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Conditions under which adjustability lowers the cost of a robust linear program

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Keywords

Robust counterpart, Adjustable robust counterpart, Affinely adjustable robust counterpart, Box uncertainty sets

Disciplines

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Conditions under which adjustability lowers the cost of a robust linear program

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Abstract The adjustable robust counterpart (ARC) of an uncertain linear program extends the robust counterpart (RC) by allowing some decision variables to adjust to the realizations of some uncertain parameters. The ARC may produce a less conservative and costly solution than the RC does but cases are known in which it does not. While the literature documents some examples of cost savings provided by adjustability (particularly affine adjustability), it is not straightforward to determine in advance whether they will materialize. The affine adjustable robust counterpart, while having a tractable structure, still may be much larger than the original problem. We establish conditions under which affine adjustability may lower the optimal cost with a numerical condition that can be checked in small representative instances. As demonstrated in applications, the conditions provide insights into constraint relationships that allow adjustability to have its intended effect.

Keywords Robust Counterpart · Adjustable Robust Counterpart · Affinely Adjustable Robust Counterpart · Box Uncertainty Sets

1 Introduction

Robust optimization is a modeling strategy in which an uncertainty set describes the possible values of some parameters of a mathematical program. The goal in this optimization approach is to find a best solution that is feasible for all parameter values within the uncertainty set. In the original formulation by Soyster (1972) the solution was often observed to be very conservative and costly. The

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approach was further developed by Ben-Tal and Nemirovski (1998, 1999, 2000) as well as El Ghaoui and Lebret (1997) and El Ghaoui et al (1998) independently. These papers proposed tractable solution approaches to special cases of the robust counterpart (RC) in the form of conic quadratic problems with less conservative results.

In the RC formulation, the values of all decision variables are determined before the realization of uncertain parameters (i.e., treated as "here and now" decisions). However, there are applications in which some variables, including auxiliary variables such as slack or surplus variables ("wait and see" decisions), could be decided after realization of (some of) the uncertain parameters. Ben-Tal et al (2004) proposed an adjustable robust counterpart (ARC) for models with adjustable variables that tune themselves to the realized values of the uncertain parameters. They introduced the ARC concept with two types of recourse: fixed, where the coefficients of adjustable variables are deterministic; and uncertain, where they are not. Because ARC formulations may not be computationally tractable, they also proposed an affinely adjustable robust counterpart (AARC) to approximate the ARC by restricting the adjustable variables to be affine functions of the uncertain parameters. Similar techniques of considering linear adjustability to uncertain parameters have also been employed for tractability in linear stochastic optimization under the label of linear decision rules such as in Kuhn et al (2009); Chen and Zhang (2009); Bertsimas et al (2010, 2013).

The ARC formulation is appealing because it avoids unnecessary conservatism by allowing adjustability, but it is generally harder than the RC to solve. The AARC, while having a tractable structure, still may be much larger and more time-consuming to solve than the RC. The challenge in applications is that it is not always straightforward to determine when the ARC or AARC might be less conservative than the RC formulation and, thus, worth the additional effort. We assume a minimization objective and denote the optimal objective value for a "model" by Z_{model}. Published applications where $Z_{AARC} < Z_{RC}$ include project management (Cohen et al, 2007), inventory control (Ben-Tal et al, 2009; de Ruiter et al, 2016; Buhayenko and den Hertog, 2017), telecommunication (Ouorou, 2013), production planning (Solyali, 2014), renewable energy (Liu and Gao, 2017) and supply chain network design (Haddadsisakht and Ryan, 2018). But several papers establish conditions under which $Z_{ARC} = Z_{RC}$ or $Z_{AARC} = Z_{RC}$. Ben-Tal et al (2004), Bertsimas and Goyal (2010), Bertsimas et al (2011), Bertsimas and Goyal (2013), Bertsimas et al (2015), Marandi and den Hertog (2017), and Awasthi et al (2018) defined conditions under which $Z_{ARC} = Z_{RC}$. In some important applications that are not covered by these papers' assumptions, we find $Z_{AARC} = Z_{RC}$, while $Z_{AARC} < Z_{RC}$ in others. Although various methods to reduce the conservatism of the RC formulation are used in applications (Zokaee et al, 2014; Thorsen and Yao, 2015; Kang et al, 2015; Orgut et al, 2018), the ARC formulation is not commonly applied because of its intractability and doubts about the value of adjustability.

The goal of this paper is to help determine whether an adjustability gap exists; i.e., $Z_{ARC} < Z_{RC}$, under less restrictive assumptions than exist in related papers. The existence of such a gap could motivate the use of the ARC or more tractable AARC formulation in additional applications. Because $Z_{ARC} \leq Z_{AARC}$, we study conditions under which $Z_{AARC} < Z_{RC}$ as a sufficient condition for $Z_{ARC} < Z_{RC}$. Our conditions include the presence of at least two binding constraints at optimality of the RC formulation, and an adjustable variable in both constraints with implicit bounds from above and below with different extreme values in the uncertainty set. Using the dual of the RC, which is explored in Beck and Ben-Tal (2009), we show how RC formulations can be tested in small instances to identify whether affine adjustability lowers the optimal cost. In this paper, we restrict attention to models with fixed recourse and box uncertainty sets.

In the next section, the required preliminary definitions and explanations are presented. Section 3 provides the proposition in detail with illustrative instances. Examples taken from applications in the literature are illustrated in Section 4. Conclusions and future research directions are provided in Section 5.

2 Preliminaries

Consider a linear program (LP):

$$\min_{w\geq 0} c^T w : \quad A'w \leq b, \tag{1}$$

where $w \in \mathbb{R}^n_+, c \in \mathbb{R}^n, A' \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$. The RC of (1) was proposed by Ben-Tal et al (2004) as follows:

$$\min_{w\geq 0} \max_{\zeta\in\mathcal{Z}} \left\{ c^T w : \quad A'w - b \leq 0, \quad \forall \zeta = [c, A', b] \in \mathcal{Z} \right\},$$

where $\mathcal{Z} \subset \mathbb{R}^n \times \mathbb{R}^{m\times n} \times \mathbb{R}^m$ is a given uncertainty set. We can decompose the decision variables w into non-adjustable variables x and adjustable variables y. In addition, if the costs of some non-adjustable variables are affected by uncertainty then we reformulate as in (2) to move all uncertainty to the constraints:

$$\min_{u,x,y\geq 0} \left\{ u : c_x^T x + c_y^T y - u \leq 0, Ax + Dy \leq b,\ \forall \zeta = [c, A, D, b] \in \mathcal{Z} \right\}, \tag{2}$$

where $x \in \mathbb{R}^{n-p}_+, y \in \mathbb{R}^p_+, A \in \mathbb{R}^{m\times(n-p)}, D \in \mathbb{R}^{m\times p}, b \in \mathbb{R}^m, \mathcal{Z} \subset \mathbb{R}^n \times \mathbb{R}^{m\times(n-p)} \times \mathbb{R}^{m\times p} \times \mathbb{R}^m$. Upon this reformulation (if necessary), we can state the robust counterpart as:

$$Z_{RC} = \min_{x,y\geq 0} \left\{ c_x^T x + c_y^T y : Ax + Dy \leq b,\ \forall \zeta = [A, D, b] \in \mathcal{Z} \right\}. \tag{3}$$

Henceforth, we assume all uncertain parameters appear in the constraints. The ARC corresponding to (3), where the adjustable variable y is decided after realization of the uncertain parameters, is:

$$Z_{ARC} = \min_{x,y(\zeta)\geq 0, \forall \zeta\in\mathcal{Z}} \left\{ c_x^T x + \max_{\zeta\in\mathcal{Z}} c_y^T y(\zeta) : Ax + Dy(\zeta) \leq b,\ \forall \zeta = [A, D, b] \in \mathcal{Z} \right\}. \tag{4}$$

Ben-Tal et al (2004) assumed, without loss of generality, that the uncertainty set $\mathcal{Z}$ is affinely parameterized by a perturbation vector ξ varying in a given non-empty convex compact perturbation set $\chi \subset \mathbb{R}^L$:

$$\mathcal{Z} = \left\{ [A, D, b] = \left[A^0, D^0, b^0\right] + \sum_{l=1}^{L} \xi^l \left[A^l, D^l, b^l\right] : \xi \in \chi \right\}. \tag{5}$$

In the case of fixed recourse, the coefficients of the adjustable variables are deterministic (i.e., $D^l = 0$ for $l = 1, ..., L$). If we define $a_i^l \in \mathbb{R}^{n-p}$ as the i^{th} row of A^l, $d_i \in \mathbb{R}^p$ as the i^{th} row of D^0 and $b_i^l \in \mathbb{R}$ as the i^{th} element of vector b^l, the RC formulation with fixed recourse is as follows:

$$Z_{RC} = \min_{x,y \geq 0} \left\{ c_x^T x + c_y^T y : \left(a_i^0 + \sum_{l=1}^{L} \xi^l a_i^l \right) x + d_i y \leq b_i^0 + \sum_{l=1}^{L} \xi^l b_i^l, \ \forall \xi \in \chi, \ i = 1, ..., m \right\}, \tag{6}$$

and the fixed recourse version of ARC is:

$$Z_{ARC} = \min_{x, y(\xi) \geq 0, \forall \xi \in \chi} \left\{ c_x^T x + \max_{\xi \in \chi} c_y^T y(\xi) : \left(a_i^0 + \sum_{l=1}^{L} \xi^l a_i^l \right) x + d_i y(\xi) \leq b_i^0 + \sum_{l=1}^{L} \xi^l b_i^l, \ \forall \xi \in \chi, \ i = 1, ..., m \right\}. \tag{7}$$

The AARC is an approximation of the ARC in which the adjustable variables are restricted to be affine functions of the uncertain parameters. In this approximation, if $\mathcal{Z}$ is affinely parameterized as defined in equation (5), the adjustable variables y are restricted to affinely depend on ξ:

$$y = \pi^0 + \sum_{l=1}^{L} \xi^l \pi^l \geq 0, \tag{8}$$

where $\pi^l \in \mathbb{R}^p$ for $l = 0, ..., L$. The fixed recourse AARC formulation corresponding to (7) is:

$$Z_{AARC} = \min_{x \geq 0, \pi} \left\{ c^T x + \max_{\xi \in \chi} c_y^T \left(\pi^0 + \sum_{l=1}^{L} \xi^l \pi^l \right) : \left(a_i^0 + \sum_{l=1}^{L} \xi^l a_i^l \right) x + d_i \left(\pi^0 + \sum_{l=1}^{L} \xi^l \pi^l \right) \leq b_i^0 + \sum_{l=1}^{L} \xi^l b_i^l, \ \forall \xi \in \chi, \quad i = 1, ..., m; \ \pi^0 + \sum_{l=1}^{L} \xi^l \pi^l \geq 0, \ \forall \xi \in \chi \right\}. \tag{9}$$

In practice, π^l would be forced to equal zero if y were not adjustable to the l^{th} perturbation for some $l \in \{1, ..., L\}$. The AARC (9) is computationally tractable.

Even when the coefficients of the adjustable variables are uncertain, it can be approximated by an explicit semi-definite program if the uncertainty set is an intersection of concentric ellipsoids (Ben-Tal et al, 2004). However, because of its computational challenge, the AARC formulation is not considered in this paper. In addition, only box uncertainty sets (10) are considered here to avoid the complexity of interactions among uncertainties. That is, we define

$$\chi = \left\{ \xi : |\xi^l| \leq \rho^l, l = 1, ..., L \right\}, \tag{10}$$

where, without loss of generality, we assume $\rho^l = 1$ for all $l = 1, ..., L$.

3 Conditions for $Z_{ARC} < Z_{RC}$

Because $Z_{ARC} \leq Z_{AARC} \leq Z_{RC}$, conditions under which $Z_{AARC} < Z_{RC}$ are sufficient for $Z_{ARC} < Z_{RC}$ as well. The behavior of the solution to the AARC formulation depends on how the uncertain parameters interact in the RC constraints. As detailed below, adjustability may lower the cost if there are at least two constraints that are binding at an optimal RC solution for different values of the same uncertain parameter. In addition, a decision variable that could be made adjustable appears in both constraints, one of which bounds it from above at one extreme of the uncertainty set and the other bounds it from below at the opposite extreme of the uncertainty set. By allowing the variable to adjust to that uncertain parameter, there is a possible improvement from using AARC formulation and, therefore, the more general ARC formulation.

Several papers provided conditions for the absence of an adjustability gap or provided bounds on Z_{ARC} based on Z_{RC}. Ben-Tal et al (2004) and Marandi and den Hertog (2017) proved that for models with constraint-wise uncertainty, $Z_{RC} = Z_{ARC}$. But they did not explicate how the interaction of the same uncertain parameter in separate constraints might allow adjustability to lower the optimal cost.

In other papers, some limitations prevent identification of models with unequal values of Z_{RC} and Z_{ARC}. Bertsimas and Goyal (2010) and Bertsimas et al (2011) approximated a two-stage stochastic model and an adjustable robust counterpart along with the robust counterpart. They considered both objective coefficient and constraint right-hand side uncertainty and proved that, for a hypercube uncertainty set when uncertainty is in the objective and right-hand side, the robust solution is equal to the fully adjustable solution. Using a generalized notion of symmetry for general convex uncertainty sets, Bertsimas et al (2011) extended the Bertsimas and Goyal (2010) static robust solution performance in two-stage stochastic optimization problems. Bertsimas and Goyal (2011) also compared the optimal affine policy to the optimal fully adaptable solution but did not compare RC and ARC. These studies are limited by assumptions of nonnegative right-hand sides of non-strict “greater than” constraints, which prevents them from modeling upper bounds on decision variables.

Bertsimas and Goyal (2013), Bertsimas et al (2015) and Awasthi et al (2018) extended models to allow uncertainty in both constraint and objective coefficients. They approximated the Z_{ARC} with the Z_{RC} to handle packing constraints such

Table 1 The limitations considered in the papers and this research for the comparison between RC and ARC objectives in LP

Paper	Uncertain parameters	Limitations of the comparison between RC and ARC
Ben-Tal et al (2004)	All parameters	Constraint-wise uncertainty
Marandi and den Hertog (2017)	All parameters	Constraint-wise uncertainty
Bertsimas and Goyal (2011, 2010)	b and c_y	$x, y \geq 0$, and $c, b \geq 0$
Bertsimas et al (2011)	b and c_y	$b \geq 0$, and $\mathcal{Z} \geq 0$
Bertsimas and Goyal (2013)	D and c_y	$x, y \geq 0$, and $c, A, D, b \geq 0$
Bertsimas et al (2015)	D and c_y	$y \geq 0$, and $c, D \geq 0$
Awasthi et al (2018)	D	$x, y \geq 0$,and $c, D \geq 0$
This paper	A and b	Box uncertainty set and $x, y \geq 0$

as in revenue management or resource allocation problems. Their assumptions, however, did not allow the adjustable variable to have a lower-bound because of the assumptions of non-strict "less than" constraints. Moreover, Bertsimas and Goyal (2013) assumed that objective and constraints are convex and the constraint functions should be convex regarding positive decision variables, and also concave and increasing with respect to uncertain parameters of the positive compact convex set. Bertsimas et al (2015) assumed a linear objective and constraint functions with tighter bounds and fewer positivity-restricted parameters compared to Bertsimas and Goyal (2013). Awasthi et al (2018) considered only constraint coefficients to be uncertain in a model with constraint-wise and column-wise uncertainty. However, they still assumed constraint coefficients, second-stage objective coefficients and decision variables to be positive which rules out lower bounds on second-stage decision variables.

Table 1 compares the restrictions existing in the literature to our model for the comparison between RC and ARC optimal objective values in linear programming. The implicit lower and upper bounds imposed by constraints on adjustable decision variables are important aspects of the conditions for $Z_{AARC} < Z_{RC}$ to be stated below.

We identify numerical conditions under which the use of the AARC formulation produces less conservative solutions than the RC. To be able to apply these conditions, we must solve the RC in a representative instance for its optimal primal and dual values. Duality in robust optimization has been studied recently by Beck and Ben-Tal (2009), Soyster and Murphy (2013), Soyster and Murphy (2014), and Bertsimas and Ruiter (2015). The dual of (6) can be written as (Beck and Ben-Tal, 2009):

$$D_{RC} = \max_{\lambda} \Bigg\{ \sum_{i=1}^{m} \lambda_i \left[b_i^0 + \sum_{l=1}^{L} \hat{\xi}_i^l b_i^l \right] : \\ \lambda a^s \leq c_x^s, \quad \lambda d^{s'} \leq c_y^{s'}, \quad \lambda \leq 0, \quad s = 1, ..., n-p, s' = 1, ...p \Bigg\}, \tag{11}$$

where a^s and $d^{s'}$, respectively, denote column s of $A^0 + \sum_{l=1}^{L} \hat{\xi}^l A^l$ and column s' of D, and $\hat{\xi}_i^l$ is the value of ξ for which constraint i is binding (see Definition 2) in the optimal solution to the RC. The dual of the RC is the same as the optimistic counterpart of the dual of the original linear program (1), as mentioned in Beck and Ben-Tal (2009).

The feasible region of the RC (6) can be expressed as a convex set $\bigcap_{i=0}^{m} \mathcal{F}_{RC}^{i}$ where (Beck and Ben-Tal, 2009):

$$\mathcal{F}_{RC}^{i} = \left\{ x, y \geq 0 : \left(a_i^0 + \sum_{l=1}^{L} \xi^l a_i^l \right) x + d_i y \leq b_i^0 + \sum_{l=1}^{L} \xi^l b_i^l, \ \forall \xi \in \chi \right\}, \quad i = 1, ..., m. \tag{12}$$

Likewise, the feasible region of AARC (9) is given by $\bigcap_{i=0}^{m} \mathcal{F}_{AARC}^{i}$, where

$$\mathcal{F}_{AARC}^{i} = \left\{ x \geq 0, \pi : \left(a_i^0 + \sum_{l=1}^{L} \xi^l a_i^l \right) x + d_i \left(\pi^0 + \sum_{l=1}^{L} \xi^l \pi^l \right) \leq b_i^0 + \sum_{l=1}^{L} \xi^l b_i^l, \right. \\ \left. \forall \xi \in \chi; \ \left(\pi^0 + \sum_{l=1}^{L} \xi^l \pi^l \right) \geq 0, \ \forall \xi \in \chi \right\}, i = 1, ..., m. \tag{13}$$

From Ben-Tal et al (2004) we know that $\bigcap_{i=0}^{m} \mathcal{F}_{RC}^{i} \subseteq \bigcap_{i=0}^{m} \mathcal{F}_{AARC}^{i}$ because the AARC differs from the RC only by the inclusion of variables $\pi^l, l = 1, ..., L$. Moreover, (12) can be obtained from (13) by forcing π^l for $l = 1, ..., L$ to be zero. However, a larger robust feasible set does not necessarily improve the objective. If the parameters of distinct constraints are affected by different perturbations, the (affine) adjustable counterpart may be equivalent to the robust counterpart. The following definition formalizes this concept.

Definition 1 (Ben-Tal et al, 2004) Uncertainty in the RC is **constraint-wise** if $[A, b] \in \mathcal{Z}$ consists of non-overlapping sub-vectors $(a_i^l, b_i^l)_{l=1}^{L}$ for $i = 1, ..., m$ such that $(a_i^0 + \sum_{l=1}^{L} \xi^l a_i^l)x + d_i y \leq b_i^0 + \sum_{l=1}^{L} \xi^l b_i^l$ depends on $(a_i^l, b_i^l)_{l=1}^{L}$ only. Moreover, if $\exists l \in \{1, ..., L\} : [a_i^l, b_i^l] \neq 0$,then $[a_j^l, b_j^l] = 0 \ \forall j \neq i$.

The following result identified some conditions under which the RC and ARC are equivalent.

Theorem 1 (See Theorem 2.1 of Ben-Tal et al (2004)) *The objective values of RC (3) and ARC (4) are equal if:*

- *The uncertainty is constraint-wise, and*
- *Whenever x is feasible for ARC (4), there exists a compact set V_x such that for every A, D, b where $\zeta \in \mathcal{Z}$, the relation $Ax + Dy \leq b$ implies that $x \in V_x$.*

However, the more interesting question of when adjustability would result in $Z_{ARC} < Z_{RC}$ was not explored. The challenge of determining whether the ARC is more advantageous than the RC formulation in real applications is compelling because it is not always evidently determined beforehand. Up to now, it can be determined only by directly solving the full-scale AARC formulation. In some cases the AARC does not produce any better solution than the RC formulation. The proposition below establishes conditions under which the objective values of AARC (9) and RC (6) are not equal. The following are definitions necessary for stating the conditions.

Definition 2 If $(\hat{x}^{RC}, \hat{y}^{RC})$ is any optimal solution of the RC (6), we say that constraint $i \in \{1, ..., m\}$ is **binding** at $(\hat{x}^{RC}, \hat{y}^{RC})$ if $a_i^0 \hat{x}^{RC} + d_i \hat{y}^{RC} = b_i^0 + \sum_{l=1}^{L} \hat{\xi}_i^l (b_i^l - a_i^l \hat{x}^{RC})$ where $\hat{\xi}_i \equiv \text{argmin}_{\xi} \left(b_i^0 + \sum_{l=1}^{L} \xi^l (b_i^l - a_i^l \hat{x}^{RC}) \right)$ is the worst-case value of ξ with respect to constraint i.

When the uncertainty is not constraint-wise, at least one component $l = 1, ..., L$ is involved in more than one constraint. However, the worst-case value of ξ^l can differ across constraints.

Definition 3 Constraint i is said to be **relaxed** by changing some parameter values a_i^l, d_i, b_i^l to $a_i'^l, d_i', b_i'^l$, if the result is a feasible region $\mathcal{F}_{RC}'^i \subset \mathcal{F}_{RC}^i$.

Proposition 1 *Considering the RC formulation of equation (6) and* $(\hat{x}^{RC}, \hat{y}^{RC})$ *to be any optimal solution, suppose:*

1. *There exist two binding constraints indexed by* $j, k \in \{1, ..., m\}, j \neq k$*, where relaxing either of these constraints strictly improves* Z_{RC}*, and* $\hat{\xi}_j \neq \hat{\xi}_k$*, where* $\hat{\xi}_j$ *and* $\hat{\xi}_k$ *are defined in Definition 2.*
2. *The uncertainty is not constraint-wise with respect to the constraints* j *and* k *identified in condition 1. Specifically,* $\exists q \in \{1, ..., L\}$ *such that the* q^{th} *parameters are non-zero in both constraints:* $[a_j^q, b_j^q] \neq 0$ *and* $[a_k^q, b_k^q] \neq 0$.
3. *There is a component* y_r *with objective coefficient* $c_{y_r} \in \mathbb{R}$ *that is basic in* $(\hat{x}^{RC}, \hat{y}^{RC})$ *such that*
   a. $d_{jr} d_{kr} < 0$ *for the constraints* j *and* k *identified in condition 1, and*
   b. y_r *is adjustable to the perturbation* ξ^q *in AARC where* q *is defined in condition 2. In other words, in equation (8)* $\pi_r^q \in \mathbb{R}$ *is not forced to be zero.*

Assume that $[a_i^q, b_i^q] = 0$ *for* $i \neq \{j, k\}$*. Let* λ^* *be an optimal dual solution corresponding to* $(\hat{x}^{RC}, \hat{y}^{RC})$ *as defined by Beck and Ben-Tal (2009), and* j *and* k *index two constraints of RC as defined in condition 1. Then*

$$\left| \lambda_j^* (b_j^q - a_j^q \hat{x}^{RC}) \right| + \left| \lambda_k^* (b_k^q - a_k^q \hat{x}^{RC}) \right| > \left| \lambda_j^* (b_j^q - a_j^q \hat{x}^{RC} - d_{jr} \delta) \right|$$
$$+ \left| \lambda_k^* (b_k^q - a_k^q \hat{x}^{RC} - d_{kr} \delta) \right| + |c_{y_r} \delta| + \sum_{\substack{i=1 \\ i \neq \{j,k\}}}^{m} \left| \delta \lambda_i^* d_{ir} \right|. \tag{14}$$

for some $\delta \neq 0$ *implies* $Z_{RC} > Z_{AARC}$.

Proof Consider the intersection of the feasible regions defined by constraints j and k, $\mathcal{F}_{RC}^i \cap \mathcal{F}_{RC}^k$, and focus on the perturbation ξ^q. Condition 1 implies:

$$a_j^0 \hat{x}^{RC} + d_j \hat{y}^{RC} = b_j^0 + \sum_{\substack{l=1 \\ l \neq q}}^{L} \hat{\xi}_j^l \left(b_j^l - a_j^l \hat{x}^{RC} \right) + \hat{\xi}_j^q \left(b_j^q - a_j^q \hat{x}^{RC} \right) \text{ and}$$

$$a_k^0 \hat{x}^{RC} + d_k \hat{y}^{RC} = b_k^0 + \sum_{\substack{l=1 \\ l \neq q}}^{L} \hat{\xi}_k^l \left(b_k^l - a_k^l \hat{x}^{RC} \right) + \hat{\xi}_k^q \left(b_k^q - a_k^q \hat{x}^{RC} \right),$$

where, based on condition 1, $\hat{\xi}_j^q \neq \hat{\xi}_k^q$.

From Bazaraa et al (2010), if surplus variables s are added to linear program (1) converting the inequalities to equalities, we have

$$z^* = \min_{w' \geq 0} c^T w' : \quad A'w' = b, \tag{15}$$

where $w'^T = [w, s]$ can be partitioned into w'_B as basic variables and w'_N as non-basic variables in a given basic solution. In addition, if B^* and N are the corresponding optimal basic and non-basic matrices, respectively, the objective and the optimal values of the basic variables can be written as $z^* - c_N^T w'_N = c_{B^*}^T w'_{B^*}$, where $w'_{B^*} = B^{*-1}b - B^{*-1}Nw'_N$. That is, we can rewrite objective z^* as

$$z^* = c_{B^*}^T B^{*-1} b + w_N'^T (c_N - c_{B^*}^T B^{*-1} N) = \lambda^* b + w_N'^T (c_N - \lambda^* N), \tag{16}$$

where $\lambda^* = c_{B^*}^T B^{*-1}$ is an optimal dual vector corresponding to the particular optimal solution w'^*. If we define $\Delta_{B^*} = (x_{B^*}, y_{B^*})$ as basic variables and $\Delta_N = (x_N, y_N)$ as the non-basic variables in $(\hat{x}^{RC}, \hat{y}^{RC})$ then we can write:

$$Z_{RC}(\Delta_N) = \sum_{i=1}^{m} \lambda_i^* \left[b_i^0 + \sum_{l=1}^{L} \hat{\xi}_i^l b_i^l \right] + \Delta_N (c_N - \lambda^* N), \tag{17}$$

where $Z_{RC}(\Delta_N)$ is the objective value of RC as a function of non-basic variables Δ_N and $Z_{RC}(0) = Z_{RC}$. By subtracting the constant $\sum_{i=1}^{m} \lambda_i^* \left[\sum_{l=1}^{L} \hat{\xi}_i^l (a_i^l \hat{x}^{RC}) \right]$ from $Z_{RC}(\Delta_N)$, we have:

$$z(\Delta_N) \equiv Z_{RC}(\Delta_N) - \sum_{i=1}^{m} \lambda_i^* \left[\sum_{l=1}^{L} \hat{\xi}_i^l (a_i^l \hat{x}^{RC}) \right] =$$

$$\sum_{i=1}^{m} \lambda_i^* \left[b_i^0 + \sum_{l=1}^{L} \hat{\xi}_i^l \left(b_i^l - a_i^l \hat{x}^{RC} \right) \right] + \Delta_N \left(c_N - \lambda^* N \right). \tag{18}$$

From condition 3, we know $y_r \in \Delta_{B^*}$. Therefore, based on (6) and (9) we can identify π_r^q as a new variable with constraint column N_r and objective coefficient $\mathcal{C}_r$ as follows:

$$N_r = \left[\dots \, d_{jr} \hat{\xi}_j^q \, \dots \, d_{kr} \hat{\xi}_k^q \, \dots \right]^T, \mathcal{C}_r = \xi^q c_{y_r}. \tag{19}$$

Recall that $\hat{\xi}_j^q$ and $\hat{\xi}_k^q$ are the worst-case values of ξ in equation (6) for constraints j and k, respectively, and c_{y_r} is the objective function coefficient of y_r. Based on equation (18), for $\Delta_N = (0, \dots, \pi_r^q)^T$ where π_r^q has been appended to the set of non-basic variables, we have:

$$z\left((0, \dots, \pi_r^q)^T \right) = \sum_{i=1}^{m} \lambda_i^* \left[b_i^0 + \sum_{l=1}^{L} \hat{\xi}_i^l (b_i^l - a_i^l \hat{x}^{RC}) \right] + \pi_r^q (\mathcal{C}_r - \lambda^* N_r). \tag{20}$$

Following equation (20) we can isolate j and k and also substitute (19):

$$z\left((0,...,\pi_r^q)^T\right) = \sum_{\substack{i=1\\ i\neq\{j,k\}}}^{m} \lambda_i^* \left[b_i^0 + \sum_{l=1}^{L} \hat{\xi}_i^l (b_i^l - a_i^l \hat{x}^{RC})\right] +$$
$$\lambda_j^* \left[b_j^0 + \sum_{\substack{l=1\\ l\neq q}}^{L} \hat{\xi}_j^l (b_j^l - a_j^l \hat{x}^{RC}) + \hat{\xi}_j^q (b_j^q - a_j^q \hat{x}^{RC})\right] +$$
$$\lambda_k^* \left[b_k^0 + \sum_{\substack{l=1\\ l\neq q}}^{L} \hat{\xi}_k^l (b_k^l - a_k^l \hat{x}^{RC}) + \hat{\xi}_k^q (b_k^q - a_k^q \hat{x}^{RC})\right] +$$
$$\pi_r^q \left(\xi^q c_{y_r} - \lambda^* \left[... \; d_{jr}\hat{\xi}_j^q \; ... \; d_{kr}\hat{\xi}_k^q \; ...\right]^T\right) \tag{21}$$

Upon rearranging terms, denoting a value of π_r^q by δ, and also based on the assumption of $[a_i^q, b_i^q] = 0$ for $i \neq \{j,k\}$, we have:

$$z\left((0,...,\delta)^T\right) = \sum_{\substack{i=1\\ i\neq\{j,k\}}}^{m} \lambda_i^* \left[b_i^0 + \sum_{l=1}^{L} \hat{\xi}_i^l (b_i^l - a_i^l \hat{x}^{RC})\right] +$$
$$\lambda_j^* \left[b_j^0 + \sum_{\substack{l=1\\ l\neq q}}^{L} \hat{\xi}_j^l (b_j^l - a_j^l \hat{x}^{RC}) + \hat{\xi}_j^q (b_j^q - a_j^q \hat{x}^{RC} - d_{jr}\delta)\right] +$$
$$\lambda_k^* \left[b_k^0 + \sum_{\substack{l=1\\ l\neq q}}^{L} \hat{\xi}_k^l (b_k^l - a_k^l \hat{x}^{RC}) + \hat{\xi}_k^q (b_k^q - a_k^q \hat{x}^{RC} - d_{kr}\delta)\right]$$
$$+ \delta \xi^q c_{y_r} - \sum_{\substack{i=1\\ i\neq\{j,k\}}}^{m} \delta \lambda_i^* d_{ir} \xi_i^q. \tag{22}$$

Let $z(0) = Z_{RC} - \sum_{i=1}^{m} \lambda_i^* \left[\sum_{l=1}^{L} \hat{\xi}_i^l (a_i^l \hat{x}^{RC})\right] \equiv z\left((0,...,\delta)^T\right)$ for $\delta = 0$. The inequality $z(0) > z\left((0,...,\delta)^T\right)$ is equivalent to:

$$\hat{\xi}_j^q \lambda_j^* (b_j^q - a_j^q \hat{x}^{RC}) + \hat{\xi}_k^q \lambda_k^* (b_k^q - a_k^q \hat{x}^{RC}) > \hat{\xi}_j^q \lambda_j^* (b_j^q - a_j^q \hat{x}^{RC} - d_{jr}\delta)$$
$$+ \hat{\xi}_k^q \lambda_k^* (b_k^q - a_k^q \hat{x}^{RC} - d_{kr}\delta) + \xi c_{y_r}\delta - \sum_{\substack{i=1\\ i\neq\{j,k\}}}^{m} \xi_i^q \delta \lambda_i^* d_{ir}. \tag{23}$$

Based on Definition 2 and assumption (10), $\hat{\xi}_i^q (b_i^q - a_i^q \hat{x}^{RC}) = -\left|b_i^q - a_i^q \hat{x}^{RC}\right|$. Then, since $\lambda_i \leq 0$ in the dual of RC (11), we have

$$\hat{\xi}_j^q \lambda_j^*(b_j^q - a_j^q \hat{x}^{RC}) + \hat{\xi}_k^q \lambda_k^*(b_k^q - a_k^q \hat{x}^{RC}) = \left|\lambda_j^*(b_j^q - a_j^q \hat{x}^{RC})\right| + \left|\lambda_k^*(b_k^q - a_k^q \hat{x}^{RC})\right| \tag{24}$$

In addition,

$$\begin{aligned} &\hat{\xi}_j^q \lambda_j^*(b_j^q - a_j^q \hat{x}^{RC} - d_{jr}\delta) + \hat{\xi}_k^q \lambda_k^*(b_k^q - a_k^q \hat{x}^{RC} - d_{kr}\delta) + \xi c_{y_r}\delta - \sum_{\substack{i=1 \\ i \neq \{j,k\}}}^{m} \xi_i^q \delta \lambda_i^* d_{ir} \\ &\leq \left|\lambda_j^*(b_j^q - a_j^q \hat{x}^{RC} - d_{jr}\delta)\right| + \left|\lambda_k^*(b_k^q - a_k^q \hat{x}^{RC} - d_{kr}\delta)\right| + |c_{y_r}\delta| + \sum_{\substack{i=1 \\ i \neq \{j,k\}}}^{m} |\delta \lambda_i^* d_{ir}| \end{aligned} \tag{25}$$

Therefore, from the right-hand sides of (24) and (25), if (14) holds considering box uncertainty set (10), there exists $\delta \neq 0$ such that $z(0) > z\left((0, ..., \delta)^T\right)$(expressed as inequality (23)). Recall that $z(0) + \sum_{i=1}^{m} \lambda_i^* \left[\sum_{l=1}^{L} \hat{\xi}_i^l (a_i^l \hat{x}^{RC})\right] = Z_{RC}$. Because the AARC could have multiple adjustable variables, $Z_{AARC} \leq z\left((0, ..., \delta)^T\right) + \sum_{i=1}^{m} \lambda_i^* \left[\sum_{l=1}^{L} \hat{\xi}_i^l (a_i^l \hat{x}^{RC})\right]$. Therefore, inequality (14) implies $Z_{RC} > Z_{AARC}$.
□

Remark 1 For simplicity in the proof, we focus on only two constraints j and k that have the same uncertain parameter in (14), and consider y_r as adjustable to a single perturbation ξ^q where $[a_i^q, b_i^q] = 0$ for $i \neq \{j, k\}$. The result can be extended using the same intuition if there exist similar constraints to j or k that satisfy conditions 1 - 3 with no assumption that $[a_\kappa^q, b_\kappa^q] = 0$. Expressions of the form $|\lambda_\kappa^*(b_\kappa^q - a_\kappa^q \hat{x}^{RC})|$ and $|\lambda_\kappa^*(b_\kappa^q - a_\kappa^q \hat{x}^{RC} - d_{\kappa r}\delta)|$ for such constraints $\kappa \in \{1, ..., m\}$ would be added to the left- and right-hand sides of (14), respectively, and index κ should be excluded from the sum $\sum_{\substack{i=1 \\ i \neq \{j,k\}}}^{m} |\delta \lambda_i^* d_{ir}|$. The extension of (14) when considering all constraints is as follows:

$$\sum_{i=1}^{m} \left|\lambda_i^*(b_i^q - a_i^q \hat{x}^{RC})\right| > \sum_{i=1}^{m} \left|\lambda_i^*(b_i^q - a_i^q \hat{x}^{RC} - d_{ir}\delta)\right| + |c_{y_r}\delta| \tag{26}$$

Examples 4 and 5 illustrate the use of this expanded inequality.

Next we show the importance of condition 1, which has been ruled out some previous results. Condition 1 holds if there are two binding constraints with different values of the uncertain parameter at the optimal RC solution. The variable that is adjustable to the uncertain parameter in both constraints is effectively bounded above and below by these constraints based on condition 3. One of these bounds is unfavorable for the objective but can be relaxed by adjustability in a direction that lowers the objective value.

Remark 2 Suppose conditions 2 and 3 of Proposition 1 hold but $\hat{\xi}_j^q = \hat{\xi}_k^q = \hat{\xi}^q$. The coefficient of π_r^q in (20) is reformulated by inserting (19) as:

$$(\mathcal{C}_r - \lambda^* N_r) = \left(\xi^q c_{y_r} - c_{B^*} B^{*-1} \left[...\ d_{jr} \hat{\xi}_j^q\ ...\ d_{kr} \hat{\xi}_k^q\ ... \right]^T \right). \tag{27}$$

The left-hand-side of (27) equals:

$$\left(\xi^q c_{y_r} - c_{B^*} B^{*-1} \left[...\ d_{jr}\ ...\ d_{kr}\ ... \right]^T \hat{\xi}^q \right). \tag{28}$$

If $N_r = N'_r \hat{\xi}^q$ in (28) where $N'_r = \left[...\ d_{jr}\ ...\ d_{kr}\ ... \right]^T$, since N'_r equals the r^{th} column of B^*, multiplying B^{*-1} and N'_r yields the r^{th} column of identity matrix I_n. Following (28) and since the r^{th} element of c_{B^*} is c_{y_r}, we have:

$$\left(\xi^q c_{y_r} - c_{B^*}^T \left[0\ ...\ \hat{\xi}^q\ ...\ 0 \right]^T \right) = c_{y_r}(\xi^q - \hat{\xi}^q), \tag{29}$$

where (29) expresses the coefficient of π_r^q in (20) as a function of ξ. The parameter ξ^q can take on a value that forces the coefficient of π_r^q in (20) to equal 0. Therefore, for any value of π_r^q, $z\left((0, ..., \pi_r^q)^T\right) = z(0)$ and $Z_{AARC} = Z_{RC}$.

Note also that if condition 2 does not hold, then the uncertainty is constraint-wise, and Z_{ARC} equals Z_{RC} (Ben-Tal et al, 2004; Marandi and den Hertog, 2017).

To illustrate the proposition, Examples 1-3 are provided based on the following LP formulation:

$$\min_{x,y \geq 0} c_x x + c_y y : \quad a_1 x + d_1 y \leq b_1, a_2 x + d_2 y \leq b_2, \tag{30}$$

where $a_i = a_i^0 + \xi a_i^1$ and $b_i = b_i^0 + \xi b_i^1$ are the uncertain parameters in constraints $i = 1, 2$.

Example 1 This example illustrates equivalence of RC and AARC objective values based on Remark 2. If the parameter values of (30) are $a_1^0 = -3, a_1^1 = -1, a_2^0 = 0, a_2^1 = -1, b_1^0 = -6, b_1^1 = -1, b_2^0 = 1, b_2^1 = -1, c_x = c_y = 1, d_1 = 1$ and $d_2 = -1$ where $\xi \in [-1, 1]$, the RC formulation is as follows:

$$\begin{aligned}
&Z_{RC} = \min_{x,y \geq 0} x + y : \\
&(i = 1)\ -(3 + \xi)x + y \leq -6 - \xi, \quad \forall \xi \in [-1, 1] \\
&(i = 2)\ -\xi x - y \leq 1 - \xi, \quad \forall \xi \in [-1, 1]
\end{aligned} \tag{31}$$

Figure 1(a) illustrates the RC feasible region formed by the constraints in their respective most restrictive cases. Since the uncertainty sets are polyhedral, the RC can be converted to an explicit LP by defining additional constraints and variables $v_1 = -\min_{-1 \leq \xi \leq 1} \xi(x-1)$ in constraint 1 and $v_2 = -\min_{-1 \leq \xi \leq 1} \xi(x-1)$ in constraint 2 as follows (Ben-Tal et al, 2004):

$$\begin{aligned}
Z_{RC} = \min_{x,y,v_1,v_2 \geq 0} x + y : &-3x + y \leq -6 - v_1, \quad -v_1 \leq x - 1 \leq v_1, \\
&-y \leq 1 - v_2, \quad -v_2 \leq x - 1 \leq v_2.
\end{aligned} \tag{32}$$

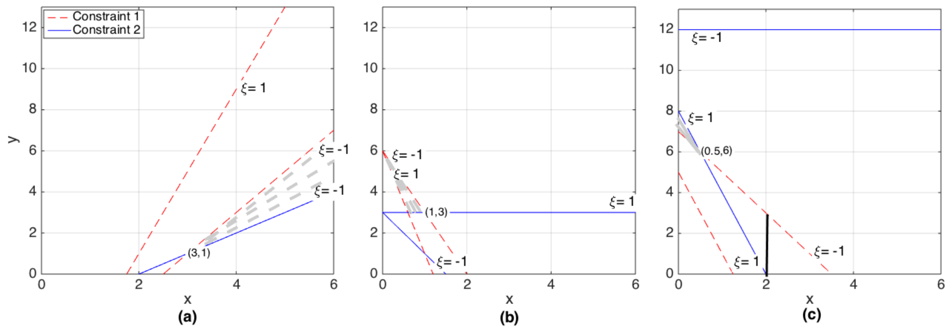

Fig. 1 The feasible regions of the RC constraints within uncertainty set $\xi \in [-1, 1]$ for (a) Example 1, (b) Example 2, (c) Example 3 are shaded with gray lines. The thick black line in (c) is $y = \frac{3}{2} - \frac{3}{2}\xi$ for $\xi \in [-1, 1]$.

The optimal values of the RC variables by solving (32) are $\hat{x}^{RC} = 3, \hat{y}^{RC} = 1, Z_{RC} = 4$. Note that the optimal solution for this particular instance could be identified with only one auxiliary variable $v \equiv v_1 = v_2$. We can identify $j = 1$ and $k = 2$ in (31) as satisfying conditions 1-3 of the proposition. The values of λ^* can be easily found using the deterministic formulation (32) for all corresponding constraints. For example, constraint $-3x + y \leq -6 - v_1$ in (32) corresponds to $i = 1$ in (31). The optimal basic variables of RC (31) are x and y. Their cost coefficients and the optimal values of the dual variables are $c_{B^*}^T = \begin{pmatrix}1 \ 1\end{pmatrix}$ and $\lambda^* = c_{B^*}^T B^{*-1} = \begin{pmatrix}\lambda_1^* \ \lambda_2^*\end{pmatrix} = \begin{pmatrix}-2 \ -3\end{pmatrix}$, respectively.

We can also obtain the values of ξ at the optimal solution in constraints j and k, by substituting the optimal values of $\hat{x}^{RC}$ and $\hat{y}^{RC}$ into constraints $j = 1$ and $k = 2$ of formulation (31) and identifying the values of ξ where the constraints hold as equalities. In this instance, we obtain $\hat{\xi}_1 = -1, \hat{\xi}_2 = -1$, which do not satisfy condition 1.

Considering the adjustable variable as an affine function $y = \pi^0 + \xi\delta$, and by inserting the corresponding parameter values and the new variable δ into (14) we obtain:

$$10 > 2\,|2 - \delta| + 3\,|2 + \delta| + |\delta|\,. \tag{33}$$

The inequality (33) cannot be satisfied by any value of δ, because its right-hand side is a convex piecewise linear expression with minimum value 10. Indeed by solving the AARC with $y = \pi^0 + \xi\delta$, we find $\hat{x}^{AARC} = 3, \hat{\pi}^0 = 1, \hat{\delta} = 0$,and $Z_{AARC} = 4 = Z_{RC}$.

Example 2 This instance shows that if condition 3(a) is not satisfied (i.e., $d_{jr}d_{kr} > 0$ but still conditions 1, 2 and 3(b) are satisfied) the objective values of RC and AARC are equal. The RC formulation of (30) with $a_1^0 = -4, a_1^1 = -1, a_2^0 = -1, a_2^1 = 1, b_1^0 = -6, b_1^1 = 0, b_2^0 = -3, b_2^1 = 0, c_x = c_y = 1, d_1 = -1$ and $d_2 = -1$ where $\xi \in [-1, 1]$ is:

$$
\begin{aligned}
&Z_{RC} = \min_{x,y \geq 0} x + y : \\
&(i=1)\ -(4+\xi)x - y \leq -6, \quad \forall \xi \in [-1,1] \\
&(i=2)\ (-1+\xi)x - y \leq -3, \quad \forall \xi \in [-1,1]
\end{aligned} \tag{34}
$$

The optimal values of RC variables following the same method of Example 1, in which we converted the RC problem to its deterministic formulation (32), are $\hat{x}^{RC} = 1, \hat{y}^{RC} = 3, Z_{RC} = 4$ (see Figure 1(b)). The two constraints $j = 1$ and $k = 2$ satisfy conditions 1, 2 and 3(b) but not 3(a). Moreover, the coefficients of the adjustable variable y for the two constraints are $d_1 = -1, d_2 = -1$. Also, $\hat{\xi}_1 = -1$ and $\hat{\xi}_2 = 1$. The optimal values of dual variables are $\begin{pmatrix}\lambda_1^* & \lambda_2^*\end{pmatrix} = \begin{pmatrix}-\frac{1}{3} & -\frac{2}{3}\end{pmatrix}$. After inserting the corresponding parameter values in equation (14), we have:

$$
1 > \frac{1}{3}\left|1+\delta\right| + \frac{2}{3}\left|-1+\delta\right| + \left|\delta\right|. \tag{35}
$$

Again, the right-hand side of inequality (35) is a convex piecewise linear expression whose minimum value is 1. The optimal values of AARC variables when $y = \pi^0 + \xi\delta$ are $\hat{x}^{AARC} = 1$, $\pi^0 = 3, \delta = 0$, and $Z_{AARC} = 4 = Z_{RC}$.

Example 3 This example illustrates the case in which all conditions of the proposition are satisfied along with (14) so that $Z_{RC} > Z_{AARC}$. In this instance, the parameter values of (30) are $a_1^0 = -3, a_1^1 = -1, a_2^0 = 1, a_2^1 = 1, b_1^0 = -6, b_1^1 = 1, b_2^0 = 5, b_2^1 = -1, c_x = c_y = 1, d_1 = -1$ and $d_2 = \frac{1}{2}$, where $\xi \in [-1,1]$. The RC formulation is:

$$
\begin{aligned}
&Z_{RC} = \min_{x,y \geq 0} x + y : \\
&(i=1)\ -(3+\xi)x - y \leq -6 + \xi, \quad \forall \xi \in [-1,1] \\
&(i=2)\ (1+\xi)x + \frac{1}{2}y \leq 5 - \xi, \quad \forall \xi \in [-1,1]
\end{aligned} \tag{36}
$$

Here the optimal values of the RC variables are: $\hat{x}^{RC} = \frac{1}{2}, \hat{y}^{RC} = 6, Z_{RC} = \frac{13}{2}$. Figure 1(c) illustrates the feasible region as well as the optimal solution of the adjustable variable $y = \pi^0 + \xi\delta = \frac{3}{2} - \frac{3}{2}\xi$ for $\xi \in [-1,1]$. In line with condition 1, the two constraints $j = 1$ and $k = 2$ are binding at $\hat{\xi}_1 = -1$ and $\hat{\xi}_2 = 1$; that is, $\hat{\xi}_1 \neq \hat{\xi}_2$. Condition 2 holds because at least one parameter depends on ξ in these two constraints. In the ARC formulation of (36), y is adjustable to ξ which has non-zero coefficients in both constraints that satisfy condition 3. The objective coefficient vector of the basic variables is $c_{B^*}^T = \begin{pmatrix}1 & 1\end{pmatrix}$ and the optimal dual variables of RC are $\lambda^* = c_{B^*}^T B^{*-1} = \begin{pmatrix}\lambda_1^* & \lambda_2^*\end{pmatrix} = \begin{pmatrix}-\frac{3}{2} & -1\end{pmatrix}$. Inequality (14) is:

$$
\frac{15}{4} > \frac{3}{2}\left|\frac{3}{2} + \delta\right| + \left|-\frac{3}{2} - \frac{1}{2}\delta\right| + |\delta|. \tag{37}
$$

If $\delta = -\frac{3}{2}$ then (37) is satisfied as $\frac{15}{4} > \frac{9}{4}$. The optimal values of the AARC variables are $\hat{x}^{AARC} = 2, y = \frac{3}{2} - \frac{3}{2}\xi$, and $Z_{AARC} = 5 < Z_{RC}$.

4 Applications

To evaluate the potential for affine adjustability to lower the cost in any application, inequality (26) (that is, the extension of (14)) can be tested in a small instance. The following examples illustrate this evaluation process in applications where the AARC approach has been applied successfully. Note that these applications are evaluated using inequality (26) before reformulation as AARC. All of the applications described in this section are much more extensive than the simplified instances explored numerically here. For example, the largest AARC instance solved in Haddadsisakht and Ryan (2018), from which Example 6 is derived, has 130,000 decision variables and 389,000 constraints. These examples illustrate how RC formulations can be tested in small-scale instances using optimal primal and dual solutions to identify whether formulating and solving the AARC might be advantageous.

Example 4 (Inventory model) Multi-stage inventory management has been solved by the AARC approach frequently (Ben-Tal et al, 2004, 2009; Adida and Perakis, 2010). Ben-Tal et al (2004) formulated the robust counterpart as:

$$
\begin{aligned}
&Z_{RC} = \min_{p} \sum_{j=1}^{J} \sum_{t=1}^{T} c_j(t) p_j(t) \\
&0 \leq p_j(t) \leq P_j(t), \quad j = 1, ..., J, \; t = 1, ..., T \\
&\sum_{t=1}^{T} p_j(t) \leq Q(j), \quad j = 1, ..., J \\
&V_{min} \leq v(1) + \sum_{j=1}^{J} \sum_{s=1}^{t} p_j(s) - \sum_{s=1}^{t} \tilde{\theta}_s(\xi) \leq V_{max}, \quad \forall \xi \in \chi, \quad t = 1, ..., T. \qquad (38)
\end{aligned}
$$

Here, J and T are the numbers of factories and periods, respectively, $p = \{p_j(t)\}$ are the production quantities with costs $c_j(t)$ and $P = \{P_j(t)\}$ are the production capacities of factory j in period t. In addition, $Q(j)$ represents the maximum cumulative capacity of factory $j, v(1)$ stands for the amount of available product at the beginning of the horizon, and V_{min} (V_{max}) are the minimum (maximum) storage capacity of the warehouse.

We assume that $\tilde{\theta}_t(\xi)$, the demand in period t, is uncertain and lies in a box uncertainty set $\tilde{\theta}_t(\xi) = \bar{\theta}_t + \hat{\theta}_t \xi^t$ where $|\xi^t| \leq \rho_t$. At the beginning of period t, the production decisions $p_j(t), j \in J$, are made given demands $\tilde{\theta}_r$ observed at periods $r \in I_t \equiv \{1, ..., t\}$, while future demands remain uncertain. The decisions p can adjust to the uncertain demands with affine decision rules as follows:

$$
p_j(t) = \pi^0_{j,t} + \sum_{r \in I_t} \pi^r_{j,t} \xi^r, \qquad (39)
$$

where π^0 and π^r are new non-adjustable variables.

Consider a simple instance where $T = 2, J = 2$ and the parameter values are:

$$c(1) = \begin{bmatrix} 9 \\ 8 \end{bmatrix}, c(2) = \begin{bmatrix} 10 \\ 9 \end{bmatrix}, P(1) = P(2) = \begin{bmatrix} 20 \\ 20 \end{bmatrix}, Q = \begin{bmatrix} 50 \\ 20 \end{bmatrix}, V_{min} = 0, V_{max} = 10$$

If the uncertain demands for two periods are $\tilde{\theta}_1(\xi) = 10+3\xi^1$ and $\tilde{\theta}_2(\xi) = 10+2\xi^2$ where $|\xi^1| \leq 1$ and $|\xi^2| \leq 1$, then the optimal solution to (38) using the same process as in Example 1 are $\hat{p}^{RC}(1) = \begin{bmatrix} 0 \ 17 \end{bmatrix}^T, \hat{p}^{RC}(2) = \begin{bmatrix} 5 \ 3 \end{bmatrix}^T$ with $Z_{RC} = 213$.

By considering $p_1(1)$ as adjustable to the first perturbation ξ^1 using (44), the following represents how to evaluate the RC optimum solution based on the general inequality (26).

Only three constraints have non-zero corresponding dual values $\lambda^* = (-1, -10, -1)^T$ as follows:

$(i = 1)\ v(1) + p_1(1) + p_2(1) \leq (\bar{\theta}_1 + \xi^1\hat{\theta}_1) + V_{max}$

$(i = 2)\ -v(1) - p_1(1) - p_2(1) - p_1(2) - p_2(2) \leq -(\bar{\theta}_1 + \xi^1\hat{\theta}_1) - (\bar{\theta}_2 + \xi^2\hat{\theta}_2) - V_{min}$

$(i = 3)\ p_2(1) + p_2(2) \leq Q(2)$

The coefficients a_i^l equal zero for all i and l while $b_1^1 = \hat{\theta}_1 = 3, b_2^1 = -\hat{\theta}_1 = -3$. Also, the coefficient vector of adjustable variable $p_1(1)$ in these constraints is $d = (1, -1, 0)^T$. Finally, the coefficient of $p_1(1)$ in the objective, denoted c_{y_r} in (26), is 9. Therefore, considering the affine function $p_1(1) = \pi^0 + \xi^1\delta$, (26) is:

$$33 > |3 - \delta| + 10|-3 + \delta| + 9|\delta| \tag{40}$$

Inequality (40) holds for values of δ including 3. Therefore, the conservatism of problem (38) would be reduced by the AARC formulation. When only $p_1(1)$ is adjustable, $Z_{AARC} = 208$. The AARC formulation when $p_i(1)$ is adjustable to ξ^1 and $p_i(2)$ is adjustable to both ξ^1 and ξ^2 yields the optimal objective value of $Z_{AARC} = 207$ in this instance.

However, a single modification to this instance renders adjustability ineffective. If V_{max} changes to 100, then the new RC solution with $Z_{RC} = 205$ is $p(1) = \begin{bmatrix} 5 \ 20 \end{bmatrix}^T, p(2) = \begin{bmatrix} 0 \ 0 \end{bmatrix}^T$. The non-zero dual values are $\lambda_2^* = -9$ and $\lambda_4^* = -1$ where constraint $i = 4$ is $p_2(1) \leq P_2(1)$. After this change, conditions 1 and 2 in the proposition do not hold because only one of the binding constraints involves an uncertain parameter. Therefore, $Z_{RC} = Z_{AARC}$.

Example 5 (Project management) A time-cost tradeoff problem (TCTP) in project management with uncertainty in the duration of activities is another application proposed by Cohen et al (2007) to solve with AARC. Given a directed acyclic graph with nodes $i \in \{1, ..., n\}$ and arcs (i, j) representing activities, the following is the RC formulation of TCTP:

$$\begin{aligned} Z_{RC} = \min_{x,y\geq 0} \max_{\xi\in\chi} \sum_{ij} \mu_{ij}\tilde{T}_{ij}(\xi) + \sum_{ij} \Phi_{ij}y_{ij} + Cx_n \\ -(x_j - x_i + y_{ij}) \leq -\tilde{T}_{ij}(\xi), \quad \forall\xi \in \chi, \quad \forall j, \forall i \in \mathcal{P}_j \\ y_{ij} \leq \tilde{T}_{ij}(\xi) - M_{ij}, \quad \forall\xi \in \chi, \quad \forall j, \forall i \in \mathcal{P}_j \\ x_1 = 0, \ x_n \leq \mathcal{D}, \end{aligned} \tag{41}$$

where x_i denotes the start time of node i. When $x_1 = 0$ then x_n is the project duration with overhead cost C, and $\mathcal{D}$ denotes its predetermined due date. The set of immediate predecessors of node j is $\mathcal{P}_j$. The decision variable y_{ij} represents the crashing of activity (i, j) with a constant marginal cost Φ_{ij}. The uncertain normal duration of each activity $\tilde{T}_{ij}(\xi)$ is assumed to belong a symmetric interval with objective coefficient μ_{ij} as the compensation of the contractor. The objective of this model is to optimize the project cost by making the optimal crashing decisions. In this example, we assume $\tilde{T}_{ij}(\xi) = \bar{T}_{ij} + \xi^{ij}\hat{T}_{ij}$ where $|\xi^{ij}| \leq \rho_{ij}$. In addition, M_{ij} represents the lower bound of activity duration ij.

In the AARC, each variable is adjustable to a subset of the uncertain parameters. The crashing level decision y_{ij} for an activity (i, j) and starting time x_i are made based upon certain information sets I_{ij} and I_i, respectively. The usual assumption is that the durations of the activities that precede node i are known. Therefore, y_{ij} and x_i are affinely adjustable as follows:

$$\begin{aligned} y_{ij} &= \pi_{ij}^0 + \sum_{kl \in I_{ij}} \pi_{ij}^{kl} \xi^{kl}, \\ x_i &= \eta_i^0 + \sum_{kl \in I_i} \eta_i^{kl} \xi^{kl}, \end{aligned} \tag{42}$$

where the coefficients $\pi_{ij}^0, \pi_{ij}^{kl}, \eta_i^0$ and η_i^{kl} are the new non-adjustable variables.

A small instance of the problem with $n = 3$ and two arcs $(i, j) \in \{(1, 2), (2, 3)\}$ is specified with the following parameter values, extracted from the same instance as in Cohen et al (2007) limited to three nodes and two sequential activities:

$$\mu = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \Phi = \begin{bmatrix} 15 \\ 2 \end{bmatrix}, M = \begin{bmatrix} 1.3 \\ 1.9 \end{bmatrix}, \bar{T} = \begin{bmatrix} 3 \\ 4.4 \end{bmatrix}, \hat{T} = \begin{bmatrix} 0.3 \\ 0.44 \end{bmatrix}, C = 15.$$

Assuming the uncertainty set surrounding the duration of each activity is $|\xi^{ij}| \leq 1$ for $(i, j) \in \{(1, 2), (2, 3)\}$, then the optimal values of the RC variables following the same process of Example 1 are $\hat{x}^{RC} = (0.0, 3.3, 6.08)^T, \hat{y}^{RC} = (0, 2.06)^T$, with $Z_{RC} = 136.02$.

We select y_{23} as the variable affinely adjustable to ξ_{23}, that is, $y_{23} = \pi^0 + \xi^{23}\delta$. The constraints with corresponding non-zero optimal dual values $\lambda^* = (-1, -15, -15, -13)^T$ of RC (41) are as follows:

$$\begin{aligned} &(i = 1) \;\; -u + Cx_n + \sum_{ij} \Phi_{ij} y_{ij} \leq -\sum_{ij} \mu_{ij} (\bar{T}_{ij} + \xi^{ij}\hat{T}_{ij}) \\ &(i = 2) \;\; -(x_2 - x_1 + y_{12}) \leq -(\bar{T}_{12} + \xi^{12}\hat{T}_{12}) \\ &(i = 3) \;\; -(x_3 - x_2 + y_{23}) \leq -(\bar{T}_{23} + \xi^{23}\hat{T}_{23}) \\ &(i = 4) \;\; y_{23} \leq (\bar{T}_{23} + \xi^{23}\hat{T}_{23}) - M_{23}. \end{aligned}$$

Here, u is an auxiliary variable introduced to convert the objective of (41) to constraint $i = 1$. All coefficients a_i^l equal zero, while $b^2 = (-2.2, 0, -0.44, 0.44)^T$. In addition, the coefficient vector d of adjustable variable y_{23} in the constraints is $(2, 0, -1, 1)^T$. Finally, the coefficient of y_{23} in the objective c_{y_r} equals zero. Substituting into inequality (26), we obtain:

$$14.52 > |-2.2 - 2\delta| + 15|-0.44 + \delta| + 13|0.44 - \delta| \tag{43}$$

In this instance, the right-hand-side of inequality (43) equals 3.08 for $\delta = 0.44$. Since the adjustability of a single variable would reduce the RC objective function, making more variables adjustable might reduce it more. Indeed, the AARC optimal objective value when both activity durations are adjustable (i.e., $y_{ij} = \pi^0_{ij} + \xi^{ij}\pi^1_{ij}$) is $Z_{AARC} = 124.58$ based on $\hat{x}^{AARC} = (0.0, 3.4, 5.3)^T, \pi^0 = (0, 2.5)^T, \pi^1 = (0.3, 0.44)^T$.

Example 6 (Supply chain network) A closed-loop supply chain network design that encompasses flows in both forward and reverse directions with multiple transportation modes under demand and carbon tax uncertainty was formulated and solved in Haddadsisakht and Ryan (2018). The model is a three-stage hybrid robust/stochastic program that combines probabilistic scenarios for the demands and return quantities with uncertainty sets for the carbon tax rates. The first stage decisions are investments to locate plants, warehouses, and collection centers; the second stage concerns the plan for distributing new and collecting returned products after realization of demands and returns, and the numbers of transportation units of various modes are the third stage decisions. The second- and third-stage decisions may adjust to the realization of the carbon tax rate. To demonstrate the use this paper's results, we present a highly simplified version of the robust counterpart assuming fixed facilities, only a single warehouse and a single retailer, and no reverse flows as follows:

$$\begin{aligned}
&Z_{RC} = \min_{t,x,u} u && \\
&\sum_{m\in\mathcal{M}} (h^m t^m + (\beta g^m + w\tilde{\alpha}(\xi)\beta\tau^m)x^m) \le u && \forall \xi \in \chi \\
&-(h^m t^m + (g^m \beta + w\tilde{\alpha}(\xi)\beta\tau^m)x^m) \le -L^m, && \forall \xi \in \chi, m \in \mathcal{M} \\
&wx^m \le W_m t^m && \forall m \in \mathcal{M} \\
&\sum_{m\in\mathcal{M}} x^m = d && \\
&x \in \mathbb{R}^{|\mathcal{M}|}_+,\ t \in \mathbb{R}^{|\mathcal{M}|}_+ &&
\end{aligned} \tag{44}$$

Here, $\mathcal{M}$ is the set of transportation modes indexed by m, t^m is the number of units of transportation capacity of type m with fixed cost of h^m and variable cost g^m, and x^m is the amount of products carried by transportation mode m. Let d be the customer demand to be satisfied and β be the distance from the warehouse to the retailer. In addition, L^m is a lower bound on the expenditure for mode m as determined by management and W_m is the weight limit of mode m where each product unit has weight w. Also, τ^m is the carbon emission factor of mode m and $\tilde{\alpha}$ is the uncertain carbon tax rate which belongs to a box uncertainty set $\tilde{\alpha}(\xi) = \bar{\alpha} + \xi\hat{\alpha}$ where $|\xi| \le 1$.

We assume that the transportation units t^m can adjust to the uncertain carbon tax $\tilde{\alpha}$ in the AARC formulation as follows: $t^m = \pi^{0m} + \xi\pi^{1m}$, where π^{0m} and π^{1m} are the new non-adjustable variables. Assume the values of parameters in this small instance with two choices of modes are as follow: $h = \begin{bmatrix} 10 \\ 10 \end{bmatrix}, g = \begin{bmatrix} 40 \\ 50 \end{bmatrix}, w = \beta = 1, d = 10, W = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \tau = \begin{bmatrix} 14 \\ 10 \end{bmatrix}$. If $L = \begin{bmatrix} 250 \\ 0 \end{bmatrix}$ and the uncertain carbon tax is $\tilde{\alpha}(\xi) = 2 + 0.3\xi$, the optimal solution of (44), found using the same process as in Example 1, is $x = \begin{bmatrix} 3.634 \\ 6.366 \end{bmatrix}, t = \begin{bmatrix} 1.82 \\ 1.59 \end{bmatrix}$ with $Z_{RC} = 761.2$. If t^1 is adjustable to ξ then, based on Remark 1, the only three constraints that include t^1 as a variable and have nonzero dual values $\lambda^* = (-1, -0.0247, -4.877)$ are:

$(i = 1)$ $h^1t^1 + h^2t^2 + (\beta g^1 + w(\bar{\alpha} + \xi\hat{\alpha})\beta\tau^1)x^1 + (\beta g^2 + w(\bar{\alpha} + \xi\hat{\alpha})\beta\tau^2)x^2 \leq u$

$(i = 2)$ $-\Big(h^1t^1 + (g^1\beta + w(\bar{\alpha} + \xi\hat{\alpha})\beta\tau^1)x^1\Big) \leq -L^1$

$(i = 3)$ $wx^1 \leq W_1t^1$

Considering the affine function $t^1 = \pi^0 + \xi\delta$, (26) becomes:

$$34.735 > |-1(-34.358 - 10\delta)| + |-0.0247(15.26 + 10\delta)| + |-9.753\delta| \quad (45)$$

Because inequality (45) holds with a right-hand-side value of 33.978 for $\delta = -3.43$, the affine adjustability reduces the optimal objective value. In fact, $Z_{AARC} = 760.5$.

However, if $L = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ then the optimal dual values of the previously binding constraints become $\lambda^* = (-1, 0, 5)$. Thus, conditions 1 and 2 of the proposition do not hold and no value of δ can be found for which inequality (26) holds. Therefore, in the absence of nonzero lower bounds on expenditures for each transportation mode, adjustability does not reduce the optimal cost.

5 Conclusion

In some situations, reformulating the RC of an uncertain linear program as an ARC or the more tractable AARC can provide a less conservative and costly solution. The proposition provided in this paper identifies conditions under which an adjustability gap exists; i.e., the objective values of the AARC and ARC are lower than that of the RC. The conditions stipulate that the RC formulation includes at least two constraints that are binding at the optimal RC solution for different values of the same uncertain parameter. In addition, a variable to be made adjustable appears in both constraints and is bounded from above by one constraint at one extreme of the uncertainty interval and bounded from below by the other at the opposite extreme of the uncertainty interval. One of these bounds is unfavorable for the objective. By relaxing this bound, adjustability increases the feasible region of the RC in a direction that lowers the objective value.

Besides providing insights into formulations where adjustability is beneficial, we show how RC formulations can be tested in small-scale instances using dual variables to identify whether an adjustability gap exists. Some small instances demonstrate different situations that may occur. Three contrived instances illustrate that, although the models are not covered by conditions previously established for $Z_{ARC} = Z_{RC}$, nevertheless Z_{AARC} is equal to Z_{RC}. Three additional examples derived from published applications of AARC demonstrate the use of this proposition to establish that $Z_{ARC} < Z_{RC}$.

In this paper we considered the fixed recourse case only. For uncertainty-affected recourse a similar approach would introduce more computational complexity that is a subject for future research. Another extension could be including more sophisticated uncertainty sets beyond box uncertainty. For instance, ellipsoidal uncertainty, used in many applications, allows interactions among uncertain parameters to be modeled.

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