Deconstruction of Real-Linear Surjective Isometries Over Complex Vector Spaces

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Abstract
Let V and W be complex inner product spaces; let T : V → W be a surjective real-linear isometry. There exist unique mappings T1 and T2, which are complex-linear and conjugate-linear respectively, such that T = T1 + T2. We use this deconstruction in a new proof of the characterization of the isometries on the complex plane. Furthermore, we present conditions under which V = ker(T1) ⊕ ker(T2).

Terminology
Let V, W be complex inner product spaces and T : V → W a function.

- T is additive if for all x, y ∈ V then T(x + y) = T(x) + T(y).
- T is real-linear if T is additive and for all x ∈ V, λ ∈ R then T(λx) = λT(x).
- T is complex-linear if T is additive and for all x ∈ V, λ ∈ C then T(λx) = λT(x).
- T is conjugate-linear if T is additive and for all x ∈ V, y ∈ V then T(x + iy) = T(x) + iT(y).
- T is an isometry if for all x ∈ V there exists x ∈ V such that T(x) = w.
- T is an isometry if for all x ∈ V then |x| = |T(x)|.
- The kernel of T, denoted as ker(T), is the set of elements that are mapped to zero under T. Specifically, ker(T) = {x ∈ V : T(x) = 0}.
- Suppose that V1 and V2 are subspaces of V. Then V = V1 ⊕ V2 if V = V1 + V2 and V1 ∩ V2 = {0}.

Deconstruction
As observed in [1], any real-linear surjective function T can be decomposed uniquely into complex-linear and conjugate-linear mappings. Consider the mappings T1, T2 : V → W, defined as follows:

T1(x) = \frac{x + iT(x)}{\sqrt{2}} and T2(x) = \frac{x + iT(x)}{\sqrt{2}}

It turns out that T2 is complex-linear and T2 is conjugate-linear, and by inspection it follows immediately that

T(x) = T1(x) + T2(x)

Goal
The ultimate goal here is to prove that the domain of T splits up in a "nice" way, namely that V = ker(T1) ⊕ ker(T2). In [3], it is shown that ker(T1) ⊕ ker(T2) = {0}.

Thus in order to prove that V = ker(T1) ⊕ ker(T2), it suffices to show that V = ker(T1) + ker(T2). In [1] there is a proof of the statement that V = ker(T1) + ker(T2) if and only if ker(T1)[V] ∩ ker(T2)[V] = {0}.

If we are able to show that T1[V] ∩ T2[V] = {0}, then it follows immediately that V = ker(T1) ⊕ ker(T2).

Example
As an example of this behavior, consider the isometric, surjective, real-linear mapping T : C^2 → C^2, T(z, w) = (z, w).

Then by using the deconstruction, we have that

T1(z, w) = \frac{z + w}{\sqrt{2}} and T2(z, w) = \frac{z - w}{\sqrt{2}}

Observe that ker(T1) = {0} ⊕ C and ker(T2) = C(x).

Thus, it is indeed true that

C ⊕ C = {0} ⊕ C ⊕ C = ker(T1) ⊕ ker(T2)

A Characterization of Isometries on the Complex Plane
It is a well-known fact that if T : C → C is an isometry, then there exists α, β ∈ C with |α| = 1 such that T(z) = αz + β or T(z) = αz + β. In a more geometric sense, this means that every isometry of the complex plane is either a reflection and a translation, or a rotation and reflection. By using the machinery of the discussed deconstruction, we were able to construct a new proof of this characterization.

Proof Sketch
For the given T, define a new function f : C → C by f(z) = T(z) − T(0). Then f is an isometry and it holds that f(0) = 0, and because an isometry between finite dimensional vector spaces must be surjective, then by Majza-Ulam [3] it holds that f is a real-linear mapping. By applying our deconstruction we know that there exists f1 and f2 and constants α1, α2 such that

f1(z) = α1z and f2(z) = α2z

Then we have that

f(z) = f1(z) + f2(z) = α1z + α2z

By further argumentation we can show that either α1 = 0 or α2 = 0, so it follows that

f(z) = α1z or f(z) = α2z

Recalling that f(z) = T(z) − T(0), then it follows that in fact

T(z) = α1z + T(0) or T(z) = α2z + T(0)

Results
Even though this problem can be stated in regards to general complex normed vector spaces, we chose to focus on the case where the norm comes from an inner product. Because we had the powerful tool of being able to look at inner products, we thought it was prudent to attempt to relate the inner product on V with the inner product on W. This can be done by using the following remarkable formula (where x, y ∈ V):

⟨T(x), T(y)⟩ = ⟨T1(x), T1(y)⟩ + ⟨T2(x), T2(y)⟩

It makes sense to combine this result with another formula that states that for all x, y ∈ V then

⟨T(x), T(y)⟩ = 0

Recall that in order to prove that V = ker(T1) ⊕ ker(T2) it is sufficient to show that T1[V] ∩ T1[V] = {0}. Thus if x, y satisfy the property that T1(x) = T1(y), then we want to show that x − y = 0. Let x, y satisfy T1(x) = T1(y). Applying both of the above two results yields that

⟨x, y⟩ = ⟨T1(x), T1(y)⟩ + ⟨T2(x), T2(y)⟩ = 0

Thus it holds that x and y must be orthogonal. Further argumentation shows that this orthogonality is preserved under T.

Changing angles here a bit, we note that T^{-1} is also a real-linear isometric mapping. Given the fruitful results that came from looking at the deconstruction of T, it is only natural to apply this same decomposition strategy to T^{-1}. We can then look at the maps S1, S2 : W → V defined as

S1(x) = \frac{T^{-1}}{2}(x) − \frac{iT^{-1}(x)}{2} and S2(x) = \frac{T^{-1}}{2}(x) + \frac{iT^{-1}(x)}{2}

We were able to construct a variety of useful formulas using the inverse mapping decomposition. For x, y ∈ V, then the following are true:

1. ⟨x, S1(x)⟩ = 0
2. ⟨x, S1(x)⟩ = 0
3. ⟨x, S2(x)⟩ = 0
4. ⟨x, S2(x)⟩ = 0
5. ⟨x, S2(x)⟩ = 0
6. ⟨x, S2(x)⟩ = 0
7. ⟨x, S2(x)⟩ = 0
8. ⟨x, S2(x)⟩ = 0
9. ⟨x, S2(x)⟩ = 0
10. ⟨x, S2(x)⟩ = 0

Further Work
We have already shown that if x, y ∈ V satisfy T1(x) = T1(y), then x = y. If we were able to prove that not only are x and y orthogonal but in fact x = y, then we would have the desired result that V = ker(T1) ⊕ ker(T2). However, it is important to recognize that because this argument heavily relies on the fact that V and W are endowed with inner products, then it does not necessarily hold for more general spaces. In further work, we would then hope to generalize this result for any normed complex vector space.

References

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