence of α in the Hurwicz equation. This subsection provides a slightly different approach by employing an exponential utility function to capture the influence of α via the risk preference coefficient.

A utility function can represent the decision maker's attitude towards uncertainty. Many utility functions are possible, but we utilize an exponential utility function for any real number z:

$$U(z) = \begin{cases} \dfrac{1 - e^{-\gamma z}}{\gamma}, & \text{if } \gamma \neq 0 \\ z, & \text{if } \gamma = 0 \end{cases} \tag{11}$$

where γ is a parameter describing an individual's risk preference (Arrow 1965, Pratt 1964). Exponential utility assumes a constant absolute risk aversion, which means that an individual's risk attitude remains constant for all wealth values (Arrow 1965). A risk averse individual has a concave utility function if $\gamma > 0$, and a risk seeking individual has a convex utility function if $\gamma < 0$ (Chelst and Canbolat 2011).

If intervals $[a, b]$ and $[c, d]$ follow a uniform distribution, the expected utility with constant absolute risk aversion of each uncertain outcome can be calculated. This article will continue to assume an exponential utility function with constant absolute risk aversion when calculating expected utility. Other utility functions may be also be possible, as will be discussed later in the article. A uniform distribution is a natural probabilistic representation for an uncertain interval in part because uniform, or equal, probabilities maximize the entropy of a discrete uncertainty when no other information is available (Jaynes 1957, Abbas 2006). Entropy is the amount of information necessary to describe the realizations of a discrete random variable (Shannon 1948).

An expected utility decision maker should prefer $[a, b]$ and $[c, d]$ if and only if the expected utility of $[a, b]$, $EU([a, b])$, is greater than the expected utility of $[c, d]$, $EU([c, d])$. Mathematically, this can be expressed as:

$$EU([a,b]) - EU([c,d]) = \begin{cases} \dfrac{(d-c)[e^{-b\gamma} - e^{-a\gamma}] - (b-a)[e^{-d\gamma} - e^{-c\gamma}]}{\gamma^2 (b-a)(d-c)}, & \text{if } \gamma \neq 0 \\ \dfrac{a+b-c-d}{2}, & \text{if } \gamma = 0. \end{cases} \tag{12}$$

where the respective intervals follow a uniform distribution and the utility is an exponential utility function as given in Equation (11).

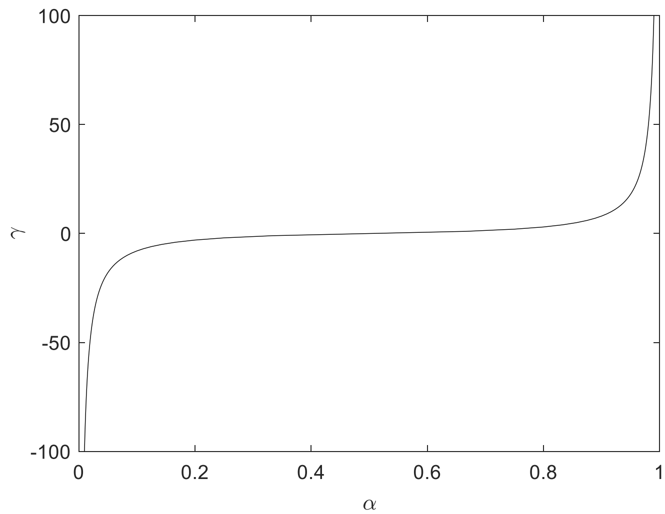
We desire a function that maps α from the Hurwicz equation to γ in the exponential utility function in order that given a value of α, there is a unique value of γ that will result in equivalent decisions. In other words, Equation (12) should be greater than 0 if and only if $\alpha a + (1-\alpha)b > \alpha c + (1-\alpha)d$.

In order to arrive at such a function, we first define the Hurwicz indifference parameter. If a decision maker is indifferent between intervals $[a, b]$ and $[c, d]$ then according to the Hurwicz decision rule, it must be true that $\alpha a + (1-\alpha)b = \alpha c + (1-\alpha)d$ for some value of $\alpha \in [0,1]$. We use this equality to make the following definition.

Definition: The Hurwicz indifference parameter $\alpha^* = \frac{d-b}{a-b+c-d}$ *is the value of* α *such that a decision maker is indifferent between* $[a, b]$ *and* $[c, d]$.

From this definition, it is clear that $\alpha^* a + (1-\alpha^*)b = \alpha^* c + (1-\alpha^*)d$. If $\alpha > \alpha^*$, the decision maker should prefer $[a, b]$ over $[c, d]$ and if $\alpha < \alpha^*$, the decision maker should prefer $[c, d]$ over $[a, b]$. The Hurwicz indifference parameter is used to establish γ in the exponential utility function.

Figure 3 Relationship between α and γ



If $\alpha = 0$, the Hurwicz decision rule is equivalent to an extreme risk-seeking decision maker who makes a decision based on the maximum value of each uncertain outcome, and $\gamma \to -\infty$. If $\alpha = 1$, the Hurwicz decision rule is equivalent to an extreme risk-averse decision maker who only focuses on the minimum outcome, and $\gamma \to \infty$. The risk neutral situation implies that $\gamma = 0$, which is identical to $\alpha = 0.5$. Thus, if $0 < \alpha \leq 0.5$, then it should be true that $-\infty < \gamma \leq 0$, and if $0.5 \leq \alpha < 1$, then it should be true that $0 \leq \gamma < \infty$. We express γ as a function of α that obeys these conditions:

$$\gamma(\alpha) = \begin{cases} \dfrac{t(2\alpha - 1)}{\alpha}, & \text{if } 0 < \alpha \leq 0.5 \\ \dfrac{t(2\alpha - 1)}{1 - \alpha}, & \text{if } 0.5 < \alpha < 1 \end{cases} \tag{13}$$

where $t > 0$ is a parameter that needs to be determined. Figure 3 displays this function.

The other condition that needs to be satisfied, which will determine the value of t, is as follows. If $\alpha = \alpha^*$ (the Hurwicz indifference parameter), then it should be true that $\gamma = \gamma^*$ where γ^* is the risk preference parameter such that the expected utility of $[a, b]$ equals the expected utility of $[c, d]$. Substituting this condition into Equation (13) results in the following parameterization of t:

$$t = \begin{cases} \dfrac{\gamma^* \alpha^*}{2\alpha^* - 1}, & \text{if } 0 \leq \alpha \leq 0.5, \alpha^* \neq 0.5 \\ \dfrac{\gamma^*(1 - \alpha^*)}{2\alpha^* - 1}, & \text{if } 0.5 < \alpha \leq 1, \alpha^* \neq 0.5 \\ 1, & \text{if } \alpha^* = 0.5 \end{cases} \tag{14}$$

Using this expression for t in the function $\gamma(\alpha)$ in Equation (13) will result in a utility function such that the expected utility of $[a, b]$ is greater than the expected utility of $[c, d]$ if and only if the decision maker prefers $[a, b]$ according to the Hurwicz equation. The following two lemmas help to prove that assertion.

Lemma 1. Given some value $\hat{\gamma} \in \mathcal{R}$ and $d > b > a > 0$,

$$\frac{d[e^{-b\gamma} - e^{-a\gamma}] - (b-a)[e^{-d\gamma} - 1]}{\gamma^2(b-a)d} < 0 \tag{15}$$

for all $\gamma \leq \hat{\gamma}$.

PROOF. If we find one value of $\hat{\gamma}$ and show that Equation (15) is valid for all $\gamma \leq \hat{\gamma}$, that completes the proof. First, let

$$\hat{\gamma} = -\frac{\log\left(\frac{d}{b-a}\right)}{d-b} \tag{16}$$

Since $d > b > a$, then $\hat{\gamma} < 0$. This implies that $de^{-b\hat{\gamma}} - (b-a)\, e^{-d\hat{\gamma}} = 0$. Since $1 < e^{-b\gamma} < e^{-d\gamma}$ for all $\gamma < 0$, then it must be true that $de^{-b\gamma} - (b-a)\, e^{-d\gamma} < 0$ for all $\gamma < \hat{\gamma}$.

Second, let

$$\hat{\gamma}_1 = -\frac{\log\left(\frac{b-a}{d}\right)}{a} \tag{17}$$

Since $e^{-a\gamma}$ is a strictly decreasing function, it must be true that $b - a < de^{-a\gamma}$ for all $\gamma < \hat{\gamma}_1$. Since $d > b > a$, then $\hat{\gamma}_1 > 0$. Since $\hat{\gamma} < 0$, then $b - a < de^{-a\gamma}$ for all $\gamma < \hat{\gamma}$. Thus $0 > de^{-b\gamma} + b - a - (b-a)e^{-d\gamma} - de^{-a\gamma}$ $= d[e^{-b\gamma}$-$e^{-a\gamma}]-(b-a)[\, e^{-d\gamma}$-1] for all $\gamma < \hat{\gamma}$. End proof.

Lemma 2. The function $f(\gamma) = d[e^{-b\gamma} - e^{-a\gamma}] - (b-a)[e^{-d\gamma} - 1]$ has at most one local minimum and one local maximum.

PROOF. We construct a proof by contradiction. Assume $f(\gamma)$ has either two local minima and one local maximum or one local minimum and two local maxima (i.e., three critical points). Then the derivative of $f(\gamma)$

$$f'(\gamma) = d[-be^{-b\gamma} + ae^{-a\gamma}] + (b-a)de^{-d\gamma} = de^{-d\gamma}[-be^{(d-b)\gamma} + ae^{(d-a)\gamma}] + (b-a)] \quad (18)$$

$$= 0$$

must have at least three solutions because $f(\gamma)$ is continuous. Since $de^{-d\gamma} \neq 0$,

$$\tilde{f}'(\gamma) = -be^{(d-b)\gamma} + ae^{(d-a)\gamma} + (b-a) = 0 \quad (19)$$

must have at least three solutions. If $\tilde{f}'(\gamma) = 0$ has at least three solutions, then it must have at least two critical points (one local minimum and one local maximum). The derivative of $\tilde{f}'(\gamma)$

$$\tilde{f}"(\gamma) = -b(d-b)e^{(d-b)\gamma} + a(d-a)e^{(d-a)\gamma} = 0 \quad (20)$$

must have at least two solutions. But $\tilde{f}"(\gamma) = 0$ can be solved directly for γ, and there is a unique solution to γ, so $\tilde{f}"(\gamma) = 0$ only has one solution. This implies that $f'(\gamma) = 0$ has two solutions and $f(\gamma)$ has only two critical points (a local minimum and a local maximum). End proof.

These lemmas will enable us to prove that there is a unique γ^* for which the decision maker will be indifferent between $[a, b]$ and $[c, d]$ according to exponential expected utility. The interval $[a, b]$ should be preferred if $\alpha > \alpha^*$ and $> \gamma^*$, and the interval $[c, d]$ should be preferred if $\alpha < \alpha^*$ and $\gamma < \gamma^*$.

Theorem 3: Assume the following:

1. *The intervals* $[a, b]$ *and* $[c, d]$ *each follow a uniform distribution where* $d > b > a > c$
2. $\alpha^* = \frac{d-b}{a-b+c-d}$
3. $EU([a, b])$ *and* $EU([c, d])$ *represent the expected utility with constant absolute risk aversion where* γ *is the risk preference parameter*
4. γ^* *is the risk preference parameter such that* $EU([a, b]) = EU([c, d])$
5. *The risk preference parameter* γ *as a function of* $\alpha \in [0,1]$ *is*

$$\gamma(\alpha) = \begin{cases} \frac{\gamma^* \alpha^* (2\alpha - 1)}{\alpha(2\alpha^* - 1)}, & 0 < \alpha < 0.5, \alpha^* \neq 0.5 \\ \frac{\gamma^* (1 - \alpha^*)(2\alpha - 1)}{(1 - \alpha)(2\alpha^* - 1)}, & 0.5 \leq \alpha < 1, \alpha^* \neq 0.5 \\ \frac{2\alpha - 1}{\alpha}, & 0 < \alpha < 0.5, \alpha^* = 0.5 \\ \frac{2\alpha - 1}{1 - \alpha}, & 0.5 \leq \alpha < 1, \alpha^* = 0.5 \end{cases} \tag{21}$$

Then $EU([a,b]) - EU([c,d]) > 0$ *for all* $\gamma > \gamma^*$ *if and only if* $a + (1-\alpha)b - \alpha c - (1-\alpha)d > 0$ *for all* $\alpha > \alpha^*$.

PROOF. First, we show that $EU([a,b]) - EU([c,d])$ is continuous from Equation (12). Using L'Hôpital's rule, the limit as $\gamma \to 0$ for the top part of Equation (12) is:

$$\lim_{\gamma \to 0} \frac{(d-c)[e^{-b\gamma} - e^{-a\gamma}] - (b-a)[e^{-d\gamma} - e^{-c\gamma}]}{\gamma^2 (b-a)(d-c)} \tag{22a}$$

$$= \lim_{\gamma \to 0} \frac{(d-c)[-be^{-b\gamma} + ae^{-a\gamma}] - (b-a)[-de^{-d\gamma} + ce^{-c\gamma}]}{2\gamma (b-a)(d-c)} \tag{22b}$$

$$= \lim_{\gamma \to 0} \frac{(d-c)[b^2 e^{-b\gamma} - a^2 e^{-a\gamma}] - (b-a)[d^2 e^{-d\gamma} - c^2 e^{-c\gamma}]}{2(b-a)(d-c)} \tag{22c}$$

$$= \frac{(b+a) - (d+c)}{2} \tag{22d}$$

which is equivalent to the difference in expected utility when $\gamma = 0$ in Equation (12).

To prove the theorem, we next show that γ^* exists and that it is unique. Without loss of generality, we can assume that $c = 0$ and all the other variables would be similarly scaled so that $d > b > a > 0$. This is valid because of the "delta" or constant absolute risk aversion property for exponential utility (Brockett and Golden 1987). Thus, the difference in expected utilities can rewritten as:

$$EU([a,b]) - EU([0,d]) = \begin{cases} \frac{d[e^{-b\gamma} - e^{-a\gamma}] - (b-a)[e^{-d\gamma} - 1]}{\gamma^2 (b-a) d}, & \text{if } \gamma \neq 0 \\ \frac{a + b - c - d}{2}, & \text{if } \gamma = 0 \end{cases} \tag{23}$$

From Lemma 1, we know that Equation (23) is less than 0 as $\gamma \to -\infty$. As $\gamma \to \infty$, $d[e^{-b\gamma} - e^{-a\gamma}] - (b-a)[e^{-d\gamma} - 1] \to b - a$ so the numerator is positive for large values of γ. Since the denominator is always positive, Equation (23) is greater than 0 for large values of γ. Since Equation (23) is continuous, there must be at least one point when Equation (23) equals 0 and this proves the existence of at least one γ^* where $EU([a,b]) = EU([0,d])$.

If γ^* is not unique (i.e., Equation (23) has more than one root), then there must be at least three roots for Equation (23). This is true because Equation (23) is less than 0 for $\gamma \to -\infty$ and Equation (23) is greater than 0 as $\gamma \to \infty$.

If Equation (23) has at least three roots, then Equation (23) must have at least one local minimum and two local maxima because the limit of Equation (23) as $\gamma \to \infty$ equals 0. From Lemma 2, the numerator of Equation (23) has at most one local minimum and one local maximum. This implies that Equation (23) has at most one local minimum and one local maximum, so Equation (23) cannot have more than two roots. However, we know that Equation (23) cannot have two roots. Thus, Equation (23) has exactly one root and γ^* is a unique value.

It is clear from $\gamma(\alpha)$ in Equation (21) that $\gamma = \gamma^*$ if and only if $\alpha = \alpha^*$; $\gamma > \gamma^*$ if and only if $\alpha > \alpha^*$; and $\gamma < \gamma^*$ if and only if $\alpha < \alpha^*$. Since Equation (23) is less than 0 as $\gamma \to -\infty$, then Equation (23) is less than 0 for all $\gamma < \gamma^*$. Since Equation (23) positive for large values of γ, then Equation (23) is greater than 0 for all $\gamma > \gamma^*$. End proof.

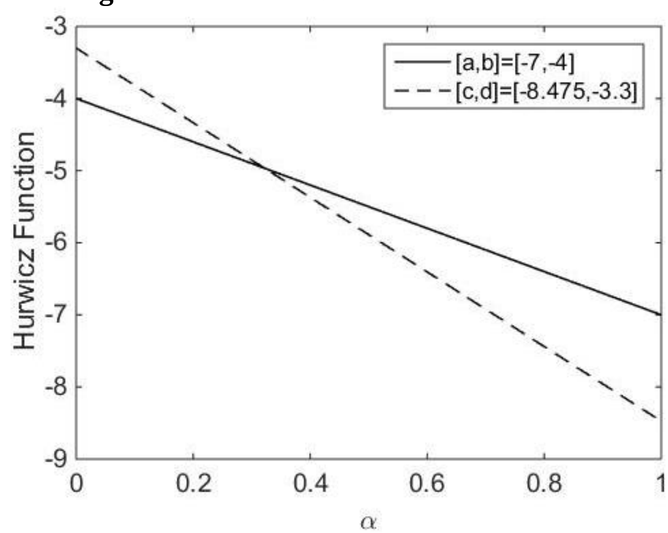
Theorem 3 signifies that if a decision maker chooses between two alternatives on the basis of intervals, that decision problem is equivalent to a problem with an exponential utility function in which each of the intervals follows a uniform distribution. The risk preference parameter in the exponential utility function can reflect the same trade-off between pessimism and optimism as in the Hurwicz equation.

5. Illustrative Example

Wilson and Barker (2012) illustrate the interval-valued decision tree approach for a maintenance, repair, and overhaul (MRO) example based on the aging U.S. Air Force fleet. Many airplanes' useful lives have been extended beyond what was originally intended. In their problem, the decision maker selects a maintenance alternative that minimizes downtime for an airplane. In this illustrative example, the decision maker assesses the uncertain downtime with intervals by defining a minimum and maximum downtime given an alternative. The decision maker can choose between replace and repair. Wilson and Barker provide details of a two-stage decision problem between replace and repair, and the decision in the first stage is reduced to a deciding between replace with a downtime between 4 and 7 hours and repair with a downtime between 3.3 and 8.475 hours. Interested readers are referred to Wilson and Barker for the details.

Since our previous discussion assumes a maximization problem, the positive downtime hours are represented as negative numbers. The following discussion sets $[a, b] = [-7, -4]$ for replace and $[c, d] = [-8.475, -3.3]$ for repair so that $c < a < b < d$. If $\alpha = 0.322$, the decision maker is indifferent between replace and repair (i.e., $\alpha^* = 0.322$). If $\alpha < \alpha^*$, the decision maker will prefer repair over replace, as depicted in Figure 4. If $\alpha > \alpha^*$, the decision maker will prefer replace over repair since the $[a, b]$ interval will be preferred to the $[c, d]$ interval.

Figure 4 Hurwicz Function for Two Intervals



We first illustrate how these intervals can be transformed to triangle distributions and beta distributions and how the Hurwicz decision rule corresponds to a risk-neutral decision with these distributions. Second, we demonstrate the relation between the Hurwicz decision rule and exponential expected utility.

5.1. Triangle Distribution

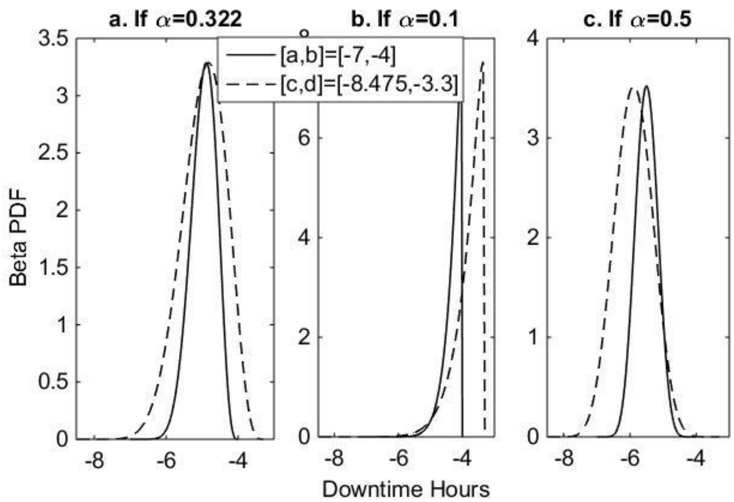
The two intervals are $[a, b] = [-7, -4]$ and $[c, d] = [-8.475, -3.3]$. The parameters of the triangle distribution depend on the value of α in the Hurwicz equation. If $\alpha = 0.322$, which is less than 1/3, we need to calculate $\tilde{a} > a$ and $\tilde{c} > c$ for these intervals as Theorem 1 indicates. If $\alpha = 0.322$, $\tilde{a} = -6.89$ and $\tilde{c} = -8.29$ based on Equations (2b) and (2c). The interval $[a, b]$ should be represented by a triangle distribution with parameters (-6.89,-4,-4) and $[c, d]$ should be represented by a triangle distribution with parameters (-8.29,-3.3,-3.3). These distributions imply that the probability that the downtime from replacement will be between 6.89 and 7 equals 0 and the probability that the downtime from repair will be between 8.475 and 8.29 equals 0. The expected downtime equals 4.96 hours for both of these triangle distributions.

If α decreases, $\tilde{a}$ and $\tilde{c}$ will increase, which implies less likelihood of a longer downtime. If $\alpha < 0.322$, the expected downtime of replace is more (i.e., more negative) than the expected downtime of repair, and the decision maker should choose repair. If $\alpha = 0$, $\tilde{a} = -4$ and $\tilde{c} = -3.3$. The decision maker is implicitly assuming certainty: replace will result in a downtime of 4 hours and repair will result in a downtime of 3.3 hours. In the Hurwicz equation $\alpha = 0$ corresponds to a best-case decision rule.

If α increases (up to $\alpha < 1/3$), $\tilde{a}$ and $\tilde{c}$ will decrease, which implies more likelihood of longer downtimes. The expected downtime of replace is less (i.e., less negative) than the expected downtime of repair, and the decision maker should choose replace. If $1/3 \leq \alpha \leq 2/3$, the lower and upper bounds of the triangle distributions correspond to the bounds of the interval. Given the mode of the triangle distribution as calculated via Equation (1), replace should be preferred to repair based on the resulting expected downtimes. If $\alpha > 2/3$, the upper bounds of the triangle distributions change, which implies less likelihood of shorter downtimes. Again, the decision maker should choose replace over repair.

5.2. Beta Distribution

Figure 5 Beta Distributions for Repair and Replace for Different α



We consider the same two intervals to form the beta distribution: $[a, b] = [-7, -4]$ for replace and $[c, d] = [-8.475, -3.3]$ for repair. We arbitrarily set the parameters α_x and α_y (alpha values for the beta distributions where α_x corresponds to $[a, b]$ and α_y corresponds to $[c, d]$) to equal 10. As depicted in Figure 5a, if $\alpha = 0.322$, $\beta_x = \beta_y = 4.74$ from Eqs. (8) and (9). The expected downtime for both alternatives is 4.96 hours. The expected values of the beta distributions must be equivalent when α equals the Hurwicz indifferent parameter.

If $\alpha = 0.1$, $\beta_x = \beta_y = 1.11$ assuming that $\alpha_x = \alpha_y = 10$ (Figure 5b). The expected downtime for replace is 4.3 hours and the expected downtime for repair is 3.8 hours, and the decision maker should choose repair. If $\alpha = 0.5$, $\beta_x = \beta_y = 10$, which means that both beta distributions are symmetric (Figure 5c). The expected downtime for replace is 5.5 hours and the expected downtime for repair is 5.88 hours, which corresponds to the midpoint of each interval (i.e., the Laplace decision rule).

5.3. Expected Utility

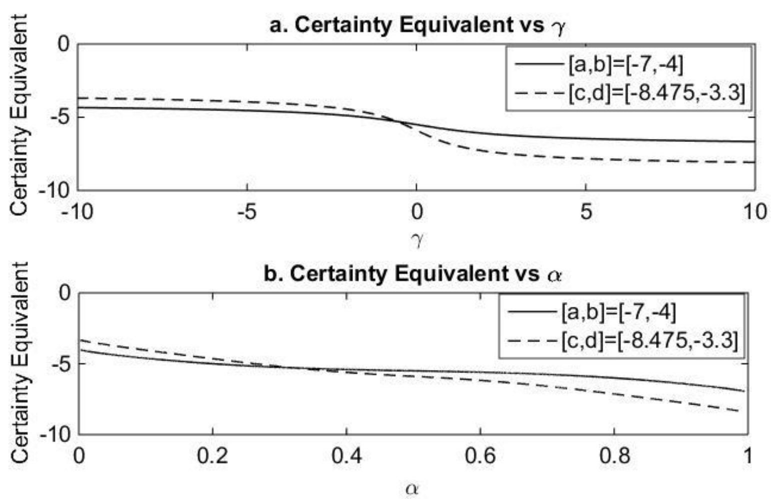
5.3.1. Two-interval Scenario

Finally, we illustrate the relationship between the Hurwicz decision rule and exponential expected utility. Figure 6 shows how the certainty equivalent (CE) varies with respect to γ and with respect to α for the two intervals $[a, b] = [-7, -4]$ and $[c, d] = [-8.475, -3.3]$. In the case of this downtime example, the CE is the number of hours that would make the decision maker indifferent between the uncertain interval and the CE for a given risk attitude γ (Garvey 2008). The CE preserves the same ordering as the expected utility. If $\gamma = -0.572$, the decision maker is indifferent between repair and replace, which makes $\gamma^* = -0.572$. This provides the mapping function from α to γ in which $\alpha^* = 0.322$ and $\gamma^* = -0.572$:

$$\gamma(\alpha) = \begin{cases} \dfrac{0.517(2\alpha - 1)}{\alpha}, & 0 < \alpha < 0.5 \\ \dfrac{1.089(2\alpha - 1)}{(1 - \alpha)}, & 0.5 \leq \alpha < 1 \end{cases} \tag{24}$$

If $\alpha < 0.322$, then $\gamma < -0.572$ and the risk-seeking decision maker prefers repair, the interval $[c, d]$, as shown in Figure 6. The decision maker is focusing on the best possible outcome. If $\alpha > 0.322$, then $\gamma > -0.572$ and the decision maker is slightly risk seeking, risk neutral, or risk averse. In these cases, the expected utility or CE of replace is greater than that of the repair.

Figure 6 Certainty Equivalent for Two Intervals



Translating the Hurwicz equation to an exponential utility decision problem provides a well-established foundation for the decision maker to determine his or her risk attitude. The function $\gamma(\alpha)$ maps α to γ, and a decision maker could use this function to determine his or her risk preference based on the optimism-pessimism trade-off or determine the optimism-pessimism trade-off based on his or her risk preference.

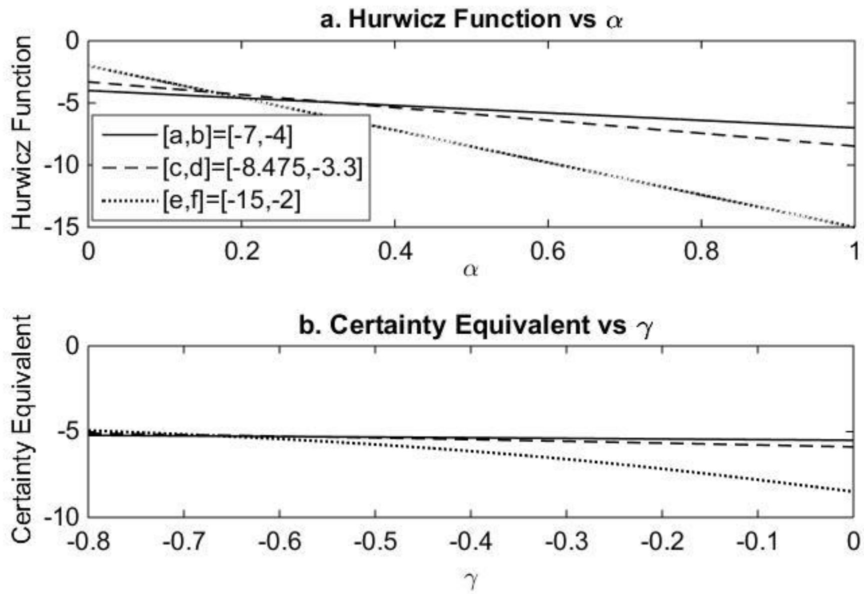
5.3.2. Three-interval Scenario

Many decisions require comparing among three or more alternatives. Although this article does not prove the results beyond decisions with two alternatives, similar rules can apply. The two-interval scenario is extended to a three-interval case for the maintenance illustration for expected exponential utility. In addition to replace with $[a, b] = [-7, -4]$ and the first repair option with $[c, d] = [-8.475, -3.3]$, we incorporate a second repair option with $[e, f] = [-15, -2]$.

The two-interval scenario has one value for α^* and γ^* to compare the intervals $[a, b]$ and $[c, d]$. In the three-interval case, there is α^* and γ^* for every two of the three intervals: $\alpha_1^* = 0.322$ and $\gamma_1^* -$ 0.572 are the points of indifference between $[a, b]$ and $[c, d]$; $\alpha_2^* = 0.166$ and $\gamma_2^* = -0.686$ are the points of indifference between $[c, d]$ and $[e, f]$; and $\alpha_3^* = 0.2$ and $\gamma_3^* = -0.658$ are the points of indifference between $[a, b]$ and $[e, f]$.

Figure 7a depicts the Hurwicz equation as a function of α, and Figure 7b depicts the CE based on the expected exponential utility as a function of γ where each interval follows a uniform distribution. The optimal alternative is the same for both decision rules. If $\alpha < \alpha_2^*$ and $\gamma < \gamma_2^*$, the decision maker should prefer the section repair option or the interval $[e, f]$, and the decision maker is risk seeking or very optimistic that the best case will occur. If $\alpha_2^* < \alpha < \alpha_1^*$ and $\gamma_2^* < \gamma < \gamma_1^*$, the decision maker should select the first repair option or interval $[c, d]$. If $\alpha > \alpha_1^*$ and $\gamma > \gamma_1^*$, the decision maker ranges from a slightly optimistic to very pessimistic or from slightly risk seeking to very risk averse. The decision maker should select the replace alternative. In this example, there are three distinct regions for α or for γ for which each of the three alternatives should be preferred.

Figure 7 Hurwicz Function and Certainty Equivalent for Three Intervals



6. Conclusion and Remarks

This article shows how the Hurwicz decision rule for non-probabilistic intervals can be transformed to equivalent decision problems that follow prescriptive decision analysis. In the first case, the interval is translated to a triangle probability distribution for a risk-neutral decision maker in which the skewness of the triangle distribution mirrors the decision maker's emphasis on the worst or best-case outcome in the Hurwicz equation. In the second case, a beta distribution with a risk-neutral decision maker is used to mirror the Hurwicz equation. In the third case, an exponential utility function that assumes a uniform distribution is used in which the risk attitude in the exponential utility function mirrors the decision maker's emphasis on the worst or best-case outcome.

The article proves that an individual who follows the Hurwicz decision rule to make a decision should choose the same alternative as (i) if he or she is an expected-value decision maker when the uncertain outcomes follow either a triangle or beta distribution and (ii) if he or she maximizes expected utility with constant absolute risk aversion when the uncertain outcomes follow a uniform distribution. Because the preferred alternatives are always the same, if an individual is willing to make a decision based on the Hurwicz decision rule, we argue that he or she should be willing to assume a probability distribution and an exponential utility function. Proponents of non-probabilistic decision making argue that utilizing probabilities may assume too much information for a given problem, especially for epistemic uncertainties. However, if subjective expected utility decision making with certain distributions and/or an exponential utility function results in the same preferred alternative as a non-probabilistic interval results, does using probabilities really assume too much information? One could logically argue that these decision rules assume the same amount of information even though the assumptions of each are different. We question the use of intervals for decision making since the Hurwicz decision rule can be translated to subjective expected utility decision rules, the latter of which is theoretically and practically well supported.

Although an interval-based decision making rule can be translated to an equivalent probability distribution and utility function, a decision maker may feel more comfortable using the Hurwicz decision rule than using probabilities or utilities. The Hurwicz equation is easy to understand, and a decision maker only needs to consider his or her belief that the best versus the worst outcome will occur. Decision making with probabilities and utilities requires that a decision maker understand probability distributions and utility functions and is comfortable answering questions to assess his or her risk attitude. Future research could conduct behavioral studies to assess if individuals prefer making decisions using the Hurwicz equation or using a triangle or beta distribution or an exponential utility function.

Future research could analyze the relationship between the Hurwicz decision rule and the expected value of other probability distributions and using utility functions other than exponential utility. The triangle and beta distributions described in this article are not exhaustive, and it appears likely that any bounded distribution in which the skewness can be manipulated could translated to the Hurwicz decision rule. This article also assumes an exponential utility function, and other utility functions could have a similar relation to the Hurwicz decision rule. The utility function should be able to translate to risk-seeking as well as risk-averse attitudes as long as a uniform distribution is used to describe the interval. For example, a utility function with hyperbolic absolute risk aversion is a general form of a utility function that encompasses exponential utility, logarithmic utility, and quadratic utility (Ingersoll 1987). Although hyperbolic absolute risk aversion is a generalization of constant absolution risk aversion, it is our understanding that the hyperbolic utility function assumes either a risk averse or risk neutral risk attitude. In order for the Hurwicz decision rule to have an equivalence with the hyperbolic utility function for every value of α, the hyperbolic utility function will also need to capture a risk-seeking attitude.

Other decision rules for intervals could arise which cannot be translated to the Hurwicz decision rule. These may not result in the same preferred alternative as probabilistic decision making with a risk neutral or an exponential utility function. For example, Gaspars-Weiloch (2014) modifies the Hurwicz decision rule to consider a trade-off between the worst and best outcome and Laplace (average). Future research could explore the relationship between this modification of the Hurwicz decision rule and subjected expected utility.

The article provides important theoretical contributions on the relationship between interval-based decision making and prescriptive decision analysis for the three cases mentioned earlier. Additionally, the theoretical contributions have practical implications. If a decision is made based on the Hurwicz equation for a given α, the results in this article could be used to determine the equivalent risk attitude γ in the exponential utility function. If the decision maker concludes that the value of γ does not match his or her risk attitude, then perhaps, the decision maker should be encouraged to reassess α.

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