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Time-Optimal Path Planning for Automated Grain Carts

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ABSTRACT

In this paper, we address a motion planning problem for an autonomous agricultural vehicle modeled as tractor-trailer system. We first present a numerical approach and a primitive-based approach for computing the time-optimal path based on given static initial and goal configurations. In the former approach, we define a value function for the entire state space. The value function is consistent with the time to reach goal configuration, and it is finally used to compute the optimal trajectory. In the latter approach, based on the regular and singular primitives, we present an algorithm to construct such primitives and derive the final path. Subsequently, we extend the results and present a dynamic motion planning strategy to accommodate the case of mobile target configuration. Finally, simulation results are provided to validate the feasibility and effectiveness of these techniques.

Key words: motion planning—autonomous—agricultural vehicle

INTRODUCTION

Logistics problem can be described as the management of resources in order to meet specific requirements, or customers. The resources to be dealt with include physical items such as materials and tools, as well as abstract items such as information and energy. Logistics problem can be embodied in many aspects, such as business, economics and agriculture. In this paper, we address the problem of logistics in path planning for agricultural vehicles.

Currently, crop-harvesting is usually performed by agricultural machines called combines (combine harvesters). Due to the limited capacities of the combines, a grain cart is involved for transporting the grains from combines to the depot. If there are sufficient number of grain carts, each combine can go alongside with a grain cart for unloading. However, if the number of combines exceeds the number of grain carts, a single grain cart has to serve multiple combines. Based on the fact that the grain cart is a tractor-trailer system, we explore the problem of time-optimal path planning for an autonomous grain cart to move from one combine to the next combine.

There have been some efforts in the past to address the problem of harvesting in large-scale farming scenarios. In (Fokkens and Puylaert 1981), a linear programming model for harvesting operations is presented. The model gives the management results of harvesting operations at the large scale grain farm. In (Foulds and Wilson 2005) and (Basnet and Foulds 2006), researchers analyze the scheduling problem of farm-to-farm harvesting operations for hay and rape seed. These works mainly focus on scheduling harvesting operations from farm to farm. In contradistinction, our research focuses on the motion planning for an unloading vehicle in a single field.

Recently, path planning of agricultural machines has received some attention in the research community. In (Makino, Yokoi, and Kakazu 1999), authors develop a motion planning system, which integrates global and local motion planning components. In (Ferentinos, Arvanitis, and Sigrimis 2002), authors propose two heuristic optimization techniques in motion planning for autonomous agricultural vehicles. In (Ali and Van Oudheusden 2009), authors address the motion planning of one combine by using integer linear programming formulation. In (Oksanen and Visala 2009), coverage path planning problem is considered. The presented algorithms not only aim to find an efficient route, but also ensure the coverage of the whole field. In (Hameed, Bochtis, and Sørensen 2013), a coverage planning approach is proposed with the consideration of the presence of obstacles. However, compared with our research, the aforementioned works do not consider the problem of path planning on unloading vehicles and do not model the vehicle as a tractor-trailer system.

There has been some previous research to plan optimal trajectories for tractor-trailer model which are prevalent in farming applications. In (Divelbiss and Wen 1994), authors present an algorithm to find a feasible path which satisfies the given non-holonomic constraints. In (Divelbiss and Wen 1997), authors propose a trajectory tracking strategy which controls a tractor-trailer system moving along a path generated off-line. In (Hao, Laxton, Benson, and Agrawal 2004), researchers present a differential flatness−based formation following for a tractor-cart moving along with a combine harvester. In (Astolfi, Bolzern, and Locatelli 2004), an application of using Lyapunov technique is proposed to design the control laws for tractor-trailer model to follow a prescribed path. Researchers introduce the notion of equivalent size and propose an approach for path planning based on genetic algorithm in (Liu, Lu, Yang, and Chen 2008). In (Yu and Hung 2012), the tractor-trailer model is regarded as a Dubins vehicle, which can only move with constant speed and turn with upper bounded curvature. The proposed algorithm is used to find the shortest path in Dubins Traveling Salesman Problem with Neighborhoods (Isaacs, Klein, and Hespanha 2011). In (Chyba and Sekhavat 1999) and (Chitsaz 2013), authors introduce the notion of regular primitive and singular primitive which are local time-optimal.

The contribution of this paper can be summarized as follows. First, according to the mathematical model of grain cart, we present a numerical approach, as well as a primitive-based approach to find the time-optimal solution to the path planning problem in different situations. To tackle the case of moving target configuration, we further propose a two-stage motion planning strategy, taking advantage of the previous results. Finally, feasibility and effectiveness of presented techniques are demonstrated by simulations.

The rest of the paper is organized as follows. In Section II, we present the mathematical models for both combine and grain cart. In Section III, we present our previous work on the scheduling of agricultural operations. Based on the scheduling scheme, we formulate the path planning problem for grain cart in Section IV. In Section V, we present a numerical approach to obtain the time-optimal path between two given configurations. In Section VI, a primitive-based approach is considered to solve the problem. In Section VII, we address the dynamic case by proposing a two-stage motion planning strategy. In Section VIII, simulation results are presented. In Section IX, we conclude with some future work.

MATHEMATICAL MODELING

In this section, we describe the mathematical models for the combine harvester and the grain cart.

Combine Harvester

Combine harvester is the machine for harvesting crops, for example, wheat, oats, rye, barley corn, soybeans and flax. Figure 1 shows a combine at work. In this active mode, the header cuts the crop and feeds it into threshing cylinder. Grain and chaff are separated from the straw when crop goes through the concaved grates. The grain, after being sieved, will be stored in the on-board tank temporarily, and the waste straw is ejected. We use C to denote the maximum capacity of on-board tank.

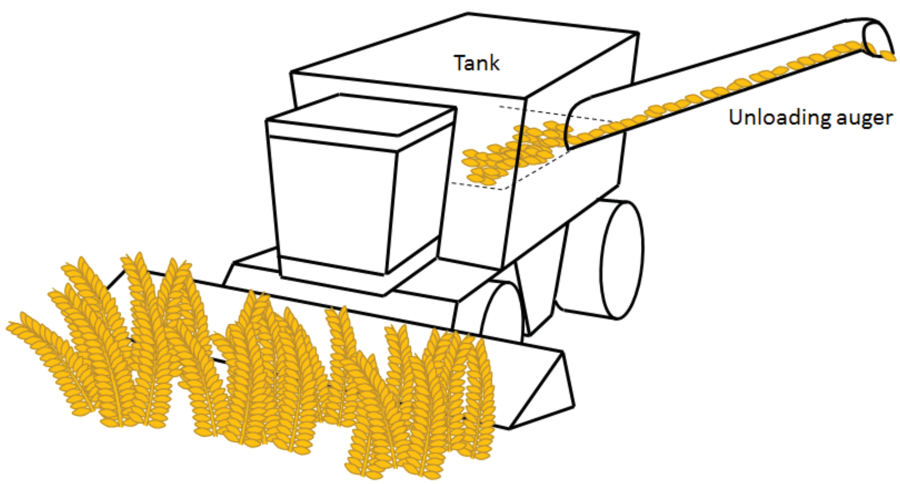

Figure 1. Combine harvester

Threshing grain loss is an important issue for combine harvester. For any combine, the quantity of threshing grain loss greatly depends on the forward speed of the harvester. In (Flufy and Stone 1983), authors show that automatic control has a better performance than manual control on the threshing grain loss. The forward speed is controlled to give a level of crop feed according to the required threshing grain loss. In this paper, we simplify the model and assume that all the combines posses identical constant speed, denoted as v_{ch}.

Since the tank does not hold a large capacity, modern combine usually has an unloading auger for removing the grains from the tank to other vehicles. For most of the combines, the auger is mounted on the left side, as shown in Figure 1. At this point, a vehicle has to be on the left side of the combine to empty the tank. Here we denote the unloading rate of the tank using auger as r_u. So when a combine proceeds with the harvesting and the unloading operations simultaneously, the unloading rate is $r_u - r_f$ $(r_u > r_f)$.

Grain Cart

A grain cart, also known as chaser bin, is a trailer towed by a tractor. In this paper, we use term grain cart to represent the system including both the tractor and the trailer. Figure 2(a) shows the appearance of a grain cart. Because of the larger capacity, one can use it to collect grains from multiple combines so that the combines could work without interruption.

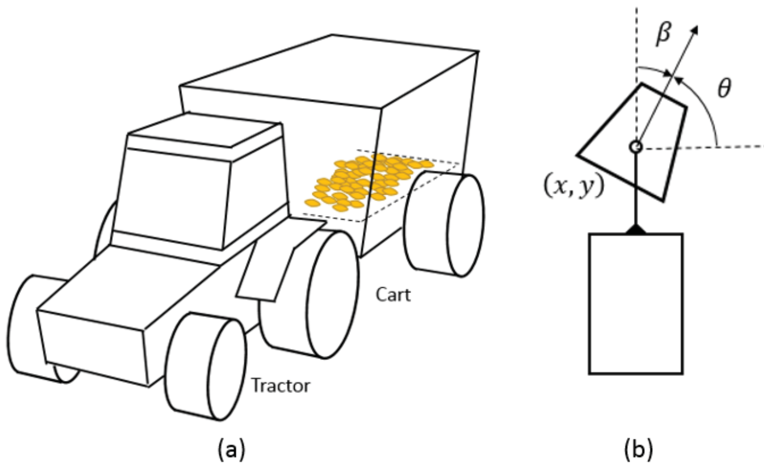

Figure 2. Grain cart

Figure 2(b) shows a grain cart. We model the grain cart as a trailer attached to a car-like robot. The robot is hitched by the trailer at the center. The equations of motion for the grain cart are as follows.

$$\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} v\cos\theta \\ v\sin\theta \\ \omega \\ -v\sin\beta + \omega \end{pmatrix}$$

where $q = (x, y, \theta, \beta) \in \mathbb{R}^2 \times S^1 \times S^1$ is the configuration, $u = (v, \omega) \in U = [-1, 1]^2$ is the control. In the configuration q, (x, y) is the coordinate of robot's center, θ is the robot's orientation, β is the angle between trailer orientation, and the robot orientation. v and ω denote the speed and angular velocity of the robot, respectively (Chyba and Sekhavat 1999) (Chitsaz 2013).

PREVIOUS WORK

In previous work, we addressed the logistics scheduling problem of grain cart during harvesting operation. A scheduling scheme was proposed for an arbitrary number of combines with a single grain cart. Based on the mathematical models and proposed scheme, the grain cart could serve multiple combines without stopping harvesting operation. In the scheme, the grain cart was scheduled to serve the combines sequentially, so that it needed to move along a trajectory from one combine to the next combine. Figure 3 shows an example of the path planning. Based on the scheduling scheme of N-combine case, we have obtained

$$\Delta T \le \frac{(r_u - r_f)C - (N-1)r_f C}{r_f (r_u - r_f)(2N-2)} \tag{1}$$

where ΔT denotes the travel time for the grain cart to switch. Since the travel time is constrained, in this paper we focus on finding the time-optimal solution to the path planning problem. In the next section, we provide an elaborated problem description.

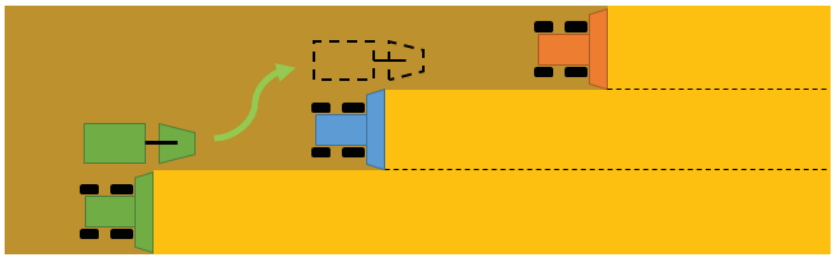

Figure 3. Path planning of the grain cart

PROBLEM DESCRIPTION

In this section, we formulate the problem. Consider a grain cart moving from one combine to the next combine, we would like to find the time-optimal solution for the grain cart to perform the path planning. In other words, given initial configuration, denoted as q_i and goal configuration, denoted as q_g, we intend to compute the path from q_i to q_g minimizing the total travel time. In the following sections, we first consider the case when initial and final configurations are static, following which, we extend the result to a dynamic target configuration case.

NUMERICAL APPROACH

In this section, we present a numerical approach to solve the navigation problem between two given static configurations. Before computing the trajectory, we first define a value function and establish Hamilton-Jacobi equation based on tractor-trailer model. Then we present an update scheme for computing the value function. Finally, the trajectory is computed using the obtained results.

Hamilton-Jacobi Equation

Denote the set of admissible path from the configuration q_i as $\mathcal{A}_{x_i, y_i, \theta_i, \beta_i}$. Given a goal configuration q_g, we define the corresponding value function $u : q \to \mathbb{R}^+ \cup \{0\}$ as follows (Takei, Tsai, Shen, and Landa 2010).

$$u(q(T)) = \inf\{T : q(t) \in \mathcal{A}_{x_i, y_i, \theta_i, \beta_i}, q(T) = q_g\} \tag{2}$$

The value function can be regarded as the optimal cost-to-go for the tractor-trailer model with certain constraints, an initial configuration and a final configuration. By applying dynamic programming principle for (2), we have

$$u(q(t)) = \inf\{u(q(t+\Delta t)) + \Delta t : q(t) \in \mathrm{A}_{x_i, y_i, \theta_i, \beta_i}\} \tag{3}$$

Dividing the terms by Δt and taking $\Delta t \to 0$, we are able to derive

$$-1 = \inf\{\bigtriangledown u \cdot \dot{q} : |v| = 1, |\omega| \leq 1\} \tag{4}$$

With the equations of motion of the grain cart, *Hamilton-Jacobi-Bellman equation* is obtained as follows.

$$-1 = \cos(\theta)\frac{\partial u}{\partial x} + \sin(\theta)\frac{\partial u}{\partial y} - \sin(\beta)\frac{\partial u}{\partial \beta} + \inf_{|\omega| \leq 1}\{\dot{\theta}(\frac{\partial u}{\partial \theta} + \frac{\partial u}{\partial \beta})\} \tag{5}$$

The last term in (5) can be eliminated by applying bang-bang principle $w = \pm 1$. Since q_i is the goal configuration which has no cost-to-go, we have $u(q_g) = 0$. For the points located in the obstacle or outside the space, we define the cost-to-go to be infinity. In the next section, we present an update scheme for the defined value function.

Update Scheme

In order to find the time-optimal path satisfying (5), we apply fast sweeping method and propose an update scheme for the value function $u(q)$ in the entire space. The basic idea is to employ the fact that the value function has zero cost-to-go at the goal configuration, and to compute the value function from the nodes close to the goal configuration, to the nodes at farther positions.

With this in mind, we first set up a four dimensional uniform Cartesian grid with refinement $(h_x, h_y, h_\theta, h_\beta)$. Let $u_{a,b,c,d} = u(q_{a,b,c,d}) = u(ah_x, bh_y, ch_\theta, dh_\beta)$ be the approximation of the solution on the grid nodes. Moreover, we discretize ω in the range of $[-1,1]$ and further define $u^*_{a,b,c,d}$ as follows.

$$u^*_{a,b,c,d} = \min_{\omega_i \in [-1,1]}\{u(q_{a,b,c,d} + \dot{q}\Delta t)\} + \Delta t \tag{6}$$

where $\dot{q} = (\cos(ch_\theta), \sin(ch_\theta), \omega_i, -\sin(dh_\beta) + \omega_i))^T$, ω_i is the *i*th element in the discretization and Δt is the length of time step. For the value of $u(q_{a,b,c,d} + \dot{q}\Delta t)$, we take the value directly if it is lying on the presented grid, otherwise it is approximated by taking the average value of the adjacent nodes.

Finally, the update scheme can be described in the following way.

$$u^{n+1}_{a,b,c,d} = \min\{u^n_{a,b,c,d}, u^{*n}_{a,b,c,d}\} \tag{7}$$

where the superscripts denote the iteration. We set up the termination condition of the computation as follows.

$$(\| u^{n+1}_{a,b,c,d} - u^{n}_{a,b,c,d} \|_2)^2 < \epsilon \tag{8}$$

where $\epsilon > 0$ is an arbitrary number.

Computing Trajectory

By using the obtained value function, we are able to derive the time-optimal path from any initial configuration q_i to the goal configuration q_g. According to (5), the control law can be summarized as follows.

$$\begin{aligned}
\dot{x} &= \cos\theta \\
\dot{y} &= \sin\theta \\
\dot{\theta} &= -sgn(\frac{\partial u}{\partial \theta} + \frac{\partial u}{\partial \beta}) \\
\dot{\beta} &= -\sin\beta + \dot{\theta}.
\end{aligned} \tag{9}$$

Note that the partial derivative in (9) is obtained by applying central difference approximation. In the computation of trajectory, the values of u which are not on the grid are computed using a nearest-neighbor interpolation.

The numerical approach computes the time-optimal trajectory efficiently if the corresponding value function is provided. The main time consumption is in computing the value function of the final configuration. But in real implementation, one can compute the value function beforehand. Therefore, the time cost will not influence the real-time operation on path planning. In the next section, we consider another approach to address the situation when we do not have such value function.

PRIMITIVE-BASED APPROACH

Related Work

Based on the model presented in Section II, the time-optimal trajectory satisfies Pontryagin Maximum Principle. In (Chitsaz 2013), authors define adjoint variables $\lambda = (\lambda_x, \lambda_y, \lambda_\theta, \lambda_\beta)$ and the Hamiltonian as follows.

$$H(q,\lambda,u) = \lambda_x \dot{x} + \lambda_y \dot{y} + \lambda_\theta \dot{\theta} + \lambda_\beta \dot{\beta} = v(\lambda_x \cos\theta + \lambda_y \sin\theta - \lambda_\beta \sin\beta) + \omega(\lambda_\theta + \lambda_\beta) \tag{10}$$

Additionally, the switching functions are defined as

$$\begin{aligned}
\phi_v &= \lambda_x \cos\theta + \lambda_y \sin\theta - \lambda_\beta \sin\beta \\
\phi_\omega &= \lambda_\theta + \lambda_\beta
\end{aligned} \tag{11}$$

Depending on ϕ_v and ϕ_ω, the optimal trajectory, which is called *extremal*, consists of two categories, namely, *regular* and *singular*. On one hand, an extremal is called regular if the times at which $\phi_v = 0$ or $\phi_\omega = 0$ have zero measure. On the other hand, an extremal is called *abnormal* if it has both $\phi_v \equiv 0$ and $\phi_\omega \equiv 0$. A singular is an extremal which contains a positive measure along which $\phi_v \equiv 0$ or $\phi_\omega \equiv 0$. The subtrajectories of a regular and a singular extremal are called *regular primitive* and *singular primitive*, respectively.

In the regular primitive, $\phi_v \neq 0$ and $\phi_\omega \neq 0$. Based on the state equations, a regular primitive satisfies

$$\begin{aligned}
\theta(t) &= \omega t + \theta(t_0) \\
x(t) &= x(t_0) + (v/\omega)(\sin(\theta) - \sin(\theta(t_0))) \\
y(t) &= y(t_0) - (v/\omega)(\cos(\theta) - \cos(\theta(t_0))) \\
\beta(t) &= 2(v/\omega)\arctan(\frac{t - 2v + K_1}{t + K_1}) \\
K_1 &= \frac{2}{v - \omega\tan(\beta(t_0)/2)}
\end{aligned} \tag{12}$$

where t_0 is the starting time instant and t is the passing time.

In singular primitive, we have either $\phi_v \equiv 0$ or $\phi_\omega \equiv 0$. Here since the grain cart has a constant forward speed, we only consider the case of $\phi_\omega \equiv 0$. It has been proved by (Chitsaz 2013) that if a ϕ_ω-singular primitive contains a straight line segment, either the entire primitive is a straight line segment, or

$$\begin{gathered}
\alpha(t) = \pm 2\beta(t) \\
\omega(t) = d = \pm 2\sin(\beta(t)) \\
-\frac{\pi}{6} \leq \beta(t) \leq \frac{\pi}{6} \; or \; \frac{5\pi}{6} \leq \beta(t) \leq \frac{7\pi}{6}
\end{gathered} \tag{13}$$

in which α denotes the angle between the robot orientation and the line, d denotes the distance between robot's center and the line. The path for the latter case is called a *merging curve*.

Path Planning of Grain Cart

In our case, when the grain cart finishes unloading one combine, it should follow a path to the second combine, and move parallel to it. This implies that the final path is supposed to end with a singular primitive, denoted as S_g. Considering the fact that a merging curve has the constraints (13), we propose a combination of the path containing two regular primitives and two ϕ_ω-singular primitives, as shown in Figure 4. The grain cart initially proceeds with a regular primitive R_1. S_1 is a singular primitive connecting to R_1. R_2 and S_g are the following regular primitive and goal singular primitive, respectively.

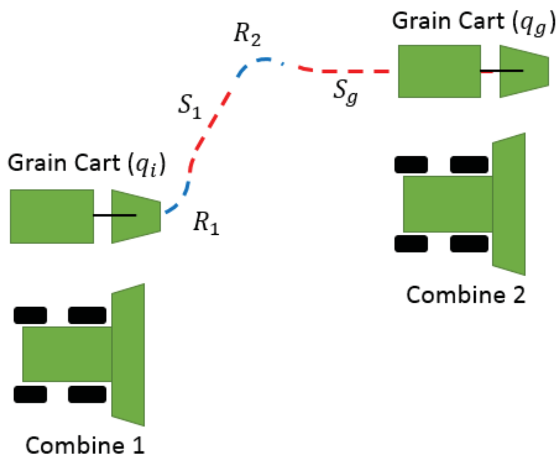

Figure 4. The path consists of 2 singular primitives (red lines) and 2 regular primitives (blue lines)

In order to minimize the travel time for the grain cart, we consider using straight lines instead of merging curves for ϕ_ω-singular primitives. Furthermore, since q_i and q_g has the same θ, it is apparent that R_1 and R_2 should have the same length, and the same central angle γ as well. Therefore, with a given γ, the slope of S_1 can be computed. Since q_i and q_g are known, the entire path can be derived. To minimize the travel time, we change the central angle corresponding to the regular primitives, and take the trajectory with minimum time in all feasible trajectories as the final path. The complete algorithm can be found in the following algorithm.

Trajectory computation using primitive-based approach:

Declare γ to be the central angle of regular primitives, $T_f = \infty$ to be the travel time of final path, P_f to be the final path

For $\gamma = 0 \rightarrow \pi$

  Compute the path P_γ starting from q_i

  If P_γ reaches q_g

    If travel time of $P_\gamma < T_f$

      $T_f =$ Travel time of P_γ, $P_f = P_\gamma$

    End if

  End if

End for

DYNAMIC MOTION PLANNING

In this section, we consider the case when the target configuration is dynamic. To accommodate this situation, we present a motion planning strategy that consists of two successive stages. In the first stage, we estimate a fixed goal configuration and perform the static path planning using aforementioned approaches. In the second stage, we apply a PID control for the grain cart to move synchronizing with the combine. Next, we elaborate both stages in detail.

Since the target configuration is mobile, we consider q_g as a function of time, i.e. $q_g(t) = (x_g(t), y_g(t), \theta_g(t), \beta_g(t))^T$. Figure 5 indicates the basic idea of the estimation on the static goal configuration. In the figure, t_0 denotes the starting time. Thus $q_g(t_0)$ could be considered as the goal configuration at the beginning. At this point, we attempt to find a ΔT satisfying the ideal case that the grain cart can successfully reach the desired goal configuration $q_g(t_0 + \Delta T)$. With this in mind, we approximate the ΔT by using a lower bound on the length of the path, which is $v\Delta T$, as shown in Figure 5. Based on geometry, the approximated ΔT could be solved using the following equation.

$$
\begin{gathered}
(x_g(t_0) - x_i + v_{ch}\Delta T)^2 + (y_g(t_0) - y_i)^2 = (v\Delta T)^2 \\
\Delta T = \frac{\sqrt{d_1^2 v^2 + d_2^2 (v^2 - v_{ch}^2)} + d_1 v_{ch}}{v^2 - v_{ch}^2}
\end{gathered}
\tag{14}
$$

where $d_1 = x_g(t_0) - x_i$ and $d_2 = y_g(t_0) - y_i$.

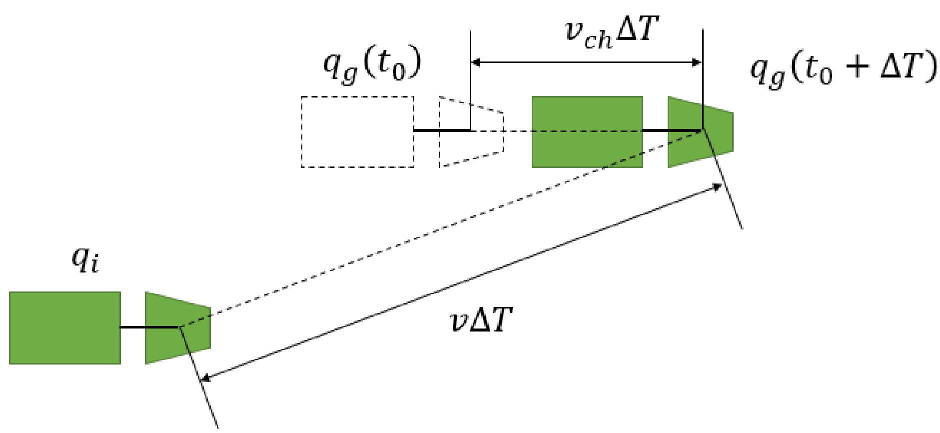

Figure 5. Dynamic motion planning

We let q_i and $q_g(t_0 + \Delta T)$ to be the initial and goal configuration in performing static path planning. Due to the fact that $v\Delta T$ is the lower bound on the path length for a non-holonomic vehicle, the grain cart will lag behind the combine when it reaches the goal configuration. Therefore, in the second stage, we apply a PID feedback control for grain cart to catch up with the combine. Figure 6 shows the block diagram for the presented control system. Position of the combine is considered to be the reference input. PID control is applied to the control of the grain cart's speed.

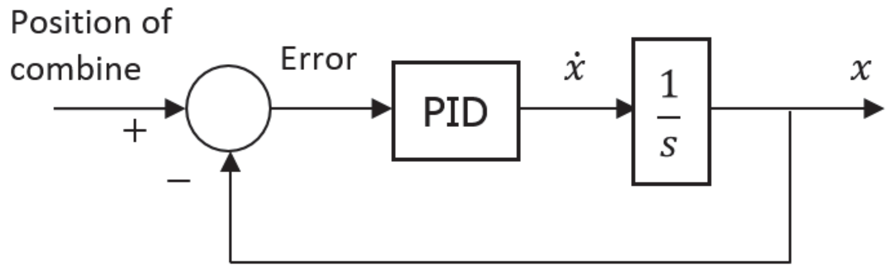

Figure 6. Block diagram for PID control system

SIMULATION

In this section, we present simulation results for the aforementioned techniques.

Figure 7 illustrates the paths obtained using numerical approach and primitive-based approach. Because of the fact that the tractor-trailer model has four state variables, it is hard to visualize the variations of all these variables. For this reason, in the simulation we only show the path of (x, y), which represents the physical location of grain cart in the environment. A colored arrow is added to show the final orientation. In this simulation, initial and goal configuration are set to be $q_i = (1,1,0,0)^T$ and $q_g = (4,4,0,0)^T$, respectively. For numerical approach, Table 1 lists all the refinement parameters in the computation of value function. In both approaches, the path computation terminates when the state of the grain cart reaches a small range of the goal configuration.

Table 1. Refinement parameters

Name	Value
h_x	0.2
h_y	0.2
h_θ	0.251 (radium)
h_β	0.251 (radium)
Δt	0.05
ò	0.005

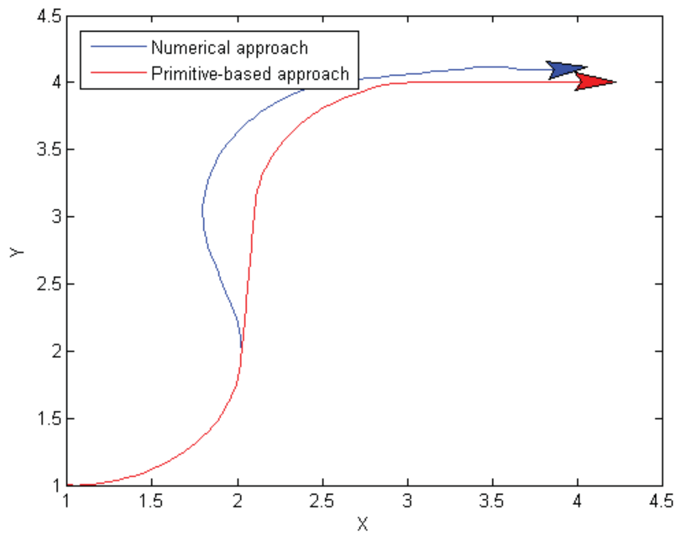

Figure 7. Simulation results with given initial and final configurations

Simulation results show that both paths finally reach the goal configuration which validates the proposed approaches. In the numerical approach, the path could be affected by the error of using inappropriate refinement parameters. With proper refinement parameters, the path could be more accurate, but it will lead to higher time consumption in the computation of value function.

In the second simulation, we compare the performance of using two proposed approaches. We keep $y_i = y_g - 4$ and plot the travel time with respect to the distance ratio $\frac{x_g - x_i}{y_g - y_i}$, as shown in Figure 8. The results show that with a given ratio, travel time of the two approaches are very close. Since the numerical approach provides us the time-optimal solution, it can be seen that the primitive-based approach also has a good performance. Note that in some cases, the performance of primitive-based approach is better because of the effect of the error in numerical approach.

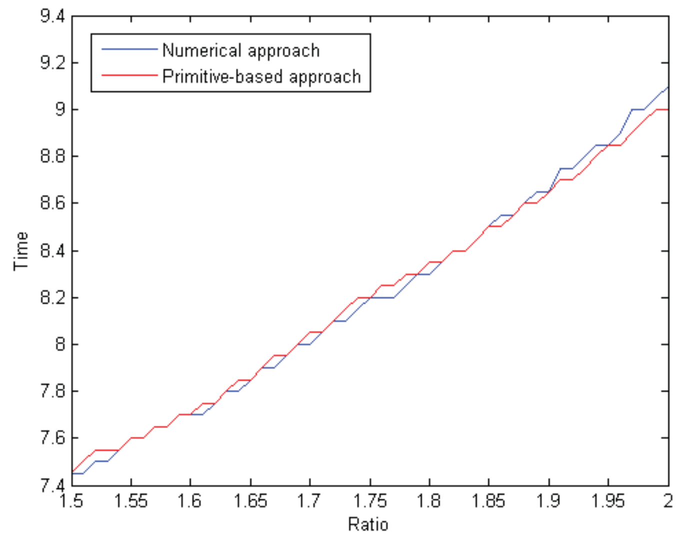

Figure 8. Travel time of using two approaches

Figure 9 illustrates the last simulation, which is an implementation of using dynamic motion planning. In this simulation, initial configuration of grain cart and initial position of combine are set to be $(1,5,0,0)^T$ and $(5,9)$, respectively. The speed of combine $v_{ch} = 0.4$, whereas the speed of grain cart v has the maximum 1. The simulation shows the paths of grain cart in both stages, which demonstrate the feasibility of the proposed dynamic motion planning strategy.

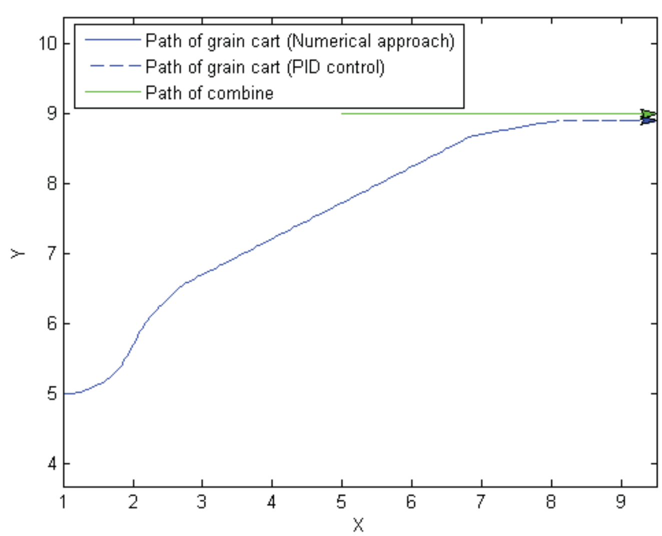

Figure 9. Dynamic motion planning

CONCLUSION

In this paper, we addressed the problem of finding the time-optimal path for the grain cart to navigate from one combine to the adjacent combine. Firstly, a numerical approach for computing the path based on given static initial and final configuration was presented.

According to the tractor-trailer model, we presented an update scheme for computing the value function corresponding to the entire state space. Finally, the trajectory could be computed with the help of this value function. Subsequently, we presented a primitive-based approach to obtain the solution. The path was considered to be constructed by regular and singular primitives that have local time optimality. Furthermore, we studied the case when target configuration was dynamic, and presented a motion planning strategy adopting the results from aforementioned approaches. Finally, simulation results were provided for these techniques. For the future work, we will consider a case of multiple grain carts, and try to coordinate grain carts so that the harvesting efficiency could be further improved.

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