

3-1992

The influence of nonlocal nonlinearities on the long time behavior of solutions of Burgers' equation

Keng Deng

University of Southwestern Louisiana

Man Kam Kwong

Argonne National Laboratory

Howard A. Levine

Iowa State University, halevine@iastate.edu

Follow this and additional works at: http://lib.dr.iastate.edu/math_pubs



Part of the [Dynamical Systems Commons](#), and the [Partial Differential Equations Commons](#)

The complete bibliographic information for this item can be found at http://lib.dr.iastate.edu/math_pubs/164. For information on how to cite this item, please visit <http://lib.dr.iastate.edu/howtocite.html>.

This Article is brought to you for free and open access by the Mathematics at Iowa State University Digital Repository. It has been accepted for inclusion in Mathematics Publications by an authorized administrator of Iowa State University Digital Repository. For more information, please contact digirep@iastate.edu.

The influence of nonlocal nonlinearities on the long time behavior of solutions of Burgers' equation

Disciplines

Applied Mathematics | Dynamical Systems | Partial Differential Equations

Comments

This is a manuscript of an article published as Deng, Keng, Man Kam Kwong, and Howard A. Levine. "The influence of nonlocal nonlinearities on the long time behavior of solutions of Burgers' equation." *Quarterly of applied mathematics* 50, no. 1 (1992): 173-200. DOI: [10.1090/qam/114663](https://doi.org/10.1090/qam/114663). Posted with permission.

THE INFLUENCE OF NONLOCAL NONLINEARITIES ON THE LONG-TIME BEHAVIOR OF SOLUTIONS OF BURGERS' EQUATION

KENG DENG, MAN KAM KWONG, AND HOWARD A. LEVINE *

Abstract

We study the long-time behavior of solutions of Burgers' equation with nonlocal nonlinearities $u_t = u_{xx} + \varepsilon uu_x + \frac{1}{2} (a \|u(\cdot, t)\|^{p-1} + b) u$, $0 < x < 1$, $a, \varepsilon \in \mathbb{R}$, $b > 0$, $p > 1$, subject to $u(0, t) = u(1, t) = 0$. A stability–instability analysis is given in some detail, and some finite-time blow-up results are given.

*The research of Deng and Levine was supported by NSF Grant DMS–8822788, that of Kwong by the Applied Mathematical Sciences subprogram of the Office of Energy Research, U. S. Department of Energy, under Contract W-31-109-Eng-38.

1 Introduction

In this paper we consider the following initial-boundary value problem:

$$\begin{cases} u_t = u_{xx} + \varepsilon uu_x + \frac{1}{2} \left(a \|u(\cdot, t)\|^{p-1} + b \right) u, & 0 < x < 1, \quad t > 0, \\ u(0, t) = u(1, t) = 0, & t > 0, \\ u(x, 0) = u_0(x), & 0 \leq x \leq 1. \end{cases} \quad (\text{P})$$

Here a , b , ε , and p are given constants, with $\varepsilon \geq 0$ (without loss of generality), $b > 0$, and $p > 1$; and $u_0(\cdot)$ is a continuous function with $u_0(0) = u_0(1) = 0$. Moreover, we take

$$\|u(\cdot, t)\| = \left(\int_0^1 |u(x, t)|^q dx \right)^{1/q}.$$

In this paper, we require that $u_0 \geq 0$. Then by the maximum principle, $u(\cdot, t) \geq 0$ for all t in the existence interval.

Our interest in (P) is twofold. First, (P) is closely related to a one-dimensional turbulence model proposed by Burgers ([2], [3]) and studied by Horgan and Olmstead [8]. (See also Drazin and Reid [5].) The major difference between (P) and the earlier model is the use of the L^q norm in (P) rather than the L^2 norm. A number of authors ([1], [6], [8], [12]) have investigated nonlocal problems as models for local problems; they also restricted their attention to the case $q = 2$. To the best of our knowledge, no one has considered problems in which a convective term (εuu_x) is present. Yet, convective terms have a remarkable effect on the dynamical behavior of solutions of equations. For example, consider

$$\begin{cases} u_t = u_{xx} + \varepsilon uu_x + \frac{1}{2}bu, & 0 < x < 1, \quad t > 0, \\ u(0, t) = u(1, t) = 0, & t > 0, \\ u(x, 0) = u_0(x) \geq 0, & 0 \leq x \leq 1. \end{cases}$$

When $\varepsilon = 0$ and $b > 2\pi^2$, this problem possesses the exponentially growing solution $e^{(b/2-\pi^2)t} \sin \pi x$, whereas for $\varepsilon > 0$, all solutions are bounded for any b (since a supersolution of the form $M(1 - \alpha x)$ with large M and

$\alpha \in (0, 1)$ exists). We shall see that this phenomenon persists, although to a somewhat less pronounced effect, for (P).

The second reason for our interest in (P) is that (with $b = 0$) it is closely related to the same initial-boundary value problem for the equation

$$u_t = u_{xx} + \varepsilon uu_x + a|u|^{p-1}u, \quad (1.1)$$

where the nonlocal nonlinearity is replaced by the more standard local term. This problem was studied extensively in [4] and [11], where the stabilizing effect of convective terms was noted. The study of (P) was taken up with the objective of obtaining analogous results for a closely related problem. It turns out that the results we derived for (P) are more complete than those for (1.1) obtained in [4] and [11].

In [14], Straughan et al. consider, from a computational point of view, the same initial-boundary value problem for

$$Lu = u_t + 2uu_x - R^{-1}u_{xx},$$

where

$$Lu = \begin{cases} u - Ru\|u\|^2, & \text{or} \\ u, & \text{or} \\ u + Ru\|u\|. \end{cases}$$

Here $\|\cdot\|$ denotes the L^2 norm, and $R > 0$ can be thought of as the Reynolds number. Under appropriate scaling, this problem is included in (P) when $q = 2$. In particular, we are able to verify theoretically for (P) all except one of the numerical observations made in [14].

The plan of our paper is as follows. In Section 2, we state the results in the absence of convection. In Section 3, we characterize the set of stationary solutions of (P) when $\varepsilon > 0$. In Section 4, we discuss the long-time behavior of solutions of (P), including stability, asymptotic stability, global existence, and nonexistence. The necessary local existence theorems and comparison theorems are discussed in Appendix A. These are standard but we could not find any reference to such results for nonlocal problems of the type considered here. In Appendix B, we gave the proof of a technical inequality, namely, that of Lemma 3.2.

2 Discussion for the Case $\varepsilon = 0$

In this section, we consider the following problem:

$$\begin{cases} u_t = u_{xx} + \frac{1}{2} \left(a \|u(\cdot, t)\|^{p-1} + b \right) u, & 0 < x < 1, \quad t > 0, \\ u(0, t) = u(1, t) = 0, & t > 0, \\ u(x, 0) = u_0(x), & 0 \leq x \leq 1. \end{cases} \quad (\text{P}^0)$$

Because (P^0) is an explicitly resolvable problem, we can easily examine the questions of stability, global existence, and nonexistence of nonnegative solutions.

First, for the stationary solutions of (P^0) , we need to solve

$$\begin{cases} v'' + \frac{1}{2} \left(a \|v\|^{p-1} + b \right) v = 0, & 0 < x < 1, \\ v(0) = v(1) = 0. \end{cases} \quad (\text{S}^0)$$

The nonnegative solutions of (S^0) must be in the form $v(x) = c_1 \varphi_1(x) = c_1 \sin \pi x$. Substituting such v in the equation, we find

$$-\pi^2 + \frac{1}{2} \left(a c_1^{p-1} \|\varphi_1\|^{p-1} + b \right) = 0. \quad (2.1)$$

Thus, for $a > 0$, there is no positive solution when $b \geq 2\pi^2$ and one positive solution when $b < 2\pi^2$. For $a < 0$, there exists a unique positive solution when $b > 2\pi^2$ but none when $b \leq 2\pi^2$.

Next, for (P^0) , we seek solutions of the form

$$u(x, t) = \sum_{n=1}^{\infty} a_n(t) \varphi_n(x), \quad (2.2)$$

where $\varphi_n(x) = \sin n\pi x$. Let $f(t) = \|u(\cdot, t)\|^{p-1} = \left(\int_0^1 \left| \sum_{n=1}^{\infty} a_n \varphi_n \right|^q dx \right)^{\frac{p-1}{q}}$.

Through a straightforward computation, we have

$$f(t) = (h(t))^{p-1} e^{\frac{a(p-1)}{2} \int_0^t f(\eta) d\eta}, \quad (2.3)$$

where

$$h(t) = e^{\frac{b}{2}t} \left(\int_0^1 \left| \sum_{n=1}^{\infty} a_n(0) e^{-n^2\pi^2 t} \varphi_n(x) \right|^q dx \right)^{\frac{1}{q}}.$$

From (2.3), after a quadrature, we can rewrite $f(t)$ as follows:

$$f(t) = (h(t))^{p-1} \left/ \left[1 - \frac{a(p-1)}{2} \int_0^t (h(\eta))^{p-1} d\eta \right] \right. . \quad (2.4)$$

Let

$$H(t) = \int_0^t h^{p-1}(\eta) d\eta. \quad (2.5)$$

If $b < 2\pi^2$, it follows that $H(t) \rightarrow H_\infty < \infty$ for some constant H_∞ which is proportional to $|a_N(0)|$ where N is the smallest integer such that $a_n \neq 0$. Thus, if $a(p-1)H_\infty < 2$, which will be the case if $a_N(0)$ is sufficiently small, then $f(t) \rightarrow 0$ as $t \rightarrow \infty$, and the trivial solution is stable in L^q . On the other hand, if $a(p-1)H_\infty > 2$, then clearly $f(t) \rightarrow \infty$ in finite time, so some solutions of (P^0) blow up in finite time. Finally, if $a(p-1)H_\infty = 2$, then $f(t)$ can have a finite, nonzero limit as $t \rightarrow \infty$. This situation occurs, for example, when u is a stationary solution of (P^0) .

To show that $v(x)$ is unstable, we look for solutions of (P^0) of the form

$$u(x, t) = a_1(t)\varphi_1(x) = \alpha(t)v(x),$$

where $\alpha(t) = a_1(t)/c_1$ and $\alpha(0) \neq 1$. Then, using (S^0) , we see that

$$\alpha'(t) = c^2 \left(\alpha^{p-1}(t) - 1 \right) \alpha(t),$$

where $c^2 = \frac{1}{2}a\|v\|_q^{p-1}$. If $\alpha(0) < 1$, then $\alpha'(t) < 0$ for small $t > 0$ and consequently for all t (since $p > 1$). Therefore $\alpha(t) < \alpha(0)$ and $\alpha'(t) < -c^2(1 - \alpha^{p-1}(0))\alpha(t)$, so $\alpha(t) \rightarrow 0$ as $t \rightarrow \infty$. If $\alpha(0) > 1$, it is easy to see that $\alpha' > 0$ for all t for which $\alpha(t)$ exists. It follows that $\alpha(t)$ blows up in finite time. Thus, for any $r > 1$,

$$\liminf_{t \rightarrow \infty} \|u(\cdot, t) - v\|_r > 0,$$

and consequently v is not stable (in any norm).

If $b \geq 2\pi^2$, then zero is unstable. To see this, we note that if $a_1(0) \neq 0$, then, in view of the asymptotic behavior of $h(t)$ near $+\infty$, $H(t)$ grows either exponentially or linearly so that it passes $2/a(p-1)$ in finite time. Consequently, whenever $(u_0, \varphi_1) \neq 0$, solutions of (P^0) blow up in finite time.

For the case $a < 0$, we again have subcases. If $b < 2\pi^2$, zero is the only nonnegative stationary solution. From the form of $h(t)$ and the fact that $a < 0$, we see that $h(t) \rightarrow 0$ as $t \rightarrow \infty$, and consequently $\|u(\cdot, t)\| \rightarrow 0$ also.

If $b = 2\pi^2$, then $h(t) \rightarrow |a_1(0)| \|\varphi_1\|$. If $|a_1(0)| = 0$, then from (2.4), since $a < 0$, we have $f(t) \rightarrow 0$ as $t \rightarrow \infty$. If $|a_1(0)| \neq 0$, then $H(t)$ behaves like $(|a_1(0)| \|\varphi_1\|)^{p-1} t$, so $f(t) \rightarrow 0$ in this case too. Therefore zero is globally asymptotically stable in this case also.

If $b > 2\pi^2$, we claim that zero is unstable and that the positive stationary solution, $v(x)$, is stable. For the first of these, let $u(x, t) = \varepsilon(t)v(x)$. Then, as before,

$$\varepsilon'(t) = c^2 \varepsilon(t) \left(1 - \varepsilon^{p-1}(t)\right),$$

where $c^2 = |a| \|v\|^{p-1}$. Thus, if $0 < \varepsilon(0) < 1$, $\varepsilon' > 0$ for small t , and hence for all t for which $\varepsilon(0) \leq \varepsilon(t) \leq 1$. It follows that if $(0, T)$ is the largest interval on which $\varepsilon'(t) > 0$, then $\varepsilon(t) \rightarrow 1^-$ as $t \nearrow T$ and $T = +\infty$. This establishes the instability of the null solution.

To show that $v(x)$ is stable for $a < 0$ and $b > 2\pi^2$, we write

$$u(x, t) = a_1(t)\varphi_1(x) + \sum_{n=2}^{\infty} a_n(t)\varphi_n(x), \quad (2.6)$$

where

$$a_1(0) = (1 + \delta)c_1 \quad \text{and} \quad |\delta| \ll 1, \quad \delta \neq 0.$$

and where $v(x) = c_1\varphi_1(x)$ solves (S^0) . Then a tedious, but routine, computation yields

$$a_n(t) = a_n(0)e^{(-n^2\pi^2 + \frac{1}{2}b)t} \left[1 - \frac{a(p-1)}{2}H(t)\right]^{-\frac{1}{p-1}} \quad (2.7)$$

and

$$u(x, t) - v(x) = (a_1(t)/c_1 - 1)v(x) + \sum_{n=2}^{\infty} a_n(t)\varphi_n(x) \quad (2.8)$$

or

$$\begin{aligned} u(x, t) - v(x) &= \left[(1 + \delta)e^{(\frac{1}{2}b - \pi^2)t} \left[1 - \frac{a(p-1)}{2}H(t)\right]^{-\frac{1}{p-1}} - 1\right] v(x) \\ &\quad + e^{(\frac{1}{2}b - \pi)t} \left[1 - \frac{a(p-1)}{2}H(t)\right]^{-\frac{1}{p-1}} \sum_{n=2}^{\infty} a_n(0)e^{-(n^2-1)\pi^2 t} \varphi_n(x). \end{aligned} \quad (2.9)$$

Since $a_1(0) \neq 0$, we see that $h(t) \sim (1 + \delta)\|v\|e^{(\frac{1}{2}b - \pi^2)t}$ as $t \rightarrow \infty$. This, together with (2.1), shows us that

$$\begin{aligned} \left[1 - \frac{1}{2}a(p-1)H(t)\right]^{\frac{1}{p-1}} &\sim \left(\frac{-a\|v\|^{p-1}}{b-2\pi^2}\right)^{\frac{1}{p-1}} (1 + \delta)e^{(\frac{b}{2} - \pi^2)t} \\ &\sim (1 + \delta)e^{(\frac{b}{2} - \pi^2)t} \end{aligned}$$

as $t \rightarrow +\infty$. Using these estimates in (2.9) and the triangle inequality, we see that

$$\begin{aligned} \|u(\cdot, t) - v\|_q &\leq \varepsilon_1(t)\|v\|_q \\ &\quad + e^{-\pi^2 t} (1 - \varepsilon_2(t)) \left[\int_0^1 \left| \sum_{n=2}^{\infty} a_n(0) e^{-(n^2-2)\pi^2 t} \varphi_n(x) \right|^q dx \right]^{1/q} \end{aligned}$$

where $\varepsilon_1(t), \varepsilon_2(t) \rightarrow 0$ as $t \rightarrow \infty$. Consequently $\|u(\cdot, t) - v\|_q \rightarrow 0$ as $t \rightarrow +\infty$.

3 The Stationary Solutions When $\varepsilon > 0$

If we write $u = \sigma v$ with $\sigma > 0$ such that $\sigma^{p-1}|a| = 1$, and if we replace ε by $\sigma\varepsilon$, we can reduce (P) to an equivalent problem for v with $a = 1$ or $a = -1$. For this reason, we seek all positive solutions of

$$\begin{cases} w_{xx} + \varepsilon w w_x + \frac{1}{2} (\delta \|w\|^{p-1} + b) w = 0, & 0 < x < 1, \\ w(0) = w(1) = 0, \end{cases} \quad (\text{S})$$

where $\delta = 1$ or $\delta = -1$. We let

$$\|w\| = \|w\|_q = \left(\int_0^1 w^q dx \right)^{1/q}$$

and set

$$y = \int_0^x w(\eta) d\eta, \quad Y = \int_0^1 w(\eta) d\eta. \quad (3.1)$$

Under this change of variable, we have

$$\frac{d}{dx} = w \frac{d}{dy}.$$

Thus, in lieu of (S), with $v(y) = w^2(x(y))$, we find

$$v_{yy} + \varepsilon v_y + \delta Y_q^{p-1} + b = 0, \quad (3.2a)$$

$$v(0) = v(Y) = 0, \quad (3.2b)$$

where $Y_q = \|w\|_q$ and $Y_1 = Y$. The positive solutions of (3.2a,b) are given by

$$v(y, Y) = \varepsilon^{-1} Y (\delta Y_q^{p-1} + b) J(\varepsilon y, \varepsilon Y), \quad (3.3a)$$

where

$$J(y, z) = \frac{1 - e^{-y}}{1 - e^{-z}} - \frac{y}{z}. \quad (3.3b)$$

It follows that $\delta Y_q^{p-1} + b > 0$, since $v = w^2$.

The values of Y, Y_q are then determined by solving the nonlinear system

$$1 = \int_0^1 dx = \int_0^Y \frac{dy}{\sqrt{v(y)}}, \quad (3.4a)$$

$$Y_q^q = \int_0^1 w^q(x) dx = \int_0^Y v^{\frac{1}{2}(q-1)}(y) dy. \quad (3.4b)$$

We define, for $s \geq 0$, $z > 0$,

$$H_s(z) = \int_0^1 [J(\sigma z, z)]^{\frac{1}{2}(s-1)} d\sigma.$$

A routine calculation shows that with

$$\begin{aligned} Z_q &= \varepsilon Y_q, \\ Z_1 &= Z = \varepsilon Y, \end{aligned} \quad (3.5)$$

we have from (3.4a,b), for $1 \leq q < \infty$,

$$Z_q \equiv Z_q(Z) = Z \left[H_q(Z) H_0^{q-1}(Z) \right]^{1/q} \quad (Z \geq 0), \quad (3.6)$$

while

$$b = Z H_0^2(Z) - \delta \varepsilon^{-(p-1)} Z_q^{p-1}(Z) = b(Z). \quad (3.7)$$

When $q = +\infty$,

$$Z_q(Z) = Z^{-1/2} \left[\frac{e^{-Z} - (1-Z)}{1-e^{-Z}} + \ln \left(\frac{1-e^{-Z}}{Z} \right) \right]^{1/2} H_0(Z).$$

Useful asymptotic formulas are, for $1 \leq q < \infty$,

$$\frac{Z_q(Z)}{Z} = \begin{cases} 2/(1+q)^{1/q}, & Z \rightarrow \infty, \\ \left[\pi^{q-1} \Gamma^2 \left(\frac{1}{2}(1+q) \right) / \Gamma(1+q) \right]^{1/q}, & Z \rightarrow 0^+, \end{cases}$$

and, for $q = \infty$,

$$\frac{Z_\infty(Z)}{Z} = \begin{cases} 2, & Z \rightarrow \infty, \\ \pi/2, & Z \rightarrow 0^+. \end{cases}$$

We have plotted $Z_q(Z)$ and $Z_q(Z)/Z$ against Z for various q in Figure 1 and Figure 2, respectively.

For fixed b , the cardinality of the set of stationary solutions turns out to be the same as the cardinality of the set of solutions of (3.7). To see this, we need some lemmas.

Lemma 3.1. *For any $q \geq 1$, $p > 1$, all the positive solutions of (S) are concave.*

Proof. The proof follows immediately from (3.3a,b). We have

$$w_{xx} = w \frac{d}{dy} \left(w \frac{dw}{dy} \right) = \frac{1}{2} \sqrt{v(y)} v_{yy}(y).$$

Clearly, from (3.3a,b) $v_{yy} < 0$.

This establishes the concavity of the steady states found in [14] numerically. ■

Lemma 3.2. *The function $h(Z) := ZH_0^2(Z)$ ($Z > 0$) is strictly increasing.*

The proof of this lemma is long and is therefore included in Appendix B.

Lemma 3.3. *The function $h_q(Z) := Z^q H_q(Z) H_0^{q-1}(Z)$ is strictly increasing for $q \geq 1$.*

Proof. We can write

$$h_q(Z) = Z^{\frac{1}{2}(q+1)} H_q(Z) (h(Z))^{\frac{1}{2}(q-1)}.$$

Therefore, by Lemma 3.2, it suffices to show that $H_q(Z)$ is strictly increasing. This in turn will hold if

$$\frac{d}{dz} J(\sigma z, z) = [\sigma e^{-\sigma z} (1 - e^{-z}) - e^{-z} (1 - e^{-\sigma z})] / (1 - e^{-z})^2 \geq 0.$$

If we write $x = e^{-z}$, then $x \in (0, 1]$, and

$$G(\sigma, x) \equiv \frac{d}{dz} J(\sigma z, z) = \left[\sigma x^\sigma - \frac{x(1 - x^\sigma)}{1 - x} \right] / (1 - x).$$

In view of the convexity of $1 - x^\sigma$, we find

$$\sigma x^{\sigma-1} \geq \frac{1 - x^\sigma}{1 - x}.$$

Hence $G(\sigma, x) \geq 0$, and the lemma is true. ■

Lemma 3.4. *Let w_1, w_2 be two nonnegative stationary solutions of (P) with $\|w_1\|_1 \leq \|w_2\|_1$. Then either $w_1 < w_2$ on $(0, 1)$ or $w_1 \equiv w_2$.*

Proof. Any such solution can be written in the form

$$\begin{aligned} \varepsilon^2 v \left(z, \frac{Z}{\varepsilon} \right) &= Z \left(\delta \varepsilon^{-(p-1)} Z_q^{p-1}(Z) + b \right) J(z, Z) \\ &= Zh(Z)J(z, Z), \end{aligned}$$

where $z = \varepsilon y$, $Z = \varepsilon \|w\|_1$ and $v(y) = w^2(x)$, in view of (3.7).

A straightforward computation shows that

$$\begin{aligned} \varepsilon^2 \frac{\partial v}{\partial Z} \left(z, \frac{Z}{\varepsilon} \right) &= \varepsilon v_{,2} \left(z, \frac{Z}{\varepsilon} \right) \\ &= h'(Z)ZJ(z, Z) + h(Z) \frac{\partial}{\partial Z} (ZJ(z, Z)) \\ &= h'(Z)ZJ(z, Z) + \frac{h(Z)(1 - e^{-z})(1 - e^{-Z} - Ze^{-Z})}{(1 - e^{-Z})^2} \\ &> 0. \end{aligned}$$

Thus, if $Y_i = \|w_i\|_1$, we have $Y_1 \leq Y_2$ and

$$v(y, Y_1) \leq v(y, Y_2).$$

We need to show that if $w_1 \not\equiv w_2$, then $w_1 < w_2$ on $(0, 1)$. Note that $w_i^2(x) = v(y_i, Y_i)$, and $\frac{dy_i}{dx} = w_i(x)$ for $i = 1, 2$. Thus, for any $x \in (0, 1)$,

$$x = \int_0^{y_1(x)} (v(\sigma, Y_1))^{-1/2} d\sigma = \int_0^{y_2(x)} (v(\sigma, Y_2))^{-1/2} d\sigma,$$

where

$$y_i(x) = \int_0^x w_i(\eta) d\eta.$$

We see that if $Y_1 = Y_2$, then $y_1(x) \equiv y_2(x)$, so that $w_1(x) \equiv w_2(x)$. Hence, we may assume $Y_1 < Y_2$.

Since $v(\sigma, Y_1) < v(\sigma, Y_2)$, if $\sigma > 0$, we must have $y_1(x) < y_2(x)$ on $(0, 1]$. Let \bar{x} denote the unique x in $(0, 1)$ where $w_2'(\bar{x}) = 0$ (Lemma 3.1). Then for

$x \in (0, \bar{x})$, $y_2(x) < y_2(\bar{x})$. Now

$$\begin{aligned} & v(y_2(x), Y_2) - v(y_1(x), Y_1) \\ &= v(y_2(x), Y_2) - v(y_1(x), Y_2) + v(y_1(x), Y_2) - v(y_1(x), Y_1). \end{aligned}$$

Since $v(\cdot, Y_2)$ is increasing on $(0, y_2(\bar{x}))$ and $v_{,2} > 0$, the differences on the right are positive on $(0, \bar{x}]$. Thus $w_1 < w_2$ on $(0, \bar{x}]$.

A similar argument using the change of variable

$$y = \int_x^1 w(\eta) d\eta$$

yields (3.2a) with ε replaced by $-\varepsilon$ and, in place of (3.3a),

$$v(y, Y) = \varepsilon^{-1}(\delta Y_q^{p-1} + b) \left[y - \frac{Y(e^{\varepsilon y} - 1)}{(e^{\varepsilon Y} - 1)} \right].$$

Again, one finds that $v_{,2} > 0$ and ultimately that $w_1 < w_2$ if $1 - x < 1 - \bar{x}$ or $x > \bar{x}$. ■

These lemmas allow us to conclude the following theorem.

Theorem 3.5. *Let $w(x)$ be a nonnegative solution of (S). Then, if $Z = \varepsilon \|w\|_1$, Z solves (3.7). Conversely, if $Z > 0$ is a solution of (3.7), then $w(x) = (v(y))^{1/2}$ solves (S) with $dx/dy = (v(y, Y))^{-1/2}$, $Y = Z/\varepsilon$, and v is given by (3.3a,b).*

We next count the solutions of (3.7) for fixed $\varepsilon > 0$. Let $Z(b)$ be a branch of solutions of (2.7). By the implicit function theorem, as long as

$$Q(Z) := \frac{\partial}{\partial Z} \left[ZH_0^2(Z) - \delta \varepsilon^{-(p-1)} Z_q^{p-1}(Z) \right] \neq 0$$

on this branch, Z will be a C^1 function of b and

$$Z'(b)Q(Z(b)) = 1.$$

For fixed b , the number of solutions of (3.7) is then the same as the number of sign changes of $Q(Z)$ on $(0, \infty)$. However, the sign analysis of Q is

complicated when $\delta > 0$, since, in principle, a function of bounded variation (the difference of two increasing functions) can have infinitely many zeros.

The case $\delta < 0$ is easy to treat. The function $b(Z)$ given by (3.7) is strictly increasing for $Z \geq 0$ and has the range $[2\pi^2, \infty)$. Therefore, for each $b > 2\pi^2$, there is exactly one positive solution of (S), and there is no nontrivial solution for $b \leq 2\pi^2$.

We observe that $b(Z)$ has the following asymptotic properties:

$$b(Z) = \begin{cases} 4Z - \delta \varepsilon^{-(p-1)} \varepsilon_\infty^{p-1} Z^{p-1} & Z \rightarrow \infty, \quad 1 \leq q \leq \infty, \\ 2\pi^2 - \delta \varepsilon^{-(p-1)} \varepsilon_0^{p-1} Z^{p-1} & Z \rightarrow 0^+, \quad 1 \leq q \leq \infty, \end{cases}$$

where

$$\varepsilon_\infty = \begin{cases} 2(1+q)^{-1/q} & 1 \leq q < \infty, \\ 2 & q = \infty, \end{cases}$$

and

$$\varepsilon_0 = \begin{cases} \left[\pi^{q-1} \Gamma^2\left(\frac{1}{2}(1+q)\right) / \Gamma(1+q) \right]^{1/q} & 1 \leq q < \infty, \\ \frac{\pi}{2} & q = \infty. \end{cases}$$

These asymptotic formulas, together with the numerical results of Figures 4–11, allow us to assert the following for $\delta > 0$:

(N–1) If $1 < p < 2$, there is $b(p, q, \varepsilon) < 2\pi^2$ such that

- (a) if $b < b(p, q, \varepsilon)$, there are no positive stationary solutions;
- (b) if $b = b(p, q, \varepsilon)$ or $b > 2\pi^2$, there is exactly one solution; and
- (c) if $b(p, q, \varepsilon) < b < 2\pi^2$, there are exactly two solutions.

(N–2) If $p = 2$, we have the following:

- (a) If $\varepsilon < \frac{1}{4}\varepsilon_\infty$, there is exactly one positive stationary solution for each $b < 2\pi^2$, none for $b \geq 2\pi^2$.
- (b) If $\varepsilon = \frac{1}{4}\varepsilon_\infty$, there is $b(\varepsilon_\infty) < 2\pi^2$ such that
 - (i) if $b \leq b(\varepsilon_\infty)$ or $b > 2\pi^2$, there are no positive stationary solutions; and
 - (ii) if $b(\varepsilon_\infty) < b < 2\pi^2$, there is exactly one solution.

- (c) If $\varepsilon > \frac{1}{4}\varepsilon_\infty$, there is $b(\varepsilon_\infty) < 2\pi^2$ such that
 - (i) if $b < b(\varepsilon_\infty)$, there are no positive stationary solutions;
 - (ii) if $b = b(\varepsilon_\infty)$ or $b > 2\pi^2$, there is exactly one solution; and
 - (iii) if $b(\varepsilon_\infty) < b < 2\pi^2$, there are exactly two solutions.
- (N-3)** If $p > 2$, there is a critical number $\varepsilon_1(p, q)$ such that
 - (a) if $\varepsilon \leq \varepsilon_1$, we have the following:
 - (i) If $b \geq 2\pi^2$, there are no positive stationary solutions.
 - (ii) If $b < 2\pi^2$, there is exactly one solution.
 - (b) if $\varepsilon > \varepsilon_1$, there is $b(p, q, \varepsilon) > 2\pi^2$ such that
 - (i) if $b > b(p, q, \varepsilon)$, there are no positive stationary solutions;
 - (ii) if $b = b(p, q, \varepsilon)$ or $b < 2\pi^2$, there is exactly one solution; and
 - (iii) if $2\pi^2 < b < b(p, q, \varepsilon)$, there are exactly two solutions.

A word of explanation about the figures is in order. In Figures 4–6, we have set $q = 2$ and chosen $p = 1.5, 2,$ and 3 , respectively, plotting the solution set of (2.7) for various ε . The assertions above are based on these figures, the asymptotic formulas following (3.7), and (H). In Figures 7–11, we have fixed ε and p and plotted the solution sets as functions of Q .

4 Stability and Global Nonexistence When $\varepsilon > 0$

We next consider the following problem equivalent to (P):

$$\begin{cases} u_t = u_{xx} + \varepsilon u u_x + \frac{1}{2} (\delta \|u(\cdot, t)\|^{p-1} + b) u, & 0 < x < 1, \quad 0 < t < T, \\ u(0, t) = u(1, t) = 0, & 0 < t < T, \\ u(x, 0) = u_0(x), & 0 \leq x \leq 1. \end{cases} \quad (\text{P}')$$

For simplicity, we let $D_T = (0, 1) \times (0, T)$ and $D_T \cup \Gamma_T = [0, 1] \times [0, T)$. Our primary interest is in the stability properties of the steady states and in the asymptotic behavior of solutions of (P') for a given initial datum u_0 . To pursue this interest, we first establish a relationship between solutions of (P') and those of (S).

Lemma 4.1. *If u is a bounded monotone (in time) solution of (P'), then u tends to a solution of (S) as $t \rightarrow \infty$.*

Proof. First, we note that such a solution must be global in time, by the continuation statement in Appendix A. Suppose that $\lim_{t \rightarrow \infty} u(x, t) = \varphi(x)$, and let

$$F(x, t) = \int_0^1 G(x, y) u(y, t) dy, \quad (4.1)$$

where

$$G(x, y) = \begin{cases} x(1-y), & 0 \leq x \leq y \leq 1, \\ y(1-x), & 0 \leq y \leq x \leq 1, \end{cases}$$

is the Green's function for $-\frac{d^2}{dy^2}$ with Dirichlet boundary conditions.

Under the assumptions for u , F is bounded in $[0, 1] \times [0, \infty)$ and

$$\begin{aligned}
F_t(x, t) &= \int_0^1 G(x, y)u_t(y, t)dy \\
&= -u(x, t) - \frac{\varepsilon}{2} \int_0^x u^2(y, t)dy + \frac{\varepsilon}{2}x \int_0^1 u^2(y, t)dy \\
&\quad + \frac{1}{2} \left(\delta \|u(\cdot, t)\|^{p-1} + b \right) \int_0^1 G(x, y)u(y, t)dy \quad (4.2) \\
&\rightarrow -\varphi(x) - \frac{\varepsilon}{2} \int_0^x \varphi^2(y)dy + \frac{\varepsilon}{2}x \int_0^1 \varphi^2(y)dy \\
&\quad + \frac{1}{2} \left(\delta \|\varphi\|^{p-1} + b \right) \int_0^1 G(x, y)\varphi(y)dy
\end{aligned}$$

as $t \rightarrow \infty$. This limit has a constant sign that depends on whether $u_t \geq 0$ or $u_t \leq 0$. In actual fact, the limit is zero for $x \in [0, 1]$; otherwise F would not have a finite limit as $t \rightarrow \infty$. Therefore

$$\begin{aligned}
\varphi(x) &= -\frac{\varepsilon}{2} \int_0^x \varphi^2(y)dy + \frac{\varepsilon}{2}x \int_0^1 \varphi^2(y)dy \\
&\quad + \frac{1}{2} \left(\delta \|\varphi\|^{p-1} + b \right) \int_0^1 G(x, y)\varphi(y)dy,
\end{aligned}$$

and hence φ is a solution of (S). ■

By means of this lemma, we can obtain a complete result for stability and instability of stationary solutions of (P') with $\delta = 1$. This time, we treat the solution of (S) as a function depending on the parameter b and denote it by $w(x, b)$.

Theorem 4.2. *Let $w(x, b)$ be a continuously differentiable positive solution of (S) with $\delta = 1$ on some b interval $[\alpha, \beta]$, and let $Z(b)$ be the corresponding solution of (3.7). Then if $Z'(b) > 0$ on $[\alpha, \beta]$, the solutions are stable, whereas they are unstable if $Z'(b) < 0$.*

Proof. For the case $Z'(b) > 0$ we first show that $w(x, b_1) < w(x, b_2)$ on $(0, 1)$ for $\alpha \leq b_1 < b_2 \leq \beta$.

From $Z'(b) > 0$, it follows that $Z(b_1) < Z(b_2)$ if $\alpha \leq b_1 < b_2 \leq \beta$. In view of (3.7), $Z(b) = \varepsilon \|w(\cdot, b)\|_1$ and $v(\varepsilon y, Z/\varepsilon, b) = w^2(x, b)$. Using the

form of $v(\varepsilon y, Z/\varepsilon, b)$ in Lemma 3.3, we see that

$$\begin{aligned}
\varepsilon^2 v\left(\varepsilon y, \frac{Z(b_1)}{\varepsilon}, b_1\right) &= \left(\varepsilon^{-(p-1)} Z_q^{p-1}(Z(b_1)) + b_1\right) Z(b_1) J(\varepsilon y, Z(b_1)) \\
&< \left(\varepsilon^{-(p-1)} Z_q^{p-1}(Z(b_2)) + b_1\right) Z(b_2) J(\varepsilon y, Z(b_2)) \\
&< \left(\varepsilon^{-(p-1)} Z_q^{p-1}(Z(b_2)) + b_2\right) Z(b_2) J(\varepsilon y, Z(b_2)) \\
&= \varepsilon^2 v\left(\varepsilon y, \frac{Z(b_2)}{\varepsilon}, b_2\right),
\end{aligned}$$

since $\frac{\partial v}{\partial Z} > 0$ and $b_1 < b_2$.

Thus, letting $Y(b) = \|w(\cdot, b)\|_1$, we have

$$v(y, Y(b_1), b_1) < v(y, Y(b_2), b_2) \quad \text{for } \alpha \leq b_1 < b_2 \leq \beta.$$

Then following the same reasoning as in Lemma 3.3, we find that

$$w(x, b_1) < w(x, b_2) \quad \text{on } (0, 1) \quad \text{for } \alpha \leq b_1 < b_2 \leq \beta.$$

Let $u(x, t, b_1)$ be a solution of (P') with $u_0(x, b_1) = w(x, b_2)$. Then, on $(0, 1)$, we have

$$\begin{aligned}
&u_0'' + \varepsilon u_0 u_0' + \frac{1}{2} (\|u_0\|^{p-1} + b_1) u_0 \\
&= w_{xx}(x, b_2) + \varepsilon w(x, b_2) w_x(x, b_2) + \frac{1}{2} (\|w(\cdot, b_2)\|^{p-1} + b_1) w(x, b_2) \\
&< w_{xx}(x, b_2) + \varepsilon w(x, b_2) w_x(x, b_2) + \frac{1}{2} (\|w(\cdot, b_2)\|^{p-1} + b_2) w(x, b_2) \\
&= 0.
\end{aligned}$$

Hence, recalling the Corollary in Appendix A, we have $u_t < 0$ in D_T . From the comparison theorem in Appendix A and the monotonicity of u , we also have, on $(0, 1)$

$$w(x, b_1) < u(x, t, b_1) \leq w(x, b_2).$$

By Lemma 4.1, $\varphi(x, b_1) = \lim_{t \rightarrow \infty} u(x, t, b_1)$ exists, and $w(x, b_1) \leq \varphi(x, b_1) \leq w(x, b_2)$. Letting $b_2 \rightarrow b_1^+$ yields $\varphi(x, b_1) \equiv w(x, b_1)$, which shows that $w(x, b_1)$ is stable from above. We can also prove similarly that $w(x, b_1)$ is stable from below.

If $Z'(b) < 0$, then $Z(b_2) < Z(b_1)$, and consequently, $\|w(\cdot, b_2)\|_1 < \|w(\cdot, b_1)\|_1$ for $\alpha \leq b_1 < b_2 \leq \beta$. Thus there is a subinterval $[x_0, x_1]$ contained in $[0, 1]$ such that $w(x, b_2) < w(x, b_1)$ on $[x_0, x_1]$. Let $u(x, t, b_2)$ be a solution of (P') with $u_0(x, b_2) = w(x, b_1)$. Then, on $(0, 1)$, we find that

$$\begin{aligned} & u_0'' + \varepsilon u_0 u_0' + \frac{1}{2} (\|u_0\|^{p-1} + b_2) u_0 \\ &= w_{xx}(x, b_1) + \varepsilon w(x, b_1) w_x(x, b_1) + \frac{1}{2} (\|w(\cdot, b_1)\|^{p-1} + b_2) w(x, b_1) \\ &> w_{xx}(x, b_1) + \varepsilon w(x, b_1) w_x(x, b_1) + \frac{1}{2} (\|w(\cdot, b_1)\|^{p-1} + b_1) w(x, b_1) \\ &= 0. \end{aligned}$$

Therefore, $u_t > 0$ in D_T . Hence, $u(x, t, b_2)$ is increasing in t , and it follows that $w(x, b_2)$ is unstable from above. Similarly, $w(x, b_2)$ is unstable from below when $b_1 > b_2$.

Using this theorem combined with the characterization of the stationary solutions in Section 3, we obtain the following stability and instability results for the case $\delta = 1$.

(C-1) For $1 < p < 2$, if $b(p, q, \varepsilon) < b < 2\pi^2$, there are two branches of solutions of (S) — one stable, the other unstable; if $b > 2\pi^2$, the unique solution is stable.

(C-2) For $p = 2$:

- (a) If $\varepsilon < \frac{1}{4}\varepsilon_\infty$ and $b < 2\pi^2$, $w(x, b)$ is unstable.
- (b) If $\varepsilon = \frac{1}{4}\varepsilon_\infty$ and $b(\varepsilon_\infty) < b < 2\pi^2$, $w(x, b)$ is unstable.
- (c) If $\varepsilon > \frac{1}{4}\varepsilon_\infty$, and
 - (i) if $b(\varepsilon_\infty) < b < 2\pi^2$, there are two branches — one stable, the other unstable; and
 - (ii) if $b > 2\pi^2$, $w(x, b)$ is stable.

(C-3) For $p > 2$:

- (a) If $\varepsilon \leq \varepsilon_1$ and $b < 2\pi^2$, $w(x, b)$ is unstable.
- (b) If $\varepsilon > \varepsilon_1$ and
 - (i) if $b < 2\pi^2$, $w(x, b)$ is unstable; and
 - (ii) if $2\pi^2 < b < b(p, q, \varepsilon)$, there are two branches — one stable, the other unstable.

In particular, we see from Figures 4–6 that, as ε increases, the portion of the bifurcation curve along which $Z'(b) > 0$ becomes more pronounced. This illustrates the stabilizing effect of the convection term in the dynamical equation.

From Figures 7–11, we see that, for fixed ε , the choice $q = 1$ leads to the “most stability” while the choice $q = +\infty$ leads to the “least stability” in the sense of the preceding paragraph. Increasing q has the effect of decreasing the set $\{b \mid Z'(b) > 0\}$.

The case $\delta = -1$, because of the lack of a comparison principle, is not so amenable to analysis. However, bearing in mind numerical evidence (see [14]), we may conjecture that: the positive stationary solution branch (where it exists) is stable.

Next we discuss the asymptotic stability of the trivial stationary solution of (P'). Henry [7] used a linearization method based on semigroup theory to analyze the asymptotic stability. His principle has wide application, but it does not readily extend to the current nonlocal problem. Therefore, we adopt another approach.

Let $F_2(t) = \int_0^1 u^2 dx$ to find

$$\begin{aligned} F_2'(t) &= 2 \int_0^1 uu_{xx} dx + 2\varepsilon \int_0^1 uu_x dx + bF_2 + \delta \|u\|^{p-1} F_2 \\ &= -2 \int_0^1 u_x^2 dx + (b + \delta \|u\|^{p-1}) F_2 \\ &\leq (-2\pi^2 + b + \delta \|u\|^{p-1}) F_2, \end{aligned}$$

since $\pi^2 \int_0^1 u^2 dx \leq \int_0^1 u_x^2 dx$.

For $\delta = 1$, for any $b < 2\pi^2$ and sufficiently small initial value u_0 , we have $F_2(t) \leq F_2(0)e^{-kt}$, which means that $v \equiv 0$ is an asymptotically stable solution of (S) in $L_2(0, 1)$. Similarly, $v \equiv 0$ is a globally asymptotically stable solution of (S) in $L_2(0, 1)$ when $b \leq 2\pi^2$ for $\delta = -1$. ■

Finally, we investigate the global existence and nonexistence of solutions of (P'). We first give the following theorem for the case $\delta = 1$.

Theorem 4.3. For $1 < p < 2$ and any $\varepsilon > 0$, or $p = 2$ and $\varepsilon > \frac{1}{2}(1+q)^{-1/q}$, the solutions of (P') are uniformly bounded on $[0, 1] \times [0, \infty)$.

Proof. We look for a supersolution v in the form $v(x) = M(1 - \sigma x)$ with $M > 0$ and $0 < \sigma < 1$ to be chosen. We shall succeed if M and σ satisfy

$$-\varepsilon \sigma M^2(1 - \sigma x) + \frac{1}{2} \left[b + M^{p-1} (\alpha(1+q))^{-\frac{p-1}{q}} \right] M(1 - \sigma x) \leq 0 \quad (4.3)$$

and

$$M(1 - \sigma) \geq \max_{0 \leq x \leq 1} u_0(x). \quad (4.4)$$

The inequality (4.3) is equivalent to

$$M^{p-1} \left[\varepsilon \sigma M^{2-p} - \frac{1}{2} (\alpha(1+q))^{-\frac{p-1}{q}} \right] \geq \frac{1}{2} b. \quad (4.5)$$

For $1 < p < 2$, (4.4) and (4.5) will hold if M is sufficiently large; while if $p = 2$, letting $\varepsilon = \frac{1}{2}(1 + \gamma)(1 + q)^{-1/q}$ ($\gamma > 0$) \sum_f , we have that $\sigma^{\frac{1+q}{q}} \leq (1 + \gamma)^{-1}$, i.e. $\sigma \leq (1 + \gamma)^{-\frac{q}{1+q}} < 1$. For such σ , we can choose M so large that (4.4) holds.

For the case $\delta = -1$, since the solution of (P') is a subsolution of (P') with $\delta = 1$ and $1 < p < 2$, all solutions are bounded on $[0, 1] \times [0, \infty)$. ■

Next for $p > 2$ or $p = 2$ with small ε , we prove that with sufficiently large initial data, problem (P') with $\delta = 1$ does not have global solutions. To show this, we employ two different arguments.

Theorem 4.4A. Let $p > 2$. Then there exists $c_0 = c_0(\varepsilon, p, b) < \infty$ such that if $u_0 \geq c_0 \sin \pi x$, the solution of (P') blows up in finite time. The same is true for $p = 2$ if ε is small enough.

Proof. We seek a subsolution $w(x, t)$ in the form $w(x, t) = h(t) \sin \pi x$ with $h(t)$ becoming unbounded in finite time. To obtain this, we need

$$h'(t) \leq -\pi^2 h(t) + \pi \varepsilon h^2(t) \cos \pi x + \frac{1}{2} (A_0 h^p(t) + b h(t)), \quad (4.6)$$

where $A_0 = \left[\int_0^1 (\sin \pi x)^q dx \right]^{\frac{p-1}{q}}$.

Set

$$Q(s) = \frac{A_0}{2} s^{p-2} - \pi \varepsilon \quad (4.7a)$$

for $b > 2\pi^2$, or set

$$Q(s) = \frac{A_0}{2} s^{p-2} - \pi \varepsilon - \left(\pi^2 - \frac{b}{2} \right) s^{-1} \quad (4.7b)$$

for $b \geq 2\pi^2$.

For $p > 2$, if s_0 is the largest positive root of Q , then $Q(s), Q'(s) > 0$ for $s > s_0$.

Choosing $c_0 = s_0$ and letting $h(t)$ be the solution of the following problem

$$\begin{aligned} h'(t) &= Q(h)h^2(t), \quad t > 0 \\ h(0) &= c_0, \end{aligned} \quad (4.8)$$

one easily sees that (3.6) is satisfied and $h(t)$ blows up in finite time.

For $p = 2$ and sufficiently small $\varepsilon > 0$, the above discussion also holds.

■

Remark. The result for $p = 2$ is in contrast to that in [4], where it is shown that for any $\varepsilon > 0$, the solution of a local problem with our nonlocal term replaced by $|u|^{p-1}u$ remains bounded on $[0, 1] \times [0, \infty)$ when $p = 2$.

For any large initial value, with more restriction on p and q , we also have the following theorem.

Theorem 4.4B. *Let $p \geq 3$ and $q \geq 2$. Then there exists $c_1 = c_1(\varepsilon, p, b)$ such that if $\int_0^1 u_0(x) dx > c_1$, the solution of (P') blows up in finite time.*

Proof. Now we use a variation of the eigenfunction method [9].

Set

$$J(t) = \int_0^1 u(x, t) \psi(x) dx, \quad (4.9)$$

where $\psi(x) = \frac{\pi}{2} \sin \pi x$.

A routine calculation shows that

$$\begin{aligned}
J'(t) &\geq -\pi^2 \int_0^1 u(x,t)\psi(x)dx - \varepsilon \int_0^1 u^2(x,t)\psi'(x)dx \\
&\quad + \frac{1}{2} \left[\int_0^1 u^q(x,t)dx \right]^{\frac{p-1}{q}} \int_0^1 u(x,t)\psi(x)dx \\
&\quad + \frac{b}{2} \int_0^1 u(x,t)\psi(x)dx.
\end{aligned} \tag{4.10}$$

Using Hölder's inequality, we see that

$$\varepsilon \int_0^1 u^2(x,t)\psi'(x)dx \leq A_1 \left[\int_0^1 u^q(x,t)dx \right]^{2/q} \tag{4.11}$$

$$\int_0^1 u(x,t)\psi(x)dx \leq A_2 \left[\int_0^1 u^q(x,t)dx \right]^{1/q} \tag{4.12}$$

with $A_1 = \varepsilon \|\psi'\|_{L^\infty(0,1)}$ and $A_2 = \pi/2$.

Let

$$R(s) = A_3 s^{(p-1)/q} - A_1 s^{2/q} \tag{4.13a}$$

for $b > 2\pi^2$, or let

$$R(s) = A_3 s^{(p-1)/q} - A_1 s^{2/q} - \left(\pi^2 - \frac{b}{2} \right) s^{1/q}, \tag{4.13b}$$

for $b \leq 2\pi^2$, where $A_3 = \frac{1}{2} \int_0^1 u_0(x)\psi(x)dx$.

If s_1 is the largest positive root of R , we find that $R(s), R'(s) > 0$ for $s > s_1$. Letting $c_1 = s_1$ and $\int_0^1 u_0(x)dx > c_1$ implies that $\int_0^1 u^q(x,t)dx > s_1$ for all t . Hence,

$$J'(t) \geq R \left(\int_0^1 u^q(x,t)dx \right) \geq R \left(\frac{2}{\pi} J^q(t) \right). \tag{4.14}$$

Since $q > 1$, $J(t)$ must blow up in finite time. ■

Appendix A

Here we establish the comparison principle and local existence of solutions for the following general problem:

$$\begin{cases} u_t = u_{xx} + (f(u))_x + g(u, \|u\|), & 0 < x < 1, \quad 0 < t < T, \\ u(0, t) = u(1, t) = 0, & & 0 < t < T, \\ u(x, 0) = u_0(x), & & 0 \leq x \leq 1, \end{cases} \quad (\text{G})$$

where $\|u\| = \left[\int_0^1 |u|^q dx \right]^{1/q}$.

First, we define the subsolution and supersolution of (G). As in Section 4, we let $D_T = (0, 1) \times (0, T)$ and $D_T \cup \Gamma_T = [0, 1] \times [0, T]$.

Definition. A function $u(x, t)$ is called a *subsolution* of (G) on D_T if $u \in C^{2,1}(D_T) \cap C(D_T \cup \Gamma_T)$, satisfying

$$\begin{aligned} u_t &\leq u_{xx} + (f(u))_x + g(u, \|u\|) & 0 < x < 1, \quad 0 < t < T, \\ u(0, t) &\leq 0, \quad u(1, t) \leq 0 & & 0 < t < T, \\ u(x, 0) &\leq u_0(x) & & 0 \leq x \leq 1. \end{aligned} \quad (\text{G}')$$

A *supersolution* is defined by (G') with each “ \leq ” replaced by “ \geq ”.

Comparison Theorem. *Suppose that f and g are continuously differentiable and that $g_2 \geq 0$. Let u and v be a nonnegative supersolution and a nonnegative subsolution, respectively, of (G), with $u(x, 0) \geq v(x, 0)$ for $x \in (0, 1)$. Then $u \geq v$ in $D_T \cup \Gamma_T$.*

Proof. For every $t \in (0, T)$ and every nonnegative $\varphi(x, t) \in C^{2,1}(\overline{D}_T)$ with $\varphi(0, t) = \varphi(1, t) = 0$, the subsolution v satisfies the following integral inequality:

$$\begin{aligned} \int_0^1 v(x, t) \varphi(x, t) dx &\leq \int_0^1 v_0(x) \varphi(x, 0) dx \\ &\quad + \int_0^t \int_0^1 [v \varphi_\tau - (v_x + f(u)) \varphi_x + g(v, \|v\|) \varphi] dx d\tau. \end{aligned}$$

The supersolution u satisfies the above with reversed inequality.

We integrate by parts in both the above inequality and that satisfied by u and subtract the two resultant expressions. Then we have

$$\begin{aligned}
& \int_0^1 (v(x, t) - u(x, t)) \varphi(x, t) dx \\
& \leq \int_0^1 (v(x, 0) - u(x, 0)) \varphi(x, 0) dx \\
& \quad + \int_0^t \int_0^1 (v - u) (\varphi_\tau + \varphi_{xx} - A(x, \tau) \varphi_x + B(x, \tau) \varphi) dx d\tau \\
& \quad + \int_0^t \int_0^1 \varphi(x, \tau) \int_0^1 C(s, \tau) (v - u) ds dx d\tau,
\end{aligned} \tag{A1}$$

where

$$A(x, t) = f'(\theta_1(x, t)),$$

$$B(x, t) = g_{,1}(\theta_2(x, t), \|v\|),$$

$$C(x, t) = g_{,2}(u, \theta_3(t)) \theta_4^{\frac{1}{q}-1}(t) \theta_5^{q-1}(x, t),$$

with $\theta_1, \theta_2, \theta_5$ between u and v , and θ_3, θ_4 between $\|u\|$ and $\|v\|$. ■

Note that by the hypotheses for f and g , A , B , and C are bounded on $\overline{D_T}$ in the uniform norm. We denote the bound by M_0 .

Now we define two sequence $\{A_n\}$ and $\{B_n\}$ in such a way that

- (i) $A_n, B_n \in C^\infty(\overline{D_T})$,
- (ii) $|A_n| \leq M_0, |B_n| \leq M_0$,
- (iii) $A_n \rightarrow A, B_n \rightarrow B$ in (D_T) as $n \rightarrow \infty$,

and we set up a backward problem on D_t :

$$\begin{cases} \varphi_{n\tau} + \varphi_{nxx} - A_n \varphi_{nx} + B_n \varphi = 0, & 0 < x < 1, \quad 0 < \tau < t, \\ \varphi_n(0, \tau) = \varphi_n(1, \tau) = 0, & & 0 < \tau < t, \\ \varphi_n(x, t) = \chi(x), & 0 \leq x \leq 1. \end{cases} \tag{G*}$$

Here, $\chi(x) \in C_0^\infty(0, 1)$, $0 \leq \chi \leq 1$.

Recalling standard theory (in [10] for example), we find that $\varphi = \lim_{n \rightarrow \infty} \varphi_n$ is a solution of (G*) with A_n, B_n replaced by A, B , and $\varphi \in C^{2,1}(\overline{D_T})$. The initial and boundary values for φ_n imply that $\varphi \geq 0$ in D_T .

Substituting φ in (A1) yields

$$\begin{aligned} \int_0^1 (v(x, t) - u(x, t)) \chi(x) dx &\leq M_1 \int_0^1 (v(x, 0) - u(x, 0))^+ dx \\ &\quad + \int_0^t \int_0^1 \varphi(x, \tau) \int_0^1 C(s, \tau)(v - u) ds dx d\tau, \end{aligned}$$

where $M_1 = \sup_{\overline{D_T}} |\varphi|$.

Since this inequality holds for every χ , we can choose a sequence $\{\chi_n\}$ on $(0, 1)$ converging to

$$\chi = \begin{cases} 1, & \text{if } v(x, t) - u(x, t) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Noting that $C \geq 0$, $v - u = (v - u)^+ - (v - u)^-$, we find that

$$\begin{aligned} \int_0^1 (v(x, t) - u(x, t))^+ dx &\leq M_1 \int_0^1 (v(x, 0) - u(x, 0))^+ dx \\ &\quad + M_0 M_1 \int_0^t \int_0^1 (v(x, \tau) - u(x, \tau))^+ dx d\tau, \end{aligned}$$

which leads, by Gronwall's inequality, to

$$\int_0^1 (v(x, t) - u(x, t))^+ dx \leq M_1 \left(1 + e^{Mt}\right) \int_0^1 (v(x, 0) - u(x, 0))^+ dx.$$

Thus, the conclusion follows from the condition on initial data.

Corollary. *If $u_0'' + f'(u_0)u_0' + g(u_0, \|u_0\|) \geq 0$ (≤ 0) on $(0, 1)$, then $u_t(x, t) \geq 0$ (≤ 0) in D_T .*

Proof. The condition on u_0 implies that u_0 is a subsolution (supersolution) of (G). Thus $u(x, t) \geq u_0(x)$ ($\leq u_0(x)$) in $D_T \cup \Gamma_T$. Let $v(x, t) = u(x, t + h)$ ($h > 0$). Then v is a supersolution (subsolution) of (G), and

therefore $u(x, t + h) \geq u(x, t)$ ($\leq u(x, t)$). Since h is arbitrary, u is increasing (decreasing) in t for fixed x , and hence $u_t \geq 0$ (≤ 0).

Next we establish the existence of solutions of (G) on $D_T \cup \Gamma_T$ for sufficiently small T and certain initial values. This time we assume only that f and g are continuously differentiable. We shall also define $f_M \equiv \sup_{|u| \leq M} |f(u)|$ and $g_M \equiv \sup_{|u| \leq M} |g(u, \|u\|)|$.

Let $G(x, y; t)$ denote the Green's function for

$$Lu = u_t - u_{xx}, \quad 0 < x < 1, \quad t > 0,$$

with boundary conditions

$$u(0, t) = u(1, t) = 0, \quad t > 0,$$

that is,

$$G(x, y; t) = 2 \sum_{n=1}^{\infty} \sin(n\pi x) \sin(n\pi y) e^{-n^2 \pi^2 t}.$$

Then

$$G(x, 0; t) = G(x, 1; t) = 0.$$

Also, u is a solution of (G) on $D_T \cup \Gamma_T$ if and only if for $(x, t) \in D_T \cup \Gamma_T$

$$\begin{aligned} u(x, t) &= \int_0^1 G(x, y; t) u_0(y) dy \\ &\quad + \int_0^t \int_0^1 \left[G(x, y; t - \eta) g(u(y, \eta), \|u(\cdot, \eta)\|) \right. \\ &\quad \left. - G_y(x, y; t - \eta) f(u(y, \eta)) \right] dy d\eta \\ &\equiv u(x, t). \end{aligned} \tag{A2}$$

To show that (A2) is solvable for sufficiently small T , we use a contraction mapping argument. To this end, we define

$$u_1(x, t) = 0 \tag{A3a}$$

and

$$u_{n+1}(x, t) = u_n(x, t). \tag{A3b}$$

■

Local Existence Theorem. *Let the initial datum for problem (G) be continuous on $[0, 1]$ and satisfy*

$$0 < m < \int_0^1 \int_0^1 G(x, y; t) u_0(y) dy dx \quad (\text{A4})$$

for $0 \leq t \leq 1$, say. Then for sufficiently small T , (G) has a unique solution that satisfies

$$\|u\| \geq \frac{m}{2} \quad (\text{A5})$$

on $[0, T]$. The solution is C^1 in t and C^2 in x on D_T and continuous on \overline{D}_T .

Proof. First, we define

$$\begin{aligned} M_0 &= \sup_{0 \leq x \leq 1} |u_0(x)|, \\ \mu(t) &= \sup_{\substack{0 \leq x \leq 1 \\ 0 \leq \tau \leq t}} \int_0^\tau \int_0^1 G(x, y; \tau - \eta) d\eta, \\ \nu(t) &= \sup_{\substack{0 \leq x \leq 1 \\ 0 \leq \tau \leq t}} \int_0^\tau \int_0^1 G_y(x, y; \tau - \eta) d\eta, \end{aligned}$$

Clearly, $\mu(t)$ and $\nu(t)$ tend to zero as $t \rightarrow 0^+$. For fixed $M > M_0$, choose T so small that $T \leq 1$ and

$$\nu(T) f_M + \mu(T) g_M < \max \left(M - M_0, \frac{1}{2} m \right), \quad (\text{A6})$$

$$\begin{aligned} \alpha &\equiv \nu(T) \sup_{|\xi| \leq M} |f'(\xi)| \\ &+ \mu(T) \left(\sup_{\substack{|\xi| \leq M \\ |\eta| \leq M}} |g_\xi(\xi, \eta)| + \sup_{\substack{|\xi| \leq M \\ |\eta| \leq M}} |g_\eta(\xi, \eta)| \left(\frac{m}{2} \right)^{\frac{1}{q}-1} M^{q-1} \right) \quad (\text{A7}) \\ &< 1. \end{aligned}$$

It then follows from (A3a), (A3b), and (A6) and induction that on \overline{D}_T

$$\|u_n\|_{L^\infty(\overline{D}_T)} \leq M$$

for all $n = 1, 2, \dots$. Moreover, we have from (A3b) and (A6) that

$$\|u_n\| \geq \|u_n\|_{L^1(\overline{D}_T)} > \frac{m}{2} \quad (\text{A8})$$

for all $n = 2, 3, \dots$. Using (A7) and letting $\beta_n \equiv \|u_{n+1} - u_n\|_{L^\infty(\overline{D}_T)}$, we can see that

$$\beta_{n+1} \leq \alpha \beta_n \leq \dots \leq \alpha^n \beta_1.$$

Therefore, $\{u_n\}$ is uniformly convergent on \overline{D}_T , and

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t) \quad (\text{A9})$$

solves (A2) with $\|u\| \geq m/2$.

The asserted interior regularity follows from the properties of G and the continuity of u in \overline{D}_T . We omit the standard arguments. ■

The above result also allows us to make the following continuation statement: If u is a classical solution on D_T , bounded on \overline{D}_T , then u may be extended to $\overline{D_{T+\delta}}$ for some $\delta > 0$.

Appendix B

Here we establish Lemma 3.2.

Lemma 3.2. *The integral*

$$\int_0^1 \frac{z^{1/2}}{\left(\frac{1-e^{-\sigma z}}{1-e^{-z}} - \sigma\right)^{1/2}} d\sigma$$

is increasing in z for $z \geq 0$.

We see that the statement is equivalent to showing that

$$\begin{aligned} & \int_0^{1/2} \left[\frac{z^{1/2}}{\left(\frac{1-e^{-\sigma z}}{1-e^{-z}} - \sigma\right)^{1/2}} + \frac{z^{1/2}}{\left(\frac{1-e^{-(1-\sigma)z}}{1-e^{-z}} - (1-\sigma)\right)^{1/2}} \right] d\sigma \\ &= \int_0^{1/2} \left[\left(\frac{z}{f(z, \sigma)}\right)^{1/2} + \left(\frac{z}{g(z, \sigma)}\right)^{1/2} \right] d\sigma \end{aligned}$$

is increasing in z for $z \geq 0$.

To establish this assertion, we shall show that

$$\left(\frac{z}{f}\right)^{1/2} + \left(\frac{z}{g}\right)^{1/2}$$

is increasing in z , for $\sigma \in [0, 1/2]$.

Reduction 1. The presence of square roots in the integrand causes difficulties. By taking the square, we see that the lemma is a consequence of showing that

$$\frac{z}{f} + \frac{z}{g} \quad \text{and} \quad \frac{z^2}{fg}$$

are increasing.

Reduction 2. Let us assume that we already know that a^2/fg is increasing. We have

$$\frac{z}{f} + \frac{z}{g} = \frac{z}{(fg)^{1/2}} \cdot \left(\left(\frac{f}{g}\right)^{1/2} + \left(\frac{g}{f}\right)^{1/2} \right). \quad (\text{B1})$$

The first factor is increasing by assumption. We therefore have (B1) once we can show that the second factor is also increasing. We claim that this is equivalent to showing that f/g is increasing in z , for $\sigma \in [0, 1/2]$. Let

$$F^2(z) = \frac{f}{g}. \quad (\text{B2})$$

Using L'Hopital's rule, we obtain

$$F(0) = \lim_{z \rightarrow 0} \frac{f}{g} = 1. \quad (\text{B3})$$

The derivative of the second factor in (B1) is

$$\frac{F'(z) [F^2(z) - 1]}{F^2(z)}. \quad (\text{B4})$$

If we can show that $F(z)$ is increasing in z , then, in view of (B3), both factors in the numerator are nonnegative (and hence the second factor in (B1) is increasing).

We have thus reduced the problem to proving that f/g is increasing and z^2/fg is increasing in z for all $\sigma \in [0, 1/2]$.

Proof that f/g is increasing

After simplification and the substitution $y = e^{-z}$, we have

$$\frac{f}{g} = \frac{1 - y^\sigma - \sigma(1 - y)}{1 - y^{1-\sigma} - (1 - \sigma)(1 - y)},$$

$y \in [0, 1], \sigma \in [0, 1/2]$. Note that y decreases as z increases. Hence, we have to show that f/g is decreasing in y . The numerator of its derivative, namely, $f'g - fg'$, is

$$X(y) = (4\sigma - 2) + (\sigma^2 - 2\sigma + 1)y^\sigma - \sigma^2 y^{\sigma-1} + (\sigma^2 - 2\sigma + 1)y^{-\sigma} - \sigma^2 y^{1-\sigma}.$$

We need to show that it is nonpositive. Note that $X(1) = 0$; hence, it suffices to show that $X'(y) \geq 0$ in $[0, 1]$, or equivalently that

$$Y(y) = \frac{yX'}{\sigma(1 - \sigma)} = -\sigma y^{1-\sigma} - (1 - \sigma)y^{-\sigma} + (1 - \sigma)y^\sigma + \sigma y^{\sigma-1} \geq 0.$$

Again we see that $Y(1) = 0$; hence, it remains to show that $Y'(y) \leq 0$ in $[0,1]$. But

$$Y'(y) = -\frac{\sigma(1-\sigma)(1-y)(1-y^{1-2\sigma})}{y^{2-\sigma}} \leq 0,$$

as desired (recall that $0 < \sigma < 1/2$). ■

Now it remains to prove that z^2/fg is increasing, or rather its reciprocal

$$K(z, \sigma) = \frac{(1 - e^{-\sigma z} - \sigma(1 - e^{-z})) (1 - e^{-(1-\sigma)z} - (1-\sigma)(1 - e^{-z}))}{z^2(1 - e^{-z})^2} \quad (\text{B5})$$

is decreasing in $z \geq 0$ for $\sigma \in [0, 1/2]$.

Reduction 3. Note that

$$K(z, 0) \equiv 0.$$

Hence, trivially,

$$K_z(z, 0) = 0.$$

If we can show that

$$K_{z\sigma}(z, \sigma) \leq 0, \quad (\text{B6})$$

for $z \geq 0$, $\sigma \in [0, 1/2]$, then we know that for a fixed z , K_z is a decreasing function in σ . Since K_z starts out at 0 when $\sigma = 0$, K_z must be nonpositive for all $\sigma > 0$ up to $\sigma = 1/2$, and (B5) is then proved. The inequality (B6) is equivalent to showing that K_σ is decreasing in $z \geq 0$ for $\sigma \in [0, 1/2]$. After differentiating K with respect to σ , we can separate out those terms in the numerator that have the extra factor z :

$$K_\sigma = H + L,$$

where

$$H = \frac{(1 - 2\sigma)(1 - e^{-z})^2 + e^{-(1-\sigma)z} + e^{-(1+\sigma)z} - e^{-\sigma z} - e^{-(2-\sigma)z}}{z^2(1 - e^{-z})^2}, \quad (\text{B7})$$

and

$$L = \frac{\sigma e^{\sigma z} - \sigma e^{-(2-\sigma)z} + (1-\sigma)e^{-(1+\sigma)z} - (1-\sigma)e^{-(1-\sigma)z}}{z(1 - e^{-z})^2}. \quad (\text{B8})$$

We have thus made the final reduction to proving that H and L are decreasing in z for $z \geq 0$ and all $\sigma \in [0, 1/2]$.

Proof that H is decreasing. The crucial observation is that the numerator of H has the factor $(1 - e^{-z})$. After cancellation, we have

$$\begin{aligned} H &= \frac{(1 - 2\sigma)(1 - e^{-z}) + e^{-(1-\sigma)z} - e^{-\sigma z}}{z^2(1 - e^{-z})} \\ &= \frac{(1 - 2\sigma)(e^{z/2} - e^{-z/2}) + e^{-(1-\sigma)z/2} - e^{(1-\sigma)z/2}}{z^2(e^{z/2} - e^{-z/2})}. \end{aligned} \quad (\text{B9})$$

The next step involves an obvious change of variables

$$t = 1 - \sigma, \quad y = z/2,$$

and the use of hyperbolic functions. We have

$$H = \frac{t \sinh(y) - \sinh(ty)}{4y^2 \sinh^2(y)}, \quad (\text{B10})$$

$y \geq 0$, $t \in [0, 1]$. Note that

$$H(y, 0) = H(y, 1) = 0.$$

Hence,

$$H_y(y, 0) = H_y(y, 1) = 0. \quad (\text{B11})$$

Let us fix a y and see how $R(t) = H_y(\cdot, t)$ changes with t . We need to show that $R(t) \leq 0$ in order to establish that H is decreasing. By (B11),

$$R(0) = R(1) = 0. \quad (\text{B12})$$

By differentiating (B10) twice with respect to t and once with respect to y , we see that

$$R''(t) = \left(-\frac{\sinh(ty)}{4 \sinh(y)} \right)_y = \frac{\cosh(y) \sinh(ty) - t \sinh(y) \cosh(ty)}{4 \sinh^2(y)}, \quad (\text{B13})$$

where the subscript y outside the parentheses denotes partial differentiation in y . If we can show that $R''(t) \geq 0$, then $R(t)$ is a convex function of t and

then by virtue of (B12), $R(t) \leq 0$ as required. The nonnegativity of R'' is implied by

$$\cosh(y) \sinh(ty) \geq t \sinh(y) \cosh(ty),$$

or equivalently

$$\frac{y \cosh(y)}{\sinh(y)} \geq \frac{(ty) \cosh(ty)}{\sinh(ty)}, \quad t \in [0, 1]. \quad (\text{B14})$$

This is a consequence of the elementary fact that $y \cosh(y)/\sinh(y)$ is an increasing function in y . ■

Proof that L is decreasing in z . We follow the same line of attack. Denote $S(\sigma) = L_z(z, \sigma)$ for a fixed z . Note that

$$L(z, 0) = L(z, 1/2) \equiv 0, \quad (\text{B15})$$

implies that

$$S(0) = S(1/2) = 0. \quad (\text{B16})$$

Unlike the function $R(t)$, the second derivative of $S(\sigma)$ does not have a simple form from which we can deduce $S''(\sigma) \geq 0$. We first rewrite L in the following form:

$$L(z, \sigma) = \frac{\sigma \sinh(1 - \sigma)z - (1 - \sigma) \sinh(\sigma z)}{z \sinh^2(z/2)}. \quad (\text{B17})$$

Taking the derivative twice with respect to σ , we have

$$L_{\sigma\sigma} = -2 \frac{\cosh(1 - \sigma)z - \cosh \sigma z}{\sinh^2\left(\frac{z}{2}\right)} + z^2 L. \quad (\text{B18})$$

Using the sum and product formula on the first term, we obtain the simplified equation:

$$L_{\sigma\sigma} = -4 \frac{\sinh\left(\left(\frac{1}{2} - \sigma\right)z\right)}{\sinh\left(\frac{z}{2}\right)} + z^2 L. \quad (\text{B19})$$

The first term on the righthand side is an increasing function of z , since it is really the same function occurring in R'' in (B13) (take $y = z/2$, and $t = (1 - 2\sigma) < 1$). Now take the z derivative of (B19) to obtain

$$S''(\sigma) = \left[\left(-\frac{\sinh\left(\left(\frac{1}{2} - \sigma\right)z\right)}{\sinh\left(\frac{z}{2}\right)} \right)_z + 2zL \right] + z^2 S(\sigma). \quad (\text{B20})$$

The terms inside the square brackets are positive. Now suppose that L is not decreasing. Then for some z , $S(\sigma)$ must have a strictly positive maximum in $[0, 1/2]$. At this maximum, the lefthand side of (B20) is nonpositive while the righthand side is strictly positive, giving a contradiction. ■

References

- [1] N. W. Bazley and J. Weyer, *Explicitly resolvable equations with functional non-linearities*, Math. Meth. Appl. Sci. **10** (1988), 477–485.
- [2] J. M. Burgers, *Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion*, Trans. Roy. Neth. Acad. Sci. Amsterdam **17** (1939), 1–53.
- [3] _____, *A mathematical model illustrating the theory of turbulence*, in Advances in Applied Mechanics (R. von Mises and T. von Kármán, editors) **1**, Academic Press, New York, 1948, pp. 171–199.
- [4] T. F. Chen, H. A. Levine, and P. E. Sacks, *Analysis of a convective reaction diffusion equation (I)*, Nonlinear Anal., TMA **12** (1988), 1349–1370.
- [5] P. G. Drazin and W. H. Reid, **Hydrodynamic Stability**, Cambridge University Press, Cambridge, 1982.
- [6] B. Fiedler and P. Poláčik, *Complicated dynamics of scalar reaction diffusion equations with a nonlocal term*, preprint Nr. 511, 1990.
- [7] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, **840**, Springer-Verlag, Berlin, 1981.
- [8] C. O. Horgan and W. E. Olmstead, *Stability and uniqueness for a turbulence model of Burgers*, Quart. Appl. Math. **36** (1978), 121–127.
- [9] S. Kaplan, *On the growth of solutions of quasilinear parabolic equations*, Comm. Pure Appl. Math. **16** (1963), 305–330.
- [10] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Ural’ceva, *Linear and Quasi-linear Equations of Parabolic Type*, Translations of Mathematical Monographs, **23**, 1968.

- [11] H. A. Levine, L. E. Payne, P. E. Sacks, and B. Straughan, *Analysis of a convective reaction-diffusion equation (II)*, SIAM J. Math. Anal. **20** (1989), 133–147.
- [12] P. D. Miletta, *An evolution equation with non-local non-linearities: Existence, uniqueness and asymptotic behavior*, Math. Meth. Appl. Sci. **10** (1988), 407–425.
- [13] L. E. Payne, **Improperly Posed Problems in Partial Differential Equations**, SIAM, Philadelphia, 1975.
- [14] B. Straughan, R. E. Ewing, P. G. Jacobs, and M. J. Djomehri, *Non-linear instability for a modified form of Burgers' equation*, Num. Meth. for PDE **3** (1987), 51–64.

HOWARD A. LEVINE
 Department of Mathematics
 Iowa State University
 Ames, IA 50011

KENG DENG
 Department of Mathematics
 University of Southwestern Louisiana
 Lafayette, LA 70504

MAN K. KWONG
 Mathematics and Computer Science
 Argonne National Laboratory
 Argonne, IL 60439-4801