Deconstruction of Real-Linear Surjective Isometries Over Complex Vector Spaces

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Abstract

Let V and W be complex inner product spaces; let $T: V \to W$ be a surjective real-linear isometry. There exist unique mappings T_1 and T_2, which are complex-linear and conjugate-linear respectively, such that $T = T_1 + T_2$. We use this deconstruction in a new proof of the characterization of the isometries on the complex plane. Furthermore, we present conditions under which $V = \ker(T_1) \oplus \ker(T_2)$.

Terminology

Let V, W be complex inner product spaces and $T: V \to W$ a function.

- T is additive if for all $x, y \in V$ then $T(x+y) = T(x) + T(y)$
- T is real-linear if T is additive and for all $x \in V, \lambda \in \mathbb{R}$ then $T(\lambda x) = \lambda T(x)$
- T is complex-linear if T is additive and for all $x \in V, \lambda \in \mathbb{C}$ then $T(\lambda x) = \lambda T(x)$
- T is conjugate-linear if T is additive and for all $x \in V, \lambda \in \mathbb{C}$ then $T(\lambda x) = \bar{\lambda} T(x)$
- T is a surjection if for all $w \in W$, there exists $x \in V$ such that $T(x) = w$
- T is an isometry if for all $x \in V$ then $\|x\| = \|T(x)\|$.
- The kernel of T, denoted as $\ker(T)$, is the set of elements that are mapped to zero under T. Specifically, $\ker(T) = \{x \in V : T(x) = 0\}$
- Suppose that V_1 and V_2 are subspaces of V. Then $V = V_1 \oplus V_2$ if $V = V_1 + V_2$ and $V_1 \cap V_2 = \{0\}$.

Deconstruction

As observed in [1], any real-linear surjective function T can be decomposed uniquely into complex-linear and conjugate-linear mappings. Consider the mappings $T_1, T_2 : V \to W$, defined as follows:

$$T_1(x) = \frac{T(x) - iT(ix)}{2} \text{ and } T_2(x) = \frac{T(x) + iT(ix)}{2}$$

It turns out that T_1 is complex-linear and T_2 is conjugate-linear, and by inspection it follows immediately that

$$T(x) = T_1(x) + T_2(x)$$

Goal

The ultimate goal here is to prove that the domain of T splits up in a "nice" way, namely that $V = \ker(T_1) \oplus \ker(T_2)$. In [1] it is shown that

$$\ker(T_1) \cap \ker(T_1) = \{0\}$$

Thus in order to prove that $V = \ker(T_1) \oplus \ker(T_2)$, it suffices to show that $V = \ker(T_1) + \ker(T_2)$. In [1] there is a proof of the statement that

$$V = \ker(T_1) + \ker(T_2) \text{ if and only if } T_1[V] \cap T_2[V] = \{0\}$$

If we are able to show that $T_1[V] \cap T_2[V] = \{0\}$, then it follows immediately that $V = \ker(T_1) \oplus \ker(T_2)$

Example

As an example of this behavior, consider the isometric, surjective, real-linear mapping

$$T: \mathbb{C}^2 \to \mathbb{C}^2, T(z, w) = (z, \bar{w})$$

Then by using the deconstruction, we have that

$$T_1(z, w) = (z, 0) \text{ and } T_2(z, w) = (0, \bar{w})$$

Observe that

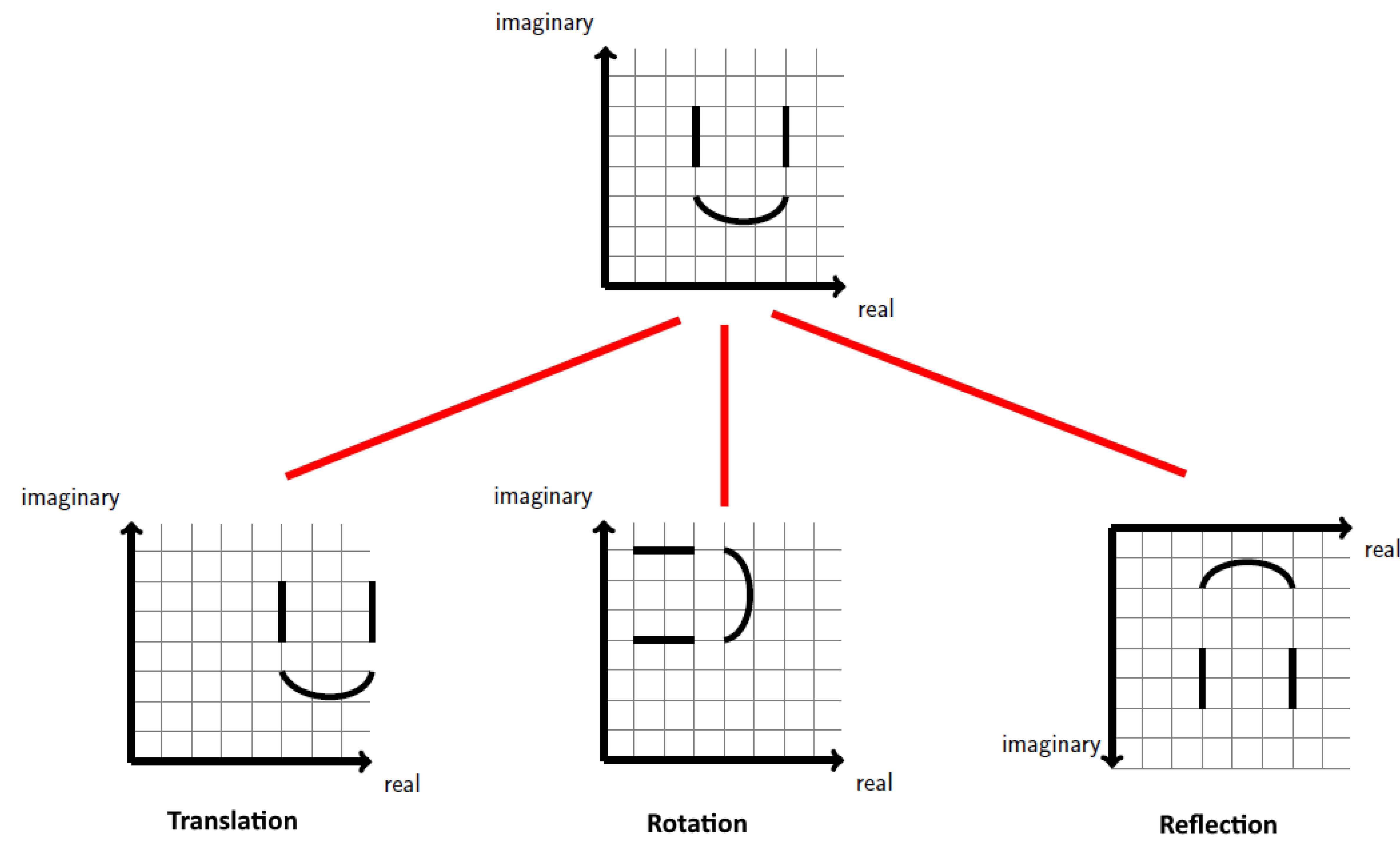
$$\ker(T_1) = \{0\} \times \mathbb{C} \text{ and } \ker(T_2) = \mathbb{C} \times \{0\}$$

Thus, it is indeed true that

$$\mathbb{C} \times \mathbb{C} = \{0\} \times \mathbb{C} \oplus \mathbb{C} \times \{0\} = \ker(T_1) \oplus \ker(T_2)$$

A Characterization of Isometries on the Complex Plane

It is a well-known fact that if $T: \mathbb{C} \to \mathbb{C}$ is an isometry, then there exists $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = 1$ such that $T(z) = \alpha z + \beta$ or $T(z) = \alpha \bar{z} + \beta$. In a more geometric sense, this means that every isometry of the complex plane is either a reflection and a translation, or a rotation and reflection. By using the machinery of the discussed deconstruction, we were able to construct a new proof of this characterization.



Proof Sketch

For the given T, define a new function $f: \mathbb{C} \to \mathbb{C}$ by $f(z) = T(z) - T(0)$. Then f is an isometry and it holds that $f(0) = 0$, and because an isometry between finite dimensional vector spaces must be surjective, then by Majur-Ulam [3] it holds that f is a real-linear mapping. By applying our deconstruction we know that there exists f_1 and f_2 and constants α_1, α_2 such that

$$f_1(z) = \alpha_1 z \text{ and } f_2(z) = \alpha_2 \bar{z}$$

Then we have that

$$f(z) = f_1(z) + f_2(z) = \alpha_1 z + \alpha_2 \bar{z}$$

By further argumentation we can show that either $\alpha_1 = 0$ or $\alpha_2 = 0$, so it follows that

$$f(z) = \alpha_1 z \text{ or } f(z) = \alpha_2 \bar{z}$$

Recalling that $f(z) = T(z) - T(0)$, then it follows that in fact

$$T(z) = \alpha_1 z + T(0) \text{ or } T(z) = \alpha_2 \bar{z} + T(0)$$

Results

Even though this problem can be stated in regards to general complex normed vector spaces, we chose to focus on the case where the norm comes from an inner product. Because we had the powerful tool of being able to look at inner products, we thought it was prudent to attempt to relate the inner product on V with the inner product on W. This can be done by using the following remarkable formula (where $x, y \in V$):

$$\langle x, y \rangle = \langle T_1(x), T_1(y) \rangle + \overline{\langle T_2(x), T_2(y) \rangle}$$

It makes sense to combine this result with another formula that states that for all $x \in V$ then

$$\langle T_1(x), T_2(x) \rangle = 0$$

Recall that in order to prove that $V = \ker(T_1) \oplus \ker(T_2)$ it is sufficient to show that $T_1[V] \cap T_2[V] = \{0\}$. Thus if x, y satisfy the property that $T_1(x) = T_2(y)$, then we want to show that $x = y = 0$. Let x, y satisfy $T_1(x) = T_2(y)$. Applying both of the above two results yields that

$$\langle x, y \rangle = \langle T_2(y), T_1(y) \rangle + \overline{\langle T_2(x), T_1(x) \rangle} = 0$$

Thus it holds that x and y must be orthogonal. Further argumentation shows that this orthogonality is preserved under T.
Changing angles here a bit, we note that T^{-1} is also a real-linear isometric mapping. Given the fruitful results that came from looking at the deconstruction of T, it is only natural to apply this same decomposition strategy to T^{-1}. We can then look at the maps $S_1, S_2 : W \to V$ defined as

$$S_1(x) = \frac{T^{-1}(x) - iT^{-1}(ix)}{2} \text{ and } S_2(x) = \frac{T^{-1}(x) + iT^{-1}(ix)}{2}$$

We were able to construct a variety of useful formulas using the inverse mapping decomposition. For $x, y \in V$, then the following are true:

- $\langle x, S_1 T_2(x) \rangle = 0$
- $\langle x, S_2 T_1(x) \rangle = 0$
- $\langle x, S_1 T_1(y) \rangle = \overline{\langle T_1(x), T_1(y) \rangle}$
- $\langle x, S_2 T_2(y) \rangle = \overline{\langle T_1(x), T_1(y) \rangle}$
- $S_1 T_1(x) + S_2 T_2(x) = x$
- $S_1 T_2(x) + S_2 T_1(x) = 0$

Further Work

We have already shown that if $x, y \in V$ satisfy $T_1(x) = T_2(y)$, then $x \perp y$. If we were able to prove that not only are x and y orthogonal but in fact $x = y = 0$, then we would have the desired result that $V = \ker(T_1) \oplus \ker(T_2)$. However, it is important to recognize that because this argument heavily relies on the fact that V and W are endowed with inner products, then it does not necessarily hold for more general spaces. In further work, we would then hope to generalize this result for any normed complex vector space.

References

- [1] C. Dziak, S. Lambert, A. Luttman "Characterizing Surjective Isometries Between Complex Normed Vector Spaces: A Complex Mazur-Ulam Theorem" (Preprint)
- [2] T. Miura "Surjective isometries between function spaces," Contemp. Math., Vol. 645 (2015), pp. 231-240.
- [3] S. Mazur, S. Ulam, Surles transformationes isometriques despaces vectoriels normes, C. R. Acad. Sci. 194 (1932) 946-948.

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