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Keywords
core, coalition, production, blocking, core convergence, nonconvexity, stochastic blocking

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Stochastic Blocking and Core Convergence in Nonconvex Production Economies*

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Abstract

In production economies, the extent to which non-equilibria are blocked depends on specific rules that allocate authority among shareholders, because a blocking coalition's resources are affected by the firms it jointly owns with outsiders. Based on a notion of stochastic blocking, we extend Anderson's (1978) core convergence theorem to production economies where preferences and technologies are not necessarily convex.

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1 Introduction

The core convergence theorem, mainly due to Debreu and Scarf [2] and Anderson [1], has been the justification for competitive equilibrium as the solution concept for privatized exchange economies. The corresponding literature on production economies, however, is still preliminary. With production, a coalition may be interlocked with its outsiders through a firm they jointly own, so the feasibility of a blocking plan depends on how the control right on the firm is distributed between the two parties. The central question is how to enrich the Arrow-Debreu notion of "corporate share" with a rule of corporate governance that allocates the control on firms.

In a recent paper, Xiong and Zheng [4], we enrich corporate share with a deterministic interpretation, generalized from realistic special cases such as majority rules: whether a coalition gets to control a firm or not is predetermined by its total share of the firm; once a firm's production plan is decided, a coalition is entitled and obligated to carry out a fraction of the plan in the proportion of the coalition's total share. Based on that interpretation, as well as the replication framework à la Debreu and Scarf relying on strict convexity assumptions,[1] we prove in that paper a core convergence theorem for privatized production economies.

In this paper, we offer an alternative, stochastic interpretation for corporate shares. This interpretation is new in the literature and uncommon in the real world. The strength of this alternative is that the notion of blocking based on it, *stochastic blocking*, has led to stronger core convergence results than the previous approach. We present such a result in this paper. It is a production-analog of Anderson [1], i.e., a core convergence theorem for production economies without the convexity assumption, dispensing with the artificial replication framework.

The idea of stochastic blocking is to think of blocking as an event that the corporate partnership between a blocking coalition and its outsiders has to be dissolved. Here the stake is the physical production machine underlying the firm that is about to be dissolved. To avoid triviality, consider only the case where the machine is indivisible. Then which party should get the machine after the firm is dissolved? Borrowing the idea from auction theory, we propose an institutional setting where the machine is awarded to either the coalition or its

[1] Although the constant-return-to-scale technology is allowed in Debreu and Scarf [2], they assume that the technology is available to every possible coalition instead of being privatized.

outsiders with probabilities equal to their shares of the firm. Then a blocking coalition either has complete control and ownership of a firm or has zero obligation to it. Since the control rights on firms are ex ante uncertain, the total resources available to a blocking coalition may be random, but they converge to their expected value as the size of the coalition goes to infinity. Thus, if a coalition can "block" with the expected value of its total resources, a sufficiently large replication of the coalition can really block the status quo because the total resources available to the enlarged coalition approximate the expected value.

To apply this idea to *possibly nonconvex economies*, where preferences and technologies are not necessarily strictly convex, we need to overcome the problem How to make the approximation uniform across all allocations and all economies. Had preferences and technologies been strictly convex, core allocations would have the equal treatment property (ETP), so we can talk about the same allocation across economies of various sizes, and it is meaningful to say: "For every non-equilibrium allocation, we can block it by making the economy sufficiently large." Without such strict convexity, by contrast, the ETP no longer holds, so it is meaningless to talk about the same allocation across economies. Now the task is to find a sufficiently large economy within which any allocation that sufficiently deviates from equilibrium conditions is blocked.

This task is achieved by our theorem. The main sacrifice we make for the aforementioned uniformity is the restriction to the core allocations where the consumption disparity among individuals has a uniform upper bound across all economies. Nevertheless, our core convergence result is true for any uniform upper bound for disparity.

Our proof uses two main techniques: (i) the treatment of nonconvexity that Anderson [1] developed in his pure exchange model, and (ii) a modified notion of approximate core that we introduce in §2.5 of this paper.

2 Definitions and Assumptions

2.1 The Primitives

There are a finite set I of individuals, a finite set J of firms, and a finite number l of goods. Let i be the index for individuals and j for firms. Let $\mathbb{R}^l_+$ be the consumption set of each individual, let $\succeq_i$ be i's preference relation on $\mathbb{R}^l_+$, u_i his utility function, ω_i

($\in \mathbb{R}^l_+$) his endowment, and Y_j ($\subseteq \mathbb{R}^l$) the production set of firm j. Denote an allocation by $(x, y) := ((x_i)_{i\in I}, (y_j)_{j\in J})$, meaning individual i consumes bundle x_i and firm j's production plan is y_j. An allocation (x, y) is *feasible* if $x_i \in \mathbb{R}^l_+$ and $y_j \in Y_j$ for each individual i and each firm j and $\sum_{i\in I} x_i = \sum_{i\in I} \omega_i + \sum_{j\in J} y_j$. A *coalition* S is a nonempty set of individuals. A firm is not a player but is an indivisible technology jointly owned by shareholders.

We make the following **assumptions** throughout the paper without mentioning them in our theorem or lemmas. Every individual i is von Neumann Morgenstern rational with utility function u_i strongly monotone on the consumption set $\mathbb{R}^l_+$. For every firm j, $\mathbf{0} \in Y_j$.

Let $\mathcal{E}$ denote the above economy, with I the set of individuals and J the set of firms. Let us measure the *size* of the economy by $n(\mathcal{E}) := |I|$, i.e., the number of individuals in $\mathcal{E}$, assuming that $|J|$ grows in the same rate as $|I|$. The sets I and J may vary across economies, but the number l of the kinds of goods is constant.

2.2 Partnership Dissolution

The interlocking problem between a blocking coalition and its outsiders is due to their partnership in the firms they jointly own. Let us consider a mechanism that disentangles such interlock by allowing shareholders to dissolve their partnership: If shareholder i dissolves his partnership in firm j, the technology of the firm, being indivisible, is allocated to either i or someone else, and is allocated to i with probability θ_{ij}. Here the *winning probability* $\theta_{ij} \in [0..1]$ is exogenous and may be unequal to the profit share in the Arrow-Debreu model. After partnership dissolution, the winner of the technology has complete discretion of what to do with it and complete ownership of its net output.

For any individual $i \in I$ and any firm $j \in J$, define

$$\mathbf{z}^i_j := \begin{cases} 1 & \text{if individual } i \text{ wins firm } j \text{ in case of partnership dissolution} \\ 0 & \text{else.} \end{cases} \quad (1)$$

Assumption 1 *The random vector* $(\mathbf{z}^i_j)_{i\in I}$ *is independent across firms* j. *For each firm* j, $\text{Prob}(\mathbf{z}^i_j = 1) = \theta_{ij}$.

2.3 Stochastic Blocking

A coalition may deviate from a status quo allocation by its members dissolving partnerships in the firms where they hold shares. For such deviation to be feasible, the coalition needs to come up with plans contingent on the allocation of the technologies of the dissolved firms. A coalition *wins* a firm j if and only if one of its members wins j in partnership dissolution.

A *contingent blocking plan* $\big((\mathbf{x}'_i)_{i\in S}, (\mathbf{y}'_j)_{j\in J}\big)$ is a mapping from a realization of $((\mathbf{z}^i_j)_{j\in J})_{i\in I}$ to a consumption-production plan $((x'_i)_{i\in S}, (y'_j)_{j\in J})$ such that, for every realization of $((\mathbf{z}^i_j)_{j\in J})_{i\in I}$, $x'_i \in \mathbb{R}^l_+$ for all $i \in S$ and $y'_j \in Y_j$ for each firm j, and $y'_j = \mathbf{0}$ if j is not won by S (if S loses firm j, the outsiders solely own the output of the firm, so j's production plan from the perspective of S is equal to $\mathbf{0}$).

Abusing notations, let ω_i also denote the mapping that constantly associates individual i's endowment to every realization of $((\mathbf{z}^i_j)_{j\in J})_{i\in I}$. Let $\mathbb{E}$ denote expected values.

Definition 1 (stochastic blocking) *A feasible allocation* (x, y) *is* stochastically blocked *by a coalition* S *if there is a contingent blocking plan* $\big((\mathbf{x}'_i)_{i\in S}, (\mathbf{y}'_j)_{j\in J}\big)$ *such that* $\mathbb{E}[u_i(\mathbf{x}'_i)] > u_i(x_i)$ *for all* $i \in S$ *and (for every realization of* $(\mathbf{z}^i_j)_{j\in J}$*)*

$$\sum_{i\in S} \mathbf{x}'_i = \sum_{i\in S} \omega_i + \sum_{j\in J} \mathbf{y}'_j. \tag{2}$$

Thus, the resources within a blocking coalition are stochastic and ex post consist of its total endowments plus the entire technology of every firm won by the coalition, with winning probability determined by θ_{ij}. Note that the outsiders of a blocking coalition can always come up with a feasible consumption plan, because through partnership dissolution neither party has any duty in the firms controlled by the other party.

2.4 An Illustrative Example

To see why stochastic blocking works, consider the following replica economy borrowed from Xiong and Zheng [4].

Example 1 *There are two goods, two individual-types, and one firm-type. In each prototype of the economy, the endowment for the individual of type-1 is* $e_1 = (1, 0)$*, and that for type-2 is* $e_2 = (0, 1)$*. For each type, the utility from consumption bundle* (x_{i1}, x_{i2}) *is*

$$u(x_{i1}, x_{i2}) := x_{i1} + 10\sqrt{x_{i2}}.$$

Each firm is equally shared by a type-1 and a type-2 individuals, and its production set is

$$Y := \left\{(y_{j1}, y_{j2}) \in \mathbb{R}^2 : y_{j2} \le \sqrt{-y_{j1}}; y_{j1} \le 0\right\}.$$

As shown in Xiong and Zheng [4], In any replica of the economy, there is a unique Walras equilibrium allocation (represented type-wise):

$$x_1^* = (0, 3/4)\,; \quad x_2^* = (0, 5/4)\,; \quad y^* = (-1, 1); \quad p^* = (1, 2),$$

but the the type-wise represented allocation

$$x_1^o = (0, 1/2)\,; \quad x_2^o = (0, 3/2)\,; \quad y^o = (-1, 1) \tag{3}$$

cannot be blocked given any deterministic rule corporate control allocation in [4].

Now suppose that the type-1 shareholder can dissolve his partnership in the firm. Suppose further that the mechanism of partnership dissolution is that the technology embodied by the firm, being an indivisible object, has to be awarded to one of the two shareholders, and the probability with which the type-1 shareholder wins is some parameter $\hat{\theta}_1 \in [0..1]$ (not necessarily equal to his profit share θ_1). Suppose the winner of the technology has complete discretion of what to do with it and complete ownership of its output. Then a 3-member coalition consisting of a prototype economy and a single type-1 individual from another unit would rather deviate from the status quo to the following plan:

i. The single type-1 person dissolves his partnership from the firm of his unit.

ii. If the firm is awarded to him, the coalition controls two firms and produces $(-1, 1)$ in each firm; the type-2 coalition member consumes $(0, 3/2)$ as in the status quo, and each of the type-1 member consumes $(0, 3/4)$.

iii. If the firm is not awarded to the single type-1 person, the coalition controls one firm and produces $(-2, \sqrt{2})$; the type-2 member consumes $(0, 3/2)$ as in the status quo, and each type-1 member consumes $(0, (\sqrt{2} - 1/2)/2)$.

This plan is feasible for the coalition at each possible state, since the total endowment of the coalition is $(2, 1)$. With this plan, each type-1 coalition member is better-off if the probability $\hat{\theta}_1$ of winning the firm is greater than 0.17:

$$\hat{\theta}_1 > 0.17 \quad \Rightarrow \quad \hat{\theta}10\sqrt{3/4} + (1 - \hat{\theta})10\sqrt{(\sqrt{2} - 1/2)/2} > 10\sqrt{1/2}.$$

Then they can make the type-2 member also better-off by giving him a small positive bundle.

2.5 Cushion Blocking

The next notion of blocking does not take into account the interlock between a coalition and its outsiders. Although our theorem is not based on this blocking notion, our proof uses an important fact about it, Lemma 3. This blocking notion has two features. First, the coalition needs to have at least N individuals. Second, the blocking plan is cushioned: for every coalition member, the consumption in the blocking plan is preferred to the status quo consumption even after we reduce the former by ϵ and increase the latter by δ.[2]

Let $\mathbf{1}$ denote the l-vector that is equal to one in every coordinate.

Definition 2 (cushion blocking) *For any* $N = 1, 2, \ldots$, *any* $\epsilon > 0$ *and any* $\delta > 0$, *a feasible allocation* $((x_i)_i, (y_j)_{j\in J})$ *is* cushion-blocked with (N, ϵ, δ) *if there is a coalition* S *with size* $|S| \geq N$ *such that, for every* $i \in S$ *and for every* $j \in J$, *there exist* $x'_i \in \mathbb{R}^l_+ + \epsilon\mathbf{1}$ *and* $y'_{ij} \in Y_j$ *such that*

$$u_i(x'_i - \epsilon\mathbf{1}) > u_i(x_i + \delta\mathbf{1}); \tag{4}$$

$$\textstyle\sum_{i\in S} x'_i = \sum_{i\in S} \omega_i + \sum_{j\in J} \sum_{i\in S} \theta_{ij} y'_{ij}. \tag{5}$$

Let $C_{\text{cush}}(\mathcal{E}; N, \epsilon, \delta)$ be the set of allocations in economy $\mathcal{E}$ that are not cushion-blocked with (N, ϵ, δ). Note that $C_{\text{cush}}(\mathcal{E}; N, \epsilon, \delta) \subseteq C_{\text{cush}}(\mathcal{E}; M, \epsilon, \delta)$ if $M \geq N$.

2.6 Measuring the Deviation from Equilibrium Conditions

A *virtual competitive equilibrium* in economy $\mathcal{E}$ is defined by the same condition that defines a competitive equilibrium except that the profit share in the standard definition is replaced by the winning probability θ_{ij} here. A virtual competitive equilibrium may be different from a standard competitive equilibrium because the winning probabilities may be unequal to the profit shares. Our core convergence theorem asserts that the deviation from the conditions for *virtual* competitive equilibrium shrinks to zero as the size of the economy goes to infinity. We shall define measures for such deviation here.

For every allocation (x, y) and every $i \in I$, let

$$\mathbb{U}_i(x_i, y) := \left\{ x'_i - \omega_i - \sum_{j\in J} \theta_{ij} y_j : x'_i \in \mathbb{R}^l_+; u_i(x'_i) > u_i(x_i) \right\}, \tag{6}$$

[2]Although the first margin ϵ is similar to the notion of ϵ-blocking in the literature, the second margin δ appears to be a new construct.

which is i's strict upper contour set relative to x_i translated by $-\omega_i - \sum_{j\in J}\theta_{ij}y_j$.

Let $\Delta := \{p \in \mathbb{R}^l_+ : \sum_{k=1}^l p_k = 1\}$ denote the set of normalized prices. For any allocation $(x, y) := ((x_i)_{i\in I}, (y_j)_{j\in J})$ in $\mathcal{E}$ and for any $p \in \Delta$, define:

$$\mathscr{D}_{\text{bdg}}(\mathcal{E}; x, y; p) := \frac{1}{n(\mathcal{E})}\sum_{i\in I}\left|p\cdot\left(x_i - \omega_i - \sum_{j\in J}\theta_{ij}y_j\right)\right|; \tag{7}$$

$$\mathscr{D}_{\text{pref}}(\mathcal{E}; x, y; p) := \frac{1}{n(\mathcal{E})}\sum_{i\in I}\left|\inf p\cdot\mathbb{U}_i\left(x_i, y\right)\right|; \tag{8}$$

$$\mathscr{D}_{\text{pft}}(\mathcal{E}; x, y; p) := \frac{1}{n(\mathcal{E})}\sum_{j\in J}\left(\sup p\cdot Y_j - p\cdot y_j\right). \tag{9}$$

The first and second measures are obvious analogs of Anderson's [1] on the deviation from budget constraints and deviation from consumer optimization. The third is an obvious measure for the per capita deviation from profit maximization.

2.7 Additional Assumptions

The essence of core convergence theorems is the idea that the influence of an individual shrinks as the economy enlarges. Clearly, this idea would fail if some individual has a disproportionally large amount of resources. The next three assumptions are to rule out these failing cases by imposing bounds on an individual's endowed resources uniform across economies. Assumption 2 says that an individual cannot have a disproportionally large endowment of goods. This assumption has been used by Anderson [1] in a slightly different form. Assumption 3 says that a firm cannot be disproportionally large, which is aligned with the idea that firms are small in competitive markets (Mas-Colell, Whinston and Green [3, p630]). This assumption implies that an individual cannot gain a disproportionally big influence through his control of a disproportionally big firm. Assumption 4 says that an individually cannot influence a disproportionally large set of firms. Assumptions 3–4 are not needed by Anderson, as his model is pure exchange.

Assumption 2 (small consumers) *There exists $\bar{\omega} \in \mathbb{R}_+$ such that, for any economy $\mathcal{E}$ and for any individual i in $\mathcal{E}$, every coordinate of ω_i is less than $\bar{\omega}$, i.e., $\|\omega_i\|_\infty < \bar{\omega}$.*

Assumption 3 (small firms) *There exists $\bar{y} \in \mathbb{R}_+$ such that, for any economy $\mathcal{E}$, for any firm j in $\mathcal{E}$ and for any $y_j \in Y_j$, $\|y_j\|_\infty < \bar{y}$.*

Assumption 4 (big world) *There exists positive integers c_s and c_f such that, in any economy $\mathcal{E}$, $|\{i \in I : \theta_{ij} > 0\}| \leq c_s$ for each firm $j \in J$, and $|\{j \in J : \theta_{ij} > 0\}| \leq c_f$ for each individual $i \in I$.*

The next assumption, as well as the uniform disparity bound in §2.8, is needed because a blocking coalition is collectively taking a gamble. To explain that, suppose that a blocking coalition wishes to achieve a target of increasing each member's consumption by ϵ of each good. Due to the uncertain outcome of partnership dissolution, the actual consumption they can come up with is different from the target, so each member needs to scale down his expected gain by the probability with which his actual consumption is sufficiently near to the target. Hence each potential coalition member has a threshold of such probability for him to join the coalition. The problem is that different individuals have different thresholds because the utilities and marginal utilities at the status quo allocation are different across individuals. That is troubling because: to approach the target in probability, the coalition needs to be sufficiently large; but in enlarging its membership, the coalition needs to reach the probability thresholds of more and more people; but to reach these thresholds, the coalition needs to be even bigger, ad infinitum. To stop this infinite loop, the next assumption imposes a uniform bound on individuals' marginal utilities, and our uniform-disparity restriction on core allocations, §2.8, implies a uniform bound on their utilities at core allocations.

Assumption 5 *There are constants $\underline{u}$, $\bar{\nu}$ and $\underline{\nu}$ such that, in any economy $\mathcal{E}$, for every individual i and any $x_i \in \mathbb{R}^l_+$, (i) $0 \leq u_i(\mathbf{0}) \leq \underline{u}$ and (ii) the directional derivative $u_i'(x_i; \mathbf{1})$ along the vector $\mathbf{1}$ exists and $0 < \underline{\nu} \leq u_i'(x_i; \mathbf{1}) \leq \bar{\nu} < \infty$.*

For an example of Assumption 5, consider a world with two kinds of commodities and suppose any utility function belongs to the family of functions $u_{(a,b)}$ parametrized by a pair $(a, b) \in [1..2] \times [3..4]$, such that for every $(x_{i1}, x_{i2}) \in \mathbb{R}^2_+$,

$$u_{(a,b)}(x_{i1}, x_{i2}) := a x_{i1} + (x_{i2} + 1)^{1/b}.$$

Then $u'_{(a,b)}(x_{i1}, x_{i2}; \mathbf{1}) = a + \frac{1}{b}(x_{i2} + 1)^{1/b-1}$. Hence we can pick $\underline{\nu} = 1$, $\bar{\nu} = 3$, and $\underline{u} = 1$.

2.8 The Disparity among Individuals

As explained immediately prior to Assumption 5, we shall restrict attention to the core allocations where the disparity among individuals is uniformly bounded across all economies.

Nevertheless, our convergence result is true given any uniform bound of disparity.

The *disparity* of an allocation $(x, y) := ((x_i)_{i\in I}, (y_j)_{j\in J})$ in economy $\mathcal{E}$ is defined to be

$$\max_{i\in I} \left\| x_i - \frac{1}{n(\mathcal{E})} \sum_{i'\in I} x_{i'} \right\|_\infty . \tag{10}$$

For any $d > 0$, let $C^d(\mathcal{E})$ be the set of all the feasible allocations in $\mathcal{E}$ which are not stochastically blocked and whose disparities are all less than or equal to d.

For any $d > 0$, the *non-competitive deviation* of an economy $\mathcal{E}$ subject to a disparity bound d is defined to be

$$\mathscr{D}^d(\mathcal{E}) := \sup \left\{ \inf_{p\in\Delta} \max \left\{ \mathscr{D}_{\text{bdg}}(\mathcal{E}; x, y; p), \mathscr{D}_{\text{pref}}(\mathcal{E}; x, y; p), \mathscr{D}_{\text{pft}}(\mathcal{E}; x, y; p) \right\} : (x, y) \in C^d(\mathcal{E}) \right\} . \tag{11}$$

3 The Core Convergence Theorem

Theorem: *If Assumptions 1 –5 hold, then for any* $d \in \mathbb{R}_{++}$, $\lim_{n(\mathcal{E})\to\infty} \mathscr{D}^d(\mathcal{E}) = 0$.

The conclusion of the theorem says: For any arbitrarily large upper bound d of consumption-disparity, for any arbitrarily small $\epsilon > 0$, there exists a sufficiently large integer N such that in any economy of size bigger than N, for any stochastic-core allocation (x, y) with disparity less than d, there exists a price vector p such that the deviation between $(x, y; p)$ and the virtual equilibrium condition is less than ϵ in all three measures.

From a social planner's viewpoint, it says that, if the planner enforces an upper bound on the disparity among individuals (as a feasibility condition for all allocations), the decentralized markets will behave more and more like virtual competitive equilibria as the markets include more and more individuals. The planner can pick any arbitrarily large disparity upper bound, as long as she maintains it constant across economies.

The theorem is proved in two steps. First, we show that a stochastic-core allocation can be cushion-blocked *only* by sufficiently small coalitions. Second, for every stochastic-core allocation (x, y), we exclude a maximal-size cushion-blocking coalition so that no other coalition can cushion-block (x, y); then an extension of Anderson's [1] proof implies that the non-competitive deviation of (x, y) is arbitrarily small.

3.1 Step 1: Only a Minority Can Cushion-Block Stochastic Core

This step is a significant extension of the approximation technique we developed in Xiong and Zheng [4]: The bigger a blocking coalition, the more probably it can achieve its target. The challenge is to make the minimum size of blocking coalitions uniform across all economies.

The next lemma is a weak law of large numbers (LLN) with bounded correlation. The correlation is bounded due to the big-world assumption. Slightly different from the standard version of LLN, the convergence here is uniform across all subsequences of random variables.

Lemma 1 *Suppose Assumptions 1, 3, and 4. Then: for any $\epsilon > 0$ and for any $\alpha \in [0..1)$, there exists an N such that, for any integer $n \geq N$, in any economy $\mathcal{E}$ with size bigger than n and production sets $(Y_j)_{j \in Y}$, for any n distinct individuals $i_1, \ldots, i_n$ in $\mathcal{E}$ and for any $((y_{kj})_{j\in J})_{k=1}^n \in \left(\prod_{j\in J} Y_j\right)^n$,*

$$\text{Prob}\left(\frac{1}{n}\left\|\sum_{k=1}^{n}\sum_{j\in J}\left(\mathbf{z}_j^{i_k} y_{kj} - \theta_{i_k j} y_{kj}\right)\right\|_\infty < \epsilon\right) \geq \alpha. \tag{12}$$

Proof Appendix A. ■

The next proof shows the roles of the uniform disparity bound and Assumption 5.

Lemma 2 *Suppose Assumptions 2–5. For any $d \in \mathbb{R}_{++}$ and for any $\epsilon > 0$, there exists $\alpha \in (0..1)$ such that, in any economy $\mathcal{E}$, for any $(x, y) \in C^d(\mathcal{E})$ and for any individual i,*

$$\alpha u_i(x_i + \epsilon\mathbf{1}) \geq u_i(x_i). \tag{13}$$

Proof First, we claim that for any $d \in \mathbb{R}_{++}$, there is a constant $K \in \mathbb{R}_{++}$ such that, for any $(x, y) \in C^d(\mathcal{E})$ and for any individual i, $\|x_i\|_\infty \leq K$. To see that, recall from the definition of $C^d(\mathcal{E})$ that $\left\|x_i - \frac{1}{n(\mathcal{E})}\sum_{i'\in I} x_{i'}\right\|_\infty$ is uniformly bounded by d. By aggregate feasibility of core allocations, $\sum_I x_{i'} = \sum_I \omega_{i'} + \sum_J y_j$, hence

$$\left\|x_i - \frac{1}{n(\mathcal{E})}\sum_I \omega_{i'} - \frac{1}{n(\mathcal{E})}\sum_J y_j\right\|_\infty \leq d.$$

Then the triangular inequality implies

$$\|x_i\|_\infty - \frac{1}{n(\mathcal{E})}\sum_I \|\omega_{i'}\|_\infty - \frac{1}{n(\mathcal{E})}\sum_J \|y_j\|_\infty \leq d.$$

By Assumption 2 (small consumers), $\sum_I \|\omega_{i'}\|_\infty / n(\mathcal{E}) \leqq \bar{\omega}$. By Assumption 4 (big world), $|J| \leq c_f |I| = c_f n(\mathcal{E})$, so Assumption 3 (small firms) implies $\sum_J \|y_j\|_\infty / n(\mathcal{E}) \leqq c_f \bar{y}$. Thus, $K := d + \bar{\omega} + c_f \bar{y}$ is the desired constant.

Pick any $d \in \mathbb{R}_{++}$ and let K be the associated upper bound. Let

$$\bar{u} := \bar{\nu} K + \underline{u}.$$

For any $\epsilon > 0$, let $\alpha := \bar{u} / (\bar{u} + \underline{\nu}\epsilon)$. Then, for any $\mathcal{E}$, in any core allocation $(x, y) \in C^d(\mathcal{E})$, the consumption x_i of any individual i is bounded from above by K; thus, as the directional derivative $u_i'(\cdot, \mathbf{1})$ exists and $u_i'(\cdot, \mathbf{1}) \leq \bar{\nu}$ on $\mathbb{R}^l_+$ and $u_i(\mathbf{0}) \leq \underline{u}$ (Assumption 5),

$$u_i(x_i) \leq u_i(K, \ldots, K) \leq u_i(\mathbf{0}) + K \sup u_i'(\cdot; \mathbf{1}) \leq u_i(\mathbf{0}) + K\bar{\nu} \leq \underline{u} + K\bar{\nu} = \bar{u};$$

as $u_i'(\cdot, \mathbf{1}) \geq \underline{\nu}$ on $\mathbb{R}^l_+$ (Assumption 5 again), we have

$$u_i(x_i + \epsilon \mathbf{1}) - u_i(x_i) \geq \epsilon \underline{\nu} \geq \frac{u_i(x_i)}{\bar{u}} \underline{\nu} \epsilon = (1/\alpha - 1) u_i(x_i).$$

Hence (13) holds. ■

The next lemma is crucial. It says that there is a uniform bound on the size of the coalitions that can cushion-block a stochastic core allocation. The intuition is that the larger a cushion-blocking coalition is, the more probable its actual resources are nearby its desired target, and the greater expected utility each coalition member would gain from blocking.

Lemma 3 *Suppose Assumptions 1–5. For any $d \in \mathbb{R}_{++}$ and for any $\epsilon > 0$, there exists a sufficiently large integer N such that, in any economy $\mathcal{E}$ with $n(\mathcal{E}) \geq N$, $C^d(\mathcal{E}) \subseteq C_{\text{cush}}(\mathcal{E}; N, \epsilon/2, \epsilon/2)$.*

Proof Pick any $d \in \mathbb{R}_{++}$. By Lemma 2,

$$\forall \epsilon > 0\ \exists \alpha \in (0..1)\ \forall \mathcal{E}\ \forall (x, y) \in C^d(\mathcal{E})\ \forall i \in I,\ \alpha u_i\left(x_i + \frac{\epsilon}{2}\mathbf{1}\right) \geq u_i(x_i). \tag{14}$$

Hold the ϵ and α in (14) fixed. By Lemma 1, there exists an integer N such that, for any $n \geq N$, for any n distinct individuals $i_1, \ldots, i_n$ in any economy $\mathcal{E}$ and for any $(y'_{kj})_{j \in J})_{k=1}^n \in \left(\prod_{j \in J} Y_j\right)^n$,

$$\text{Prob}\left(\frac{1}{n}\left\|\sum_{k=1}^n \sum_{j \in J} \left(\mathbf{z}_j^{i_k} y'_{kj} - \theta_{i_k j} y'_{kj}\right)\right\|_\infty < \frac{\epsilon}{2}\right) \geq \alpha. \tag{15}$$

We shall show that this N is what we need. Let $(x, y) \in C^d(\mathcal{E})$; we shall prove $(x, y) \in C_{\text{cush}}(\mathcal{E}; N, \epsilon/2, \epsilon/2)$. Suppose not. Then some coalition S, with $|S| \geq N$ and a blocking plan $(x'_i, (y'_{ij})_{j\in J})_{i\in S}$, cushion-blocks (x, y) with margins $(\epsilon/2, \epsilon/2)$. To reach a contradiction, we shall prove that S stochastically blocks the allocation (x, y) with a contingent blocking plan $((\mathbf{x}''_i)_{i\in S}, (\mathbf{y}''_j)_{j\in J})$. Define

$$\Lambda := \left\{ (\mathbf{z}^i_j)^{i\in S}_{j\in J} : \frac{1}{|S|} \left\| \sum_{i\in S} \sum_{j\in J} \left(\mathbf{z}^i_j y'_{ij} - \theta_{ij} y'_{ij} \right) \right\|_\infty < \frac{\epsilon}{2} \right\}.$$

By the choice of N, $\text{Prob}(\Lambda) \geq \alpha$. If event Λ occurs, define for every $i \in S$ and every $j \in J$,

$$\begin{aligned} \mathbf{x}''_i &:= x'_i + \frac{1}{|S|} \sum_{i\in S} \sum_{j\in J} \left(\mathbf{z}^i_j y'_{ij} - \theta_{ij} y'_{ij} \right); \\ \mathbf{y}''_j &:= \begin{cases} y'_{ij} & \text{if } \mathbf{z}^i_j = 1 \text{ for some } i \in S \\ \mathbf{0} & \text{else.} \end{cases} \end{aligned}$$

If the event Λ does not occur, define for every $i \in S$ and every $j \in J$,

$$\mathbf{x}''_i := \omega_i; \qquad \mathbf{y}''_j := \mathbf{0}.$$

It is easy to verify that this contingent blocking plan is feasible at every possible realized state (by (5)). We are done if every $i \in S$ prefers this plan to the allocation (x, y). Conditional on the event Λ, we know for every $i \in S$, $\mathbf{x}''_i \gg x'_i - \frac{\epsilon}{2}\mathbf{1}$ and so $\mathbf{x}''_i \succ_i x'_i - \frac{\epsilon}{2}\mathbf{1}$; as (x', y') cushion-blocks (x, y) by margin $(\epsilon/2, \epsilon/2)$, (4) implies

$$\Lambda \text{ occurs} \Longrightarrow \mathbf{x}''_i \succ_i x'_i - \frac{\epsilon}{2}\mathbf{1} \succ_i x_i + \frac{\epsilon}{2}\mathbf{1}.$$

Then it follows from (14) and the supposition $(x, y) \in C^d(\mathcal{E})$ that

$$\Lambda \text{ occurs} \Longrightarrow \alpha u_i(\mathbf{x}''_i) > \alpha u_i \left(x_i + \frac{\epsilon}{2}\mathbf{1} \right) \geq u_i(x_i).$$

As every individual i is von Neumann Morgenstern rational and since $\text{Prob}(\Lambda) \geq \alpha$, her expected utility from the contingent blocking plan is greater than or equal to

$$\alpha \mathbb{E}\left[u_i(\mathbf{x}''_i) \mid \Lambda \right] + (1 - \alpha) 0 > \alpha u_i \left(x_i + \frac{\epsilon}{2}\mathbf{1} \right) \geq u_i(x_i),$$

which implies the desired contradiction that (x, y) is stochastically blocked. ■

3.2 Step 2: The Diminishing Effect of Excluding Minorities

By step one, a stochastic-core allocation say (x, y) cannot be cushion-blocked by more than N individuals. Hence if we exclude a maximal-size cushion-blocking coalition from the economy—there are at most N such individuals—nobody else can cushion-block (x, y). Then, for the other individuals, an argument similar to Anderson's yields a supporting price p for all but at most l individuals (l being a constant). Hence totally at most $N + l$ individuals are not supported, and the error from excluding them is at most in the order of $(N + l)/n(\mathcal{E})$ per capita. The other error comes from the cushion-blocking condition, which means that the sets supported by p differ from the upper contour sets by a multiple of the margin ϵ. The sum of these two errors is the non-competitive deviation of $(x, y; p)$. As the margin ϵ can be arbitrarily small, this sum of errors can be arbitrarily small as the economy enlarges.

Lemma 4 *Suppose Assumptions 2–4. For any $\epsilon > 0$, any $N = 1, 2, \ldots$, and any economy $\mathcal{E}$ with $n(\mathcal{E}) \geq N$, if $(x, y) \in C_{\text{cush}}(\mathcal{E}; N, \epsilon/2, \epsilon/2)$ then there is $p \in \Delta$ such that:*

$$\mathscr{D}_{\text{bdg}}(\mathcal{E}; x, y; p) \leq \frac{2(N + l)}{n(\mathcal{E})} (\bar{\omega} + c_f \bar{y}) + 2\epsilon; \tag{16}$$

$$\mathscr{D}_{\text{pft}}(\mathcal{E}; x, y; p) \leq \frac{N + l}{n(\mathcal{E})} (\bar{\omega} + \bar{y} c_f) + \epsilon; \tag{17}$$

$$\mathscr{D}_{\text{pref}}(\mathcal{E}; x, y; p) \leq \frac{2(N + l)}{n(\mathcal{E})} (\bar{\omega} + c_f \bar{y}) + \left(2 + \frac{\bar{\nu}}{\underline{\nu}}\right) \epsilon. \tag{18}$$

Proof Let $(x, y) \in C_{\text{cush}}(\mathcal{E}; N, \epsilon/2, \epsilon/2)$, so (x, y) cannot be cushion-blocked by any coalition of size at least N with margin $(\epsilon/2, \epsilon/2)$. Let T be a maximum-size coalition among all coalitions that can cushion-block (x, y) with margin $(\epsilon/2, \epsilon/2)$, so $|T| < N$. Define:

$$\Phi_i := \{\mathbf{0}\}, \ \forall i \in T; \tag{19}$$

$$\Phi_i := \{\mathbf{0}\} \cup \left\{ x'_i - \omega_i - \sum_{j \in J} \theta_{ij} y'_{ij} : \begin{array}{c} \mathbf{0} \leqq x'_i - (\epsilon/2)\mathbf{1} \succ_i x_i + (\epsilon/2)\mathbf{1}; \\ \forall j \in J, y'_{ij} \in Y_j \end{array} \right\}, \ \forall i \notin T; \tag{20}$$

$$\Psi := \frac{1}{n(\mathcal{E})} \sum_{i \in I} \Phi_i. \tag{21}$$

First, we claim that if $G \in \Psi$ then it is impossible to have $G \ll \mathbf{0}$. Suppose not, hence $\frac{1}{n(\mathcal{E})} \sum_{i \in I} g_i \ll \mathbf{0}$ for some $(g_i)_{i \in I} \in (\Phi_i)_{i \in I}$. Then, as in Anderson's [1] proof, the nonempty coalition $B := \{i \in I : g_i \neq \mathbf{0}\}$ cushion-blocks (x, y) with margin $(\epsilon/2, \epsilon/2)$. As $B \cap T = \varnothing$ by (19), $B \cup T$ also cushion-blocks (x, y) with the same margin. Since $|B \cup T| = |B| + |T| > |T|$, this contradicts the fact that T is a maximum-size cushion-blocking coalition.

Second, we prove existence of a separating price vector. Pick any vector z from the convex hull $\operatorname{con}\Psi$. The Shapley-Folkman theorem implies that

$$z = \frac{1}{n(\mathcal{E})}\sum_{i\in I_*} g_i + \frac{1}{n(\mathcal{E})}\sum_{i\in I\setminus I_*} g_i$$

for some $(g_i)_{i\in I}$ such that $g_i \in \operatorname{con}\Phi_i$ for all $i \in I$ and $g_i \in \Phi_i$ for all i but the set I_* such that $|I_*| \le l$. For any $i \in I \setminus T$ and any $g'_i \in \Phi_i$, (20) implies that

$$g'_i \geqq -\omega_i - \sum_{j\in J}\theta_{ij}y'_{ij} \geqq -\bar{\omega}\mathbf{1} - \bar{y}\mathbf{1}\sum_{j\in J}\theta_{ij} \geqq -(\bar{\omega}+\bar{y}c_f)\mathbf{1},$$

where the second inequality follows from Assumptions 2 and 3, and the last inequality from Assumption 4. Thus, as in Anderson, the first claim implies

$$\operatorname{con}\Psi \cap \left\{ z \in \mathbb{R}^l : z \ll -\frac{l}{n(\mathcal{E})}\left(\bar{\omega}+c_f\bar{y}\right)\mathbf{1} \right\} = \varnothing,$$

and hence there exists $p \in \Delta$ such that

$$\inf p\cdot\Psi \;\geq\; -\frac{l(\bar{\omega}+\bar{y}c_f)}{n(\mathcal{E})}; \tag{22}$$

$$\inf p\cdot\Phi_i \;\leq\; 0, \qquad \forall i \in I. \tag{23}$$

Third, with this price p, we shall prove (16)–(18). Denote

$$v_i \;:=\; p\cdot\left(x_i - \omega_i - \sum_{j\in J}\theta_{ij}y_j\right), \quad \forall i \in I;$$

$$S_* \;:=\; \{i \in I : v_i < 0\}.$$

The next inequality is the counterpart of the second displayed formula in Anderson [1, p1486]:

$$\frac{1}{n(\mathcal{E})}\sum_{i\in S_*} v_i \geq -\frac{|T|+l}{n(\mathcal{E})}(\bar{\omega}+\bar{y}c_f) - \epsilon. \tag{24}$$

To prove it, note the following facts:

$$\textstyle\sum_{i\in I\setminus S_*} v_i = -\sum_{i\in S_*} v_i; \tag{25}$$

$$v_i \geq -(\bar{\omega}+c_f\bar{y}), \quad \forall i \in I; \tag{26}$$

$$v_i \geq \inf p\cdot\Phi_i - \epsilon, \quad \forall i \in I\setminus T. \tag{27}$$

Eq. (25) follows from the aggregate feasibility of (x, y); inequality (26) follows from the fact that $-\omega_i - \sum_j \theta_{ij}y_j \geqq -(\bar{\omega}+c_f\bar{y})\mathbf{1}$; and (27) follows from the fact that, for every $i \in I \setminus T$,

$x_i + (\epsilon+\eta)\mathbf{1} - \omega_i - \sum_j \theta_{ij} y_j \in \Phi_i$ for any arbitrarily small $\eta > 0$. Now we prove (24):

$$\begin{aligned}
\frac{1}{n(\mathcal{E})}\sum_{S_*\setminus T} v_i + \frac{1}{n(\mathcal{E})}\sum_{S_*\cap T} v_i &\geq \frac{1}{n(\mathcal{E})}\sum_{S_*\setminus T}(\inf p\cdot\Phi_i - \epsilon) - \frac{1}{n(\mathcal{E})}\sum_{S_*\cap T}(\bar{\omega} + c_f\bar{y}) \\
&\geq \frac{1}{n(\mathcal{E})}\left(\sum_{S_*\setminus T}\inf p\cdot\Phi_i + \sum_{T} p\cdot\mathbf{0} + \sum_{I\setminus(S_*\cup T)}\inf p\cdot\Phi_i\right) \\
&\quad -\frac{|S_*\setminus T|}{n(\mathcal{E})}\epsilon - \frac{|S_*\cap T|}{n(\mathcal{E})}(\bar{\omega}+c_f\bar{y}) \\
&= \inf p\cdot\Psi - \frac{|S_*\setminus T|}{n(\mathcal{E})}\epsilon - \frac{|S_*\cap T|}{n(\mathcal{E})}(\bar{\omega}+c_f\bar{y}),
\end{aligned}$$

where the first inequality uses (26) and (27), the second inequality uses (23), and the equality uses the definition of Ψ and (19). Then (24) follows from (22).

Based on (24) and the fact $|T| < N$, we prove (16)–(18):

Proof of (16) By the definitions of $\mathscr{D}_{\text{bdg}}(\mathcal{E}; x, y; p)$ and v_i,

$$\begin{aligned}
\mathscr{D}_{\text{bdg}}(\mathcal{E}; x, y; p) &= \frac{1}{n(\mathcal{E})}\sum_{i\in S_*}|v_i| + \frac{1}{n(\mathcal{E})}\sum_{i\in I\setminus S_*}|v_i| \\
&= -\frac{1}{n(\mathcal{E})}\sum_{i\in S_*} v_i - \frac{1}{n(\mathcal{E})}\sum_{i\in S_*} v_i \\
&\leq \frac{2(N+l)}{n(\mathcal{E})}(\bar{\omega}+c_f\bar{y}) + 2\epsilon,
\end{aligned}$$

where the second equality follows from the definition of S_* and (25), and the inequality follows from (24). □

Proof of (17) For each firm j, let

$$\pi_j^* := \sup p\cdot Y_j := \sup_{y_j'\in Y_j} p\cdot y_j'.$$

By the same reason for (27), we have

$$\forall i \in I\setminus T, \quad p\cdot(x_i - \omega_i) - \sum_{j\in J}\theta_{ij}\pi_j^* \geq \inf p\cdot\Phi_i - \epsilon.$$

Thus, summing across $i \in I\setminus T$, we have

$$\sum_{I\setminus T} p\cdot x_i \geq \sum_{I\setminus T} p\cdot\omega_i + \sum_{I\setminus T}\sum_{j\in J}\theta_{ij}\pi_j^* + \sum_{I\setminus T}\inf p\cdot\Phi_i - \epsilon|I\setminus T|.$$

This inequality, together with the aggregate feasibility condition of (x, y), implies that

$$\begin{aligned}\sum_I p\cdot\omega_i + \sum_J p\cdot y_j \;&\geq\; \sum_T p\cdot x_i + \sum_I p\cdot\omega_i + \sum_J \pi_j^* - \left(\sum_T p\cdot\omega_i + \sum_T\sum_J \theta_{ij}\pi_j^*\right)\\ &\quad + \left(\sum_{I\setminus T}\inf p\cdot\Phi_i + \sum_T p\cdot\mathbf{0}\right) - \epsilon|I\setminus T|\\ &\geq\; \sum_I p\cdot\omega_i + \sum_J \pi_j^* - |T|(\bar\omega + c_f\bar y) + n(\mathcal{E})\inf p\cdot\Psi - n(\mathcal{E})\epsilon,\end{aligned}$$

where the last inequality follows from the fact that $p\cdot x_i \geq 0$, $-p\cdot\omega_i - \sum_j \theta_{ij}\pi_j^* \geq -(\bar\omega + c_f\bar y)$ (as for (26)), and from the definition of Φ_i. Hence inequalities (17) follows from (22). $\square$

Proof of (18) First, we claim

$$\inf p\cdot\mathbb{U}_i(x_i, y) \geq \inf p\cdot\Phi_i - \frac{\epsilon}{2}\left(\frac{\bar\nu}{\underline{\nu}} + 1\right), \quad \forall i\in I\setminus T. \tag{28}$$

This inequality follows from the fact that, for any $x_i'\in\mathbb{R}^l_+$, $u_i(x_i') > u_i(x_i)$ implies that

$$u_i\left(x_i' + \frac{\epsilon}{2}\frac{\bar\nu}{\underline{\nu}}\mathbf{1}\right) \geq u_i(x_i') + \frac{\epsilon}{2}\frac{\bar\nu}{\underline{\nu}}\underline{\nu} = u_i(x_i') + \frac{\epsilon}{2}\bar\nu > u_i(x_i) + \frac{\epsilon}{2}\bar\nu \geq u_i\left(x_i + \frac{\epsilon}{2}\mathbf{1}\right),$$

where the first and last inequalities follow from the assumption that the directional derivative $u_i'(\cdot;\mathbf{1})$ is bounded within the interval $[\underline{\nu}..\bar\nu]$ (Assumption 5).

Also note the following facts:

$$\inf p\cdot\mathbb{U}_i(x_i, y) \;\leq\; v_i, \quad \forall i\in I; \tag{29}$$

$$\inf p\cdot\mathbb{U}_i(x_i, y) \;\geq\; 0 \quad\Longrightarrow\quad i\notin S_*. \tag{30}$$

Here (29) follows from the fact that $x_i + \eta\mathbf{1} \succ_i x_i$ for any arbitrarily small $\eta > 0$, and (30) follows directly from (29) and the definition of S_*.

To prove (18), denote

$$\mathbb{U}_i := \mathbb{U}_i(x_i, y).$$

For any sentence s, let $[s] := 1$ if s is true and $[s] := 0$ if s is false. Decompose the sum

$$\begin{aligned}\sum_I |\inf p\cdot\mathbb{U}_i| \;&=\; \sum_I [\inf p\cdot\mathbb{U}_i \geq 0]\inf p\cdot\mathbb{U}_i - \sum_I [\inf p\cdot\mathbb{U}_i < 0]\inf p\cdot\mathbb{U}_i\\ &\leq\; \sum_{i\notin S_*} [\inf p\cdot\mathbb{U}_i \geq 0]\inf p\cdot\mathbb{U}_i - \sum_T [\inf p\cdot\mathbb{U}_i < 0]\inf p\cdot\mathbb{U}_i\\ &\quad - \sum_{I\setminus T} [\inf p\cdot\mathbb{U}_i < 0]\inf p\cdot\mathbb{U}_i,\end{aligned}$$

with the inequality due to (30). By (29), $\sum_{i\notin S_*}[\inf p\cdot \mathbb{U}_i \geq 0]\inf p\cdot\mathbb{U}_i \leq \sum_{i\notin S_*} v_i$; since $\mathbb{U}_i$ is bounded from below by $-\bar{\omega}-\bar{y}c_f$, we get $-\sum_T [\inf p\cdot \mathbb{U}_i < 0]\inf p\cdot\mathbb{U}_i \leq |T|(\bar{\omega}+c_f\bar{y})$; by (28), $-\sum_{I\setminus T}[\inf p\cdot\mathbb{U}_i < 0]\inf p\cdot\mathbb{U}_i \leq -\sum_{I\setminus T}[\inf p\cdot\mathbb{U}_i<0]\left(\inf p\cdot\Phi_i - \frac{\epsilon}{2}\left(\frac{\bar{\nu}}{\underline{\nu}}+1\right)\right)$. Thus,

$$\begin{aligned}
\sum_I |\inf p\cdot\mathbb{U}_i| &\leq \sum_{i\notin S_*} v_i + |T|(\bar{\omega}+c_f\bar{y}) - \sum_{I\setminus T}[\inf p\cdot\mathbb{U}_i<0]\left(\inf p\cdot\Phi_i - \frac{\epsilon}{2}\left(\frac{\bar{\nu}}{\underline{\nu}}+1\right)\right)\\
&= -\sum_{i\in S_*} v_i + |T|(\bar{\omega}+c_f\bar{y}) - \sum_{I\setminus T}[\inf p\cdot\mathbb{U}_i<0]\inf p\cdot\Phi_i\\
&\quad -\sum_{I\setminus T}[\inf p\cdot\mathbb{U}_i\geq 0]\,0 - \sum_T p\cdot\mathbf{0} + \sum_{I\setminus T}[\inf p\cdot\mathbb{U}_i<0]\frac{\epsilon}{2}\left(\frac{\bar{\nu}}{\underline{\nu}}+1\right)\\
&\leq -\sum_{i\in S_*} v_i + |T|(\bar{\omega}+c_f\bar{y}) - |I|\inf p\cdot\Psi + |I|\frac{\epsilon}{2}\left(\frac{\bar{\nu}}{\underline{\nu}}+1\right)\\
&\leq (|T|+l)(\bar{\omega}+c_f\bar{y}) + |I|\epsilon + |T|(\bar{\omega}+c_f\bar{y}) + l(\bar{\omega}+c_f\bar{y}) + |I|\frac{\epsilon}{2}\left(\frac{\bar{\nu}}{\underline{\nu}}+1\right)\\
&< 2(|T|+l)(\bar{\omega}+c_f\bar{y}) + \left(2+\frac{\bar{\nu}}{\underline{\nu}}\right)\epsilon|I|.
\end{aligned}$$

The equality follows from (25). The second inequality follows from (23) and the definitions of Φ_i and Ψ; the third inequality follows from (24) and (22). Dividing this chain of inequalities by $n(\mathcal{E})$, we obtain (18). □

Hence the lemma is proved. ■

3.3 Proof of the Theorem

For any disparity bound $d > 0$ and any $\eta > 0$, pick any sufficiently small $\epsilon > 0$ such that

$$\left(2+\frac{\bar{\nu}}{\underline{\nu}}\right)\epsilon < \frac{\eta}{2}.$$

By Lemma 3, there exists a sufficiently large integer N such that, in any economy $\mathcal{E}$ with $n(\mathcal{E}) \geq N$, $C^d(\mathcal{E}) \subseteq C_{\text{cush}}(\mathcal{E}; N, \epsilon/2, \epsilon/2)$. Then pick an integer N_* so large that

$$\frac{2(N+l)}{N_*}(\bar{\omega}+c_f\bar{y}) < \frac{\eta}{2}.$$

By Lemma 4, we have, in any economy $\mathcal{E}$ with size $n(\mathcal{E}) > N_*$,

$$\mathscr{D}^d(\mathcal{E}) \leq \frac{2(N+l)}{n(\mathcal{E})}(\bar{\omega}+c_f\bar{y}) + \left(2+\frac{\bar{\nu}}{\underline{\nu}}\right)\epsilon < \frac{\eta}{2}+\frac{\eta}{2} = \eta.$$

This proves the theorem. ■

A The Proof of Lemma 1

By Assumption 4 (big world), an individual can have positive winning probabilities in at most c_f firms. Thus, in any economy $\mathcal{E}$, for any individual i and for any $(y_{ij})_{j\in J} \in (Y_j)_{j\in J}$, the sum $\sum_{j\in J} \mathbf{z}_j^i y_{ij}$ (defined in (1)) contains at most c_f nondegenerate terms. By Assumption 1,

$$\mathbb{E} \sum_{j\in J} \mathbf{z}_j^i y_{ij} = \sum_{j\in J} \theta_{ij} y_{ij}. \tag{31}$$

By the same assumption, for any two individuals i and m in any economy and for any $(y_{ij})_{j\in J}, (y_{mj})_{j\in J} \in (Y_j)_{j\in J}$, the two sums $\sum_{j\in J} \mathbf{z}_j^i y_{ij}$ and $\sum_{j\in J} \mathbf{z}_j^m y_{mj}$ are correlated *only* if i and m are *related* in the sense that $\theta_{ij}\theta_{mj} > 0$ for some firm j. By Assumption 4, an individual can have positive winning probabilities in at most c_f firms, and each firm can have at most c_s such individuals; thus, for any individual, the number of individuals to whom he is related is at most

$$b := c_f c_s.$$

Pick any coordinate $\ell = 1, \ldots, l$, so that the ℓth coordinate of a vector say y_{ij} is denoted by $(y_{ij})_\ell$. For any individual i and for any profile $(y_{ij})_{j\in J} \in (Y_j)_{j\in J}$ of production plans, let

$$h_i := \sum_{j\in J} \mathbf{z}_j^i (y_{ij})_\ell - \mathbb{E} \sum_{j\in J} \mathbf{z}_j^i (y_{ij})_\ell.$$

Since $\sum_{j\in J} \mathbf{z}_j^i (y_{ij})_\ell$ has at most c_f nondegenerate terms and each term is uniformly bounded in the interval $[-\bar{y}..\bar{y}]$ across all i and all economies (Assumptions 3 and 4), we have, for any individuals i and m in any economy and for any $(y_{ij})_{j\in J}, (y_{mj})_{j\in J} \in (Y_j)_{j\in J}$,

$$\mathbb{E}\left[h_i h_m\right] \begin{cases} = 0 & \text{if } i \text{ and } m \text{ are not related} \\ \leq c_f \bar{y}^2 & \text{if } i \text{ and } m \text{ are related.} \end{cases} \tag{32}$$

For any sentence s, denote $[s] = 1$ if s is true and $[s] = 0$ otherwise. For any $n = 1, 2, \ldots$, for any n distinct individuals $i_1, \ldots, i_n$ in any economy, and for any $((y_{i_k j})_{j\in J})_{k=1}^n \in \left(\prod_{j\in J} Y_j\right)^n$,

$$\mathbb{E}\left[\left(\sum_{k=1}^n h_{i_k}\right)^2\right] = \sum_{k=1}^n \sum_{m=1}^n [i^k \text{ and } i^m \text{ are related}] \mathbb{E}\left[h_{i_k} h_{i_m}\right] \leq n b c_f \bar{y}^2$$

by (32) and the fact that an individual i_k is related to at most $b = c_f c_s$ individuals. It follows from Chebyshev's inequality that

$$\forall \epsilon > 0,\ \text{Prob}\left(\left|\sum_{k=1}^n h_{i_k}\right| \geq n\epsilon\right) \leq \frac{1}{(n\epsilon)^2} \mathbb{E}\left[\left(\sum_{k=1}^n h_{i_k}\right)^2\right] \leq \frac{n b c_f \bar{y}^2}{n^2 \epsilon^2} = \frac{b c_f \bar{y}^2}{n \epsilon^2}.$$

Thus, for any $\epsilon > 0$ and any $\alpha \in (0..1)$, pick $N := bc_f\bar{y}^2/(\epsilon^2(1-\alpha)/l)$. Then for any $n \geq N$ and for any n distinct individuals $i_1, \ldots, i_n$ in any economy,

$$\text{Prob}\left(\left|\sum_{k=1}^{n}\sum_{j\in J}\mathbf{z}_j^{i_k}(y_{i_kj})_\ell - \mathbb{E}\sum_{j\in J}\mathbf{z}_j^{i_k}(y_{i_kj})_\ell\right| \geq n\epsilon\right) = \text{Prob}\left(\left|\sum_{k=1}^{n}h_{i_k}\right| \geq n\epsilon\right) \leq \frac{bc_f\bar{y}^2}{n\epsilon^2} \leq \frac{1-\alpha}{l}.$$

As this is true for every coordinate $\ell = 1, \ldots, l$, it follows from (31) that

$$\begin{aligned}
&\text{Prob}\left(\frac{1}{n}\left\|\sum_{k=1}^{n}\sum_{j\in J}\left(\mathbf{z}_j^{i_k}y_{kj} - \theta_{i_kj}y_{kj}\right)\right\|_\infty < \epsilon\right) \\
= \;& 1 - \text{Prob}\left(\bigcup_{\ell=1}^{l}\left\{\left(\left(\mathbf{z}_j^{i_k}\right)_{k=1}^{n}\right)_{j\in J} : \frac{1}{n}\left|\sum_{k=1}^{n}\sum_{j\in J}\mathbf{z}_j^{i_k}(y_{i_kj})_\ell - \mathbb{E}\sum_{j\in J}\mathbf{z}_j^{i_k}(y_{i_kj})_\ell\right| \geq \epsilon\right\}\right) \\
\geq \;& 1 - \sum_{\ell=1}^{l}\frac{1-\alpha}{l} \\
= \;& \alpha.
\end{aligned}$$

Hence (12) is true. ■

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